Extended state observer-based control for systems with locally Lipschitz uncertainties: LMI-based stability conditions

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Abstract

This paper deals with the closed-loop stability of an Extended State Observer (ESO)-based control for systems with locally Lipschitz uncertainties. Novel stability conditions are developed, in terms of Linear Matrix Inequalities, in order to prove local/global Input-to-State stability or local/global Exponential stability, respectively. The stability conditions presented in this paper does not require neither the uncertainty to satisfy the so-called matched condition, nor the system to satisfy some special internal structure, such as the canonical integral chain form. Various LMI-based optimization methodologies are developed in order to optimize the presented results.

Keywords: Extended State Observer, Nonlinear Uncertainty, Disturbance Rejection, Linear Matrix Inequalities.

1. Introduction

It is known that all industrial systems are affected by external disturbances and/or internal uncertainties that bring adverse effects in the controller performance, degrading its nominal behavior, or even causing instability [1, 2, 3, 4]. In order to deal with them, different Disturbance/Uncertainty Estimation and Attenuation (DUEA) techniques have been proposed [1], from which the Extended State Observer (ESO) [5, 6] has become of notably interest. The ESO-based control was proposed as a methodology to estimate, and compensate, for these unknown uncertainties. Since it was proposed, it has been subject to theoretical developments [7, 8, 9], it has been successfully applied in different practical applications [10, 11, 12, 13] and it has become the main core of the Active Disturbance Rejection Control (ADRC) [5, 14].

In spite of these results, there is a lack of numerical methods to guarantee its closed-loop stability when the system is affected by state-dependent uncertainties and external disturbances, which widely appear in practice. Initially, the ESO was proposed for a system expressed as a chain of n integrators, with the uncertainty, $f(\cdot)$, and the control action, $u(t)$, satisfying the so-called matched condition [6]. Its closed-loop stability was firstly guaranteed under the assumption of global boundedness of $\frac{d}{dt}(f(\cdot))$ [15]; an strong assumption that was also taken in the works that followed [16, 17, 18, 19], and was not relaxed until 2009 and 2011 in [20] and [21], respectively. Recent results have indicated that the stability of an ESO-based controller can be guaranteed if the partial derivatives of $f(\cdot)$ are bounded [14, 22, 23].

However, to consider that the system is expressed as a chain of n integrators, satisfying the matched condition, is a strong restriction that cannot be always considered as pointed out in [1, 8, 9]. Therefore, it was recently developed in [8, 9], a Generalized ESO (GESO) for systems with possibly mismatched uncertainties. However, due to the technical difficulties, its closed-loop stability was proved by recovering the restrictive assumption of global boundedness of $\frac{d}{dt}(f(\cdot))$. The main problem with this assumption is that, in general, it cannot be strictly guaranteed (a priory) if $f(\cdot)$ is dependent on the system state. Previous works need to consider that the system is originally stable [21], or that the dependency of $f(\cdot)$ on the system state is weak-enough [8, 9], in order to assure the boundedness of $\frac{d}{dt}(f(\cdot))$. However, these are restrictive assumptions that may not hold in some applications.

This paper presents LMI-based stability conditions for the GESO-based control when system is affected by locally Lipschitz uncertainties. In contrast to previous results, the stability conditions do not rely on the assumption of global boundedness of $\frac{d}{dt}(f(\cdot))$. Instead, simple local requirements over its partial derivatives are taken. Also, different LMI-based optimization methodologies, which can be used to get numerical results of the closed-loop response, are given. The presented stability conditions are also be valid for the conventional ESO, or the linear ADRC, since the so-called matched condition, or the plant being expressed in the canonical integral chain form, are particular cases of the problem being considered.

The rest of the paper is structured as follows. Section 1.1 presents the main notation. Section 2 introduces the problem being considered. The main results are given in Sec-

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tion 3, where the stability theorems are developed. In Section 4, different LMI-based optimization methodologies are introduced. Finally, Sections 5 and 6 contain numerical examples and the main conclusions, respectively.

1.1. Notation

Through the paper, $\mathbb{R}^n$ denote the n-dimensional Euclidean space with vector norm $\|\cdot\|$. $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices. The superscript 'T' denotes matrix transposition, while the notation $P \succ 0$ means that P is positive definite. The symmetric elements of a symmetric matrix are denoted by $(*)$, while the maximum and minimum eigenvalues of a given matrix, P, are denoted by $\overline{\lambda}(P)$ and $\underline{\lambda}(P)$, respectively.

