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Two new methods to construct fuzzy metrics from metrics

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Abstract

In the last years, the interest in the notion of fuzzy metric has been growing in such a way that many works have focused their efforts on the study of their topological properties and their applications to Engineering problems. However, the applicability of fuzzy metrics is limited due to lack of examples in the literature. Motivated, on the one hand, by these facts and, on the other hand, by the fact that most of the instances of fuzzy metrics in the literature are constructed from classical metrics, in this paper we introduce two new techniques which allow us to construct systematically fuzzy metrics from metrics in such a way that the celebrated classical method for constructing indistinguishability operators from metrics is retrieved as a particular case. Hence, we construct strong fuzzy metrics from a given classical one considering continuous Archimedean t-norms and the pseudo-inverse of their additive generators acting on the metric modified by a positive real function. Moreover, we extend this technique tackling the particular case of the minimum t-norm, which is continuous but non-Archimedean. In such a construction, two non-negative real functions are now involved in order to modify the classical metric and one of them must be superadditive. In this case, the fuzzy metric obtained is not strong in general. Furthermore, the new methods are illustrated by means of different examples which, in addition, show that some celebrated examples of fuzzy metrics can be retrieved as a particular case through them. Finally, in the light of the developed theory, an open problem about strong fuzzy metrics is solved completing the partial solutions that can be found in the literature.

Keywords: Fuzzy metric space; (strict) continuous Archimedean t-norm; Additive generator; Pseudo-inverse; Minimum t-norm; Strong fuzzy metric; Superadditivity

1. Introduction

The notion of metric has become a mathematical tool very useful in many real problems in which is necessary to measure a dissimilarity or similarity between objects. Nevertheless, such problems often require measurements which cannot be provided by a metric due to the fact that its axiomatic turns out to be too restrictive. This evidence motivated the appearance in the literature of various generalizations of the concept of metric. A particular case of

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such generalizations are those framed in fuzzy setting. In this direction, among others, we can find the notion of fuzzy metric in which we focus our attention throughout the paper (see Definitions 2.6 and 2.7).

Kramosil and Michalek introduced in [34] a concept of fuzzy metric and, subsequently, George and Veeramnai provided a modification of it in [9]. Since then, the interest in both notions of fuzzy metric has been growing in such a way that many works have focused their efforts on the study of their topological properties (see, for instance, [1,10,11,13,16,21,24–26,30,35,46,51,52]). The aforesaid fuzzy metrics and their properties have shown to be useful in Engineering problems as, among others, image filtering (see, for instance, [4,18,39,40]), in modelling multi-agent systems (see [2,7,8,27–29]) and in model estimation (see [37,42,43]). However, the applicability of fuzzy metrics is limited due to lack of examples in the literature. So, a wider range of examples of fuzzy metrics can enlarge their applicability. Different theoretical studies on fuzzy metrics have provided several examples of this kind of fuzzy measurements in the literature. In this sense, studies on completion [20,25,26], on convergence [14,15,19]), on the "strong" property (see Definition 2.9) [5,31] and on fixed point theory [17,36] have been carried out. Nonetheless, the aforementioned introduced examples are still few and very repetitive. Therefore, providing examples of fuzzy metric spaces becomes an interesting issue. Motivated, in part, by this fact, in [23] several new concrete examples were introduced. Most of the exposed examples in the mentioned references are constructed from classical metrics and, in addition, they only provide specific instances of fuzzy metrics but it does not give any method to construct systematically fuzzy metrics from metrics. Taking into account this fact, two methods were developed in order to generate fuzzy metrics from classical ones in [22,38].

Also framed in the fuzzy setting, we can find the notion of indistinguishability operator introduced by Trillas in [49] (see Definition 2.5) which provides a way to measure a degree of similarity between objects. Several authors have contributed to the study of such kind of operators (see [45] and references therein). Among all explored properties, it is worth noting the duality relationship between indistinguishability operators and classical metrics. A sign of the interest that such property arouses is given by the fact that many authors have tried to delve into it (see [6,12,32,33,44,45,50]). In this direction, in the preceding references mainly a technique has been explored deeply in order to clarify the aforementioned duality relationship. Such a technique allows to construct indistinguishability operators by means of the use of pseudo-inverses of additive generators of continuous Archimedean triangular norms evaluated on metrics (see Section 2 for a detailed discussion). Although the notion of indistinguishability operator was introduced with the aim of defining a graded equivalence, and thus a degree of similarity, between elements of a set, fuzzy metric presents an advantage with respect to indistinguishability operators. Indeed, the former yield a degree of similarity between two elements relative to the value of a parameter which is involved in the definition of fuzzy metric. This relative measurement has played a distinguished role in the success of fuzzy metrics in applications. Since there exists a similarity between indistinguishability operators and fuzzy metrics (see Remark 3) in this paper we propose two new methods of constructing fuzzy metrics from a classical metric which are inspired in the aforesaid method for generating indistinguishability operators from metrics via pseudo-inverses of additive generators. Hence, on the one hand, we construct fuzzy metrics from a given classical one considering continuous Archimedean t-norms and the pseudo-inverse of their additive generators acting on the metric modified by a positive real function defined on $(0, \infty)$. It must be stressed that the resultant fuzzy metric satisfies the property of being strong. This new construction retrieves as a particular case the aforementioned classical one when indistinguishability operators are considered. On the other hand, we extend the technique tackling the particular case of the minimum t-norm, which is continuous but non-Archimedean. In such a construction, two non-negative real functions are evaluated in order to modify the classical metric, one must be superadditive and is defined on $[0, \infty]$ and another defined on $(0, \infty)$. In this case, the fuzzy metric obtained is not strong. Both techniques are approached for the notion due to Kramosil and Michalek as well as the George and Veeramani's one. Besides, the new methods of constructing fuzzy metrics are illustrated by means of different examples which, in addition, show that well-known examples of fuzzy metrics can be retrieved as a particular case through them. Finally, in the light of the developed theory, a discussion on strong fuzzy metrics is exposed in such a way that a problem posed in [31] is solved completing the partial solution provided in [5].

The paper is organized as follows. Section 2 is devoted to the main results and notions used throughout the paper. Concretely, the duality relationship between indistinguishability operators and classical metrics and the technique for generating indistinguishability operators by means of the use of pseudo-inverses of additive generators of continuous Archimedean t-norms is exposed. In Section 3 a method of constructing a fuzzy metric from a classical one is provided. In such a technique both kind of fuzzy metrics, the Kramosil and Michalek and the George and Veeramani's one, are generated when a continuous Archimedean t-norm is under consideration. Section 4 tackles the case of the minimum t-norm and provides a discussion on strong fuzzy metrics solving the aforementioned open problem.

2. Preliminaries

This section is devoted to the collection of the concepts and results which will be useful throughout the paper. It has been divided in three different subsections. The first and the second one are devoted to introduce the fundamentals about t-norms, indistinguishability operators and the duality relationship between them and classical metrics. The last one recalls the basics about fuzzy metrics.

2.1. Triangular norms

Below we recall only those notions related to t-norms that will be key in our subsequent discussions. For a fuller treatment we refer the reader to [33].

Definition 2.1. A *triangular norm* (briefly, t-norm) is a binary operation $*$ on $[0, 1]$ such that, for all $x, y, z \in [0, 1]$, the following axioms are satisfied:

(T1) $x * y = y * x$; (Commutativity)
(T2) $x * (y * z) = (x * y) * z$; (Associativity)
(T3) $x * y \geq x * z$, whenever $y \geq z$; (Monotonicity)
(T4) $x * 1 = x$. (Boundary Condition)

An interesting subclass of t-norms for our next study is the so called Archimedean which is defined as follows.

Definition 2.2. A t-norm $*$ fulfills the Archimedean property if for each $x, y \in]0, 1[$ there exists $n \in \mathbb{N}$ such that $x^{(n)} < y$, where $x^{(n)} = x * \cdots * x$ n-times. In such a case, we will say that $*$ is an Archimedean t-norm.

It must be stressed that continuous Archimedean t-norms are characterized by the property $x * x < x$ for each $x \in]0, 1[$. Of course the continuity of the t-norm $*$ refers to the continuity as function $* : [0, 1]^2 \to [0, 1]$.

