Research Article

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Relating the super domination and 2-domination numbers in cactus graphs

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Abstract: A set $D \subseteq V(G)$ is a super dominating set of a graph G if for every vertex $u \in V(G)\setminus D$, there exists a vertex $v \in D$ such that $N(v)\setminus D = \{u\}$. The super domination number of G, denoted by $\gamma_{sp}(G)$, is the minimum cardinality among all super dominating sets of G. In this article, we show that if G is a cactus graph with $k(G)$ cycles, then $\gamma_{sp}(G) \le \gamma_2(G) + k(G)$, where $\gamma_2(G)$ is the 2-domination number of G. In addition, and as a consequence of the previous relationship, we show that if T is a tree of order at least three, then $\gamma_{sp}(T) \le \alpha(T) + s(T) - 1$ and characterize the trees attaining this bound, where $\alpha(T)$ and $s(T)$ are the independence number and the number of support vertices of T, respectively.

Keywords: super domination number, 2-domination number, cactus graphs, trees

MSC 2020: 05C69

1 Introduction

Throughout this article, we consider $G = (V(G), E(G))$ as a connected simple graph of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. Given a vertex v of G, $N(v)$ represents the *open neighbourhood* of v, i.e. $N(v) = \{u \in V(G) : uv \in E(G)\}$, and the *degree* of v is the cardinality of $N(v)$. A *leaf* of G is a vertex of degree 1. Moreover, a *support vertex* of G is a vertex adjacent to a leaf. The set of leaves is denoted by $\mathcal{L}(G)$, and the set of support vertices is denoted by $\mathcal{S}(G)$. The values $l(G)$ and $s(G)$ represent the number of leaves and the number of support vertices of G, respectively, i.e. $l(G) = |\mathcal{L}(G)|$ and $s(G) = |\mathcal{S}(G)|$. For any two vertices u and v of G, the *distance* $d(u, v)$ between u and v is the length of a shortest u–v path in G. The diameter of G is the maximum distance among pairs of vertices in G. A diametral path in G is a shortest path whose length equals the diameter of the graph. If D is a set of vertices of G, then the *open neighbourhood* of D is $N(D) = \cup_{v \in D} N(v)$. The graph obtained from G by removing all the vertices in $D \subseteq V(G)$ (and all the edges incident with a vertex in D) will be denoted by $G - D$. Analogously, the graph obtained from G by removing all the edges in $U \subseteq E(G)$ will be denoted by $G - U$. For any other terminology, we follow the books [1] and [2].

A set $D \subseteq V(G)$ is a *super dominating set* of G if for every vertex $u \in V(G)\setminus D$, there exists a vertex $v \in D$ such that $N(v)\setminus D = \{u\}$. The *super domination number* of G, denoted by $\gamma_{sp}(G)$, is the minimum cardinality among all super dominating sets of G. A $\gamma_{sp}(G)$-*set* is a super dominating set of G of cardinality $\gamma_{sp}(G)$. This concept was introduced by Lemańska et al. in [3] and studied further in [4–9]. Moreover, a set $S \subseteq V(G)$ is a 2-*dominating set* of G if $|N(v) \cap S| \ge 2$ for every vertex $v \in V(G)\setminus S$. The minimum cardinality among all 2-dominating sets of G, denoted by $\gamma_2(G)$, is the 2-*domination number* of G. A $\gamma_2(G)$-*set* is a 2-dominating set of G of cardinality $\gamma_2(G)$. For more information about 2-domination in graphs, we suggest the recent works [10–13].

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To illustrate the previous parameters, we consider the cactus graph (connected graph where each edge is contained in at most one cycle) shown in Figure 1.