Let $\xi \triangleq [x^T, e_o^T]^T \in \mathbb{R}^{2n+q}$, with $x \in \mathbb{R}^n$ and $e_o \in \mathbb{R}^{n+q}$. A symmetric matrix $0 \prec P_i \in \mathbb{R}^{(2n+q)\times(2n+q)}$, defines an ellipsoid in $\mathbb{R}^{2n+q}$ given by

$$\mathcal{E}_i \triangleq \left\{\xi \in \mathbb{R}^{2n+q} \,\middle|\, \xi^T P_i \xi \le k_i,\ k_i > 0\right\},$$

whose projection onto $\mathbb{R}^n$, $\mathcal{E}_i^{\perp}$, is automatically defined by $P_i^{\perp} \in \mathbb{R}^n$:

$$\mathcal{E}_i^{\perp} \triangleq \left\{x \in \mathbb{R}^n \,\middle|\, x^T P_i^{\perp} x \le k_i,\ k_i > 0\right\}.$$

2. Problem formulation

Let us consider the following class of non-linear systems:

$$\begin{cases} \dot{x} = Ax + B_u u + B_f f(x,\omega(t)), \\ y = Cx \end{cases} \tag{1}$$

where $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ is the system state; $u \in \mathbb{R}^m$ is the control action; $y \in \mathbb{R}^p$ is the measurable output; $A \in \mathbb{R}^{n\times n}$, $B_u \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ are the nominal system matrices; $\omega(t) : \mathbb{R}_{\ge 0} \to \mathbb{R}^r$ is a differentiable time-varying function representing the external disturbances; $f : \mathcal{A}\times\mathbb{R}^r \to \mathbb{R}^q$ is a possibly non-linear function, differentiable in $\mathcal{A}\times\mathbb{R}^r$, for some domain $\mathcal{A} \subseteq \mathbb{R}^n$ containing the origin; while $B_f \in \mathbb{R}^{n\times q}$ indicates which state derivatives are affected by $f(x,\omega(t))$.

The function $f(x,\omega(t))$ represents an unknown term that contains the internal uncertainties and external disturbances. The main control purpose is to stabilize system (1), while being actively compensating for $f(x,\omega(t))$. To this purpose, the next control law is considered [8]:

$$u = K_x\hat{x} + K_f\hat{f}, \tag{2}$$

where K_x is a state feedback gain, K_f is a disturbance feed-forward gain and $\hat{x}$, $\hat{f}$ are estimates of x and $f(x,\omega(t))$, respectively.

In order to get the estimates $\hat{x}$, $\hat{f}$, note that, for all $x \in \mathcal{A}$, system (1) can be equivalently represented in the extended-state form:

$$\begin{cases} \dot{\eta} = \bar{A}\eta + \bar{B}_u u + \bar{B}_f \dot{f}(x,\omega(t)), \\ y = \bar{C}\eta, \end{cases} \tag{3}$$

where

$$\begin{aligned} &\eta \triangleq [x^T,\ f^T(x,\omega(t))]^T, \\ &\bar{A} \triangleq \begin{bmatrix} A & B_f \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_u \triangleq \begin{bmatrix} B_u \\ 0 \end{bmatrix}, \quad \bar{B}_f \triangleq \begin{bmatrix} 0 \\ I_q \end{bmatrix}, \\ &\bar{C} \triangleq \begin{bmatrix} C & 0 \end{bmatrix}, \\ &\dot{f}(x,\omega(t)) \triangleq \frac{\partial f}{\partial x}\big(x,\omega(t)\big)\dot{x} + \frac{\partial f}{\partial \omega}\big(x,\omega(t)\big)\dot{\omega}(t). \end{aligned}$$

This allows to construct the following ESO [8], which provides the desired estimates:

$$\dot{\hat{\eta}} = \bar{A}\eta + \bar{B}_u u + L(y - \bar{C}\hat{\eta}), \tag{4}$$

where $L \in \mathbb{R}^{(n+q)\times p}$ is the observer gain.

Let us consider the following assumptions:

Assumption 1. The pair (A, B_u) is controllable.

Assumption 2. The pair (A, C) is observable and $\operatorname{rank}\left(\begin{bmatrix} A & B_f \\ C & 0 \end{bmatrix}\right) = n + q$.

Assumption 1 guarantees that K_x can be found such that $(A + B_u K_x)$ is Hurwitz, while Assumption 2 guarantees the observability of $(\bar{A}, \bar{C})$, as it was shown in [24].

3. Closed-loop stability

Let us define the observation error as

$$e_o \triangleq \begin{bmatrix} e_{o,x} \\ e_{o,f} \end{bmatrix} \triangleq \begin{bmatrix} x - \hat{x} \\ f(x,\omega(t)) - \hat{f} \end{bmatrix} = \eta - \hat{\eta}. \tag{5}$$

By differentiating (5) and substituting (3) and (4), the observation error dynamics are given by

$$\dot{e}_o = \big(\bar{A} - L\bar{C}\big)e_o + \bar{B}_f\dot{f}(x,\omega(t)). \tag{6}$$

The control action in (2) can be rewritten as

$$\begin{aligned} u = K_x\hat{x} + K_f\hat{f} &= K_x x + K_f f(x,\omega(t)) - K_x e_{o,x} - K_f e_{o,f} \\ &= K_x x + K_f f(x,\omega(t)) + E_o e_o \end{aligned} \tag{7}$$

where $E_o \triangleq -\big[K_x,\ K_f\big]$.