Two well-known examples of continuous Archimedean t-norms are the usual product, i.e. $x *_P y = x \cdot y$, and the Luckasievicz t-norm, given by $x *_L y = \max\{x + y - 1, 0\}$. However, an example of continuous t-norm which is non-Archimedean is the minimum t-norm, i.e. $x *_M y = \min\{x, y\}$. Notice that not all Archimedean t-norms are continuous. Indeed, a well-known example of non-continuous Archimedean t-norm is the so called Drastic product t-norm, which is given by $x *_D y = 0$ if $x, y \in [0, 1[$ and $x *_D y = \min\{x, y\}$ otherwise.

Continuous Archimedean t-norms can be represented by means of a real function called additive generator. In order to state how such t-norms can be represented we recall the notion of pseudo-inverse. With this aim, notice that a function $f : A \subseteq [0, \infty] \to [0, \infty]$ is said to be (strictly) decreasing provided that $f(y) \leq f(x)$ ($f(y) < f(x)$) whenever $x < y$.

Definition 2.3. Let $f : [0, 1] \to [0, \infty]$ be a decreasing function. Then the *pseudo-inverse* $f^{(-1)} : [0, \infty] \to [0, 1]$ of f is defined by

$$f^{(-1)}(y) = \sup\{x \in [0, 1] : f(x) > y\}. \tag{1}$$

Observe that in the foregoing definition we assume that $\sup \emptyset = 0$. It must be stressed that if the function f is in addition continuous, then the pseudo-inverse $f^{(-1)}$ is given as follows:

$$f^{(-1)}(y) = \begin{cases} f^{-1}(y) & \text{if } 0 \leq y < f(0) \\ 0 & \text{if } f(0) \leq y \leq \infty \end{cases}. \tag{2}$$

In the light of the preceding concept the next theorem allows us to construct t-norms as follows:

Theorem 2.1. *Let $f : [0, 1] \to [0, \infty]$ be a strictly decreasing function which is right-continuous at 0 and satisfies $f(1) = 0$ and, in addition,*

$$f(x) + f(y) \in Ran(f) \cup [f(0), \infty] \tag{3}$$

for all $x, y \in [0, 1]$. Then the binary operator $*_f$ on $[0, 1]$ is a t-norm, where $*_f$ is given by

$$x *_f y = f^{(-1)}(f(x) + f(y)), \text{ for each } x, y \in [0, 1]. \tag{4}$$

This result leads to the concept of additive generator which we recall next.

Definition 2.4. An *additive generator* of a t-norm $*$ is a strictly decreasing function $f_* : [0, 1] \to [0, \infty]$ which is also right-continuous at 0 and satisfies that $f_*(1) = 0$ and, in addition, that

$$f_*(x) + f_*(y) \in Ran(t) \cup [f_*(0), \infty] \tag{5}$$

and

$$x * y = f_*^{(-1)}(f_*(x) + f_*(y)) \tag{6}$$

for all $x, y \in [0, 1]$.

Given a t-norm $*$, the continuity of an additive generator f_* is equivalent with its left-continuity at 1 and with the continuity of $*$. Moreover, observe that if a t-norm $*$ has an additive generator f_*, then such an additive generator is uniquely determined up to a non-zero positive multiplicative constant.

Each t-norm with an additive generator is Archimedean. Nevertheless, there are Archimedean t-norms which have not additive generators (see [33, Example 3.21]). The next theorem guarantees the existence of additive generators for continuous t-norms.

Theorem 2.2. *A binary operation $*$ on $[0, 1]$ is a continuous Archimedean t-norm if and only if there exists a continuous additive generator f_* such that*

$$x * y = f_*^{(-1)}(f_*(x) + f_*(y)), \text{ for all } x, y \in [0, 1].$$

2.2. Indistinguishability operators

In this subsection, we recall the basics about indistinguishability operators and the duality relationship between them and classical metrics. Our main references on these topics are [33,45,49].

Definition 2.5. [45,49] Let X be a non-empty set and let $*$ be a t-norm, we will say that a fuzzy set $E : X \times X \to [0, 1]$ is an indistinguishability operator for $*$ if it satisfies, for each $x, y, z \in X$, the following axioms:

(E1) $E(x, x) = 1$; (Reflexivity)
(E2) $E(x, y) = E(y, x)$; (Symmetry)
(E3) $E(x, z) \geq E(x, y) * E(y, z)$. (Transitivity)

If, in addition, E satisfies for all $x, y \in X$ the following condition:

(E1') $E(x, y) = 1$ implies $x = y$,

we will say that E is an indistinguishability operator that separates points.

If confusion does not arise, we will refer to both as indistinguishability operator.

The following is a well-known result which provides a method to construct an indistinguishability operator from a (pseudo-)metric (see for instance [33, Theorem 12.4]).

Theorem 2.3. *Let X be a non-empty-set, let $*$ be a continuous Archimedean t-norm with a continuous additive generator $f_* : [0,1] \to [0,\infty]$ and let d be a pseudo-metric on X. Then the fuzzy set $E_d^* : X \times X \to [0,1]$ is an indistinguishability operator for $*$, where $E_d^*(x,y) = f_*^{(-1)}(d(x,y))$ for each $x, y \in X$. Furthermore, E_d^* is an indistinguishability operator that separates points if and only if d is a metric on X.*

2.3. Fuzzy metric spaces

This last subsection is devoted to the basics of fuzzy metrics that will play a crucial role in our subsequent work.

Kramosil and Michalek introduced a notion of fuzzy metric space in [34], which nowadays is commonly used by the reformulation presented by Grabiec in [13]. Nevertheless, taking into account the research that we will conduct throughout this article, we present this concept in accordance with Miñana and Valero's reinterpretation given in [38].

Definition 2.6. A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying for all $x, y, z \in X$ and $s, t \in (0,\infty)0$, the following conditions:

(KM1) $M(x,y,t) = 1$ for all $t \in (0,\infty)$ if and only if $x = y$;
(KM2) $M(x,y,t) = M(y,x,t)$;
(KM3) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(KM4) The assignment $M_{x,y} : (0,\infty) \to [0,1]$ is a left-continuous function, where $M_{x,y}(t) = M(x,y,t)$ for each $t \in (0,\infty)$.

Remark 1. Observe that, given a fuzzy metric space $(X, M, *)$ (as defined above), we only need to define, for each $x, y \in X$, $M(x,y,0) = 0$ in order to obtain a fuzzy metric on X as introduced by Grabiec in [13].

Later on, in [9], George and Veeramani introduced a concept of fuzzy metric space by modifying a few axioms in the previous definition.

Definition 2.7. A GV-fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying, for all $x, y, z \in X$ and $s, t \in (0,\infty)$, axioms **(KM2)**, **(KM3)** and the following ones:

(GV0) $M(x,y,t) > 0$;
(GV1) $M(x,y,t) = 1$ if and only if $x = y$;
(GV4) The assignment $M_{x,y} : (0,\infty) \to [0,1]$ is a continuous function, where $M_{x,y}(t) = M(x,y,t)$ for each $t \in (0,\infty)$.

Note that each GV-fuzzy metric space is a fuzzy metric space as defined in Definition 2.6.

Remark 2. On account of the previous definitions, we have the following immediate consequence: if $(X, M, *)$ is a (GV-)fuzzy metric space and $\diamond$ is a continuous t-norm satisfying $* \geq \diamond$ (i.e. $a * b \geq a \diamond b$, for each $a, b \in [0,1]$), then $(X, M, \diamond)$ is also a (GV-)fuzzy metric space.

Similar to the classical case, we will say that $(X, M, *)$ is a *fuzzy pseudo-metric space* if it satisfies all the axioms in Definition 2.6 except **(KM1)** and instead it fulfills the following weaker one:

(KM1') $M(x,x,t) = 1$ for all $t \in (0,\infty)$.

Similarly, by replacing axiom **GV1** in the Definition 2.7 by **(KM1')** we also get the concept of GV-fuzzy pseudo-metric space.

If $(X, M, *)$ is a $(GV$-)fuzzy (pseudo-)metric space, we will say that $(M, *)$ is a *(GV-)fuzzy (pseudo-)metric* on X. Also, if confusion does not arise, we will say that (X, M) is a $(GV$-)fuzzy (pseudo-)metric space or M is a $(GV$-)fuzzy (pseudo-)metric on X.

The following are two well-known examples of GV-fuzzy metric spaces provided by George and Veeramani in [9].

Example 1. Let (X, d) be a metric space.

