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On metrization of fuzzy metrics and application to fixed point theory

Juan-José Miñana [a,b,*], Alexander Šostak [c], Oscar Valero [a,b]

[a] *Departament de Ciències Matemàtiques i Informàtica, Universitat de les Illes Balears, 07122 Palma de Mallorca (Illes Balears), Spain*
[b] *Institut d' Investigació Sanitària Illes Balears (IdISBa), Hospital Universitari Son Espases, 07120 Palma de Mallorca (Illes Balears), Spain*
[c] *Institute of Mahematics and CS, University of Latvia, Riga LV-1459, Latvia*

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Abstract

It is a well-known fact that the topology induced by a fuzzy metric is metrizable. Nevertheless, the problem of how to obtain a classical metric from a fuzzy one in such a way that both induce the same topology is not solved completely. A new method to construct a classical metric from a fuzzy metric, whenever it is defined by means of an Archimedean t-norm, has recently been introduced in the literature. Motivated by this fact, we focus our efforts on such a method in this paper. We prove that the topology induced by a given fuzzy metric M and the topology induced by the metric constructed from M by means of such a method coincide. Besides, we prove that the completeness of the fuzzy metric space is equivalent to the completeness of the associated classical metric obtained by the aforementioned method. Moreover, such results are applied to obtain fuzzy versions of two well-known classical fixed point theorems in metric spaces, one due to Matkowski and the other one proved by Meir and Keeler. Although such theorems have already been adapted to the fuzzy context in the literature, we show an inconvenience on their applicability which motivates the introduction of these two new fuzzy versions.

Keywords: Fuzzy metric space; Archimedean t-norm; Additive generator; Metrization; Completeness; Fixed point theorem

1. Introduction

In 1975, I. Kramosil and J. Michalek introduced a notion of fuzzy metric space in [20]. Later on, such a concept was slightly modified by A. George and P. Veeramani in [7]. Besides, in [7] it was proved that each fuzzy metric induces a topology which has as a base the induced family of open balls. Such a fact can be retrieved for fuzzy metrics in the sense of Kramosil and Michalek. Since then, several authors have addressed topological issues of both notions of fuzzy metric space. Indeed currently it is still a topic of research activity (see, for instance [1,2,12,10,17,29,32]).

In [14], a significant topological result for fuzzy metrics in the sense of George and Veeramani was proved by V. Gregori and S. Romaguera. Specifically, they showed that the class of metrizable topological spaces coincides with the class of fuzzy metrizable topological ones, i.e., those topological spaces $(X, \mathcal{T})$ whose topology can be induced

* Corresponding author.
E-mail addresses: jj.minana@uib.es (J.-J. Miñana), aleksandrs.sostaks@lu.lv (A. Šostak), o.valero@uib.es (O. Valero).

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by a fuzzy metric on X. Again, this result can be obtained for fuzzy metric spaces due to Kramosil and Michalek. Therefore, both notions of fuzzy metric are topologically equivalent to classical metrics. Nevertheless, we can find some differences between fuzzy metrics and metrics. For instance, there exist fuzzy metrics, in the sense of George and Veeramani, which are not completable (see [15,11]). A topic of interest, in which fuzzy metrics, in both senses, and metrics differ is fixed point theory. Many works have tackled the problem of adapting a classical metric fixed point theorem to the fuzzy framework, which is currently an intensive field of research (see, for instance, [5,9,18,25,26,31] and references therein). Unlike to the classical case, fuzzy adaptations of classical results usually need to assume some additional requirements to the completeness in order to ensure the existence of fixed point.

Going back to the metrizability of fuzzy metric spaces, an interesting issue is to establish a technique to obtain a metric from a given fuzzy one in such a way that both induced topologies are the same. A few works have provided advances in this direction. Hence particular techniques to construct a metric from a fuzzy one by means of the use of auxiliary functions were introduced in [28,3]. In the aforesaid references the both induced topologies were shown to be the same and, in addition, the completeness of the constructed metric and the completeness of the given fuzzy metric were shown to be equivalent. Such a fact allowed to prove new fixed point theorems, among them a Caristi fixed point type theorem, in the fuzzy context using the classical counterparts.

Another approach to the aforesaid issue has been recently stated in [24]. However, this new approach presents great differences from that developed in [28,3]. Now the metric is obtained using uniquely the own structure of the fuzzy metric space, whenever the t-norm that defines the fuzzy metric is Archimedean and, thus, it is generated by means of an additive generator (see Theorem 3.3 in Section 2). However, the relationship between the topologies induced by both metrics, the classical and the fuzzy one, has not been studied yet. Moreover, the relationship between the completeness of both metric structures remains open.

Inspired by the preceding facts, in this paper we continue the work exposed in [24] with the aim of providing a solution to the general posed problem about the equivalence between the topologies generated by the fuzzy metric and its induced classical metric. Specifically, we show that both topologies coincide when the method given in [24] (Theorem 3.3 in Section 2) is used. After that we show that completeness of the fuzzy metric space is equivalent to completeness of the constructed metric space. Moreover, we take the advantage of such an equivalence in order to apply the new results to fixed point theory in fuzzy metric spaces. Concretely, we have focused our attention on two fixed point theorems appeared in the literature. On the one hand, D. Mihet introduced the notion of fuzzy ψ-contractive mapping in [23] and he proved an existence of fixed point result for this kind of contractive mappings (see Theorem 4.2). Later V. Gregori and J.J. Miñana characterized the class of fuzzy ψ-contractive mappings with a unique fixed point (see Theorem 4.3). On the other hand, D. Zheng and P. Wang have recently provided the notion of fuzzy Meir-Keeler contractive mapping in [33] and, in addition, they have characterized such mappings with a unique fixed point (see Theorem 4.12). However, both characterizations present an inconvenience related to their applicability, since in some cases we need to know exactly the fixed point of the self-mapping in order to be able to check the condition ensuring the existence of such a fixed point (see Example 4.4). On account of this drawback, we strengthen both contractive conditions providing new fuzzy notions of both aforementioned contractions in order to obtain, on the basis of the previously exposed theory, the corresponding characterizations without such an inconvenience and (contrary to the results given in [28,3]) with the completeness as the only request on the fuzzy metric space (see Corollary 4.11). The proofs of these two characterizations follow from two fixed point theorems in the classical metric context due, on the one hand, to Matkowski ([21]) and, on the other hand, to Meir and Keeler ([22]).

The rest of the paper is organized as follows. In Section 2, we recall the basics of the theory of t-norms and fuzzy metrics which will play a central role in our study. Section 3 is devoted to the study of the relationship between the topology induced by a fuzzy metric and the topology induced by the metric constructed from it, as well as the completeness. Thus, we show that the aforesaid topologies are the same and that the completeness of both are equivalent. Finally, in Section 4 we prove two new fixed point theorems in fuzzy metric spaces by means of the use of the metrization results given in Section 3. Concretely, we prove existence of fixed point for two classes of contractive mappings, the so-called fuzzy ψ-contractive mappings and the fuzzy Meir-Keeler contractive mappings. The uniqueness of fixed point is also characterized in both cases.

2. Preliminaries

In this section we compile the basic notions about t-norms and fuzzy metrics that will be useful throughout the paper.

We begin recalling those concepts about t-norms. For a detailed treatment of the topic we refer the reader to [19].

Definition 2.1. A t-norm is a function $*: [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms are satisfied:

(T1) $x * y = y * x$; (Commutativity)
(T2) $x * (y * z) = (x * y) * z$; (Associativity)
(T3) $x * y \leq x * z$ whenever $y \leq z$; (Monotonicity)
(T4) $x * 1 = x$. (Boundary Condition)

If in addition, the t-norm $*$ is a continuous function on $[0,1]^2$ (the continuity is considered with respect to the usual topology on $[0,1]^2$), then it is said to be continuous. Moreover, a t-norm is called Archimedean if for each $x, y \in]0,1[$ there exists $n \in \mathbb{N}$ such that $x^{(n)} < y$, where $\mathbb{N}$ stands for the set of positive integer numbers and $x^{(n)}$ is defined as follows: $x^{(1)} = x$ and $x^{(n+1)} = x^{(n)} * x$ for all $n \in \mathbb{N}$. According to [19], the following proposition characterizes the class of continuous Archimedean t-norms.

Proposition 2.2. *A continuous t-norms $*$ is Archimedean if and only if it satisfies $x * x < x$ for each $x \in]0,1[$.*

Two paradigmatic and well-known examples of continuous Archimedean t-norms are the usual product $*_P$ and the Luckasievicz t-norm $*_L$, where $x *_P y = x \cdot y$ and $x *_L y = \max\{x + y - 1, 0\}$ for all $x, y \in [0,1]$. An example of continuous t-norm which is non-Archimedean is the minimum t-norm $\wedge$, i.e., $x \wedge y = \min\{x, y\}$ for all $x, y \in [0,1]$.

Another example of continuous Archimedean t-norm, which will play a crucial role in our subsequent discussion, is the so-called Hamacher product $*_H$, where $0 *_H 0 = 0$ and $x *_H y = \frac{xy}{x+y-xy}$ elsewhere.

Essential concepts, in our work, are the notions of additive generator and pseudo-inverse. Let us recall that, given a strictly decreasing continuous function $f: [0,1] \to [0,\infty]$ such that $f(1) = 0$, the pseudo-inverse $f^{(-1)}$ of f is the decreasing function $f^{(-1)}: [0,\infty] \to [0,1]$ defined as follows:

$$f^{(-1)}(y) = \begin{cases} f^{-1}(y), & \text{if } y < f(0) \\ 0, & \text{elsewhere} \end{cases}.$$

Moreover, given a t-norm $*$, a strictly decreasing continuous function $f_*: [0,1] \to [0,\infty]$ is said to be an additive generator of $*$ provided that $f_*(1) = 0$ and

$$x * y = f_*^{(-1)}(f_*(x) + f_*(y))$$

for all $x, y \in [0,1]$, where $f_*^{(-1)}$ is the pseudo-inverse of the additive generator f_*. In case the preceding equality holds, the t-norm $*$ is said to be generated by f_* or, equivalently, f_* is said to be an additive generator of $*$.

It must be stressed that when a t-norm $*$ is generated by means of a continuous additive generator, then the continuity of the t-norm $*$ and the continuity of the additive generator f_* are equivalent. Note that if a t-norm $*$ is generated by an additive generator f_*, then this additive generator is uniquely determined up to a non-zero positive multiplicative constant. It is known that each t-norm generated by an additive generator is Archimedean. Nevertheless, the converse of this assertion is not true in general (see [19, Example 3.21] for instance). The next celebrated result, the proof of which can be found in [19], states that continuous Archimedean t-norms are always generated by additive generators.