Finally, by substituting (7) into (1) and incorporating (6), the following closed-loop is obtained:

$$\dot{\xi} = \Phi_c\xi + \Gamma_1 f(x,\omega(t)) + \Gamma_2\dot{f}(x,\omega(t)), \tag{8}$$

where

$$\begin{aligned} &\xi \triangleq [x^T,\ e_o^T]^T, \quad \Phi_c \triangleq \begin{bmatrix} A + B_u K_x & B_u E_o \\ 0 & \bar{A} - L\bar{C} \end{bmatrix}, \\ &\Gamma_1 \triangleq \begin{bmatrix} B_u K_f + B_f \\ 0 \end{bmatrix}, \qquad \Gamma_2 \triangleq \begin{bmatrix} 0 \\ \bar{B}_f \end{bmatrix}. \end{aligned}$$

In the next sections, the local/global stability of the closed-loop (8) is analyzed. To this purpose, let us define an n-dimensional ball, $\mathcal{B}_r \triangleq \left\{x \in \mathbb{R}^n \,\middle|\, \|x\| \le r\right\} \subseteq \mathcal{A}$, $r > 0$, and let us state the following assumption, being needed for well-posedness problem formulation:

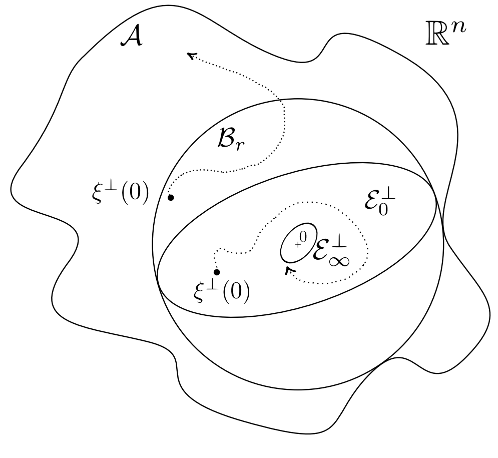


Figure 1: Illustration of the sets being considered in this problem.

Assumption 3. Under Assumptions 1-2 and the control law (2), (4), and in the absence of external disturbances (i.e. $\omega(t) = 0$, $\dot{\omega}(t) = 0$), the state $x^* = 0$ is the unique equilibrium point of (8) in $\mathcal{B}_r$. $\triangledown$

3.1. Input-to-State Stability.

In order to prove ISS, let us consider that

Assumption 4. There exist scalars, $\beta_f \geq 0$, $\beta_{\dot{\omega}} \geq 0$, $\beta_{dx} \geq 0$, $\beta_{d\omega} \geq 0$, such that,

$$\|f(x, \omega(t))\| \leq \beta_f, \qquad \|\dot{\omega}(t)\| \leq \beta_{\dot{\omega}}$$
$$\left\|\frac{\partial f}{\partial x}(x, \omega(t))\right\| \leq \beta_{dx}, \quad \left\|\frac{\partial f}{\partial \omega}(x, \omega(t))\right\| \leq \beta_{d\omega},$$

for all $x \in \mathcal{B}_r$, $t \geq 0$. $\triangledown$

Assumption 4 states that $f(x, \omega(t))$ and its partial derivatives are bounded in $\mathcal{B}_r \times \mathbb{R}_{\geq 0}$ (not necessarily globally bounded). This also implies that $f(x, \omega(t))$ is Lipschitz in $\mathcal{B}_r \times \mathbb{R}_{\geq 0}$ and, if the control action is chosen such that $x(t)$ does not leave $\mathcal{B}_r$, it also ensures the existence and uniqueness of the solution of (1) for all $t \geq 0$ [25]. Note that, in contrast to previous works [8, 21], whose stability results rely on different assumptions that imply the boundedness of $\dot{f}(x, \omega(t))$; Assumption 4 just guarantees the following worst-case upper bound: $\|\dot{f}(x, \omega(t))\| \leq \beta_{d\omega}\beta_{\dot{\omega}} + \beta_{dx}\dot{x}(t)$, for all $x \in \mathcal{B}_r$ and $t \geq 0$.

Now, let us recall the next well-known result, being needed for the subsequent analysis:

Lemma 1. *(ISS). Define $V(\xi(t)) = \xi(t)^T P \xi(t)$, with $P \succ 0$. Let $\bar{V}(t) \triangleq V(\xi(t))$ be absolutely continuous and let $g_1(x(t), t)$, $g_2(x(t), t)$ be essentially bounded functions, i.e. $\|g_1(x(t), t)\| \leq \alpha_1$, $\|g_2(x(t), t)\| \leq \alpha_2$, for all $t \geq 0$, with $\alpha_1 \geq 0$, $\alpha_2 \geq 0$. If there exist $\delta > 0, \gamma_1 \geq 0, \gamma_2 \geq 0$ such that*

$$\dot{\bar{V}}(t) + \delta\bar{V}(t) - \gamma_1\|g_1(x(t), t)\|^2 - \gamma_2\|g_2(x(t), t)\|^2 \leq 0, \ \forall t \geq 0$$

then, the ellipsoid $\mathcal{E} \triangleq \left\{\xi \in \mathbb{R}^{2n+q} \,|\, \xi^T P \xi \leq \frac{\gamma_1\alpha_1^2 + \gamma_2\alpha_2^2}{\delta}\right\}$, is a positively invariant and exponentially attractive set, with decay rate $\delta/2$, for $\xi(t)$.