1. Let M_d be the fuzzy set on $X \times X \times (0, \infty)$ given, for each $x, y \in X$ and $t \in (0, \infty)$, by
$$M_d(x, y, t) = \frac{t}{t + d(x, y)}. \tag{7}$$
Then, $(X, M_d, *_M)$ is a GV-fuzzy metric space. The fuzzy metric M_d is called the *standard GV-fuzzy metric* induced by d.
2. Let M_e be the fuzzy set defined on $X \times X \times (0, \infty)$, for each $x, y \in X$ and $t \in (0, \infty)$, as follows
$$M_e(x, y, t) = \mathrm{e}^{-\frac{d(x,y)}{t}}. \tag{8}$$
Then, $(X, M_e, *_M)$ is a GV-fuzzy metric space. We will call it the *exponential GV-fuzzy metric* induced by d.

Both $(X, M_d, *)$ and $(X, M_e, *)$ are fuzzy metric spaces for each continuous t-norm $*$ because of $*_M \geq *$ for each (continuous) t-norm $*$.

Now, we recall two outstanding subclasses of fuzzy metrics.

Definition 2.8. A $(GV$-)fuzzy (pseudo-)metric space $(X, M, *)$ is said to be *stationary* if M does not depend on $t \in (0, \infty)$, i.e., if, for each $x, y \in X$, the function $M_{x,y} : (0, \infty) \to [0, 1]$ is constant, where $M_{x,y}(t) = M(x, y, t)$ for all $t \in (0, \infty)$. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

Remark 3. Note that if E is an indistinguishability operator on a non-empty set X for a continuous t-norm $*$, then the fuzzy set M on $X \times X \times (0, \infty)$ given, for each $x, y \in X$ and $t \in (0, \infty)$, by
$$M(x, y, t) = E(x, y), \tag{9}$$
is a stationary fuzzy pseudo-metric on X (for the t-norm $*$). If in addition, E separates points, then M is a fuzzy metric.

Moreover, if $E(x, y) > 0$ for each $x, y \in X$, then M is a GV-fuzzy (pseudo-)metric.

Definition 2.9. A $(GV$-)fuzzy (pseudo-)metric space $(X, M, *)$ is said to be *strong* if M satisfies, in addition, for all $x, y, z \in X$ and $t \in (0, \infty)$ the following stronger version of the triangle inequality

(KM3') $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$.

Remark 4. A well-known fact, deduced directly from the axioms which defines a fuzzy (pseudo-)metric, is that for each $x, y \in X$ the function $M_{x,y}$ of the axiom **(KM4)** is increasing. Now, it is not hard to check that, a fuzzy set M on $X \times X \times (0, \infty)$ satisfying **(KM1), (KM2), (KM3')** and **(KM4)** is a strong fuzzy (pseudo-)metric if in addition the function $M_{x,y}$ is increasing, for each $x, y \in X$. Analogously, a fuzzy set M on $X \times X \times (0, \infty)$ satisfying **(GV0), (GV1), (KM2), (KM3')** and **(GV4)** is a strong GV-fuzzy (pseudo-)metric if in addition the function $M_{x,y}$ is increasing, for each $x, y \in X$.

The next example provides instances of strong fuzzy metric spaces.

Example 2. As pointed out in [47], the standard GV-fuzzy metric space $(X, M_d, *_M)$ is a strong fuzzy metric if and only if d is an ultrametric. Moreover, a straightforward computation shows that the standard GV-fuzzy metric space

$(X, M_d, *_P)$ is a strong fuzzy metric space for each metric d. It is not hard to check that the same conclusions are obtained for the exponential GV-fuzzy metric space.

We finish this section with the following proposition which characterizes strong GV-fuzzy metric spaces by means of a family of stationary GV-fuzzy metric spaces. The original version was published in [41] and it was stated only for GV-fuzzy metrics. It is easy to verify that the result remains true when we replace GV-fuzzy metrics by GV-fuzzy pseudo-metrics. So, the proposition can be stated as follows.

Proposition 1. *Let $\{(M_t, *) : t \in (0, \infty)\}$ be a family of stationary GV-fuzzy (pseudo-)metrics on a non-empty set X. Then the following assertions hold:*

(i) *If $M : X \times X \times (0, \infty) \to (0, 1]$ is defined for each $x, y \in X$ and $t \in (0, \infty)$ by $M(x, y, t) = M_t(x, y)$, then $(X, M, *)$ is a GV-(pseudo)fuzzy metric space if and only if $\{(M_t, *) : t \in (0, \infty)\}$ is an increasing family (i.e., for each $x, y \in X$, we have that $M_s(x, y) \leq M_t(x, y)$ whenever $0 < s < t$) and, in addition, for each $x, y \in X$, the assignment $M_{x,y} : (0, \infty) \to (0, 1]$ defined by $M_{x,y}(t) = M_t(x, y)$ is a continuous function.*

(ii) *If conditions of (i) are fulfilled then $(M, *)$ is strong.*

3. The new technique: continuous Archimedean t-norms

This section is devoted to the introduction of a method for constructing fuzzy metrics from a given classical one considering continuous Archimedean t-norms and the pseudo-inverse of their additive generators acting on the metric modified by a positive real function defined on $(0, \infty)$. The new resultant fuzzy metrics are strong. Moreover, the new construction retrieves as a particular case the classical technique when indistinguishability operators are considered.

We split the section into two different parts. In the first one, we develop the aforementioned technique in the context of the Kramosil and Michalek fuzzy metric spaces. Then, in the second one, we introduce an analogous of such a technique in the context of the George and Veeramnai fuzzy metrics.

3.1. Fuzzy metrics

We begin this subsection by establishing the following proposition. It constitutes an adaptation of Proposition 1, whose easy proof we omit, to the context of fuzzy metric spaces in the sense of Kramosil and Michalek.

Proposition 2. *Let $*$ be a continuous t-norm and consider a family $\{(M_t, *) : t \in (0, \infty)\}$ of stationary fuzzy (pseudo-)metrics on a non-empty set X. Then the following assertions hold:*

(i) *If $M : X \times X \times (0, \infty) \to (0, 1]$ is defined for each $x, y \in X$ and $t \in (0, \infty)$ by $M(x, y, t) = M_t(x, y)$, then $(X, M, *)$ is fuzzy (pseudo-)metric space if and only if $\{(M_t, *) : t \in (0, \infty)\}$ is an increasing family and, in addition, for each $x, y \in X$, the assignment $M_{x,y} : (0, \infty) \to [0, 1]$ defined by $M_{x,y}(t) = M_t(x, y)$ is a left-continuous function.*

(ii) *If conditions of (i) are fulfilled then $(M, *)$ is strong.*

Now, we are able to provide the announced method for the construction of a fuzzy (pseudo-)metric from a (pseudo-)metric space.

Theorem 3.1. *Let (X, d) be a pseudo-metric space, let $\varphi : (0, \infty) \to (0, \infty)$ be an increasing and left-continuous function, let $*$ be a continuous Archimedean t-norm and let f_* be an additive generator of $*$. Then $(X, M, *)$ is a strong fuzzy pseudo-metric space, where the fuzzy set $M : X \times X \times (0, \infty)$ is given, for each $x, y \in X$ and $t \in (0, \infty)$, by*

$$M(x, y, t) = f_*^{(-1)}\left(\frac{d(x, y)}{\varphi(t)}\right). \tag{10}$$

Moreover, M is a strong fuzzy metric if and only if d is a metric.

Proof. To show that $(X, M, *)$ is a strong fuzzy pseudo-metric space we will construct a family of stationary fuzzy pseudo-metrics $\{(M_t, *) : t \in (0, \infty)\}$ that satisfies the conditions of Proposition 2.

Fix $t \in (0, \infty)$. First of all note that if f_* is an additive generator of $*$, then the function $g : [0, 1] \to [0, \infty]$ given, for each $a \in [0, 1]$, by $g(a) = \varphi(t) \cdot f_*(a)$ is also an additive generator of $*$. Moreover, it is easy to verify that if $f_*^{(-1)}$ is the pseudo-inverse of f_*, then the pseudo-inverse of g is given, for each $b \in [0, \infty]$, by $g^{(-1)}(b) = f_*^{(-1)}\left(\frac{b}{\varphi(t)}\right)$.

Then, by Theorem 2.3, the fuzzy set $E^*_{d,t}$ on $X \times X$ given, for each $x, y \in X$, by

$$E^*_{d,t}(x, y) = g^{(-1)}(d(x, y)) = f_*^{(-1)}\left(\frac{d(x, y)}{\varphi(t)}\right), \tag{11}$$

is an indistinguishability operator for the t-norm $*$. Therefore, attending to Remark 3 we conclude that the fuzzy set M_t on $X \times X \times (0, \infty)$ given, for each $x, y \in X$ and $s \in (0, \infty)$, by $M_t(x, y, s) = E^*_{d,t}(x, y)$ is a stationary fuzzy pseudo-metric on X (for the t-norm $*$). From now on, we denote $M(x, y, s)$ by $M(x, y)$ for each $x, y \in X$ and each $s \in (0, \infty)$.