In general, these two previous parameters are incomparable. For instance, for the double star $S_{1,n-3}$ (a tree obtained from a star of order $n-1$ by subdividing one edge exactly once) and the complete graph K_n of order $n \geq 4$, it follows that $\gamma_{sp}(S_{1,n-3}) = n-2 < n-1 = \gamma_2(S_{1,n-3})$ and $\gamma_{sp}(K_n) = n-1 > 2 = \gamma_2(K_n)$, respectively. In such a sense, it is desirable to find specific families of graphs for which these parameters are comparable. In this article, the previous problem is addressed for the case of cactus graphs. In particular, we first show that for trees, the super domination number is bounded above by the 2-domination number, and as a consequence, we show that if T is a tree of order at least three, then $\gamma_{sp}(T) \leq \alpha(T) + s(T) - 1$ and characterize the trees attaining this bound, where $\alpha(T)$ represents the independence number of T. Finally, we extended the first previous relationship for the family of cactus graphs. For instance, we show that if G is a cactus graph with $k(G)$ cycles, then $\gamma_{sp}(G) \leq \gamma_2(G) + k(G)$.

2 Trees

We begin with the following useful lemma.

Lemma 2.1. *If G is a graph obtained from any graph G' by adding a star $K_{1,r-1}$ ($r \geq 2$) with the support vertex v attached by an edge vu at a vertex $u \in V(G')$, then the following statements hold*:

(*i*) $\gamma_{sp}(G) \leq \gamma_{sp}(G') + r - 1$.

(*ii*) $\gamma_2(G') \leq \gamma_2(G) - r + 1$.

Proof. Let G be a graph obtained from G' by adding the star $K_{1,r-1}$ and the edge vu, where $v \in \mathcal{S}(K_{1,r-1})$ and $u \in V(G')$. Now, let D' be a $\gamma_{sp}(G')$-set and let $h \in V(K_{1,r-1}) \setminus \{v\}$. From D', we define a set $D \subseteq V(G)$ as follows:

$$D = \begin{cases} D' \cup V(K_{1,r-1}) \setminus \{h\} & \text{if } u \in D', \\ D' \cup V(K_{1,r-1}) \setminus \{v\} & \text{if } u \notin D'. \end{cases}$$

By the previous definition, and considering that D' is a $\gamma_{sp}(G')$-set, it is easy to deduce that D is a super dominating set of G. Hence, $\gamma_{sp}(G) \leq |D| = \gamma_{sp}(G') + r - 1$, which completes the proof of (i).

Now, we proceed to prove (ii). Let S be a $\gamma_2(G)$-set such that $|S \cap V(K_{1,r-1})|$ is minimum. By the fact that $\mathcal{L}(G) \subseteq S$ and the minimality of $|S \cap V(K_{1,r-1})|$, it follows that $V(K_{1,r-1}) \setminus \{v\} \subseteq S$ and $v \notin S$. This implies that $S \cap V(G')$ is a 2-dominating set of G' of cardinality $|S| - |V(K_{1,r-1}) \setminus \{v\}|$. Hence, $\gamma_2(G') \leq |S \cap V(G')| = \gamma_2(G) - r + 1$, which completes the proof. □

Next, we introduce some basic and well-known definitions commonly used in trees. A *rooted tree* T is a tree with a distinguished special vertex r, called the root. A *descendant* of a vertex v of T is a vertex $u \neq v$ such that the unique $r-u$ path contains v. The set of descendants of v is denoted by $D(v)$. The *maximal subtree* at v, denoted by T_v, is the subtree of T induced by $D(v) \cup \{v\}$.

Now, we are ready to show that, for any tree, the super domination number is bounded above by the 2-domination number.

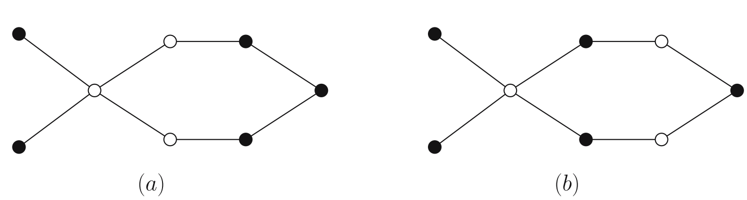

Figure 1: The set of black-coloured vertices forms a $\gamma_{sp}(G)$-set (a) and a $\gamma_2(G)$-set (b).