Proof. The proof is similar to the one presented in Lemma 4.1 of [26], where the term $b\|\omega(t)\|^2$ is substituted by $\gamma_1\|g_1(x(t), t)\|^2 + \gamma_2\|g_2(x(t), t)\|^2$. $\square$

The above lemma is employed in the next theorem, which represents conditions for the local ISS of (1) controlled by (2), (4).

Theorem 1. *(Local ISS). Let $i \triangleq \{0, \infty\}$. Under Assumptions 1-4, given any δ_i, let there exist positive definite $P_i \in \mathbb{R}^{(2n+q)\times(2n+q)}$ and scalars $\tau_i > 0$, $\gamma_{1_i} \geq 0$, $\gamma_{2_i} \geq 0$, that satisfy the following LMIs:*

$$\Psi_{iss}^i \triangleq \begin{bmatrix} \psi_{iss}^i & P_i\Gamma_1 + \tau_i\beta_{dx}^2\Delta_\xi^T\Delta_f & P_i\Gamma_2 & P_i\Gamma_2 \\ (*) & -\gamma_{1_i} + \tau_i\beta_{dx}^2\Delta_f^T\Delta_f & 0 & 0 \\ (*) & (*) & -\tau_i & 0 \\ (*) & (*) & (*) & -\gamma_{2_i} \end{bmatrix} \preceq 0, \quad (9)$$

being $\psi_{iss}^i \triangleq P_i\Phi_c + \Phi_c^T P_i + \delta_i P_i + \tau_i\beta_{dx}^2\Delta_\xi^T\Delta_\xi$, $\Delta_\xi \triangleq [(A + B_u K_x),\ B_u E_o]$ and $\Delta_f \triangleq (B_u K_f + B_f)$.

Assume additionally that

$$r_i \triangleq \sqrt{\frac{\gamma_{1_i}\beta_f^2 + \gamma_{2_i}(\beta_{d\omega}\beta_{\dot{\omega}})^2}{\underline{\lambda}(P_i^\perp)\delta_i}} < r. \quad (10)$$

Then, for all states ξ starting from the initial ellipsoid

$$\mathcal{E}_0 \triangleq \left\{\xi \in \mathbb{R}^{2n+q} \,|\, \xi^T P_0 \xi \leq \frac{\gamma_{1_0}\beta_f^2 + \gamma_{2_0}(\beta_{d\omega}\beta_{\dot{\omega}})^2}{\delta_0}\right\},$$

the solution $x(t)$ of the closed-loop system (1)-(2), (4), does not leave the ball $\mathcal{B}_r$ and it exponentially approaches, with a decay rate $\delta_\infty/2$, to the attractive ellipsoid

$$\mathcal{E}_\infty^\perp \triangleq \left\{x \in \mathbb{R}^n \,|\, x^T P_\infty^\perp x \leq \frac{\gamma_{1_\infty}\beta_f^2 + \gamma_{2_\infty}(\beta_{d\omega}\beta_{\dot{\omega}})^2}{\delta_\infty}\right\}.$$

Proof. See Appendix B. $\square$

Figure 1 represents an illustration of the sets that are being considered in this problem. The domain $\mathcal{A}$ is the region of $\mathbb{R}^n$ in which $f(x, \omega(t))$ is continuously differentiable and, therefore, is the set in which the extended state representation (3) is equivalent to the original system (1). $\mathcal{B}_r$ is the ball where Assumptions 3-4 hold. Theorem 1 states that, if (9)-(10) hold, then for any $\xi(0) \triangleq [x^T(0), e_o^T(0)]^T \in \mathcal{E}_0$, the state $x(t)$ does not leave $\mathcal{B}_r$ and it approaches to $\mathcal{E}_\infty^\perp$. If $\xi(0) \notin \mathcal{E}_0$, convergence is not guaranteed as the state could leave $\mathcal{B}_r$.

Conditions (9)-(10) have a simple meaning. Condition (9) guarantees that the obtained ellipsoids, $\mathcal{E}_0$, $\mathcal{E}_\infty$, are positively invariant and exponentially attractive, i.e. any trajectory $\xi(t)$ starting inside the ellipsoids is kept inside them for all $t \geq 0$, while trajectories starting outside approach to them. Condition (10) guarantees that $\mathcal{E}_0^\perp$ and $\mathcal{E}_\infty^\perp$ are strictly inside $\mathcal{B}_r$. It is clear that, if (9)-(10) are satisfied, then the stability result of Theorem 1 hold.

Note also that Theorem 1 defines two i-independent sets of parameters, i.e. $s_i \triangleq \{P_i, \delta_i, \tau_i, \gamma_{1_i}, \gamma_{2_i}\}$ with $i = 0, \infty$, that may satisfy (9)-(10). The parameters in s_0 define the set of allowable initial states $\mathcal{E}_0$, while the parameters in s_∞ define the terminal ellipsoid $\mathcal{E}_\infty$. This provides an additional degree of freedom so that Theorem 1 can be optimized to find $\mathcal{E}_0^\perp$ as large as possible and $\mathcal{E}_\infty^\perp$ as small as possible (as depicted in Figure 1). In Section 4, different optimization methodologies to address this issue are introduced.