Now, consider the family of stationary fuzzy metrics $\{M_t : t \in (0, \infty)\}$. We claim that it is an increasing family. Indeed, given $x, y \in X$ and $0 < s < t$, we have that $\varphi(s) \leq \varphi(t)$, since φ is increasing. So

$$M_s(x, y) = f_*^{(-1)}\left(\frac{d(x, y)}{\varphi(s)}\right) \leq f_*^{(-1)}\left(\frac{d(x, y)}{\varphi(t)}\right) = M_t(x, y), \tag{12}$$

since $f_*^{(-1)}$ is decreasing.

Finally, for each $x, y \in X$, the function $M_{x,y} : (0, \infty) \to [0, 1]$ defined by $M_{x,y}(t) = M_t(x, y)$ for each $t \in (0, \infty)$ is left-continuous, since φ is left-continuous and f_* is continuous.

Thus, the family of stationary fuzzy pseudo-metrics $\{M_t : t \in (0, \infty)\}$ satisfies the conditions of Proposition 2 and, thus, $M(x, y, t) = f_*^{(-1)}\left(\frac{d(x,y)}{\varphi(t)}\right)$ is a strong fuzzy pseudo-metric on X (for the t-norm $*$).

It remains to show that M is a strong fuzzy metric on X if and only if d is a metric X.

On the one hand, suppose that M is a strong fuzzy metric. Consider $x, y \in X$ such that $d(x, y) = 0$. Therefore, for each $t \in (0, \infty)$ we have that $M(x, y, t) = f_*^{(-1)}\left(\frac{0}{\varphi(t)}\right) = 1$. Then $M(x, y, t) = 1$ for all $t \in (0, \infty)$. So $x = y$. Thus d is a metric on X.

On the other hand, suppose that d is a metric on X and let $x, y \in X$ such that $M(x, y, t) = 1$ for all $t \in (0, \infty)$. By definition of M we have that

$$f_*^{(-1)}\left(\frac{d(x, y)}{\varphi(t)}\right) = 1 \text{ for all } t \in (0, \infty). \tag{13}$$

Observe that $f_*^{(-1)}$ is the pseudo-inverse of an additive generator of $*$ and, hence, $f_*^{(-1)}(a) = 1$ if and only if $a = 0$. Therefore, taking into account that $\varphi(t) > 0$ for all $t \in (0, \infty)$, we conclude that $M(x, y, t) = 1$ for all $t \in (0, \infty)$ implies $d(x, y) = 0$. Whence we conclude that $M(x, y, t) = 1$ for all $t \in (0, \infty)$ implies $x = y$. □

It must be stressed that Theorem 2.3 can be retrieved from Theorem 3.1 when the function φ is given by $\varphi(t) = 1$ for all $t \in (0, \infty)$.

Next we illustrate the technique provided in Theorem 3.1. Throughout the rest of this section, (X, d) will be a metric space and the function $\varphi : (0, \infty) \to (0, \infty)$ will be the identity map which is obviously increasing and left-continuous.

We begin applying our theorem to the main continuous Archimedean t-norms, which are the Lukasievicz and the product t-norms.

Example 3. (A fuzzy metric space for the Luckasievicz t-norm.)

Let (X, d) be a metric space. Consider the Luckasiewicz t-norm $*_L$. An additive generator of $*_L$ is given, for each $a \in [0, 1]$, by $f_{*_L}(a) = 1 - a$. The pseudo-inverse of f_{*_L} is given, for each $b \in [0, \infty]$, by $f_{*_L}^{(-1)}(b) = \max\{0, 1 - b\}$.

Attending to formula (10) we have, for each $x, y \in X$ and $t \in (0, \infty)$, that

$$M_L(x, y, t) = \max\left\{0, 1 - \frac{d(x, y)}{t}\right\}. \tag{14}$$

Therefore, by Theorem 3.1, we conclude that $(X, M_L, *_L)$ is a strong fuzzy metric space.

Notice that, for each $x, y \in X$ and $t \in (0, \infty)$, the preceding expression can be rewritten as follows:

$$M_L(x, y, t) = \begin{cases} 1 - \frac{d(x,y)}{t}, & \text{if } x, y \in X \text{ with } d(x, y) \leq t \\ 0, & \text{otherwise} \end{cases}. \tag{15}$$

Example 4. (A fuzzy metric space for the product t-norm.)

Let (X, d) be a metric space. Consider the product t-norm $*_P$. An additive generator of $*_P$ is given, for each $a \in [0, 1]$, by $f_{*_P}(a) = -log(a)$. Of course we assume that $\log(0) = -\infty$. The pseudo-inverse of f_{*_P} is given by $f_{*_P}^{-1}(b) = \mathrm{e}^{-b}$ for all $b \in [0, \infty]$.

On account of formula (10) we have, for each $x, y \in X$ and $t \in (0, \infty)$, that

$$M_P(x, y, t) = \mathrm{e}^{-\frac{d(x,y)}{t}}. \tag{16}$$

Therefore, Theorem 3.1 ensures that $(X, M_P, *_P)$ is a strong fuzzy metric space.

Example 4 shows that the technique provided by Theorem 3.1 allows to retrieve the exponential GV-fuzzy metric as a particular case (compare with Example 1) when we consider the product t-norm. Besides, such an example shows that the exponential GV-fuzzy metric is strong for the product t-norm, as pointed out in Example 2.

Now, we show that the standard fuzzy metric presented in Example 1 can be also obtained using our technique. With this aim, let us recall that, according to [33], the Hamacher product t-norm is defined as follows:

$$a *_H b = \begin{cases} 0, & \text{if } a = b = 0 \\ \frac{a \cdot b}{a+b-a \cdot b}, & \text{otherwise} \end{cases}. \tag{17}$$

Observe that $*_H$ is continuous and Archimedean (see [33, Example 4.5]).

Now, we are able to recover, as announced, the standard fuzzy metric from our new approach.

Example 5. (A fuzzy metric space for the Hamacher product t-norm.)

Let (X, d) be a metric space. Consider the Hamacher product t-norm $*_H$. On account of [33], an additive generator of $*_H$ is given, for each $a \in [0, 1]$, by $f_{*_H}(a) = \frac{1-a}{a}$. Of course we assume that $f_{*_H}(0) = \infty$. It is easy to verify that the pseudo-inverse of f_{*_H} is given by $f_{*_H}^{(-1)}(b) = \frac{1}{1+b}$ for all $b \in [0, \infty]$. Thus, attending to formula (10) we obtain, for each $x, y \in X$ and $t \in (0, \infty)$, that

$$M_H(x, y, t) = \frac{1}{1 + \frac{d(x,y)}{t}} = \frac{t}{t + d(x, y)}. \tag{18}$$

So, $(X, M_H, *_H)$ is a strong fuzzy metric space by Theorem 3.1.

Again, the technique provided by Theorem 3.1 gets back a well-known example of fuzzy metric, as it is so the standard GV-fuzzy metric. Moreover, it is easy to verify that $*_H \geq *_P$ and so, by Remark 2, (X, M_H) is also a strong fuzzy metric space for the product t-norm as pointed out in Example 2.

To finish this section, we show that conditions demanded on φ in the statement of Theorem 3.1 cannot be removed.

Example 6. Let d_u be the usual metric on $\mathbb{R}$. We will distinguish two cases:

Case 1. We show that we cannot omit the left-continuity of φ in Theorem 3.1.

Let $\varphi : (0, \infty) \to (0, \infty)$ be given by:

$$\varphi(t) = \begin{cases} t, & \text{if } t \in (0, 1) \\ 2t, & \text{if } t \in [1, \infty) \end{cases}. \tag{19}$$

Obviously, φ is increasing but it is not left-continuous.