Theorem 2.3. *A t-norm $*$ is continuous and Archimedean if and only if there exists a continuous additive generator f_* which generates $*$.*

After recalling the necessary basics on t-norms, we focus now our attention on the notion of fuzzy metric space introduced by Kramosil and Michalek in [20].

Definition 2.4. A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty[$ satisfying, for all $x, y, z \in X$ and for all $s, t \in]0, \infty[$, the following axioms:

(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, y, t) = 1$ for each $t \in]0, \infty[$ if and only if $x = y$;
(KM3) $M(x, y, t) = M(y, x, t)$;
(KM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(KM5) The function $M_{x,y} :]0, \infty[\to [0, 1]$ is left-continuous, where $M_{x,y}(t) = M(x, y, t)$ for each $t \in]0, \infty[$.

If $(X, M, *)$ is a fuzzy metric space, then we say that $(M, *)$, or simply M if no confusion arises, is a fuzzy metric on X.

According to [7], given a fuzzy metric space $(X, M, *)$, the value $M(x, y, t)$ can be understood as the degree of similarity between x and y with respect to a positive real parameter t. Thus axiom **(KM1)** does not play any role from a fuzzy measurement point of view. Taking this fact into account, from now on, we will consider as a fuzzy metric a fuzzy set M on $X \times X]0, \infty[$ satisfying axioms **(KM2)−(KM5)** for all $t \in]0, \infty[$. Under such a consideration, it must be pointed out that the notion of fuzzy metric space given by George and Veeramani is a particular case of fuzzy metric in the sense of Kramosil and Michalek (see [7, Definition 2.4]).

It should be noted that given a fuzzy metric space $(X, M, *)$, for each continuous t-norm $\diamond$ such that $\diamond \leq *$ (i.e. $a \diamond b \leq a * b$, for all $a, b \in [0, 1]$) we have that $(X, M, \diamond)$ is also a fuzzy metric spaces.

The arguments similar to the ones given in [7] remain valid for showing that every fuzzy metric M on X generates a topology $\mathcal{T}_M$ on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t \in]0, \infty[\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t \in]0, \infty[$. Observe that $\mathcal{T}_M$ does not depend on the t-norm that defines the fuzzy metric $(X, M, *)$.

The following are two well-known examples of fuzzy metric spaces, both involving a classical metric in their definition.

Let (X, d) be a metric space. Define the fuzzy sets M_e and M_d, respectively, on $X \times X]0, \infty[$ as follows:

$$M_e(x, y, t) = e^{-\frac{d(x,y)}{t}}, \tag{1}$$

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}. \tag{2}$$

According to [7], $(X, M_e, \wedge)$ and $(X, M_d, \wedge)$ are fuzzy metric spaces and, hence, both M_e and M_d are fuzzy metrics on X. So $(X, M_e, *)$ and $(X, M_d, *)$ are also fuzzy metric spaces for each continuous t-norm $*$, since $* \leq \wedge$. Moreover, a well-known fact is that the topologies $\mathcal{T}_{M_e}$ and $\mathcal{T}_{M_d}$ coincide with the topology $\mathcal{T}_d$ on X induced by d. According to [7], the fuzzy metric M_d is called the *standard fuzzy metric* induced by d.

The same arguments to those given in [7] remain valid to prove the next proposition which characterizes convergent sequences in fuzzy metric spaces.

Proposition 2.5. *A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ converges to x with respect to $\mathcal{T}_M$ if and only if for each $r \in]0, 1[$ and $t \in]0, \infty[$ there exists $n_0 \in \mathbb{N}$ such that $M(x, x_n, t) > 1 - r$ for each $n \geq n_0$, or equivalently,* $\lim_n M(x_n, x, t) = 1$ *for all $t \in]0, \infty[$.*

We end the section recalling the notion of Cauchy sequence in fuzzy metric spaces as well as the concept of complete fuzzy metric space as introduced in [7].

Definition 2.6. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *Cauchy*, if for each $r \in]0, 1[$ and each $t \in]0, \infty[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geq n_0$ or, equivalently, $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t \in]0, \infty[$. A fuzzy metric is said to be *complete* if each Cauchy sequence is convergent.

Observe that the preceding notions were introduced in the framework of fuzzy metric spaces in the sense of George and Veeramani. However, the same concepts remain valid for the Kramosil and Michalek fuzzy metric spaces as they are defined in Definition 2.4.

3. On the metrization of a fuzzy metric: topology and completeness

As mentioned in Section 1, the topology induced by a fuzzy metric is metrizable. However, it is not clear which metric induces this topology in general. Motivated by this fact in Theorems 3.1 and 3.2 were developed some techniques in order to induce a metric from a fuzzy metric in such a way that the topology induced by the metric coincides with the topology induced by the fuzzy metric. Moreover, the completeness of the former one is guaranteed by the completeness of the last one, and vice-versa.

Below, we provide Theorem 3.1 as a slight modification of a result established by V. Radu in [28]. We include its proof for the sake of completeness. Observe that the original proof by Radu differs from the proof presented below. In order to state such a result, let us recall that, given a fuzzy metric space $(X, M, *)$, the function $M_{x,y} :]0, \infty[\to [0, 1]$ is increasing for all $x, y \in X$ (see, for instance, [6]).

Theorem 3.1. *Let $(M, *)$ be a fuzzy metric on X. Suppose that there exists a function $\mu : [0, \infty[\to [0, \infty[$ satisfying the following conditions:*

(R1) *μ is continuous on $[0, \infty[$;*
(R2) *$\mu(t) = 0 \Leftrightarrow t = 0$;*
(R3) *$\mu(t + s) \geq \mu(t) + \mu(s)$ for all $t, s \geq 0$;*
(R4) *$M(x, y, t) > 1 - \mu(t), M(y, z, s) > 1 - \mu(s) \Rightarrow M(x, z, t + s) > 1 - \mu(t + s)$.*

Then the function $d_{R_\mu} : X \times X \to [0, \infty[$ defined by

$$d_{R_\mu}(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - \mu(t)\},$$

is a metric on X. Moreover, the topologies $\mathcal{T}(d_{R_\mu})$ and $\mathcal{T}_M$ coincide. Furthermore, (X, d_{R_μ}) is complete if and only if $(X, M, *)$ is complete.

Proof. First, we will show that d_{R_μ} is a metric on X.

Obviously, if $x = y$ then $d_{R_\mu}(x, x) = \sup\{t \geq 0 : M(x, x, t) \leq 1 - \mu(t)\} = 0$ since $M(x, x, t) = 1$ for all $t > 0$. Conversely, if $d_{R_\mu}(x, y) = 0$ then $\sup\{t \geq 0 : M(x, y, t) \leq 1 - \mu(t)\} = 0$ and so $M(x, y, t) > 1 - \mu(t)$ for all $t > 0$. Due to the fact that μ is continuous on $[0, \infty[$ we conclude that

$$\lim_{t\to 0^+} M(x, y, t) \geq 1 - \lim_{t\to 0^+} \mu(t) = 1 - \mu(0) = 1.$$

Therefore $M(x, y, t) = 1$ for all $t > 0$, because of the function $M_{x,y}$ is increasing. It follows that $x = y$.

Clearly $d_{R_\mu}(x, y) = d_{R_\mu}(y, x)$. It remains to show the triangle inequality.

Let $x, y, z \in X$ and consider $\epsilon > 0$. Then, by definition of d_{R_μ} we have

$$M(x, y, d_{R_\mu}(x, y) + \epsilon) > 1 - \mu(d_{R_\mu}(x, y) + \epsilon) \text{ and } M(y, z, d_{R_\mu}(y, z) + \epsilon) > 1 - \mu(d_{R_\mu}(y, z) + \epsilon).$$

Then, by $(R4)$, we get

$$M(x, z, d_{R_\mu}(x, y) + d_{R_\mu}(y, z) + 2\epsilon) > 1 - \mu(d_{R_\mu}(x, y) + d_{R_\mu}(y, z) + 2\epsilon).$$

Again, by definition of d_{R_μ} we have that $d_{R_\mu}(x, z) \leq d_{R_\mu}(x, y) + d_{R_\mu}(y, z) + 2\epsilon$. Due to the fact that $\epsilon > 0$ is arbitrary we conclude that $d_{R_\mu}(x, z) \leq d_{R_\mu}(x, y) + d_{R_\mu}(y, z)$.

Hence, d_{R_μ} is a metric on X.

Now, we will see that for each $x, y \in X$ and $\epsilon > 0$ we have that

$$M(x, y, \epsilon) > 1 - \mu(\epsilon) \Leftrightarrow d_{R_\mu}(x, y) < \epsilon. \tag{3}$$

On the one hand, suppose that $M(x, y, \epsilon) > 1 - \mu(\epsilon)$. Then $d_{R_\mu}(x, y) \leq \epsilon$. However, if assume $d_{R_\mu}(x, y) = \epsilon$ then $M(x, y, t) \leq 1 - \mu(t)$ for each $t \in]0, \epsilon[$. Taking limits in the preceding inequality we get, by left-continuity of $M_{x,y}$ and continuity of μ, the following

$$M(x, y, \epsilon) = \lim_{t\to\epsilon^-} M(x, y, t) \leq 1 - \lim_{t\to\epsilon^-} \mu(t) = 1 - \mu(\epsilon),$$

a contradiction. Then, $d_{R_\mu}(x,y) < \epsilon$.

On the other hand, if $d_{R_\mu}(x,y) < \epsilon$ then, by definition of d_{R_μ}, we conclude that $M(x,y,\epsilon) > 1-\mu(\epsilon)$.

In the light of the preceding fact, it is not hard to check, using (3), that $\mathcal{T}(d_{R_\mu}) = \mathcal{T}_M$ and that (X, d_{R_μ}) is complete if and only if $(X, M, *)$ is complete. □

The next result was proved by F. Castro-Company, S. Romaguera and P. Tirado in [3].