Finally, the next corollaries can be established. Corollary 1 shows that local ISS is guaranteed for weak-enough uncertainties if $A+B_uK_x$ and $\bar{A}-L\bar{C}$ are Hurwitz. Corollaries 2 and 3 represent simplified stability conditions for the cases of matched uncertainties and $\mathcal{A} \equiv \mathcal{B}_r \equiv \mathbb{R}^n$, respectively.

Corollary 1. *(Local ISS for weak-enough uncertainties). Consider that Assumption 4 is satisfied with a small-enough β_f, β_{dx} and $\beta_{d\omega}\beta_{\dot{\omega}}$. Then, if $A+B_uK_x$ and $\bar{A}-L\bar{C}$ are Hurwitz, the solution $x(t)$ of the closed-loop system* (1)-(2), (4) *is locally ISS for any $\xi(0)$ sufficiently close to the origin.*

Proof. Since $A+B_uK_x$ and $\bar{A}-L\bar{C}$ are Hurwitz, given any δ_i, there exist P_i such that $P_i\Phi_c+\Phi_c^TP_i+\delta_iP_i \prec 0$. Then, by Schur complements, $\Psi_{iss}^i \prec 0$ for large enough γ_{1_i}, γ_{2_i}, τ_i, and small enough β_{dx}. On the other hand, for a given P_i, γ_{1_i}, γ_{2_i}, δ_i, condition (10) is satisfied if β_f and $\beta_{d\omega}\beta_{\dot{\omega}}$ are sufficiently small. Therefore, if the initial state is chosen sufficiently close to the origin, then $\xi(0) \in \mathcal{E}_0$ and Theorem 1 is verified. □

Corollary 2. *(Local ISS for matched uncertainties). Consider that $B_f = B_u$, $K_f = -I$. Given any δ_i, $i \triangleq \{0,\infty\}$, let there exist positive definite $P_i \in \mathbb{R}^{(2n+q)\times(2n+q)}$ and scalars $\tau_i > 0$, $\gamma_{2_i} \geq 0$, that satisfy the following LMIs:*

$$\bar{\Psi}_{iss}^i \triangleq \begin{bmatrix} \psi_{iss}^i & P_i\Gamma_2 & P_i\Gamma_2 \\ (*) & -\tau_i & 0 \\ (*) & (*) & -\gamma_{2_i} \end{bmatrix} \preceq 0. \quad (11)$$

Assume additionally that $r_i = \sqrt{\frac{\gamma_{2_i}(\beta_{d\omega}\beta_{\dot{\omega}})^2}{\underline{\lambda}(P_i^\perp)\delta_i}} < r$. Then, for any arbitrarily large β_f, and for all states ξ starting from $\mathcal{E}_0 = \left\{\xi \in \mathbb{R}^{2n+q} \,\middle|\, \xi^TP_0\xi \leq \frac{\gamma_{2_0}(\beta_{d\omega}\beta_{\dot{\omega}})^2}{\delta_0}\right\}$, the solution $x(t)$ of the closed-loop system (1)-(2), (4), *does not leave the ball $\mathcal{B}_r$ and it exponentially approaches, with a decay rate $\delta_\infty/2$, to the attractive ellipsoid $\mathcal{E}_\infty^\perp = \left\{x \in \mathbb{R}^n \,\middle|\, x^TP_\infty^\perp x \leq \frac{\gamma_{2_\infty}(\beta_{d\omega}\beta_{\dot{\omega}})^2}{\delta_\infty}\right\}$.*

Proof. If $B_u = B_f$ and $K_f = -I$, then $\Gamma_1 \equiv 0$ and $\Delta_f \equiv 0$. In this case the LMI in (9) is reduced to (11), subject to $\gamma_{1_i} \geq 0$. As γ_{1_i} is a free parameter, it can be set $\gamma_{1_i} = 0$, which completes the proof. □

Corollary 3. *(Global ISS). Consider that $\mathcal{A} \equiv \mathcal{B}_r \equiv \mathbb{R}^n$, i.e. $r \to \infty$. Given any δ_∞, let there exist a positive definite $P_\infty \in \mathbb{R}^{(2n+q)\times(2n+q)}$ and scalars $\tau_\infty > 0$, $\gamma_{1_\infty} \geq 0$, $\gamma_{2_\infty} \geq 0$, that satisfy $\Psi_{iss}^\infty \preceq 0$. Then, for any initial state, the solution $x(t)$ of the closed-loop system* (1)-(2), (4), *exponentially approaches, with a decay rate $\delta_\infty/2$, to the attractive ellipsoid $\mathcal{E}_\infty^\perp$.*

Proof. Since $r \to \infty$, condition (10) holds in any case. Also, if there exist δ_∞, P_∞, τ_∞, γ_{1_∞}, γ_{2_∞} such that $\Psi_{iss}^\infty \preceq 0$, it can be always set $\delta_0 \leq \delta_\infty$, $P_0 = P_\infty$, $\tau_0 = \tau_\infty$, $\gamma_{1_0} = \gamma_{1_\infty}$ and $\gamma_{2_{01}} = \gamma_{2_\infty}$; which satisfy $\Psi_{iss}^0 \preceq 0$. By decreasing δ_0, the set of allowable initial states, $\mathcal{E}_0$, can be made arbitrarily large. □