A straightforward computation gives that $(\mathbb{R}, M, *_P)$ is not a fuzzy metric space, where M is the fuzzy set provided by the construction in Theorem 3.1 and given, for each $x, y \in \mathbb{R}$ and $t \in (0, \infty)$, as follows:

$$M(x,y,t)=\begin{cases} \mathrm{e}^{-\frac{d_u(x,y)}{t}}, & t\in(0,1) \\ \mathrm{e}^{-\frac{d_u(x,y)}{2t}}, & t\in[1,\infty) \end{cases}. \tag{20}$$

Indeed, the function $M_{1,2}:(0,\infty)\to[0,1]$ given by

$$M_{1,2}(t)=\begin{cases} \mathrm{e}^{\frac{-1}{t}}, & \text{if } t\in(0,1); \\ \mathrm{e}^{\frac{-1}{2t}}, & \text{if } t\in[1,\infty), \end{cases}$$

is not left-continuous at $t=1$ because $\lim_{t\to 1^-}=\mathrm{e}^{-1}\neq \mathrm{e}^{-\frac{1}{2}}=M_{x,y}(1)$. So, M does not satisfy axiom **(KM4)**.
Case 2. We show that the (increasing) monotony of φ cannot be omitted either.
Let $\varphi:(0,\infty)\to(0,\infty)$ be given by $\varphi(t)=\frac{1}{t}$. Then φ is a continuous decreasing function.
The ordered triplet $(\mathbb{R},M,*_P)$ is not a fuzzy metric space, where M is the fuzzy set provided by the construction in Theorem 3.1 and given, for each $x,y\in\mathbb{R}$ and $t\in(0,\infty)$, as follows:

$$M(x,y,t)=\mathrm{e}^{-t\cdot d(x,y)} \tag{21}$$

Indeed M fails to satisfy axiom **(KM3)**, since

$$M(0,2,1+1)=\mathrm{e}^{-2\cdot 2}=\mathrm{e}^{-4}<\mathrm{e}^{-2}=\mathrm{e}^{-1}\cdot\mathrm{e}^{-1}=M(0,1,1)*_P M(1,2,1).$$

3.2. *GV-fuzzy metrics*

Example 3 is an instance of fuzzy metric space provided by Theorem 3.1 that is not a GV-fuzzy metric space. Indeed, observe that considering $X=\mathbb{R}$ endowed with the usual metric d_u, then $M(0,t,t)=1-\frac{|t-0|}{t}=0$ for all $t\in(0,\infty)$ and so M does not satisfy axiom **(GV1)**.

Motivated by this fact, in this subsection we approach the issue of obtaining the George and Veeramani version of Theorem 3.1. Taking into account that the concept due to George and Veramani is a restriction of the Kramosil and Michalek one, we study if additional conditions on Theorem 3.1 must be demanded to get it. To this end, we recall the class of the so-called strict t-norms (see [33]).

Definition 3.1. A t-norm $*$ is called strict if it is continuous and strictly monotone (i.e $x*y>x*z$ for each $x,y,z\in(0,1]$ with $y>z$).

According to [33], when strict t-norms are, in addition, Archimedean the following characterization can be stated.

Proposition 3. *Let $*$ be a continuous Archimedean t-norm and let $f_*:[0,1]\to[0,\infty]$ be a continuous additive generator of $*$. Then $*$ is strict if and only if $f_*(0)=\infty$.*

Now, we present a method to obtain a GV-fuzzy (pseudo-)metric from a classical (pseudo-)metric based on Theorem 3.1.

Theorem 3.2. *Let (X,d) be a pseudo-metric space, let $\varphi:(0,\infty)\to(0,\infty)$ be an increasing and continuous function and let $*$ be a strict Archimedean t-norm with an additive generator f_*. Then $(X,M,*)$ is a strong GV-fuzzy pseudo-metric space, where the fuzzy set $M:X\times X\times(0,\infty)$ is given, for each $x,y\in X$ and $t\in(0,\infty)$, by*

$$M(x,y,t)=f_*^{(-1)}\left(\frac{d(x,y)}{\varphi(t)}\right). \tag{22}$$

Moreover, M is a strong GV-fuzzy metric if and only if d is a metric.

Proof. Define the fuzzy set M on $X\times X\times(0,\infty)$ given, for each $x,y\in X$ and $t\in(0,\infty)$, by

$$M(x,y,t)=f^{(-1)}\left(\frac{d(x,y)}{\varphi(t)}\right). \tag{23}$$

By Theorem 3.1 we conclude that $(X, M, *)$ is a strong fuzzy pseudo-metric space and so, M satisfies axioms **(KM1'), (KM2)**, **(KM3)** and **(KM3')**. Therefore, we just need to show that M also satisfies axioms **(GV0)** and **(GV4)**.

First we show that M satisfies axiom **(GV0)**. Since $*$ is strict and Archimedean, by Proposition 3 we conclude that $f_*^{(-1)}(b) = f_*^{-1}(b) > 0$ for each $b \in [0, \infty)$. Thus, for each $x, y \in X$ and $t \in (0, \infty)$, we have that

$$M(x, y, t) = f_*^{(-1)}\left(\frac{d(x, y)}{\varphi(t)}\right) > 0. \tag{24}$$

Next we show that axiom **(GV4)** is fulfilled. Let $x, y \in X$ and consider the assignment $M_{x,y} : (0, \infty) \to (0, 1]$ given by $M_{x,y}(t) = M(x, y, t)$, for all $t \in (0, \infty)$. Then $M_{x,y}$ is a continuous function on $(0, \infty)$, since $\varphi : (0, \infty) \to (0, \infty)$ and $f_*^{(-1)} : [0, \infty) \to (0, 1]$ are continuous functions and $\varphi(t) > 0$ for each $t \in (0, \infty)$.

Therefore $(X, M, *)$ is a strong GV-fuzzy pseudo-metric space.

To finish the proof we must show that M is a strong GV-fuzzy metric if and only if d is a metric. Again, Theorem 3.1 ensures that if M is a strong GV-fuzzy metric then d is a metric on X. It remains to prove that M satisfies **(GV1)** provided that d is exactly a metric. With this aim, observe that M is a strong GV-fuzzy metric on X if and only if, given $t \in (0, \infty)$, we have that $M(x, y, t) = 1$ implies $x = y$. So, fix $t \in (0, \infty)$ and assume that $M(x, y, t) = 1$. Thus we have that

$$M(x, y, t) = f_*^{(-1)}\left(\frac{d(x, y)}{\varphi(t)}\right) = 1. \tag{25}$$

It follows that $\frac{d(x,y)}{\varphi(t)} = 0$. Hence, since $\varphi(t) \in (0, \infty)$ we get that $d(x, y) = 0$. The fact that d is a metric on X yields that $x = y$. Whence we conclude that M is a strong GV-fuzzy metric. □

Observe that the t-norm considered in Example 3 was the Lukasievicz one which is not strict, and this is reason for which the induced fuzzy metric is not a GV-fuzzy metric.

Moreover, as pointed out before, in Examples 4 and 5 were used the product t-norm and the Hamacher product t-norm, respectively, which are strict. Besides, the function φ used in such examples is the identity map on $(0, \infty)$ which is obviously continuous. Therefore, Theorem 3.2 provides the fuzzy sets M_P and M_H as strong GV-fuzzy metrics.

Finally, the next example shows the necessity of demanding continuity on φ instead of left-continuity in statement of Theorem 3.2.

Example 7. Let (X, d) be a metric space and consider the function $\varphi : (0, \infty) \to (0, \infty)$ given by

$$\varphi(t) = \begin{cases} t, & \text{if } t \in (0, 1] \\ 2t, & \text{if } t \in (1, \infty) \end{cases},$$

which is increasing, left-continuous but not continuous at 1.

Since $*_H$ is a strict Archimedean t-norm then Theorem 3.1 warranties that $(X, M, *_H)$ is a strong fuzzy metric space. Nevertheless, M is not a GV-fuzzy metric space due to axiom **(GV4)** is not held as we show below.

It is easy to verify that the fuzzy set M provided by the construction of Theorem 3.2 is the next one for $x, y \in X$ (compare with Example 5):

$$M(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)}, & t \in (0, 1] \\ \frac{2t}{2t+d(x,y)}, & t \in (1, \infty) \end{cases}. \tag{26}$$

Then, for each $x, y \in X$ with $x \neq y$, the function $M_{x,y} : (0, \infty) \to (0, 1]$ given by $M_{x,y}(t) = M(x, y, t)$ is not continuous at $t = 1$. Indeed, given $x, y \in X$ with $x \neq y$, then

$$\lim_{t \to 1^-} M_{x,y}(t) = \frac{1}{1 + d(x, y)} \neq \frac{2}{2 + d(x, y)} = \lim_{t \to 1^+} M_{x,y}(t). \tag{27}$$

Thus, M does not satisfy axiom **(GV4)** and so M is not a (strong) GV-fuzzy metric on X.