Theorem 3.2. *Let $(M,*)$ be a fuzzy metric on X. Suppose that there exists a function $\alpha : [0,\infty[\to [0,\infty[$ satisfying the following conditions:*

(C1) *α is strictly increasing on $[0,1]$;*
(C2) *$0 < \alpha(t) \le t$ for all $t \in]0,1[$ and $\alpha(t) > 1$ for all $t > 1$;*
(C3) *$(1-\alpha(t)) * (1-\alpha(s)) \ge 1-\alpha(t+s)$ for all $t, s \in [0,1]$.*

Then the function $d_\alpha : X \times X \to [0,\infty[$ defined by

$$d_\alpha(x,y) = \sup\{t \ge 0 : M(x,y,t) \le 1-\alpha(t)\},$$

is a metric on X. If, in addition, the function α is left-continuous on $]0,1]$, then the topologies $\mathcal{T}(d_\alpha)$ and $\mathcal{T}_M$ coincide. Moreover, (X, d_α) is complete if and only if $(X, M, *)$ is complete.

It must be stressed that the construction given in the preceding results depends on the existence of auxiliary functions. Inspired by this fact, J.-J. Miñana and O. Valero introduced a new technique which allows to induce a metric from a fuzzy metric in such a way that the construction depends only on the structural aspects of the fuzzy metric. This construction is described in the following theorem.

Theorem 3.3. *Let $(M,*)$ be a fuzzy metric on X, where $*$ is a continuous Archimedean t-norm. Then the function $d_{M,f_*} : X \times X \to [0,\infty]$ defined as*

$$d_{M,f_*}(x,y) = \sup\{t \in]0, f_*(0)[: M(x,y,t) \le f_*^{(-1)}(t)\},$$

is an extended metric on X, where f_* is an additive generator of $*$.

Observe that the metrics generated via Theorem 3.3 can take the value ∞ and that such metrics are known as extended metrics in [4] (generalized metrics in [27]). Obviously the notions of convergent sequence, Cauchy sequence and completeness are defined in the same way to the classical case.

Clearly the preceding construction presents the advantage of involving only the pseudo-inverse of an additive generator of the t-norm $*$ under consideration. Despite the aforesaid benefit, it must be pointed out that such a method works only for Archimedean continuous t-norms and this excludes, for instance, the minimum t-norm.

In contrast with the construction given in Theorems 3.1 and 3.2, the problems about the coincidence of the both topologies and the equivalence of their completeness equivalence for the construction given in Theorem 3.3 remain open. Motivated by this fact, the main target of this section is to provide an answer to such problems.

To this end, we want to stress first that the function d_{M,f_*} defined in Theorem 3.3 is given by means of both, the fuzzy metric M and an additive generator f_* of the continuous Archimedean t-norm $*$. As mentioned in Section 2, the additive generator of a (continuous) Archimedean t-norm is uniquely determined up to a non-zero positive multiplicative constant. So, the additive generator that defines a t-norm is not unique. Thus, one can wonder if the metric constructed in Theorem 3.3 can be different in case we consider different additive generators of the same continuous Archimedean t-norm. The next example shows that the answer to the posed question is affirmative.

Example 3.4. Let (X,d) be a metric space and consider the fuzzy metric space $(X, M_e, *_P)$ introduced in Section 2 (see expression (1)). Next, consider the two additive generators f_{*_P} and g_{*_P} of the continuous Archimedean t-norm $*_P$ given, for each $x \in [0,1]$, by

$$f_{*_P}(x) = -\log(x) \quad \text{and} \quad g_{*_P}(x) = -2\log(x).$$

Of course we are assuming that $f_{*P}(0) = g_{*P}(0) = \infty$.

Clearly, the pseudo-inverses of f_{*P} and g_{*P} are given, for each $x \in [0, \infty]$, by $f_{*P}^{(-1)}(x) = e^{-x}$ and $g_{*P}^{(-1)}(x) = e^{\frac{-x}{2}}$, respectively. Thus one can verify that

$$d_{M_e, f_{*P}}(x, y) = \sqrt{d(x, y)} \quad \text{and} \quad d_{M_e, g_{*P}}(x, y) = \sqrt{2d(x, y)}$$

for each $x, y \in X$. Therefore $d_{M_e, f_{*P}}(x, y) \neq d_{M_e, g_{*P}}(x, y)$ for each $x, y \in X$ whenever $x \neq y$. It follows that the metric $d_{M_e, f_{*P}}$ is different from the metric $d_{M_e, g_{*P}}$. Nevertheless, one can check easily that $d_{M_e, f_{*P}}$, $d_{M_e, g_{*P}}$ and M_e generate the same topology which coincides with the topology induced by d.

In view of the fact that the above example shows that the metric d_{M, f_*} depends on the additive generator f_* of t-norm $*$ under consideration, we introduce the following notion.

Definition 3.5. Let $(X, M, *)$ be a fuzzy metric space, where $*$ is a continuous Archimedean t-norm, and let f_* be an additive generator of $*$. The function $d_{M, f_*} : X \times X \to [0, \infty]$ defined, for each $x, y \in X$, by

$$d_{M, f_*}(x, y) = \sup\{t \in]0, f_*(0)[: M(x, y, t) \leq f_*^{(-1)}(t)\},$$

will be called *the metric deduced from M and f_**.

In order to achieve the goals of this section, we will study first the relationship between the topology induced by a fuzzy metric $(M, *)$ on a set X, whenever $*$ is a continuous Archimedean t-norm, and the topology induced by the metric d_{M, f_*} deduced (in the sense of Definition 3.5) from M and f_*, where f_* is an additive generator of $*$.

The following proposition will be helpful to get the aforementioned relationship.

Proposition 3.6. *Let $(X, M, *)$ be a fuzzy metric space, where $*$ is a continuous Archimedean t-norm, and let f_* be an additive generator of $*$. Then, for each $x \in X$, the following assertions hold:*

1) *For each $r \in]0, 1[$ and $t \in]0, \infty[$ there exists $\epsilon \in]0, \infty[$ such that*

$$B_{d_{M, f_*}}(x; \epsilon) \subseteq B_M(x, r, t).$$

2) *For each $\epsilon \in]0, \infty[$ there exist $r \in]0, 1[$ and $t \in]0, \infty[$ such that*

$$B_M(x, r, t) \subseteq B_{d_{M, f_*}}(x; \epsilon).$$

Proof. 1). Fix $x \in X$. Let $r \in]0, 1[$ and $t \in]0, \infty[$. Take $\epsilon = \min\{t, f_*(1 - r)\}$. We assert that $B_{d_{M, f_*}}(x; \epsilon) \subseteq B_M(x, r, t)$. Indeed, let $y \in B_{d_{M, f_*}}(x; \epsilon)$. Then, $d_{M, f_*}(x, y) < \epsilon$. By definition of d_{M, f_*} we have that $M(x, y, \epsilon) > f_*^{(-1)}(\epsilon)$, so

$$M(x, y, t) \geq M(x, y, \epsilon) > f_*^{(-1)}(\epsilon) \geq f_*^{(-1)}(f_*(1 - r)) = 1 - r.$$

Therefore, $y \in B_M(x, r, t)$ as we claimed.

2). Fix $x \in X$. Let $\epsilon \in]0, \infty[$. We distinguish two cases:

Case 1. Suppose that $\epsilon \in]0, f_*(0)[$. Take $r = 1 - f_*^{(-1)}(\epsilon)$ and $t = \epsilon \in]0, \infty[$. By definition of $f_*^{(-1)}$ and since $\epsilon \in]0, f_*(0)[$ we have that $r \in]0, 1[$. We will see that, in such case, $B_M(x, r, t) \subseteq B_{d_{M, f_*}}(x; \epsilon)$.
Let $y \in B_M(x, r, t)$. Then,

$$M(x, y, \epsilon) = M(x, y, t) > 1 - r = f_*^{(-1)}(\epsilon).$$

Thus, by definition of d_{M, f_*}, we deduce that $d_{M, f_*}(x, y) < \epsilon$ and so $y \in B_{d_{M, f_*}}(x; \epsilon)$.
Case 2. Suppose that $\epsilon \in [f_*(0), \infty[$. Take an arbitrary $r \in]0, 1[$ and $t = \epsilon \in]0, \infty[$. Then, it is easy to verify that $B_M(x, r, t) \subseteq B_{d_{M, f_*}}(x; \epsilon)$. □

From the preceding proposition we obtain the following immediate corollary.

Corollary 3.7. *Let $(X, M, *)$ be a fuzzy metric space, where $*$ is a continuous Archimedean t-norm, and let f_* be an additive generator of $*$. Then $\mathcal{T}(d_{M,f_*}) = \mathcal{T}_M$.*

In the light of Corollary 3.7 we have that, given a fuzzy metric space $(X, M, *)$, where $*$ is a continuous Archimedean t-norm, the convergent sequences with respect to $\mathcal{T}_M$ coincide with the convergent sequences with respect to $\mathcal{T}(d_{M,f_*})$. Under such observation, it seems natural to wonder whether Cauchy sequences in $(X, M, *)$ coincide with Cauchy sequences in (X, d_{M,f_*}). The next proposition provides an affirmative response to the posed question.

Proposition 3.8. *Let $(X, M, *)$ be a fuzzy metric space, where $*$ is a continuous Archimedean t-norm, and let f_* be an additive generator of $*$. If $\{x_n\}$ is a sequence in X, then the following assertions are equivalent:*

1) *$\{x_n\}$ is Cauchy in $(X, M, *)$.*
2) *$\{x_n\}$ is Cauchy in (X, d_{M,f_*}).*

Proof. 1) $\Rightarrow$ 2). Let $\{x_n\}$ be a Cauchy sequence in $(X, M, *)$. Then, for each $r \in]0, 1[$ and $t \in]0, \infty[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for each $n, m \geq n_0$. We will show that for each $\epsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that $d_{M,f_*}(x_n, x_m) < \epsilon$ for all $n, m \geq n_1$.

Fix $\epsilon > 0$. The case $\epsilon \geq f_*(0)$ is obvious and for this reason we assume that $\epsilon \in]0, f_*(0)[$. Consider $r = 1 - f_*^{(-1)}(\epsilon) \in]0, 1[$ and $t = \epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geq n_0$.

Again, take $n_1 = n_0$ and suppose $n, m \geq n_1$. Then

$$M(x_n, x_m, \epsilon) = M(x_n, x_m, t) > 1 - r = f_*^{(-1)}(\epsilon)$$

and so $d_{M,f_*}(x_n, x_m) < \epsilon$. Thus we deduce that $d_{M,f_*}(x_n, x_m) < \epsilon$ for all $n, m \geq n_1$.

2) $\Rightarrow$ 1). Let $\{x_n\}$ be a Cauchy sequence in (X, d_{M,f_*}). We will prove that for each $r \in]0, 1[$ and $t \in]0, \infty[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geq n_0$.