3.2. Exponential Stability

In order to prove ES, let us consider that

Assumption 5. There exist scalars, $\beta_{\dot{\omega}} \geq 0$, $\beta_{dx} \geq 0$, and matrices, $\Pi_1 \in \mathbb{R}^{n\times n}$, $\Pi_2 \in \mathbb{R}^{n\times n}$, such that

$$\|f(x,\omega(t))\| \leq \|\Pi_1 x\|, \qquad \left\|\frac{\partial f}{\partial x}(x,\omega(t))\right\| \leq \beta_{dx},$$
$$\|\dot{\omega}(t)\| \leq \beta_{\dot{\omega}}, \qquad \left\|\frac{\partial f}{\partial \omega}(x,\omega(t))\dot{\omega}(t)\right\| \leq \beta_{\dot{\omega}}\|\Pi_2 x\|,$$

for all $x \in \mathcal{B}_r$, $t \geq 0$. ▽

Assumption 5 is stronger than 4 as it further restricts the class of uncertainties to those whose terms $f(x,\omega(t))$ and $\frac{\partial f}{\partial \omega}(x,\omega(t))\dot{\omega}(t)$ vanish when the state goes to zero.

In the same way, let us recall the next result being needed for the subsequent analysis:

Lemma 2. *(ES). Define $V(\xi(t)) = \xi(t)^TP\xi(t)$, with $P \succ 0$. Let $\bar{V}(t) \triangleq V(\xi(t))$ be absolutely continuous. If there exists $\delta > 0$ such that*

$$\dot{\bar{V}}(t) + \delta\bar{V}(t) \leq 0, \quad \forall\, t \geq 0 \quad (12)$$

then $\xi(t)$ is exponentially stable with decay-rate $\delta/2$.

Proof. The proof follows from Lemma 1. □

The above lemma is employed to establish the following theorem representing conditions for the local ES.

Theorem 2. *(Local ES). Let $i \triangleq \{0,\infty\}$. Under Assumptions 1-3 and 5, given any δ_i, let there exist positive definite $P_i \in \mathbb{R}^{(2n+q)\times(2n+q)}$ and scalars $\tau_{1_i} \geq 0$, $\tau_{2_i} \geq 0$, $\tau_{3_i} \geq 0$, that satisfy the following LMIs:*

$$\Psi_{es}^i \triangleq \begin{bmatrix} \psi_{es}^i & P_i\Gamma_1 + \tau_{1_i}\beta_{dx}^2\Delta_\xi^T\Delta_f & P_i\Gamma_2 & P_i\Gamma_2 \\ (*) & -\tau_{2_i} + \tau_{1_i}\beta_{dx}^2\Delta_f^T\Delta_f & 0 & 0 \\ (*) & (*) & -\tau_{1_i} & 0 \\ (*) & (*) & (*) & -\tau_{3_i} \end{bmatrix} \preceq 0, \quad (13)$$

being

$$\psi_{es}^i \triangleq P_i\Phi_c + \Phi_c^TP_i + \delta_iP_i + \tau_{1_i}\beta_{dx}^2\Delta_\xi^T\Delta_\xi + H^T(\tau_{2_i}\Pi_1^T\Pi_1 + \tau_{3_i}\beta_{\dot{\omega}}^2\Pi_2^T\Pi_2)H.$$

Then, for all states ξ starting from the initial ellipsoid $\bar{\mathcal{E}}_0 \triangleq \left\{\xi \in \mathbb{R}^{2n+q} \,\middle|\, \xi^TP_0\xi \leq \underline{\lambda}(P_0^\perp)r^2\right\}$, the solution $x(t)$ of the closed-loop system (1)-(2), (4), *does not leave the ball $\mathcal{B}_r$ and it is exponentially stable with a decay rate $\delta_\infty/2$.*

PROOF. See Appendix B. □

Theorem 2 also defines two i-independent set of parameters, i.e. $\bar{s}_i \triangleq \{P_i, \delta_i, \tau_{1_i}, \tau_{2_i}, \tau_{3_i}\}$ with $i = 0, \infty$; which may satisfy (13). The set $\bar{s}_0$ should be optimized so that the initial ellipsoid is obtained as large as possible, while the set $\bar{s}_\infty$ should be optimized so that the higher exponential decay rate is obtained. These optimization issues are discussed in Section 4.

Remark 1. The same arguments employed in Corollaries 1-3 could be reproduced for the ES.

4. Optimization issues

In this section, different optimization problems are introduced in order to check, and optimize, the stability conditions presented in Theorems 1 and 2.