4. The new technique: the minimum t-norm

In this section we extend the technique introduced in Section 3 to the non-Archimedean case. Concretely when the t-norm under consideration is the minimum t-norm $*_M$. Clearly, it is continuous. Now in the construction we need to consider two non-negative real functions in order to modify the classical metric, one defined on $[0, \infty]$ and another on $(0, \infty)$. In this case, the fuzzy metrics that we obtain are not strong. Recall that the minimum t-norm is the largest one.

Following [3], a function $\varphi : (0, \infty) \to (0, \infty)$ is said to be superadditive if it satisfies, for each $t, s \in (0, \infty)$, the following condition

$$\varphi(t + s) \geq \varphi(t) + \varphi(s). \tag{28}$$

The announced method is provided by the following theorem.

Theorem 4.1. *Let (X, d) be a pseudo-metric space, let $\varphi : (0, \infty) \to (0, \infty)$ be an increasing left-continuous superadditive function and let $g : [0, \infty] \to [0, 1]$ be a decreasing left-continuous function such that $g(0) = 1$. Then $(X, M, *_M)$ is a fuzzy pseudo-metric space, where the fuzzy set $M : X \times X \times (0, \infty)$ is given, for each $x, y \in X$ and $t \in (0, \infty)$, by*

$$M(x, y, t) = g\left(\frac{d(x, y)}{\varphi(t)}\right). \tag{29}$$

Moreover if, in addition, $g^{-1}(1) = \{0\}$ then M is a fuzzy metric if and only if d is a metric.

Proof. Define the fuzzy set $M : X \times X \times (0, \infty)$ as in formula (29). We will see that, for each $x, y, z \in X$ and $s, t \in (0, \infty)$, axioms **(KM1'), (KM2), (KM3)** and **(KM4)** are held.

(KM1') Let $x \in X$ and $t \in (0, \infty)$. Then, $M(x, x, t) = g\left(\frac{d(x,x)}{\varphi(t)}\right) = g(0) = 1$.

(KM2) The symmetry is obvious by definition of M.

(KM3) Let $x, y, z \in X$ and $t, s \in (0, \infty)$. We will show that $M(x, z, t + s) \geq M(x, y, t) *_M M(y, z, s)$.
Suppose that $\frac{d(x,y)}{\varphi(t)} \leq \frac{d(y,z)}{\varphi(s)}$. The fact that g is a decreasing function gives that

$$M(x, y, t) = g\left(\frac{d(x, y)}{\varphi(t)}\right) \geq g\left(\frac{d(y, z)}{\varphi(s)}\right) = M(y, z, s). \tag{30}$$

So, $M(x, y, t) *_M M(y, z, s) = M(y, z, s)$. Then we must show that $M(x, z, t + s) \geq M(y, z, s)$.
By our assumption we obtain that

$$\varphi(s) \cdot d(x, y) \leq \varphi(t) \cdot d(y, z). \tag{31}$$

So, in the light of the preceding inequality and the superadditivity of φ we have that

$$\begin{aligned} &\varphi(s) \cdot d(x, z) \leq \varphi(s) \cdot (d(x, y) + d(y, z)) = \varphi(s) \cdot d(x, y) + \varphi(s) \cdot d(y, z) \leq \\ &\leq \varphi(t) \cdot d(y, z) + \varphi(s) \cdot d(y, z) = (\varphi(t) + \varphi(s)) \cdot d(y, z) \leq \varphi(t + s) \cdot d(y, z). \end{aligned} \tag{32}$$

It follows that

$$\frac{d(x, z)}{\varphi(t + s)} \leq \frac{d(y, z)}{\varphi(s)}. \tag{33}$$

Now, taking into account that g is decreasing, the following inequality is satisfied

$$M(x, z, t + s) = g\left(\frac{d(x, z)}{\varphi(t + s)}\right) \geq g\left(\frac{d(y, z)}{\varphi(s)}\right) = M(y, z, s) = M(x, y, t) *_M M(y, z, s). \tag{34}$$

The contrary case $\frac{d(x,y)}{\varphi(t)} > \frac{d(y,z)}{\varphi(s)}$ is proved analogously.

(KM4) Since the functions g and φ are left-continuous and, in addition, $\varphi(t) > 0$ for all $t \in (0, \infty)$ we have that the assignment $M_{x,y} : (0, \infty) \to [0, 1]$ is a left-continuous function.

Now, we show that if $g^{-1}(1) = \{0\}$ then M is a fuzzy metric if and only if d is a metric.

On the one hand, suppose that M is a fuzzy metric on X, then $M(x, y, t) = 1$ for all $t \in (0, \infty)$ implies $x = y$. Let $x, y \in X$ such that $d(x, y) = 0$. Then, $M(x, y, t) = g(0) = 1$ for all $t \in (0, \infty)$ and so $x = y$. Thus, d is a metric on X.

On the other hand, suppose that d is a metric on X and let $x, y \in X$ such that $M(x, y, t) = g\left(\frac{d(x,y)}{\varphi(t)}\right) = 1$ for all $t \in (0, \infty)$. Hence we deduce that $\frac{d(x,y)}{\varphi(t)} = 0$, since $g^{-1}(1) = \{0\}$. Therefore, due to the fact that d is a metric, $x = y$ and, thus, M is a fuzzy metric on X. □

Remark 5. The preceding theorem also allows to construct new examples of probabilistic metric spaces in the sense of Menger (see [33,48]) whenever an additional condition is imposed to the functions g and φ, respectively. Indeed, it is not hard to check that if we impose $\lim_{t\to 0^+} g(t) = 1$ and $\lim_{t\to\infty} \varphi(t) = \infty$ in addition to the conditions demanded on g and φ in Theorem 4.1, then $(X, \mathcal{F}, \tau)$ is a probabilistic metric space for each triangle function (see [33]) with $\tau \leq *_M$ where, for each $x, y \in X$ and $t \in \mathbb{R}$, $\mathcal{F}$ is defined as follows

$$\mathcal{F}_{x,y}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0] \\ g\left(\frac{d(x,y)}{\varphi(t)}\right), & \text{if } t \in (0, \infty) \end{cases}. \tag{35}$$

Next we show that the conditions demanded in the statement of Theorem 4.1 cannot be weakened.

Define the function $g : [0, \infty] \to [0, 1]$, for each $a \in [0, \infty]$, by $g(a) = f_{*_P}^{(-1)}(a) = \mathrm{e}^{-a}$. It is not hard to verify that such function satisfies all the conditions in Theorem 4.1. Consider the functions φ provided in Cases 1 and 2 in Example 6. Clearly, the functions φ given in Case 1 and Case 2 are not left-continuous and increasing, respectively. The fuzzy sets generated via Theorem 4.1 are exactly those introduced in Example 6 and they are not fuzzy pseudo-metrics.

The next example shows that the superadditivity of φ cannot be removed in Theorem 4.1.

Example 8. Consider the metric space $(\mathbb{R}, d_u)$. Define the function $\varphi : (0, \infty) \to (0, \infty)$ by $\varphi(t) = \sqrt{t}$ and define the function $g : [0, \infty] \to [0, 1]$ by $g(x) = \frac{1}{1+x}$.

Obviously, g is a decreasing function and $g(0) = 1$. Besides, φ is increasing and continuous, but it is not superadditive. Indeed, for instance, $\varphi(1 + 1) = \sqrt{2} < 2 = \varphi(1) + \varphi(1)$.

The fuzzy set M induced by Theorem 4.1 is given, for each $x, y \in \mathbb{R}$ and $t \in (0, \infty)$, by

$$M(x, y, t) = g\left(\frac{d_u(x, y)}{\varphi(t)}\right) = \frac{1}{1 + \frac{d_u(x,y)}{\sqrt{t}}} = \frac{\sqrt{t}}{\sqrt{t} + d_u(x, y)}. \tag{36}$$

In the following we show that M does not fulfill axiom **(KM3)**. Indeed,

$$M(0, 2, 1 + 1) = \frac{\sqrt{2}}{\sqrt{2} + 2} = \frac{1}{1 + \sqrt{2}} < M(0, 1, 1) *_M M(1, 2, 1) = \frac{\sqrt{1}}{\sqrt{1} + 1} = \frac{1}{2}.$$

We focus now on the conditions required on g in Theorem 4.1. The next example justifies that all of them cannot be weakened.

Example 9. Let (X, d) be a metric space and consider $\varphi : (0, \infty) \to (0, \infty)$ given by $\varphi(t) = t$ for all $t \in (0, \infty)$. Obviously, φ satisfies all conditions required in Theorem 4.1.