Fix $r \in]0, 1[$ and $t \in]0, \infty[$. Take $\epsilon \leq \min\{f_*(1 - r), t\}$. Note that $\epsilon \in]0, f_*(0)[$. It follows that $f_*^{(-1)}(\epsilon) \geq 1 - r$. Due to the fact that $\{x_n\}$ is a Cauchy sequence in (X, d_{M,f_*}) we conclude that there exists $n_0 \in \mathbb{N}$ such that $d_{M,f_*}(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$. Then, by definition of d_{M,f_*}, we have $M(x_n, x_m, \epsilon) > f_*^{(-1)}(\epsilon)$ for all $n, m \geq n_0$.

Let $n, m \geq n_0$. Then

$$M(x_n, x_m, t) \geq M(x_n, x_m, \epsilon) > f_*^{(-1)}(\epsilon) \geq 1 - r.$$

Since $n, m \geq n_0$ are arbitrary we deduce that $\{x_n\}$ is a Cauchy sequence in $(X, M, *)$. $\square$

An immediate consequence of the above proposition and Corollary 3.7 is the next one.

Corollary 3.9. *Let $(X, M, *)$ be a fuzzy metric space, where $*$ is a continuous Archimedean t-norm, and let f_* be an additive generator of $*$. Then the following statements are equivalent:*

1) *$(X, M, *)$ is complete.*
2) *(X, d_{M,f_*}) is complete.*

4. Applications to fixed point theory in fuzzy metric spaces

In this section we take advantage of the theory exposed in Section 3 in order to present its applications to fixed point theory in fuzzy metric spaces. Concretely, we get a new result on the existence of fixed point for fuzzy ψ-contractive mappings in the sense of Mihet [23]. The proof of this result is base on a fixed point given by Matkowski in [21] established in metric spaces Conditions that guarantee the uniqueness of fixed point for this type of contractive mappings are also provided. Moreover, the existence and the uniqueness of fixed point for fuzzy Meir-Keeler contractive mappings in the sense of Zheng and Wang ([33]) are discussed. The proof of these new fixed point results depend on a fixed point result established in metric spaces given by Meir and Keeler in [22].

The aforesaid new fixed point results are obtained (in contrast with similar results from [23] and [33]) without demanding any extra condition on the fuzzy metric. To this end, as we will show later, we need to strengthen the contractive conditions used in [23] and [33].

4.1. Fuzzy ψ-contractive mappings

In 2008, Mihet in [23] introduced a general concept of contractivity in the fuzzy setting. Such a concept generalized the notion of fuzzy contractive mapping formerly given by Gregori and Sapena in [16]. According to [23], given a fuzzy metric space $(X, M, *)$, we will say that a self-mapping $T : X \to X$ is a *fuzzy ψ-contractive mapping* if there exists a non-decreasing continuous function $\psi : [0, 1] \to [0, 1]$ such that $\psi(s) > s$ for each $s \in]0, 1[$ satisfying, for all $x, y \in X$ and for all $t \in]0, \infty[$ such that $M(x, y, t) > 0$, the following:

$$M(T(x), T(y), t) \geq \psi(M(x, y, t)).$$

From now on, the class of mappings ψ satisfying all conditions in the definition of ψ-contractive mapping will be denoted by Ψ.

After introducing the previous concept, Mihet proved the following fixed point theorem in [23]. In order to state it, let us recall the following interesting class of fuzzy metrics, which was introduced in [13], that plays a crucial role in the aforesaid theorem.

Definition 4.1. Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric M (or the fuzzy metric space $(X, M, *)$) is said to be *strong* if, in addition to axioms **(KM2)−(KM5)**, it satisfies, for each $x, y, z \in X$ and each $t \in]0, \infty[$, the following one:

(KM4′) $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$.

The promised result given by Mihet can be stated as follows:

Theorem 4.2. *Let $(X, M, *)$ be a complete strong fuzzy metric space and let $T : X \to X$ be a fuzzy ψ-contractive mapping. If there exists $x \in X$ such that $M(x, T(x), t) > 0$ for each $t \in]0, \infty[$, then T has a fixed point.*

In the light of the fact that the previous result does not warrant the uniqueness of a fixed point, we provide a characterization of the existence and uniqueness of fixed point for ψ-contractive mappings in fuzzy metric spaces. It is based on [8, Theorem 3.3] which was proved for fuzzy metrics in the sense of George and Veeramani.

Theorem 4.3. *Let $(X, M, *)$ be a complete fuzzy metric space and let $T : X \to X$ be a fuzzy ψ-contractive mapping. Then the following assertions hold:*

1) *T has a fixed point if and only if there exists $x \in X$ such that $\bigwedge_{t>0} M(x, T(x), t) > 0$.*
2) *If T has a fixed point, then the fixed point of T is unique if and only if, for every fixed point $x, y \in X$, $M(x, y, t) > 0$ for all $t \in]0, \infty[$.*

Proof. 1). Follows the same arguments to those used in [8, Theorem 3.3] to demonstrate the existence of fixed point.

2) If the fixed point of T is unique, then $M(x, x, t) = 1$ for all $t > 0$ and, the desired conclusion follows. Suppose that there exist two fixed points $x, y \in X$ with $x \neq y$. Then there exists $s > 0$ such that $0 < M(x, y, s) < 1$. Hence we have that

$$M(x, y, s) = M(T(x), T(y), s) \geq \psi(M(x, y, s)) > M(x, y, s),$$

which is a contradiction. Therefore the fixed point of T is unique. □

Despite Theorem 4.3 characterizes those ψ-contractive mappings that have a unique fixed point, its applicability to ensure the existence of fixed point sometimes may be too limited. The justification for this assertion is given by the fact that there are instances of ψ-contractive mappings for which the application of Theorem 4.3 for guaranteeing the existence of fixed point requires to know exactly the point in which the self-mapping has such a fixed point.

Example 4.4 gives an instance of such a situation.

Example 4.4. Consider the discrete fuzzy metric space $(\mathbb{R}, M, *_P)$, where M is defined on $\mathbb{R} \times \mathbb{R} \times]0, \infty[$ by $M(x, x, t) = 1$ for all $t \in]0, \infty[$ and

$$M(x, y, t) = \begin{cases} M(x, y, t) = 0 & \text{if } t \in]0, 1] \text{ and } x \neq y \\ M(x, y, t) = 1 & \text{if } t \in]1, \infty[\text{ and } x \neq y \end{cases}.$$

According to [23], the fuzzy metric space $(\mathbb{R}, M, *_P)$ is strong and complete. Moreover, every self-mapping $T : \mathbb{R} \to \mathbb{R}$ is a fuzzy ψ-contractive mapping for any $\psi \in \Psi$. Consider the self-mapping $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = \frac{x}{4}$ for all $x \in \mathbb{R}$. Then T is a fuzzy ψ-contractive mapping. Clearly, the unique fixed point of T is $x = 0$. Moreover, $\bigwedge_{t>0} M(x, T(x), t) = 0$ for all $x \neq 0$ and $\bigwedge_{t>0} M(0, T(0), t) = 1$.

In conclusion, in order to guarantee the existence of a fixed point of the mapping T one requires to know that its fixed point is exactly $x = 0$, which is an important limitation for the application of Theorem 4.3.

To avoid the exposed inconvenience, we propose to introduce a new class of fuzzy contractive mappings. Such a class consists in a modified version of the class of the fuzzy ψ-contractive mappings introduced by Mihet.

Definition 4.5. Let $(X, M, *)$ be a fuzzy metric space such that $*$ is a continuous Archimedean t-norm and let $T : X \to X$ be a self-mapping. We will say that T is a *fuzzy ψ_{f_*}-contractive mapping* if there exists a function $\varphi : [0, f_*(0)] \to [0, f_*(0)]$ which is strictly increasing, right-continuous and satisfying $\lim_n \varphi^n(t) = 0$ for each $t \in]0, f_*(0)[$ such that, for each $x, y \in X$ and $t \in]0, f_*(0)[$ satisfying $M(x, y, t) > 0$, the following is held:

$$M(T(x), T(y), \varphi(t)) \geq \psi_{f_*}(M(x, y, t)),$$

where $\psi_{f_*}(s) = \left(f_*^{(-1)} \circ \varphi \circ f_*\right)(s)$ for each $s \in [0, 1]$, where f_* is an additive generator of $*$.

With the aim of stating an example of fuzzy ψ_{f_*}-contractive mappings, which motivates the introduction of the previous concept, let us prove the proposition below.

Proposition 4.6. *Let (X, d) be a metric space and let M_d be the standard fuzzy metric induced by d. Then $(\tilde{M}_d, *_H)$ is a fuzzy metric on X, where $\tilde{M}_d : X \times X \times]0, \infty[$ is given by*

$$\tilde{M}_d(x, y, t) = \begin{cases} M_d(x, y, t), & \text{if } 0 < t \leq d(x, y) \\ M_d(x, y, 2t), & \text{if } t > d(x, y) \end{cases}.$$

Proof. Let $x, y \in X$ and $t, s \in]0, \infty[$. We have to show that $\tilde{M}_d$ satisfies axioms **(KM2)** and **(KM4)** since the validity of axioms **(KM3)** and **(KM5)** holds by definition of $\tilde{M}_d$.

(KM2) Obviously, if $x = y$ then $\tilde{M}_d(x, y, t) = 1$ for each $t \in]0, \infty[$. Now, suppose that $\tilde{M}_d(x, y, t) = 1$ for each $t \in]0, \infty[$. Then $\frac{2t}{2t+d(x,y)} = 1$ for each $t > d(x, y)$ and, hence, $d(x, y) = 0$. Therefore $x = y$.

(KM4) We distinguish two possible cases.

Case 1: $t + s > d(x, z)$. In this case, $\tilde{M}_d(x, z, t + s) = M_d(x, z, 2t + 2s)$. Whence we have that

$$\tilde{M}_d(x, z, t + s) = M_d(x, z, 2t + 2s) \geq M_d(x, y, 2t) *_H M_d(y, z, 2s) \geq$$
$$\geq \tilde{M}_d(x, y, t) *_H \tilde{M}_d(y, z, s).$$

Case 2: $0 < t + s \leq d(x, z)$. Now, $\tilde{M}_d(x, z, t + s) = M_d(x, z, t + s)$. Notice that $t > d(x, y)$ and $s > d(y, z)$ cannot hold simultaneously. Indeed, the fact that $t > d(x, y)$ and $s > d(y, z)$ implies that $d(x, y) + d(y, z) < t + s \leq d(x, z)$, which is a contradiction. Next we consider three possible cases. If $0 < t \leq d(x, y)$ and $0 < s \leq d(y, z)$, then

$$\tilde{M}_d(x, z, t + s) = M_d(x, y, t + s) \geq M_d(x, y, t) *_H M_d(y, z, s) =$$

$$= \tilde{M}_d(x, y, t) *_H \tilde{M}_d(y, z, s).$$

It remains to consider the cases in which either $0 < t \le d(x, y)$ and $s > d(y, z)$ or $t > d(x, y)$ and $0 < s \le d(y, z)$. We focus on the first one, since the other is proved analogously.