Theorem 2.2. *If T is a tree, then $\gamma_{sp}(T) \leq \gamma_2(T)$.*

Proof. Let T be a tree. We proceed by induction on the order of T. If $n(T) \in \{1, 2, 3\}$, then it is easy to check the relationship $\gamma_{sp}(T) \leq \gamma_2(T)$. These particular cases establish the base cases. We assume that $n(T) \geq 4$ and that $\gamma_{sp}(T') \leq \gamma_2(T')$ for each tree T' of order $n(T') < n(T)$. Let $u_1 \cdots u_d u_{d+1}$ be a diametral path in T, and consider that T is a rooted tree with root u_1. If $d = 1$, then T is a star and it is straightforward that $\gamma_{sp}(T) = \gamma_2(T)$, as desired. From now on, we assume that $d \geq 2$. Note that $u_d \in \mathcal{S}(T)$ and that the subgraph induced by $V(T_{u_d})$ is isomorphic to a star. Let $T' = T - V(T_{u_d})$. Observe that T' is a tree of order $n(T') < n(T)$. Hence, by Lemma 2.1(i), the induction hypothesis, and Lemma 2.1(ii) we deduce the following inequality chain:

$$\gamma_{sp}(T) \leq \gamma_{sp}(T') + |N(u_d) \cap \mathcal{L}(T)| \leq \gamma_2(T') + |N(u_d) \cap \mathcal{L}(T)| \leq \gamma_2(T),$$

which completes the proof. □

A set $I \subseteq V(T)$ is an *independent set* of T if the subgraph induced by I is isomorphic to a graph with no edges. The *independence number* of T, denoted by $\alpha(T)$, is the maximum cardinality among all independent sets of T. The next result was established by Chellali in 2006 [14].

Theorem 2.3. [14] *If T is a tree of order at least three, then $\gamma_2(T) \leq \alpha(T) + s(T) - 1$.*

As an immediate consequence of Theorems 2.2 and 2.3, it follows that $\gamma_{sp}(T) \leq \alpha(T) + s(T) - 1$ for any tree T of order at least three. In the following result, we characterize the trees attaining this previous relationship. Note that the next characterization guarantees the tightness of the bound given in Theorem 2.2.

Theorem 2.4. *If T is a tree of order at least three, then*

$$\gamma_{sp}(T) \leq \alpha(T) + s(T) - 1.$$

In addition, $\gamma_{sp}(T) = \alpha(T) + s(T) - 1$ if and only if T is a star.

Proof. The inequality $\gamma_{sp}(T) \leq \alpha(T) + s(T) - 1$ holds by Theorems 2.2 and 2.3. We proceed to prove the equivalence. It is straightforward that if T is a star, then $\gamma_{sp}(T) = \alpha(T) + s(T) - 1$. Now, we suppose that T is a tree different from a star. We only need to prove that $\gamma_{sp}(T) < \alpha(T) + s(T) - 1$. For this, we proceed by induction on the order of T. Observe that $n(T) \geq 4$. If $n(T) = 4$, then T is the path P_4 and it is straightforward that $\gamma_{sp}(T) = 2 < 3 = \alpha(T) + s(T) - 1$. This particular case establishes the base case. From now on, we assume that $n(T) \geq 5$ and that $\gamma_{sp}(T') < \alpha(T') + s(T') - 1$ for each tree T' different from a star such that $4 \leq n(T') < n(T)$. Let $u_1 \cdots u_d u_{d+1}$ be a diametral path in T, and consider that T is a rooted tree with root u_1. Let $T' = T - V(T_{u_d})$. If T' is a star, then T is either a double star or a tree obtained from a double star in which its central edge is subdivided once. In both cases, it follows that $\gamma_{sp}(T) = n(T) - 2 < n(T) - 1 = \alpha(T) + s(T) - 1$, as desired. From now on, we can assume that T' is a tree different from a star of order at least four. Since $n(T') < n(T)$, it follows that $\gamma_{sp}(T') < \alpha(T') + s(T') - 1$ by the induction hypothesis. Moreover, we observe that the subgraph induced by $V(T_{u_d})$ is isomorphic to a star. Hence, by Lemma 2.1-(i), the previous inequality, and the fact that $s(T') \leq s(T)$ and $\alpha(T) \geq \alpha(T') + |N(u_d) \cap \mathcal{L}(T)|$, we deduce the following inequality chain:

$$\gamma_{sp}(T) \leq \gamma_{sp}(T') + |N(u_d) \cap \mathcal{L}(T)| < \alpha(T') + s(T') - 1 + |N(u_d) \cap \mathcal{L}(T)| \leq \alpha(T) + s(T) - 1,$$

which completes the proof. □

3 Cactus graphs

A connected graph G is a cactus graph if each edge of G is contained in at most one cycle. If G does not contain any cycles, then it is a tree. Moreover, if G contains exactly one cycle, then it is a unicyclic graph. A particular case of unicyclic graph is the cycle graph C_n of order n. For this specific graph, we have that $\gamma_2(C_n) = \lceil n/2 \rceil$ (this value is easy to compute) and its super domination number was obtained in [3].

Proposition 3.1. [3] *For any integer $n \geq 3$,*

$$\gamma_{sp}(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \equiv 0, 3 \pmod 4, \\ \left\lceil \frac{n+1}{2} \right\rceil & \text{otherwise}. \end{cases}$$

Next, we introduce some necessary definitions given in [15]. Let C^{l_1} and C^{l_2} be two cycles in the cactus graph. We define

$$d\left(C^{l_1}, C^{l_2}\right) = \min\left\{d(u,v) : u \in V\left(C^{l_1}\right), v \in V\left(C^{l_2}\right)\right\}.$$

Let $u_1 \in V(C^{l_1})$ and $u_2 \in V(C^{l_2})$ be two vertices such that $d(u_1, u_2) = d(C^{l_1}, C^{l_2})$. Then, we call u_1 and u_2 *exit-vertices* of cycles C^{l_1} and C^{l_2}, respectively. A cycle is said to be an *outer cycle* if it has at most one exit-vertex. If a cactus graph is not a tree, then by definition it must contain at least one outer cycle. Figure 2 shows a cactus graph through which the previously exposed definitions are exemplified.

Now, we present the main result of this article.

Theorem 3.2. *If G is a cactus graph with $k(G)$ cycles, then*

$$\gamma_{sp}(G) \leq \gamma_2(G) + k(G).$$

Proof. To prove the result, we will use the function $g(G) = n(G) + m(G)$ defined on every finite graph G (recall that $n(G) = |V(G)|$ and $m(G) = |E(G)|$). Observe that g is strictly monotone in the sense that if G' is a proper subgraph of G, then $g(G') < g(G)$. Let G be a cactus graph. We proceed by induction on the value of function $g(G) \geq 1$. If $g(G) \leq 3$, then G is the path P_1 or P_2, and $\gamma_{sp}(G) \leq \gamma_2(G) + k(G)$, as required. These particular cases establish the base cases. We assume that $g(G) \geq 5$ (observe that there is no connected graph G with $g(G) = 4$) and that $\gamma_{sp}(G') \leq \gamma_2(G') + k(G')$ for each cactus graph G' such that $g(G') < g(G)$. If G is a tree, then by Theorem 2.2, the result follows. On the other side, if G is a cycle, then by Proposition 3.1 and the fact that $\gamma_2(C_n) = \lceil n/2 \rceil$, the required inequality holds. From now on, we consider that G is neither a tree nor a cycle. Thus, G contains at least one cycle as a proper subgraph. We denote with C^l an outer cycle of G, where $|V(C^l)| = l \geq 3$. Hence, C^l has at most one exit vertex. If it has one, let $u \in V(C^l)$ be the exit vertex of C^l. Otherwise, we consider that $u \in V(C^l)$ is a vertex of degree at least three. We now proceed with the following claims.