(i) Let $g : [0, \infty] \to [0, 1]$ be the function given, for each $a \in [0, \infty]$, by $g(a) = \frac{1}{2} \cdot \mathrm{e}^{-a}$. Clearly, g is decreasing and continuous but $g(0) = \frac{1}{2}$.

The fuzzy set obtained via Theorem 4.1 is given, for each $x, y \in X$ and $t \in (0, \infty)$, as follows:

$$M(x, y, t) = \frac{1}{2} \cdot \mathrm{e}^{-\frac{d(x,y)}{t}}. \tag{37}$$

Observe that, for each $x, y \in X$, we have $M(x, x, t) = \frac{1}{2}$ for all $t \in (0, \infty)$. Therefore, M does not satisfy axiom **(KM1)**.

(ii) Consider the function $g : [0, \infty] \to [0, 1]$ defined as follows:

$$g(a) = \begin{cases} \mathrm{e}^{-a}, & \text{if } a \in [0, 1) \\ \mathrm{e}^{-2a}, & \text{if } a \in [1, \infty] \end{cases}. \tag{38}$$

It is easily seen that g is decreasing and $g(0) = 1$. However, g is not left-continuous at $a = 1$.
Recall that the fuzzy set M provided by Theorem 4.1 is given, for each $x, y \in X$ and $t \in (0, \infty)$, by $M(x, y, t) = g\left(\frac{d(x,y)}{\varphi(t)}\right)$.
Fix $x, y \in X$ with $x \neq y$. Observe that $\frac{d(x,y)}{t} \in [0, 1)$ if and only if $t \in (0, d(x, y))$ and, $\frac{d(x,y)}{t} \in [1, \infty)$ if and only if $t \in [d(x, y), \infty)$. So we obtain, for each $x, y \in X$ with $x \neq y$, the next expression of the fuzzy set M

$$M(x, y, t) = \begin{cases} \mathrm{e}^{-\frac{d(x,y)}{t}}, & \text{if } t \in (0, d(x, y)) \\ \mathrm{e}^{-2\frac{d(x,y)}{t}}, & \text{if } t \in [d(x, y), \infty) \end{cases}. \tag{39}$$

Note that, for each $x, y \in X$ with $x \neq y$, the assignment $M_{x,y} : (0, \infty) \to [0, 1]$, given by $M_{x,y}(t) = M(x, y, t)$, is not left-continuous at $t = d(x, y)$. Thus, M does not satisfy axiom **(KM4)**.

(iii) Define the function $g : [0, \infty] \to [0, 1]$ given by $g(0) = 1$ and $g(a) = 1 - \mathrm{e}^{-a}$ for each $a \in (0, \infty]$. It is easy to verify that g is left-continuous. Nevertheless, g is not decreasing.
The fuzzy set provided by Theorem 4.1 is given by

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y, t \in (0, \infty) \\ 1 - \mathrm{e}^{-\frac{d(x,y)}{t}}, & \text{if } x \neq y, t \in (0, \infty) \end{cases}. \tag{40}$$

Now, set $t = s = 1$ and let $x, y, z \in X$ with $x \neq y$ and $y = z$. Then, $M(x, z, t + s) = 1 - \mathrm{e}^{-\frac{d(x,z)}{2}}$, $M(x, y, t) = 1 - \mathrm{e}^{-d(x,y)}$ and $M(y, z, s) = 1$.
Since $d(x, z) = d(x, y)$ we deduce that

$$M(x, z, t + s) = 1 - \mathrm{e}^{-\frac{d(x,z)}{2}} < 1 - \mathrm{e}^{-d(x,y)} = M(x, y, t) *_M M(y, z, s). \tag{41}$$

Therefore, M does not satisfy axiom **(KM3)**.

We continue establishing an adaptation of Theorem 4.1 to the context of George and Veeramani below.

Theorem 4.2. *Let (X, d) be a pseudo-metric space, let $\varphi : (0, \infty) \to (0, \infty)$ be an increasing continuous superadditive function and let $g : [0, \infty] \to [0, 1]$ be a decreasing continuous function such that $g(0) = 1$ and $g(a) > 0$ for each $a \in (0, \infty)$. Then $(X, M, *_M)$ is a GV-fuzzy pseudo-metric space, where the fuzzy set $M : X \times X \times (0, \infty)$ is given, for each $x, y \in X$ and $t \in (0, \infty)$, by*

$$M(x, y, t) = g\left(\frac{d(x, y)}{\varphi(t)}\right). \tag{42}$$

Moreover if, in addition, $g^{-1}(1) = \{0\}$ then M is a GV-fuzzy metric if and only if d is a metric.

Proof. Theorem 4.1 ensures that M satisfies axioms **(KM1'), (KM2)** and **(KM3)**. It remains to prove that M also satisfies axioms **(GV0)** and **(GV4)**.

On the one hand, axiom **(GV0)** is fulfilled by M, for each $x, y \in X$ and $t \in (0, \infty)$, due to $g(a) > 0$ for each $a \in [0, \infty)$. On the other hand, since g and φ are continuous and $\varphi(t) > 0$ for each $t \in (0, \infty)$ we have that the assignment $M_{x,y}$ is continuous. Thus $(X, M, *_M)$ is a GV-fuzzy pseudo-metric.

Now, suppose that M is a GV-fuzzy metric and let $x, y \in X$ such that $d(x, y) = 0$. By construction of M, given $t \in (0, \infty)$, we have that $M(x, y, t) = 1$. Since M is a GV-fuzzy metric we get that $x = y$. Thus d is a metric.

Conversely, suppose that d is a metric and let $x, y \in X$ such that $M(x, y, t) = 1$ for some $t \in (0, \infty)$. It follows that

$$1 = M(x, y, t) = g\left(\frac{d(x, y)}{\varphi(t)}\right). \tag{43}$$

Taking into account that $g^{-1}(1) = \{0\}$ and $\varphi(t) > 0$ we conclude $d(x, y) = 0$. Therefore, since d is a metric we have that $x = y$ and, hence, M is a GV-fuzzy metric. □

Notice that Theorems 3.1 and 3.2 provide a technique for constructing a strong fuzzy metric and strong GV-fuzzy metric, respectively, from a classical metric for a given continuous Archimedean t-norm. Such constructions are given through the pseudo-inverse of an additive generator of the considered t-norm. Taking into account that the pseudo-inverse of an additive generator of a (strict) continuous t-norm satisfies the conditions imposed on g both in Theorems 4.1 and 4.2, the aforesaid theorems give immediately the following corollaries.

Corollary 4.3. *Let (X, d) be a pseudo-metric space, $\varphi : (0, \infty) \to (0, \infty)$ an increasing left-continuous superadditive function, and let $*$ be a continuous Archimedean t-norm with an additive generator f_*. Then $(X, M, *_M)$ is a fuzzy pseudo-metric space, where the fuzzy set $M : X \times X \times (0, \infty)$ is given, for each $x, y \in X$ and $t \in (0, \infty)$, by*

$$M(x, y, t) = f_*^{(-1)}\left(\frac{d(x, y)}{\varphi(t)}\right). \tag{44}$$

Moreover, M is a fuzzy metric if and only if d is a metric.

Corollary 4.4. *Let (X, d) be a pseudo-metric space, $\varphi : (0, \infty) \to (0, \infty)$ an increasing continuous superadditive function, and let $*$ be a strict continuous Archimedean t-norm with an additive generator f_*. Then $(X, M, *_M)$ is a (GV)-fuzzy pseudo-metric space, where the fuzzy set $M : X \times X \times (0, \infty)$ is given, for each $x, y \in X$ and $t \in (0, \infty)$, by*

$$M(x, y, t) = f_*^{(-1)}\left(\frac{d(x, y)}{\varphi(t)}\right). \tag{45}$$

Moreover, M is a GV-fuzzy metric if and only if d is a metric.

In the light of the preceding two corollaries, one can observe that imposing superadditivity on the function φ, the fuzzy sets obtained by means of the constructions provided in Theorems 4.3 and 4.4, respectively, are also fuzzy metrics for each t-norm, since the minimum t-norm is the largest one. Nevertheless, we cannot assert, in general, that they are strong fuzzy metrics. Indeed, the standard fuzzy metric is obtained using the construction provided by Corollary 4.4 (or Theorem 4.1) when the function $\varphi : (0, \infty) \to (0, \infty)$ is given by $\varphi(t) = t$ for all $t \in (0, \infty)$, which is superadditive. Nevertheless, the standard fuzzy metric is strong for the minimum t-norm if and only if d is an ultrametric, as mentioned in Example 2.