In such a case, we have that $\tilde{M}_d(x, y, t) *_H \tilde{M}_d(y, z, s) = M_d(x, y, t) *_H M_d(y, z, 2s)$. Then we need to show that the following inequality is fulfilled:

$$\frac{t+s}{t+s+d(x,z)} \ge \frac{t}{t+d(x,y)} *_H \frac{2s}{2s+d(y,z)} = \frac{2ts}{2ts+2sd(x,y)+td(y,z)}.$$

The preceding inequality is held if and only if the next one is so

$$(t+s)(2ts+2sd(x,y)+td(y,z)) \ge 2ts(t+s+d(x,z)). \tag{4}$$

Taking into account that we have supposed that $0 < t \le d(x, y)$ and $s > d(y, z)$, then $s^2 d(x, y) > std(y, z)$. Therefore,

$$\begin{aligned}
&(t+s)(2ts+2sd(x,y)+td(y,z)) \ge (t+s)(2ts+2sd(x,y)) = \\
&2t^2s+2ts^2+2tsd(x,y)+2s^2d(x,y) > \\
&2t^2s+2ts^2+2tsd(x,y)+2tsd(y,z) = 2t^2s+2ts^2+2ts(d(x,y)+d(y,z)) \ge \\
&\ge 2t^2s+2ts^2+2tsd(x,z) = 2ts(t+s+d(x,z)).
\end{aligned}$$

Thus, inequality (4) holds and we obtain that $\tilde{M}_d(x, z, t+s) \ge \tilde{M}_d(x, y, t) *_H \tilde{M}_d(y, z, s)$ in the supposed case.

Hence, since $\tilde{M}_d(x, z, t+s) \ge \tilde{M}_d(x, y, t) *_H \tilde{M}_d(y, z, s)$ in all possible cases we conclude that **(KM4)** is satisfied. □

The following example gives the promised instance of a fuzzy ψ_{f_*}-contractive mapping.

Example 4.7. Consider the fuzzy metric space $(\mathbb{R}, \tilde{M}_{d_E}, *_H)$, where $\tilde{M}_{d_E}$ is the fuzzy metric on $\mathbb{R}$ constructed in Proposition 4.6 and induced by the metric $d_E(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Let $T : \mathbb{R} \to \mathbb{R}$ be the self-mapping given by $T(x) = \frac{x}{4}$ for each $x \in \mathbb{R}$. We will see that T is a fuzzy $\psi_{f_{*H}}$-contractive mapping, where f_{*H} is the additive generator of the continuous Archimedean t-norm $*_H$ given by $f_{*H}(a) = \frac{1-a}{a}$, for each $a \in [0, 1]$. Observe that $f_{*H}(0) = \infty$ and that the pseudo-inverse of f_{*H} is given by $f_{*H}^{(-1)}(b) = \frac{1}{1+b}$ for each $b \in [0, \infty]$, where $\frac{1}{\infty}$ is assumed to be 0.

Consider $\varphi : [0, \infty] \to [0, \infty]$ given by $\varphi(t) = \frac{t}{2}$ for each $t \in [0, \infty[$ and $\varphi(\infty) = \infty$. Obviously, φ is strictly increasing, right-continuous and $\lim_n \varphi^n(t) = 0$ for each $t \in]0, \infty[$.

Observe that, for each $s \in]0, 1]$, we have that

$$\psi_{f_{*H}}(s) = \left(f_{*H}^{(-1)} \circ \varphi \circ f_{*H}\right)(s) = \left(f_{*H}^{(-1)} \circ \varphi\right)\left(f_{*H}(s)\right) = f_{*H}^{(-1)}\left(\varphi\left(\frac{1-s}{s}\right)\right) = f_*^{(-1)}\left(\frac{1-s}{2s}\right) = \frac{1}{1+\frac{1-s}{2s}} = \frac{s}{s+\frac{1}{2}(1-s)}.$$

Notice, in addition, that $\psi_{f_{*H}}(0) = 0$.

Let $x, y \in \mathbb{R}$ and $t > 0$. We distinguish three cases:

Case 1. Suppose that $0 < \frac{t}{2} \le d_E(T(x), T(y))$. Taking into account that $d_E(T(x), T(y)) = \frac{1}{4} d_E(x, y)$ we deduce that $0 < t \le d_E(x, y)$. Then

$$\tilde{M}_{d_E}(T(x), T(y), \varphi(t)) = \frac{\frac{t}{2}}{\frac{t}{2} + d_E(T(x), T(y))} = \frac{\frac{t}{2}}{\frac{t}{2} + \frac{1}{4} d_E(x, y)} = \frac{t}{t + \frac{1}{2} d_E(x, y)}$$

and

$$\tilde{M}_{d_E}(x,y,t)=\frac{t}{t+d_E(x,y)}.$$

Whence we get that

$$\tilde{M}_{d_E}(T(x),T(y),\varphi(t))=\psi_{f_{*H}}\left(\tilde{M}_{d_E}(x,y,t)\right)=\frac{t}{t+\frac{1}{2}d_E(x,y)}.$$

Case 2. Suppose that $d_E(T(x),T(y))<\frac{t}{2}$ and $d_E(x,y)<t$. Then

$$\tilde{M}_{d_E}(T(x),T(y),\varphi(t))=\frac{2\frac{t}{2}}{2\frac{t}{2}+d_E(T(x),T(y))}=\frac{t}{t+\frac{1}{4}d_E(x,y)}$$

and

$$\tilde{M}_{d_E}(x,y,t)=\frac{2t}{2t+d_E(x,y)}=\frac{t}{t+\frac{1}{2}d_E(x,y)}.$$

Thus we obtain that

$$\tilde{M}_{d_E}(T(x),T(y),\varphi(t))=\psi_{f_{*H}}\left(\tilde{M}_{d_E}(x,y,t)\right)=\frac{t}{t+\frac{1}{4}d_E(x,y)}.$$

Case 3. Suppose that $d_E(T(x),T(y))<\frac{t}{2}$ and $t\leq d_E(x,y)$. Then

$$\tilde{M}_{d_E}(T(x),T(y),\varphi(t))=\frac{t}{t+\frac{1}{4}d_E(x,y)}$$

and

$$\tilde{M}_{d_E}(x,y,t)=\frac{t}{t+d_E(x,y)}.$$

Hence we have that

$$\tilde{M}_{d_E}(T(x),T(y),\varphi(t))\geq\psi_{f_{*H}}\left(\tilde{M}_{d_E}(x,y,t)\right)=\frac{t}{t+\frac{1}{2}d_E(x,y)}.$$

Therefore T is a fuzzy $\psi_{f_{*H}}$-contractive mapping.

Since the notion of fuzzy ψ_{f_*}-contractive mapping comes from a modified version of the notion of fuzzy ψ-contractive mappings it seems natural to wonder whether every mapping ψ_{f_*}, as given in Definition 4.5, belongs to Ψ. The answer to the posed question is negative. Indeed, Example 4.7 gives an instance of mapping ψ_{f_*} which belongs to Ψ. However, a straightforward computation shows that the mapping $\psi_{f_{*L}}$ does not belong to Ψ because it satisfies all condition in Definition 4.5 but it is not continuous at 1 (it is only right continuous), where $f_{*L}(x)=1-x$ for all $x\in[0,1]$ ($f_{*L}(0)=1$) and the mapping $\varphi:[0,1]\to[0,1]$ is given by

$$\varphi(x)=\begin{cases}\frac{x}{4} & \text{if } x\in[0,1[\\ \frac{1}{2} & \text{if } x=1\end{cases}.$$

As mentioned before, the new contractive notion has been introduced in order to introduce a new fixed point theorem in the fuzzy context that avoids the mentioned inconvenience of Theorem 4.3 and that, on the one hand, it does not impose the use of restrictive constraints about completeness as in [23, Theorem 3.1] (where the fuzzy metric space is required to be strong) and, on the other hand, it does impose the use of auxiliary functions as in Theorems 3.1 and 3.2.

In order to achieve our target we will need the following auxiliary classical result.

Theorem 4.8. *Let (X,d) be a complete extended metric space and let $T:X\to X$ be a self-mapping satisfying, for each $x,y\in X$ such that $d(x,y)<\infty$, the following:*

$$d(T(x),T(y))\leq\varphi\left(d(x,y)\right),$$

where $\varphi:[0,\infty]\to[0,\infty]$ is non-decreasing and $\lim_n\varphi^n(t)=0$ for each $t\in]0,\infty[$. Then the following assertions hold:

1) *T has a fixed point if and only if there exists $x_0 \in X$ such that $d(x_0, T(x_0)) < \infty$.*
2) *If T has a fixed point, then it is unique if and only if $d(x, y) < \infty$ for all fixed points $x, y \in X$ of T.*

The proof of the preceding result runs exactly as the original proof obtained for metric spaces by Matkowski in [21, Theorem 1.2]. The difference is given by the facts that the condition "$d(x_0, T(x_0)) < \infty$" cannot be dropped and, in addition, the uniqueness is not guaranteed in general such as the next example shows.

Example 4.9. Let $X = \{x, y\}$ with $x \neq y$. Define $d(x, y) = d(y, x) = +\infty$ and $d(x, x) = d(y, y) = 0$. Then (X, d) is a complete extended metric space.

Define the self-mapping $T_1 : X \to X$ by $T_1(x) = x$ and $T_1(y) = y$. Clearly T_1 satisfies, for all $x, y \in X$ such that $d(x, y) < \infty$, the condition

$$d(T_1(x), T_1(y)) \leq \varphi\left(d(x, y)\right),$$

where $\varphi : [0, \infty] \to [0, \infty]$ is any non-decreasing function such that $\varphi(t) < \infty$ for each $t \in]0, \infty[$, $\lim_n \varphi^n(t) = 0$ for each $t \in]0, \infty[$ and $\varphi(\infty) = \infty$. Notice that $\varphi(0) \geq 0$.

Observe that x and y are fixed points of T_1 and, thus, that T_1 has not a unique fixed point.