Claim I. If there exist two adjacent vertices v_i and v_{i+1} from $V(C^l) \setminus \{u\}$ with $|N(v_i)| = |N(v_{i+1})| = 2$, then $\gamma_{sp}(G) \leq \gamma_2(G) + k(G)$.

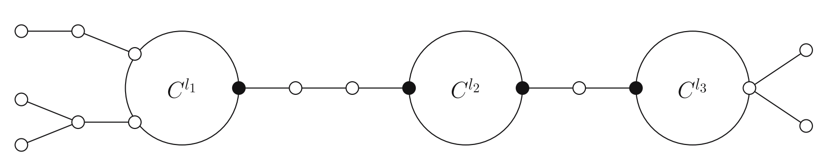

Figure 2: A cactus graph G with three cycles. The black-coloured vertices are exit vertices and the cycles C^{l_1} and C^{l_3} are outer cycles.

Proof of Claim I. Let S be a $\gamma_2(G)$-set. Observe that $S \cap \{v_i, v_{i+1}\} \neq \varnothing$. Without loss of generality, we can assume that $v_{i+1} \in S$. Next, we analyse the following two cases.

Case 1: $v_i \in S$. Let $G' = G - \{v_i v_{i+1}\}$. Since $v_i, v_{i+1} \in S$, it is straightforward that S is also a 2-dominating set of G'. Hence, $\gamma_2(G') \leq |S| = \gamma_2(G)$. Now, we observe that G' is a cactus graph with $k(G') = k(G) - 1$ and that $g(G') < g(G)$. This implies that $\gamma_{sp}(G') \leq \gamma_2(G') + k(G')$ by the induction hypothesis. Now, we proceed to prove that $\gamma_{sp}(G) \leq \gamma_{sp}(G') + 1$. Let D' be a $\gamma_{sp}(G')$-set. Next, we define a set $D \subseteq V(G)$ as follows:

$$D = \begin{cases} D' \cup \{v_i\} & \text{if } v_{i+1} \in D', \\ D' \cup \{v_{i+1}\} & \text{if } v_{i+1} \notin D'. \end{cases}$$

By the previous definition, and considering that D' is a $\gamma_{sp}(G')$-set and that $v_{i+1} \in \mathcal{L}(G')$, it is easy to deduce that D is a super dominating set of G. Hence, $\gamma_{sp}(G) \leq |D| \leq \gamma_{sp}(G') + 1$, as desired. Thus, by the previous inequalities, we obtain that

$$\gamma_{sp}(G) \leq \gamma_{sp}(G') + 1 \leq \gamma_2(G') + k(G') + 1 \leq \gamma_2(G) + k(G).$$

Case 2: $v_i \notin S$. Let $G' = G - \{v_i\}$. Observe that G' is a cactus graph with $k(G') = k(G) - 1$ and that $g(G') < g(G)$, which implies that $\gamma_{sp}(G') \leq \gamma_2(G') + k(G')$ by the induction hypothesis. In addition, we have that S is also a 2-dominating set of G'. Hence, $\gamma_2(G') \leq |S| = \gamma_2(G)$. Now, we proceed to prove that $\gamma_{sp}(G) \leq \gamma_{sp}(G') + 1$. Let D' be a $\gamma_{sp}(G')$-set. Observe that $v_{i+1} \in \mathcal{L}(G')$, and let $N(v_i)\backslash\{v_{i+1}\} = \{v_{i-1}\}$. Next, we define a set $D \subseteq V(G)$ as follows:

$$D = \begin{cases} D' \cup \{v_i\} & \text{if } D' \cap \{v_{i-1}, v_{i+1}\} \neq \varnothing, \\ D' \cup \{v_{i+1}\} & \text{if } D' \cap \{v_{i-1}, v_{i+1}\} = \varnothing. \end{cases}$$