This fact, and taking into account that Theorems 3.1 and 3.2 give fuzzy pseudo-metrics which are strong when the t-norm under consideration is $*$, inspires us to ask for the following interesting question. There exists another t-norm $\diamond$ different from $*$ for which the fuzzy metrics obtained following the constructions provided by Corollaries 4.3 and 4.4, respectively, are strong?

The following theorems provide an answer to the posed question.

Theorem 4.5. *Let $\varphi : (0, \infty) \to (0, \infty)$ be an increasing left-continuous superadditive function and let $*$ be a continuous Archimedean t-norm with an additive generator f_*. Then $(X, M, \diamond)$ is a strong fuzzy pseudo-metric space, for each pseudo-metric space (X, d), if and only if $* \geq \diamond$, where the fuzzy set $M : X \times X \times (0, \infty)$ is given, for each $x, y \in X$ and $t \in (0, \infty)$, by*

$$M(x, y, t) = f_*^{(-1)}\left(\frac{d(x, y)}{\varphi(t)}\right). \tag{46}$$

Proof. If $* \geq \diamond$, then the conclusion is obvious by Theorem 3.1. Conversely, assume that $(X, M, \diamond)$ is a strong fuzzy pseudo-metric on X, for each pseudo-metric space (X, d). Suppose that contrary to our claim there exist $a, b \in (0, 1)$ satisfying $a \diamond b > a * b$.

Fix $t_0 \in (0, \infty)$. We will construct a metric space (X, d) such that the induced fuzzy metric $(X, M, \diamond)$ is not strong.

Consider the set formed by three different elements $X = \{x, y, z\}$ and define the function $d : X \times X \to [0, \infty[$ as follows:

$$d(x, x) = d(y, y) = d(z, z) = 0;$$
$$d(x, z) = d(z, x) = \varphi(t_0) \cdot f_*(a * b);$$

$$d(x,y)=d(y,x)=\varphi(t_0)\cdot f_*(a);$$
$$d(y,z)=d(z,y)=\varphi(t_0)\cdot f_*(b).$$

Observe that $d(u,v)=0$ if and only if $u=v$, since $a,b\in]0,1[$, $f_*(a), f_*(b), f_*(a*b)>0$ and $\varphi(t_0)>0$. We only need to prove that the triangle inequality is held to see that d is a metric on X. Indeed, due to the fact that f_* is decreasing and $a\geq a*b$ as well as $b\geq a*b$ we have that $f_*(a*b)\geq f_*(a)$ and $f_*(a*b)\geq f_*(b)$. So

$$d(x,z)=\varphi(t_0)\cdot f_*(a*b)=\varphi(t_0)\cdot f_*\left(f_*^{(-1)}(f_*(a)+f_*(b))\right)\leq$$
$$\varphi(t_0)\cdot(f_*(a)+f_*(b))=\varphi(t_0)\cdot f_*(a)+\varphi(t_0)\cdot f_*(b)=d(x,y)+d(y,z),$$
$$d(x,y)=\varphi(t_0)\cdot f_*(a)\leq\varphi(t_0)\cdot(f_*(a*b)+f_*(b))=\varphi(t_0)\cdot f_*(a*b)+\varphi(t_0)\cdot f_*(b)=d(x,z)+d(z,y),$$

and

$$d(y,z)=\varphi(t_0)\cdot f_*(b)\leq\varphi(t_0)\cdot(f_*(a)+f_*(a*b))=\varphi(t_0)\cdot f_*(a)+\varphi(t_0)\cdot f_*(a*b)=d(y,x)+d(x,z).$$

Hence, the triangle inequality is satisfied and (X,d) is a metric space.

Next we show that $(X,M,\diamond)$ is not strong.

In this case, for each $t\in(0,\infty)$, the fuzzy set M is given by

$$M(x,z,t)=M(z,x,t)=f^{(-1)}\left(\frac{d(x,z)}{\varphi(t)}\right)=f^{(-1)}\left(\frac{\varphi(t_0)\cdot f(a*b)}{\varphi(t)}\right),$$
$$M(x,y,t)=M(y,x,t)=f^{(-1)}\left(\frac{d(x,y)}{\varphi(t)}\right)=f^{(-1)}\left(\frac{\varphi(t_0)\cdot f(a)}{\varphi(t)}\right),$$
$$M(y,z,t)=M(z,y,t)=f^{(-1)}\left(\frac{d(y,z)}{\varphi(t)}\right)=f^{(-1)}\left(\frac{\varphi(t_0)\cdot f(b)}{\varphi(t_0)}\right).$$

Corollary 4.3 ensures that $(X,M,\diamond)$ is a fuzzy (pseudo-)metric space. Nevertheless, $(X,M,\diamond)$ is not strong because $M(x,z,t_0)=f_*^{(-1)}(f_*(a*b))=a*b$, $M(x,y,t_0)=f_*^{(-1)}(f_*(a))=a$ and $M(y,z,t_0)=f^{(-1)}(f_*(b))=b$. Therefore

$$M(x,y,t_0)\diamond M(y,z,t_0)=a\diamond b>a*b=M(x,z,t_0), \tag{47}$$

since $a\diamond b>a*b$. Consequently **(KM3')** is not satisfied for $t_0\in(0,\infty)$. Thus, $(X,M,\diamond)$ is not strong as we claim. □

An analogous version of the previous theorem for GV-fuzzy metrics is provided below. We omit the proof because it can be obtained following similar arguments.

Theorem 4.6. *Let $\varphi:(0,\infty)\to(0,\infty)$ be an increasing continuous superadditive function and let $*$ be a strict continuous Archimedean t-norm with an additive generator f_*. Then $(X,M,\diamond)$ is a strong GV-fuzzy pseudo-metric space, for each pseudo-metric space (X,d), if and only if $*\geq\diamond$, where the fuzzy set $M: X\times X\times(0,\infty)$ is given, for each $x,y\in X$ and $t\in(0,\infty)$, by*

$$M(x,y,t)=f_*^{(-1)}\left(\frac{d(x,y)}{\varphi(t)}\right). \tag{48}$$

We end the paper stressing that Theorems 4.5 and 4.6 allow to tackle the issue proposed in [31] which consists in finding examples of non strong GV-fuzzy metrics for continuous t-norms greater than $*_P$ and different from $*_M$. Indeed, Theorem 4.6 ensures that we can construct a fuzzy metric space following Example 4 which is not strong for each t-norm greater than the product t-norm. It is worth to mention that the aforesaid issue was also approached in [5] providing an example of GV-fuzzy metric which is not strong for the Hammacher product t-norm $*_H$. In addition, in [5] it was proved that there exist continuous t-norms greater than $*_H$ for which the standard fuzzy metric is strong. Concretely, it was showed that the standard fuzzy metric is strong for the Hammacher product t-norm. Now, by Example 5 and Theorem 4.6 we conclude that the Hammacher product t-norm is the greatest one for which the standard fuzzy metric is strong, for each metric space.

5. Conclusions

Kramosil and Michalek defined a notion of fuzzy metric by means of continuous t-norms. Subsequently, George and Veeramani introduced a modification of such a concept strengthening some of its axioms. Many works have focused their efforts on the study of topological properties and their applications to Engineering problems. However, such an applicability is limited due to lack of examples. So a wide range of examples of fuzzy metrics can enlarge their applicability. Motivated by these facts and by the fact that most instances of fuzzy metrics in the literature are constructed from classical metrics, in this paper we have introduced two new techniques which allow us to construct systematically fuzzy metrics from metrics. We have shown that the celebrated classical method for constructing indistinguishability operators from metrics can be retrieved as a particular case of our new methods. The first introduced method induces strong fuzzy metrics from a given classical one considering continuous Archimedean t-norms and the pseudo-inverse of their additive generators acting on the metric modified by a positive real function. We have proved that the t-norms involved must be, in addition, strict in order to generate strong GV-fuzzy metrics. Moreover, we have extended this technique tackling the particular case of the minimum t-norm, which is continuous but non-Archimedean. In such a construction, two non-negative real functions are now involved in order to modify the classical metric and one of them must be superadditive. In this case, we have proved that the fuzzy metric obtained is not strong in general. Furthermore, the new methods have been illustrated by means of different examples which, in addition, have shown that some celebrated examples of fuzzy metrics are retrieved as a particular case of our methods. Finally, in the light of the developed theory, an open problem about strong fuzzy metrics has been solved completing the partial solutions that can be found in the literature.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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