Next define $T_2 : X \to X$ by $T_2(x) = y$ and $T_2(y) = x$. Again, the self-mapping T_2 satisfies, for all $x, y \in X$ such that $d(x, y) < \infty$, the condition

$$d(T_2(x), T_2(y)) \leq \varphi\left(d(x, y)\right),$$

where $\varphi : [0, \infty] \to [0, \infty]$ is any non-decreasing function such that $\varphi(t) < \infty$ for each $t \in]0, \infty[$, $\lim_n \varphi^n(t) = 0$ for each $t \in]0, \infty[$ and $\varphi(\infty) = \infty$. Moreover, $d(x, T_2(x)) = \infty$ for all $x \in X$. Clearly, T_2 does not have fixed point.

In view of the exposed facts we are able to prove the promised new fixed point theorem for ψ_{f_*}-contractive mappings.

Theorem 4.10. *Let $(X, M, *)$ be a complete fuzzy metric space such that $*$ is a continuous Archimedean t-norm and f_* is an additive generator of $*$. Suppose that $T : X \to X$ is a fuzzy ψ_{f_*}-contractive mapping, then the following assertions hold:*

1) *T has a fixed point if and only if there exists $x_0 \in X$ such that $M(x_0, T(x_0), t) > 0$ for some $t \in]0, f_*(0)[$.*
2) *If T has a fixed point, then the fixed point of T is unique if and only if, for every fixed point $x, y \in X$, $M(x, y, t) > 0$ for all $t \in]0, \infty[$.*

Proof. First we prove a few facts that will be useful in our subsequent reasoning when proving assertion 1) and 2). Since $T : X \to X$ is a fuzzy ψ_{f_*}-contractive mapping we have warranted the existence of a function $\varphi : [0, f_*(0)] \to [0, f_*(0)]$ strictly increasing, right-continuous and fulfilling $\lim_n \varphi^n(t) = 0$ for each $t \in]0, f_*(0)[$, such that, for each $x, y \in X$ and $t \in]0, f_*(0)[$ satisfying that $M(x, y, t) > 0$, the following is held:

$$M(T(x), T(y), \varphi(t)) \geq \psi_{f_*}\left(M(x, y, t)\right).$$

Next we show that, for each $x, y \in X$ such that $d(x, y) < \infty$, we have that

$$d_{M,f_*}(T(x), T(y)) \leq \varphi\left(d_{M,f_*}(x, y)\right),$$

where d_{M,f_*} is the extended metric deduced from M and f_*, i.e.,

$$d_{M,f_*}(x, y) = \sup\{t \in]0, f_*(0)[: M(x, y, t) \leq f_*^{(-1)}(t)\}, \text{ for each } x, y \in X.$$

With this aim we distinguish two cases:

Case 1: $d_{M,f_*}(x, y) = f_*(0) < \infty$. It follows that $M(x, y, t) > f_*^{(-1)}(t) > 0$ for each $t \in]0, f_*(0)[$, or equivalently, $f_*\left(M(x, y, t)\right) < t$ for each $t \in]0, f_*(0)[$.

Since $T : X \to X$ is a fuzzy ψ_{f_*}-contractive mapping, for each $x, y \in X$ and for each $t \in]0, f_*(0)[$ satisfying $M(x, y, t) > 0$, we have

$$M(T(x), T(y), \varphi(t)) \geq \psi_{f_*}(M(x, y, t)) = \left(f_*^{(-1)} \circ \varphi \circ f_*\right)(M(x, y, t)) = \left(f_*^{(-1)} \circ \varphi\right)(f_*(M(x, y, t))).$$

Besides, $f_*^{(-1)} \circ \varphi$ is a strictly decreasing function on $]0, f_*(0)[$. Hence $f_*^{(-1)} \circ \varphi(f_*(M(x, y, t))) > f_*^{(-1)} \circ \varphi(t)$ for each $t \in]0, f_*(0)[$, since $f_*(M(x, y, t)) < t$ for each $t \in]0, f_*(0)[$. Whence we deduce that

$$M(T(x), T(y), \varphi(t)) > \left(f_*^{(-1)} \circ \varphi\right)(t) = f_*^{(-1)}(\varphi(t)),$$

for each $t \in]0, f_*(0)[$. Thus, $d_{M,f_*}(T(x), T(y)) \leq \varphi(t)$ for each $t \in]0, f_*(0)[$ and, consequently, $d_{M,f_*}(T(x), T(y)) \leq \varphi(d_{M,f_*}(x, y))$.

Case 2: $d_{M,f_*}(x, y) \in [0, f_*(0)[$. In this case, $d_{M,f_*}(x, y) < \infty$ and $M(x, y, t) > f_*^{(-1)}(t) > 0$, for each $t \in]d_{M,f_*}(x, y), f_*(0)[$, or equivalently, $f_*(M(x, y, t)) < t$ for each $t \in]d_{M,f_*}(x, y), f_*(0)[$.

Let $t \in]d_{M,f_*}(x, y), f_*(0)[$. Again, the fact that $T : X \to X$ is a fuzzy ψ_{f_*}-contractive mapping gives, for each $x, y \in X$ and for each $t \in]0, f_*(0)[$ satisfying $M(x, y, t) > 0$, that

$$M(T(x), T(y), \varphi(t)) \geq \left(f_*^{(-1)} \circ \varphi\right)(f_*(M(x, y, t))).$$

Moreover, the facts that $f_*^{(-1)} \circ \varphi$ is a strictly decreasing function on $]d_{M,f_*}(x, y), f_*(0)[$ and $f_*(M(x, y, t)) < t$ for each $t \in]d_{M,f_*}(x, y), f_*(0)[$ yield that

$$M(T(x), T(y), \varphi(t)) > \left(f_*^{(-1)} \circ \varphi\right)(t) = f_*^{(-1)}(\varphi(t))$$

for each $t \in]d_{M,f_*}(x, y), f_*(0)[$. So $d_{M,f_*}(T(x), T(y)) \leq \varphi(t)$ for each $t \in]d_{M,f_*}(x, y), f_*(0)[$. Now, from the right-continuity of φ we obtain that $d_{M,f_*}(T(x), T(y)) \leq \varphi(d_{M,f_*}(x, y))$.

Therefore, we conclude that $d_{M,f_*}(T(x), T(y)) \leq \varphi(d_{M,f_*}(x, y))$ for each $x, y \in X$ satisfying $d_{M,f_*}(x, y) < \infty$.

Obviously, φ satisfies conditions in Theorem 4.8. Moreover, by Corollary 3.9, (X, d_{M,f_*}) is complete.

Now we prove 1).

($\Rightarrow$). If there exists a fixed point x_0 of T, then $M(x_0, T(x_0), t) = M(x_0, x_0, t) = 0$ for all $t \in]0, f_*(0)[$.

($\Leftarrow$). Assume that there exists $x_0 \in X$ such that $M(x_0, T(x_0), t) > 0$ for some $t \in]0, f_*(0)[$.

Next we show that there exists $x_0' \in X$ such that $d_{M,f_*}(x_0', T(x_0')) < \infty$. In case that $f_*(0) < \infty$, then $d_{M,f_*}(x, y) \leq f_*(0)$, for each $x, y \in X$ and, thus, we obtain the desired conclusion by setting $x_0' = x_0$. Now, suppose that $f_*(0) = \infty$. If $d_{M,f_*}(x_0, T(x_0)) = \infty$, by definition of d_{M,f_*}, we have that $M(x_0, T(x_0), t) \leq f_*^{(-1)}(t)$ for each $t \in]0, \infty[$. Taking into account that $M_{x_0,T(x_0)}$ is non-decreasing and, in addition, $f_*^{(-1)}$ is strictly decreasing and continuous (notice that in this case $f_*^{(-1)} = f_*^{-1}$) with $f_*^{(-1)}(0) = 1$ and $f_*^{(-1)}(\infty) = 0$, we conclude that $M(x_0, T(x_0), t) = 0$ for each $t > 0$ that is a contradiction.

Hence, all the hypothesis demanded in Theorem 4.8 are satisfied and the existence of fixed point is guaranteed.

It remains to prove 2).

($\Rightarrow$). If the fixed point of T is unique, then $M(x, x, t) = 1$ for all $t > 0$ as claimed.

($\Leftarrow$). First we show that $\psi_{f_*}(s) > s$ for all $s \in]0, 1[$. According to [30], we have that the function $\varphi : [0, f_*(0)] \to [0, f_*(0)]$ satisfies $\varphi(t) < t$ for all $t \in]0, f_*(0)[$. Moreover, given $s \in]0, 1[$, we have $f_*(s) \in]0, f_*(0)[$ and, thus, that $\varphi(f_*(s)) < f_*(s)$. Since $f_*^{(-1)}$ matches up with f_*^{-1} on $]0, f_*(0)[$ we have that it is strictly decreasing on $]0, f_*(0)[$ and, hence, that $f_*^{(-1)}(\varphi(f_*(s))) > f_*^{(-1)}(f_*(s)) = s$. Whence we deduce that $\psi_{f_*}(s) = \left(f_*^{(-1)} \circ \varphi \circ f_*\right)(s) > s$ for all $s \in]0, 1[$. Now, for the purpose of contradiction, suppose that there exist two fixed points $x, y \in X$ with $x \neq y$. Then there exists $s > 0$ such that $0 < M(x, y, s) < 1$. Hence we have that $\varphi(s) < s$ and that

$$M(x, y, s) \geq M(x, y, \varphi(s)) = M(T(x), T(y), \varphi(s)) \geq \psi_{f_*}(M(x, y, s)) > M(x, y, s),$$

which is a contradiction. Therefore the fixed point of T is unique. □

As a consequence of the above result we obtain the following one.

Corollary 4.11. *Let $(X, M, \diamond)$ be a complete fuzzy metric space and let $T : X \to X$ be a fuzzy ψ_{f_*}-contractive mapping for any continuous and Archimedean t-norm such that $\diamond \geq *$. Then the following assertions hold:*

1) *T has a fixed point if and only if there exists $x_0 \in X$ such that $M(x_0, T(x_0), t) > 0$ for some $t \in]0, f_*(0)[$.*
2) *If T has a fixed point, then the fixed point of T is unique if and only if, for every fixed point $x, y \in X$, $M(x, y, t) > 0$ for all $t \in]0, \infty[$.*

Proof. Since $\diamond \geq *$ we have immediately that $(X, M, *)$ is a fuzzy metric space. Moreover, the completeness of $(X, M, \diamond)$ provides the completeness of the fuzzy metric space $(X, M, *)$. The conclusions of the result follow from Theorem 4.10. □

It must be stressed that, as exposed before, the preceding results require neither to involve auxiliary functions in the spirit of Theorems 3.1 and 3.2 nor to impose another constraints in addition to completeness as in [23, Theorem 3.1] in order to guarantee the existence of a fixed point. Moreover, Theorem 4.10 and Corollary 4.11 overcome the aforementioned inconvenience of Theorem 4.3.