By the previous definition, and considering that D' is a $\gamma_{sp}(G')$-set, we deduce that D is a super dominating set of G. Hence, $\gamma_{sp}(G) \leq |D| \leq \gamma_{sp}(G') + 1$, as desired. Thus, by the previous inequalities, we obtain that

$$\gamma_{sp}(G) \leq \gamma_{sp}(G') + 1 \leq \gamma_2(G') + k(G') + 1 = \gamma_2(G) + k(G).$$

Therefore, the proof of Claim I is complete.

By Claim I, we may henceforth assume that $|N(v_i)| + |N(v_{i+1})| \geq 5$ for any two adjacent vertices v_i and v_{i+1} in $V(C^l)\backslash\{u\}$. As a consequence, there exists at least one vertex from $V(C^l)\backslash\{u\}$ of degree at least three.

Claim II. If there exists a vertex $u_1 \in V(C^l)\backslash\{u\}$ such that $|N(u_1)| \geq 3$ and $N(u_1)\backslash V(C^l) \not\subseteq \mathcal{L}(G)$, then $\gamma_{sp}(G) \leq \gamma_2(G) + k(G)$.

Proof of Claim II. Since C^l is an outer cycle, there exists a subgraph of $G - (V(C^l)\backslash\{u_1\})$ that is isomorphic to a tree T rooted at u_1. Let $h(T) = \max\{d(u_1, y) : y \in V(T)\}$. Since $N(u_1)\backslash V(C^l) \not\subseteq \mathcal{L}(G)$, it follows that $h(T) \geq 2$. Let $u_1 \cdots wxy$ be the path in T such that $d(u_1, y) = h(T)$ (if $h(T) = 2$, then $u_1 = w$). Note that $x \in \mathcal{S}(T)$ and $N(x)\backslash\{w\} \subseteq \mathcal{L}(T)$. This implies that the subgraph induced by $V(T_x)$ is isomorphic to a star. Let $G' = G - V(T_x)$. Observe that G' is a cactus graph such that $g(G') < g(G)$. Hence, $\gamma_{sp}(G') \leq \gamma_2(G') + k(G')$ by the induction hypothesis. Thus, by Lemma 2.1-(i), the previous inequality, Lemma 2.1-(ii), and the fact that $k(G) = k(G')$, we obtain that

$$\gamma_{sp}(G) \leq \gamma_{sp}(G') + |N(x) \cap \mathcal{L}(T)| \leq \gamma_2(G') + k(G') + |N(x) \cap \mathcal{L}(T)| \leq \gamma_2(G) + k(G).$$

Therefore, the proof of Claim II is complete.

Let $u_1, \ldots, u_t \in V(C^l)\backslash\{u\}$ $(t \leq l - 1)$ be the vertices in C^l with degree at least three. By Claim II, we may also henceforth assume that $N(x)\backslash V(C^l) \subseteq \mathcal{L}(G)$ for every vertex $x \in \{u_1, \ldots, u_t\}$.

Claim III. If there exist two adjacent vertices u_i and u_{i+1} from $\{u_1, \ldots, u_t\}$, then $\gamma_{sp}(G) \leq \gamma_2(G) + k(G)$.