4.1. Numerical optimization of Theorem 1

Let $s_i \triangleq \{P_i, \delta_i, \tau_i, \gamma_{1_i}, \gamma_{2_i}\}$, with $i = 0, \infty$, be the sets of parameters in Theorem 1 that should be optimized. The set s_∞ is optimized such that $\mathcal{E}_\infty$ is minimized [27]. This can be performed by solving

$$\begin{aligned} \max_{\{s_\infty, \alpha\}} \quad & \alpha \\ \text{s.t. } & \Psi_{iss}^\infty \preceq 0,\ P_\infty \succ \alpha I_{2n+q}, \\ & \gamma_{1_\infty}\beta_f^2 + \gamma_{2_\infty}(\beta_{d\omega}\beta_{\dot\omega})^2 = \delta_\infty, \\ & \alpha > 0,\ \delta_\infty > 0,\ \tau_\infty > 0,\ \gamma_{1_\infty} \geq 0,\ \gamma_{2_\infty} \geq 0. \end{aligned} \tag{14}$$

The first constrain assures that condition (9) of Theorem 1 is satisfied. The second constraint assures that $\underline{\lambda}(P_\infty) \geq \alpha$. The third constraint forces that r_∞ in (10) takes the form $r_\infty = \sqrt{\frac{1}{\underline{\lambda}(P_\infty^\perp)}} \leq \sqrt{\frac{1}{\alpha}}$. So, if α is maximized, then r_∞ is being minimized. Finally, note that the feasibility of (14) guarantees (9) but not (10), which should be checked with the obtained values in s_∞.

The set s_0 is optimized such that $\mathcal{E}_0$ is maximized. This can be done by minimizing the condition number of P_0 (so that $\mathcal{E}_0$ is as similar as possible to an sphere), while forcing condition (10) to be strictly satisfied, i.e. as an equality. In this way, the largest ellipsoid such that its projection strictly fits inside $\mathcal{B}_r$ is obtained. This can be performed by solving the next optimization problem for different fixed (and decreasing) values of α, until the following equality holds $r_0 = r$.

$$\begin{aligned} \min_{\{s_0, \alpha, \gamma\}} \quad & \gamma, \\ \text{s.t. } & \Psi_{iss}^0 \preceq 0,\ P_0 = \alpha P,\ \gamma I_{2n+q} \succeq P \succeq I_{2n+q}, \\ & \gamma_{1_0}\beta_f^2 + \gamma_{2_0}(\beta_{d\omega}\beta_{\dot\omega})^2 = \delta_0, \\ & \gamma \geq 1, \alpha > 0,\ \delta_0 > 0,\ \tau_0 > 0,\ \gamma_{1_0} \geq 0,\ \gamma_{2_0} \geq 0, \end{aligned} \tag{15}$$

The second and third constraints force $\mathcal{E}_0$ to have the following form $\mathcal{E}_0 = \left\{\xi \in \mathbb{R}^{2n+q} \,\middle|\, \xi^T(\alpha P)\xi \leq 1\right\}$. Hence, if γ is minimized, then P_0 is forced to be as similar as possible to an sphere; while, by decreasing α, $\mathcal{E}_0$ is enlarged; so α should be chosen such that (10) strictly holds.

4.2. Numerical optimization of Theorem 2.

Let $\bar{s}_i \triangleq \{P_i, \delta_i, \tau_{1_i}, \tau_{2_i}, \tau_{3_i}\}$, with $i = 0, \infty$, be the set of parameters in Theorem 2 that should be optimized. The set s_∞ is optimized such that δ_∞ is maximized. This can be performed by solving:

$$\begin{aligned} \max_{\bar{s}_\infty} \quad & \delta_\infty, \\ \text{s.t. } & \Psi_{es}^\infty \preceq 0, \\ & P_\infty \succ 0,\ \delta_\infty > 0,\ \tau_{1_\infty} > 0, \tau_{2_\infty} > 0, \tau_{3_\infty} > 0. \end{aligned} \tag{16}$$

The set s_0 is optimized in order to obtain the largest $\bar{\mathcal{E}}_0$. This can be done by minimizing the condition number of P_0 by solving:

$$\begin{aligned} \min_{s_0, \gamma} \quad & \gamma, \\ \text{s.t. } & \Psi_{es}^0 \preceq 0,\ P_0 \succ 0,\ \gamma I_{2n+q} \succeq P_0 \succeq I_{2n+q}, \\ & \delta_0 > 0,\ \tau_{1_0} > 0, \tau_{2_0} > 0, \tau_{3_0} > 0. \end{aligned} \tag{17}$$

5. Numerical examples

5.1. Example 1

Let us consider the following system:

$$\begin{cases} \dot{x}(t) = x(t) + u(t) + \beta x(t) + \omega(t), & \forall\, |x| < r, \\ y = x, \end{cases} \tag{18}$$

where $\beta \geq 0$ is an unknown parameter and $\omega(t) = \sin(t)$ represents the external disturbance.