4.2. Fuzzy Meir-Keeler contractive maprings

Recently in [33], Zheng and Wang have introduced the notion of fuzzy Meir-Keeler contractive mapping. It consists in an adaptation to the fuzzy setting of the celebrated notion of contractivity introduced by Meir and Keeler in [22]. Such an adaptation was given in the context of fuzzy metric spaces in the sense of George and Veeramani, which is a particular case of the notion due to Kramosil and Michalek. From now on, we will call such fuzzy metric spaces GV-fuzzy metric spaces.

Let us recall that, according to [7], a GV-fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying, for all $x, y, z \in X$ and $s, t \in]0, \infty[$, axioms **(KM2)**, **(KM3)** and the following ones:

(GV0) $M(x, y, t) > 0$;
(GV1) $M(x, y, t) = 1$ if and only if $x = y$;
(GV4) The assignment $M_{x,y} :]0, \infty[\to [0, 1]$ is a continuous function, where $M_{x,y}(t) = M(x, y, t)$ for each $t \in]0, \infty[$.

Following [33], given a GV-fuzzy metric space $(X, M, *)$, a self-mapping $T : X \to X$ is said to be a fuzzy Meir-Keeler contractive mapping with respect to $\delta \in \Delta$ provided that for each $\epsilon \in]0, 1[$ the following condition is satisfied for all $x, y \in X$ and $t \in]0, \infty[$:

$$\epsilon - \delta(\epsilon) < M(x, y, t) \leq \epsilon \Rightarrow M(T(x), T(y), t) > \epsilon,$$

where Δ denotes the set of all functions $\delta :]0, 1] \to]0, 1]$ such that δ is right continuous.

In [33], the following characterization was stated when GV-fuzzy metric spaces are under consideration.

Theorem 4.12. *Let $(X, M, *)$ be a complete GV-fuzzy metric space and let $T : X \to X$ be a fuzzy Meir-Keeler contractive mapping with respect to $\delta \in \Delta$. Then, T has a unique fixed point if and only if there exists $x_0 \in X$ such that $\bigwedge_{t>0} M(x_0, T(x_0), t) > 0$.*

Theorem 4.12 can be extended to the context of fuzzy metric spaces (in the sense of Kramosil and Michalek). To this end, we need to adapt the notion of fuzzy Meir-Keeler contractive mapping. Thus, given a fuzzy metric space $(X, M, *)$, a self-mapping $T : X \to X$ is said to be a fuzzy Meir-Keeler contractive mapping with respect to $\delta \in \Delta$ provided that for each $\epsilon \in]0, 1[$ the following condition is satisfied

$$\epsilon - \delta(\epsilon) < M(x, y, t) \leq \epsilon \Rightarrow M(T(x), T(y), t) > \epsilon,$$

for each $x, y \in X$ and $t \in]0, \infty[$ such that $M(x, y, t) > 0$.

In view of the preceding notion we can state the aforesaid extension of Theorem 4.12 as follows.

Theorem 4.13. *Let $(X, M, *)$ be a complete fuzzy metric space and let $T : X \to X$ be a fuzzy Meir-Keeler contractive mapping with respect to $\delta \in \Delta$. Then the following assertions hold:*

1) *T has a fixed point if and only if there exists $x_0 \in X$ such that $\bigwedge_{t>0} M(x_0, T(x_0), t) > 0$.*
2) *If T has a fixed point, then the fixed point of T is unique if and only if, for every fixed point $x, y \in X$, $M(x, y, t) > 0$ for all $t \in]0, \infty[$.*

Proof. 1) The same arguments given in the proof of Theorem 4.12 in [33] remain valid to show that T has a fixed point if and only if there exists $x_0 \in X$ such that $\bigwedge_{t>0} M(x_0, T(x_0), t) > 0$.

(2) ($\Rightarrow$). The proof follows the same arguments to those given in the proof of Theorem 4.10.

($\Leftarrow$). Suppose that there exists $x, y \in X$ such that x and y are fixed points of T and $x \neq y$. By pour assumption $M(x, y, t) > 0$ and taking $\epsilon = M(x, y, t)$ we have, applying the contractive condition, that $\epsilon \in]0, 1[$ and that

$$M(x, y, t) = M(T(x), T(y), t) > M(x, y, t),$$

which is impossible. So T has a unique fixed point. □

The following example illustrates the preceding result.

Example 4.14. Consider the complete fuzzy metric space $(X, M, \wedge)$, where $X = \{0, 1\}$ and the fuzzy metric M is given, for each $t \in]0, \infty[$, by

$$M(x, y, t) = \begin{cases} x \wedge y & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}.$$

Consider the function $\delta \in \Delta$ given by $\delta(\epsilon) = \frac{\epsilon}{2}$ for all $\epsilon \in]0, 1]$.

Next, define the self-mapping $T_1 : X \to X$ by $T_1(0) = 1$ and $T_1(1) = 0$. It is a simple matter to check that T_1 is a fuzzy Meir-Keeler contractive mapping with respect to δ. However, T_1 does not have fixed points. Observe that $\bigwedge_{t>0} M(x, T(x), t) = 0$ for all $x \in X$.

Now, define the self-mapping $T_2 : X \to X$ by $T_2(0) = 0$ and $T_2(1) = 1$. It is clear that T_2 is also a fuzzy Meir-Keeler contractive mapping with respect to δ. Clearly T_2 has two fixed points. Notice that $M(0, 1, t) = 0$ for all $t > 0$.

Although Theorem 4.13 characterizes those fuzzy Meir-Keeler contractive mappings that have a unique fixed point, its applicability to ensure the existence of fixed point sometimes may be too limited, as it happens with Theorem 4.3, because in order to warrant such an existence it requires to know exactly the point in which the self-mapping has such a fixed point. Again, with the aim of overcoming this drawback, we propose a new version of fuzzy contractivity of Meir-Keeeler and a characterization of the existence and uniqueness of fixed point.

Definition 4.15. Let $(X, M, *)$ be a fuzzy metric space such that $*$ is a continuous Archimedean t-norm and let $T : X \to X$ be a self-mapping. We will say that T is a *fuzzy f_*-Meir-Keeler contractive mapping* if, given $\epsilon \in]0, f_*(0)[$, there exists $\delta > 0$ such that $M(x, y, \epsilon) \leq f_*^{(-1)}(\epsilon)$ and $M(x, y, \epsilon + \delta) > f_*^{(-1)}(\epsilon + \delta)$ imply $M(T(x), T(y), \epsilon) > f_*^{(-1)}(\epsilon)$, where f_* is an additive generator of $*$.

Example 4.7 (see Theorem 4.19) gives instances of fuzzy f_*-Meir-Keeler contractive mappings.

In the light of the previous notion, we are able to introduce a new fixed point theorem in the fuzzy context that avoids the aforementioned inconvenience of Theorem 4.13.

In order to get our objective the following auxiliary classical result, whose proof can be found in [27], will play a central role.

Theorem 4.16. *Let* (X, d) *be a complete extended metric space and let* $T : X \to X$ *be a self-mapping satisfying the following condition:*

Given $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\epsilon \leq d(x, y) < \epsilon + \delta \text{ implies } d(T(x), T(y)) < \epsilon.$$

Then the following assertions hold:

1) *T has a fixed point if and only if there exists $x_0 \in X$ such that $d(x_0, T(x_0)) < \infty$.*
2) *If T has a fixed point, then it is unique if and only if $d(x, y) < \infty$ for all fixed points $x, y \in X$ of T.*

Now, we are able to prove the promised theorem that characterizes the class of fuzzy f_*-Meir-Keeler contractive mappings with a (unique) fixed point.

Theorem 4.17. *Let $(X, M, *)$ be a complete fuzzy metric space such that $*$ is a continuous Archimedean t-norm and f_* is an additive generator of $*$. Suppose that $T : X \to X$ is a fuzzy f_*-Meir-Keeler contractive mapping, then the following assertions hold:*

1) *T has a fixed point if and only if there exists $x_0 \in X$ such that $M(x_0, T(x_0), t) > 0$ for some $t \in]0, f_*(0)[$.*
2) *If T has a fixed point, then the fixed point of T is unique if and only if, for every fixed point $x, y \in X$, $M(x, y, t) > 0$ for all $t \in]0, \infty[$.*

Proof. 1). ($\Rightarrow$). The proof follows the by same arguments as those given in the proof of Theorem 4.10. $\Leftarrow$). Suppose that there exists $x_0 \in X$ such that $M(x_0, T(x_0), t) > 0$ for some $t \in]0, f_*(0)[$. With the aim of showing that T has a fixed point, we will see that T satisfies the conditions of Theorem 4.16 for the extended metric space (X, d_{M,f_*}). To this end, observe that, by Corollary 3.9, (X, d_{M,f_*}) is complete.

Fix $\epsilon \in]0, f_*(0)[$. Then there exists $\delta > 0$ such that $M(T(x), T(y), \epsilon) > f_*^{(-1)}(\epsilon)$ provided that $M(x, y, \epsilon) \leq f_*^{(-1)}(\epsilon)$ and $M(x, y, \epsilon + \delta) > f_*^{(-1)}(\epsilon + \delta)$.

Suppose that $\epsilon \leq d_{M,f_*}(x, y) < \epsilon + \delta$. Then the construction of d_{M,f_*} gives that $M(x, y, \epsilon) \leq f_*^{(-1)}(\epsilon)$ and $M(x, y, \epsilon + \delta) > f_*^{(-1)}(\epsilon + \delta)$. It follows that $M(T(x), T(y), \epsilon) > f_*^{(-1)}(\epsilon)$ since T is a f_*-Meir-Keeler contractive mapping. So we get again that $d_{M,f_*}(T(x), T(y)) < \epsilon$. The same arguments given in the proof of assertion 1) in Theorem 4.10 remain valid to show the existence of $x_0 \in X$ such that $d_{M,f_*}(x_0, T(x_0)) < \infty$.

Applying Theorem 4.16 we have that T has a fixed point.