Proof of Claim III. Recall that $N(u_j)\setminus V(C^l) \subseteq \mathcal{L}(G)$ for any $j \in \{i, i+1\}$. Let $N(u_i) \cap \mathcal{L}(G) = \{h_1, \ldots, h_r\}$ and $(N(u_i) \cap V(C^l))\setminus\{u_{i+1}\} = \{v_{i-1}\}$. Create G' by removing the leaves adjacent to vertex u_i. Create G'' by removing the edge between v_{i-1} and u_i in G'. That is,

$$G' = G - \{h_1, \ldots, h_r\} \quad \text{and} \quad G'' = G' - \{v_{i-1}u_i\}.$$

Observe that G'' is a cactus graph with $k(G'') = k(G) - 1$ and $g(G'') < g(G)$. Thus, $\gamma_{sp}(G'') \leq \gamma_2(G'') + k(G'')$ by the induction hypothesis. Let D'' be a $\gamma_{sp}(G'')$-set. Since $u_i \in N(u_{i+1}) \cap \mathcal{L}(G'')$, it follows that $|N(u_{i+1}) \cap \mathcal{L}(G'') \cap D''| \geq |N(u_{i+1}) \cap \mathcal{L}(G'')| - 1 \geq 1$. Hence, we can assume, without loss of generality, that $u_i \in D''$. So, $D = D'' \cup \{h_1, \ldots, h_r\}$ is a super dominating set of G, which implies that $\gamma_{sp}(G) \leq |D| = \gamma_{sp}(G'') + r$. Now, we proceed to prove that $\gamma_2(G'') \leq \gamma_2(G) - r + 1$. Let S be a $\gamma_2(G)$-set. Observe that $\{h_1, \ldots, h_r\} \subseteq S$. Next, we define a set $S'' \subseteq V(G'')$ as follows:

$$S'' = \begin{cases} (S\setminus\{h_1, \ldots, h_r\}) \cup \{v_{i-1}\} & \text{if } u_i \in S, \\ (S\setminus\{h_1, \ldots, h_r\}) \cup \{u_i\} & \text{if } u_i \notin S. \end{cases}$$

From the previous definition, it follows that S'' is a 2-dominating set of G''. Hence, $\gamma_2(G'') \leq |S''| \leq |S| - r + 1 = \gamma_2(G) - r + 1$, as desired. By the previous inequalities, we obtain that

$$\gamma_{sp}(G) \leq \gamma_{sp}(G'') + r \leq \gamma_2(G'') + k(G'') + r \leq \gamma_2(G) + k(G).$$

Therefore, the proof of Claim III is complete.

By Claim III, we may also henceforth assume that if v_i and v_{i+1} are adjacent vertices in $V(C^l)\setminus\{u\}$, then $|\{v_i, v_{i+1}\} \cap \{u_1, \ldots, u_t\}| = 1$.

Claim IV. If there exist three consecutive vertices $v_{i-1}, v_i, v_{i+1} \in V(C^l)\setminus\{u\}$ such that $\{v_{i-1}, v_i, v_{i+1}\} \cap \{u_1, \ldots, u_t\} = \{v_i\}$, then $\gamma_{sp}(G) \leq \gamma_2(G) + k(G)$.

Proof of Claim IV. Let $G' = G - \{h_1, \ldots, h_r\}$, where $\{h_1, \ldots, h_r\} = N(v_i) \cap \mathcal{L}(G)$. Observe that G' is a cactus graph with $k(G') = k(G)$ and $g(G') < g(G)$. Hence, $\gamma_{sp}(G') \leq \gamma_2(G') + k(G')$ by the induction hypothesis. Now, we observe that if D' is a $\gamma_{sp}(G')$-set, then $D = D' \cup \{h_1, \ldots, h_r\}$ is a super dominating set of G. Hence, $\gamma_{sp}(G) \leq |D| = \gamma_{sp}(G') + r$. Moreover, let S be a $\gamma_2(G)$-set. It is straightforward that $\{h_1, \ldots, h_r\} \subseteq S$. We claim that $S' = S\setminus\{h_1, \ldots, h_r\}$ is a 2-dominating set of G'. If $v_i \in S$, then we are done. Now, we consider that $v_i \notin S$. Since v_{i-1} and v_{i+1} have degree two, and both are adjacent to v_i, it follows that $v_{i-1}, v_{i+1} \in S$. This implies that S' is a 2-dominating set of G', as desired. Hence, $\gamma_2(G') \leq |S'| = \gamma_2(G) - r$. Thus, and considering the previous inequalities, we obtain that

$$\gamma_{sp}(G) \leq \gamma_{sp}(G') + r \leq \gamma_2(G') + k(G') + r \leq \gamma_2(G) + k(G).$$

Therefore, the proof of Claim IV is complete.

By Claim IV, we also may henceforth assume that if $v \in \{u_1, \ldots, u_t\}$, then $u \in N(v)$.

Considering all the assumptions derived from the previous claims, it only remains to consider the cases where the outer cycle C^l is either C_3 or C_4 (in this last case, under the condition that $N(u) \cap V(C^l) = \{u_1, u_2\}$). We can assume that Claims I and III do not hold. So if $l = 3$, then $t = 1$. Similarly, if $l = 4$, then $t = 1$ or $t = 2$. However, we can assume that Claim IV does not hold, so $t = 2$.

Claim V. If C^l is either C_3 or C_4 (in the last case, under the condition that $N(u) \cap V(C^l) = \{u_1, u_2\}$), then $\gamma_{sp}(G) \leq \gamma_2(G) + k(G)$.

Proof of Claim V. Recall that $\{u_1, \ldots, u_t\}$ is the non-empty set of vertices in $V(C^l)\setminus\{u\}$ with degree at least three. If $l = 3$ (resp. $l = 4$), then $t = 1$ (resp. $t = 2$). In addition, we observe that $\{u_1, \ldots, u_t\} \subseteq N(u) \cap V(C^l)$. Let $G' = G - \{u_1, h_1, \ldots, h_r\}$, where $\{h_1, \ldots, h_r\} = N(u_1) \cap \mathcal{L}(G)$. Note that G' is a cactus graph with $k(G') = k(G) - 1$

and $g(G') < g(G)$. Thus, $\gamma_{sp}(G') \leq \gamma_2(G') + k(G')$ by the induction hypothesis. Moreover, from any $\gamma_{sp}(G')$-set D', the set $D = D' \cup \{u_1, h_1, \ldots, h_r\}$ is a super dominating set of G. Hence, $\gamma_{sp}(G) \leq |D| = \gamma_{sp}(G') + r + 1$.

Now, we proceed to prove that $\gamma_2(G') \leq \gamma_2(G) - r$. Let S be a $\gamma_2(G)$-set. Clearly, $\{h_1, \ldots, h_r\} \subseteq S$. Let $N(u_1) \cap V(C^l)\setminus\{u\} = \{v_2\}$. Observe that $|N(v_2)| = 2$. Now, let us define a set $S' \subseteq V(G')$ as follows:

$$S' = \begin{cases} S\setminus\{h_1, \ldots, h_r\} & \text{if } u_1 \notin S, \\ (S\setminus\{u_1, h_1, \ldots, h_r\}) \cup \{u, v_2\} & \text{if } u_1 \in S. \end{cases}$$

It is left to the reader to verify that $|S'| \leq |S| - r$. In addition, by the definition of S' and the fact that $|N(v_2)| = 2$, we can deduce that S' is a 2-dominating set of G'. Hence, $\gamma_2(G') \leq |S'| \leq |S| - r = \gamma_2(G) - r$, as desired. By the previous inequalities, we obtain that

$$\gamma_{sp}(G) \leq \gamma_{sp}(G') + r + 1 \leq \gamma_2(G') + k(G') + r + 1 \leq \gamma_2(G) + k(G).$$

Therefore, the proof of Claim V is complete, which concludes the proof. □

Observe that the bound given in Theorem 3.2 is sharp. For instance, it is attained when G is a star or a cycle graph C_n with $n \equiv 2 \pmod 4$.

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