2). We only prove the implication ($\Leftarrow$). Suppose that there exist two fixed points $x, y \in X$ with $x \neq y$. Thus we have that there exists $s > 0$ such that $0 < M(x, y, s) < 1$. We distinguish two cases:

Case 1: $f_*(M(x, y, s)) < s$. Take $\varepsilon = f_*(M(x, y, s))$. Then $\varepsilon \in]0, f_*(0)[$ and, in addition, $f_*^{(-1)}(\varepsilon) = f_*^{(-1)}(f_*(M(x, y, s))) = M(x, y, s)$. Hence $M(x, y, \epsilon) \leq M(x, y, s) = f_*^{(-1)}(\epsilon)$. Moreover, taking $\delta = s - \epsilon$, we have $M(x, y, \epsilon + \delta) = M(x, y, s) = f^{(-1)}(\epsilon) > f^{(-1)}(\epsilon + \delta)$. Note that either $f_*^{(-1)}(\epsilon + \delta) = 0$ or $f_*^{(-1)}(\epsilon + \delta) = f_*^{-1}(\epsilon + \delta)$ and f_*^{-1} is strictly decreasing on $]0, f_*(0)[$. Since T is a fuzzy f_*-Meir-Keeler contractive mapping, we deduce that $M(x, y, \epsilon) = M(T(x), T(y), \epsilon) > f_*^{(-1)}(\epsilon) = M(x, y, s) \geq M(x, y, \epsilon)$, which is a contradiction.

Case 2: $s \leq f_*(M(x, y, s))$. Take $\varepsilon = s$. Then $s \in]0, f_*(0)[$ and, in addition, $M(x, y, \epsilon) = M(x, y, s) = f_*^{(-1)}(f_*(M(x, y, s)) \leq f_*^{(-1)}(s) = f_*^{(-1)}(\epsilon)$. Moreover, taking $\delta = f_*(M(x, y, s))$, we have that $M(x, y, \epsilon + \delta) = M(x, y, s + f_*(M(x, y, s))) \geq M(x, y, s) = f_*^{(-1)}(f_*(M(x, y, s)) > f_*^{(-1)}(f_*(M(x, y, s)) + s) = f_*^{(-1)}(\epsilon + \delta)$.

Since T is a fuzzy f_*-Meir-Keeler contractive mapping we deduce that $M(x, y, \epsilon) = M(T(x), T(y), \epsilon) > f_*^{(-1)}(\epsilon) \geq M(x, y, \epsilon)$, which is a contradiction.

Therefore T has a unique fixed point. □

From Theorem 4.17 we get the following consequence whose proof we omit because the reasoning given in Corollary 4.11 applies also to this case.

Corollary 4.18. *Let $(X, M, \diamond)$ be a complete fuzzy metric space and let $T : X \to X$ be a fuzzy f_*-Meir-Keeler contractive mapping for any continuous and Archimedean t-norm such that $\diamond \geq *$. Then the following assertions hold:*

1) *T has a fixed point if and only if there exists $x_0 \in X$ such that $M(x_0, T(x_0), t) > 0$ for some $t \in]0, f_*(0)[$.*
2) *If T has a fixed point, then the fixed point of T is unique if and only if, for every fixed point $x, y \in X$, $M(x, y, t) > 0$ for all $t \in]0, \infty[$.*

In [22], each fuzzy ψ-contractive mapping was shown to be also a fuzzy Meir-Keeler contractive mapping. Whence Theorem 4.3 can be obtained as a corollary of Theorem 4.13. It seems natural to wonder whether the same happens for fuzzy ψ_{f_*}-contractions and fuzzy f_*-Meir-Keeler. The answer to such a question is affirmative as the next result shows.

Theorem 4.19. *Let $(X, M, *)$ be a complete fuzzy metric space such that $*$ is a continuous Archimedean t-norm and f_* is an additive generator of $*$. If $T : X \to X$ is a fuzzy ψ_{f_*}-contractive mapping, then T is a fuzzy f_*-Meir-Keeler contractive mapping.*

Proof. Since $T : X \to X$ is a $fuzzy$ ψ_{f_*}-contractive mapping there exists a function $\varphi : [0, f_*(0)] \to [0, f_*(0)]$ strictly increasing, right-continuous, satisfying $\lim_n \varphi^n(t) = 0$ for each $t \in]0, f_*(0)[$ and, in addition, fulfilling, for all $x, y \in X$ and $t \in]0, f_*(0)[$ such that $M(x, y, t) > 0$, the following inequality:

$$M(T(x), T(y), \varphi(t)) \geq \psi_{f_*}(M(x, y, t)),$$

where $\psi_{f_*}(s) = \left(f_*^{(-1)} \circ \varphi \circ f_*\right)(s)$ for each $s \in [0, 1]$.

In the following, we will show that T is a fuzzy f_*-Meir-Keeler contractive mapping.

Fix $\epsilon \in]0, f_*(0)[$. According to [30], we have that the function $\varphi : [0, f_*(0)] \to [0, f_*(0)]$ satisfies $\varphi(t) < t$ for all $t \in]0, f_*(0)[$. So $\varphi(\epsilon) < \epsilon$. Besides, the right-continuity of φ yields the existence of $\delta > 0$ such that $\varphi(\epsilon + \delta) < \epsilon$ and besides $\epsilon + \delta \in]0, f_*(0)[$. Observe that, in such a case, $\delta < f_*(0) - \epsilon$.

Let $x, y \in X$ be such that $M(x, y, \epsilon) \leq f_*^{(-1)}(\epsilon)$ and $M(x, y, \epsilon + \delta) > f_*^{(-1)}(\epsilon + \delta)$. Since $M_{x,y}$ is non-decreasing and T is a fuzzy ψ_{f_*}-contractive mapping we have that

$$M(T(x), T(y), \epsilon) \geq M(T(x), T(y), \varphi(\epsilon + \delta)) \geq \psi_{f_*}(M(x, y, \epsilon + \delta)).$$

Moreover, we have that

$$M(T(x), T(y), \epsilon) > \psi_{f_*}\left(f_*^{(-1)}(\epsilon + \delta)\right) = \left(f_*^{(-1)} \circ \varphi \circ f_*\right)\left(f_*^{(-1)}(\epsilon + \delta)\right),$$

since $M(x, y, \epsilon + \delta) > f_*^{(-1)}(\epsilon + \delta)$ and ψ_{f_*} is a strictly increasing function.

Hence we deduce that

$$M(T(x), T(y), \epsilon) > \psi_{f_*}\left(f_*^{(-1)}(\epsilon + \delta)\right) = \left(f_*^{(-1)} \circ \varphi \circ f_*\right)\left(f_*^{(-1)}(\epsilon + \delta)\right).$$

Furthermore, $\left(f_* \circ f_*^{(-1)}\right)(\epsilon + \delta) = \epsilon + \delta$, since $\epsilon + \delta \in]0, f_*(0)[$. Whence we get that

$$M(T(x), T(y), \epsilon) > f_*^{(-1)}(\varphi(\epsilon + \delta)) > f_*^{(-1)}(\epsilon),$$

due to $\varphi(\epsilon + \delta) < \epsilon$ and since $f_*^{(-1)}$ is strictly decreasing on $]0, f_*(0)[$.

Therefore T is a fuzzy f_*-Meir-Keeler contractive mapping. □

On account of the preceding result, we wonder if the contrary implication is also satisfied. Later, we will a negative answer to this question. Before, we prove the next proposition, which establish a relationship between fuzzy Meir-Keeler contractive mappings and fuzzy f_*-Meir-Keeler contractive ones when stationary fuzzy metrics are under consideration. Recall that a fuzzy metric space $(X, M, *)$ is said to be stationary if the function $M_{x,y}$ is constant for all $x, y \in X$.

Proposition 4.20. *Let* $(X, M, *)$ *be a stationary fuzzy metric space, where* $*$ *is a continuous Archimedean* t*-norm. Then, every fuzzy Meir-Keeler contractive mappings with respect to some* $\delta \in \Delta$ *is a* f_**-Meir-Keeler contractive mapping for each additive generator* f_* *of* $*$.

Proof. Let $T : X \to X$ be a fuzzy Meir-Keeler contractive mappings with respect to some $\delta \in \Delta$ and let f_* be an additive generator of $*$. Taking into account that M is stationary, we denote $M(x, y, t) = M(x, y)$ for all $t \in]0, \infty[$.

Fix $\epsilon \in]0, f_*(0)[$ and consider $\epsilon' = f_*^{(-1)}(\epsilon) \in]0, 1[$. By our assumption we get, for all $x, y \in X$ and $t \in]0, \infty[$, the next

$$\epsilon' - \delta(\epsilon') < M(x, y, t) \le \epsilon' \Rightarrow M(T(x), T(y), t) > \epsilon'.$$

We distinguish two possibilities:

1. Suppose $\epsilon' - \delta(\epsilon') \le 0$. Then, for each $\delta' > 0$ we have that

$$\epsilon' - \delta(\epsilon') \le 0 \le f_*^{(-1)}(\epsilon + \delta') < M(x, y, \epsilon + \delta') = M(x, y, \epsilon) \le f_*^{(-1)}(\epsilon) = \epsilon'.$$

   So, $M(T(x), T(y)) > \epsilon' = f_*^{(-1)}(\epsilon)$.
2. On the contrary, assume $\epsilon' - \delta(\epsilon') > 0$. Take $\delta' = f_*(\epsilon' - \delta(\epsilon')) - \epsilon$. Then,

$$f_*^{(-1)}(\epsilon + \delta') = f_*^{(-1)}(f_*(\epsilon' - \delta(\epsilon'))) = \epsilon' - \delta(\epsilon') \le M(x, y, \epsilon + \delta') =$$
$$= M(x, y, \epsilon) \le f_*^{(-1)}(\epsilon) = \epsilon'.$$

   So, $M(T(x), T(y)) > \epsilon' = f_*^{(-1)}(\epsilon)$.

Hence, T is a fuzzy f_*-Meir-Keeler contractive mapping. □

As it was mentioned above, there exist f_*-Meir-Keeler contractive mappings which are not ψ_{f_*}-contractive. Indeed, Example 3.12 in [33] provides an instance of f_*-Meir-Keeler contractive mapping which is not ψ_{f_*}-contractive. Note that, on the one hand, the same arguments to those used in [33, Example 3.12] to show that the considered self-mapping is not a fuzzy ψ-contractive mapping, remain valid in order to show that it is not a ψ_{f_*}-contractive mapping, where the t-norm under consideration is $*_P$. On the other hand, observe that the fuzzy metric space used in such an example is stationary. Then, Proposition 4.20 shows that the mapping introduced in it is a fuzzy f_*-Meir-Keeler contractive mapping.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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