The control law (2), (4) is applied with $L = [41,\ 400]^T$, $K_x = -2$ and $K_f = -1$. Let us consider $r = 1$. For all $x \in \mathcal{B}_r$ and $t \geq 0$, Assumption 4 is satisfied with

$$\begin{aligned} &\|f(x, \omega(t))\| \leq \beta + 1, && \|\dot\omega(t)\| \leq 1 \\ &\left\|\frac{\partial f}{\partial x}(x, \omega(t))\right\| \leq \beta, && \left\|\frac{\partial f}{\partial \omega}(x, \omega(t))\right\| \leq 1, \end{aligned}$$

Theorem 1 is applied to check the closed-loop ISS. Consider an upper bound of $\beta \leq 2$. Theorem 1 is optimized according to (14)-(15). The resulting s_0, s_∞ are presented in Table 5.1. Since both s_0, s_∞ satisfy (9)-(10) (concretely (10) is satisfied with $r_\infty = 0.15 < 1$ and $r_0 = 0.99 < 1$), the results of Theorem 1 hold.

A simulation result is presented in Figure 2, where system (18), with $\beta = 2$, is controlled under (2), (4). The initial state is set to $x(0) = \hat{x}(0) = x_0$ and $\hat{f}(0) = 0$. It is verified that $\xi(0) \in \mathcal{E}_0$ for all $x_0 \leq 0.625$, so the simulation is performed with $x_0 = 0.625$. It can be seen that the the simulation results match with the results given by Theorem 1.

On the other hand, Theorem 1 can be also employed to get robustness properties of the closed-loop response

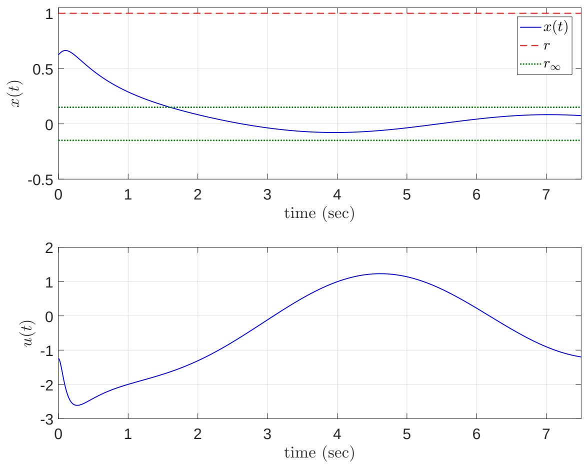


Figure 2: Simulation of Example 5.1.

P_∞	δ_∞	τ_∞	γ_{1_∞}	γ_{2_∞}
$\begin{bmatrix} 45.4 & -248 & 2.95 \\ -248 & 151e3 & -3.17e3 \\ 2.95 & -3.17e3 & 81.7 \end{bmatrix}$	1.33	5.57	0	1.33
P_0	δ_0	τ_0	γ_{1_0}	γ_{2_0}
$\begin{bmatrix} 1 & 0.14 & -0.052 \\ 0.14 & 13.4 & -0.70 \\ -0.052 & -0.70 & 0.15 \end{bmatrix}$	0.5	0.1	0	0.5

Table 1: Example 5.1. Results of the optimization for $\beta = 2$.

against the uncertain parameter β. It is verified that the closed-loop becomes unstable for $\beta \geq 10.17$. By evaluation of Theorem 1 it is found that none set, s_i, satisfying (9), can be found if $\beta > 10.04$, which is remarkable.

5.2. Example 2.

Let us consider the system presented in [8], which, in order to satisfy Assumption 3, is conveniently rewritten after translating its equilibrium point up to the origin:

$$\begin{cases} \dot{x}_1 = x_2 + f(x), \\ \dot{x}_2 = -2x_1 - x_2 + u(t), \\ y = x_1, \end{cases}$$

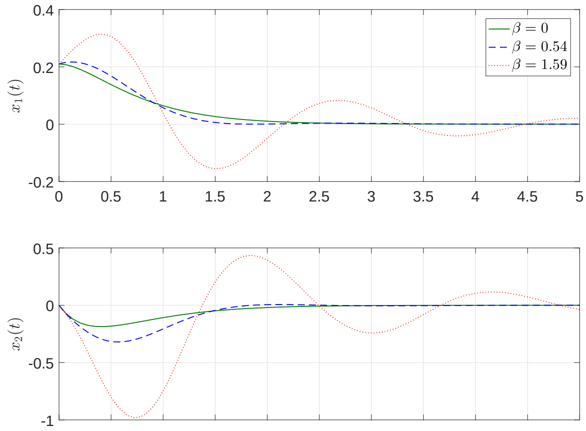
being $f(x) \triangleq e^{\beta x_1} - 1$, and $\beta \geq 0$ a constant unknown parameter.

As proposed in [8], the observer gain is set to $L = [14, -66, 125]^T$, the state-feedback gain is set to $K_x = [-4, -4]$ and the disturbance feed-forward gain is set to $K_f = -5$. Let us fix $\mathcal{B}_r$ of radius $r = 1$. For all $x \in \mathcal{B}_r$ and $t \geq 0$, Assumption 5 is satisfied with

$$\|f(x, \omega(t))\| \leq \left\| \begin{bmatrix} e^{\beta} - 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot x \right\|, \qquad \|\dot{\omega}(t)\| = 0,$$

$$\left\| \frac{\partial f}{\partial x}(x, \omega(t)) \right\| \leq \beta e^{\beta}, \quad \left\| \frac{\partial f}{\partial \omega}(x, \omega(t))\dot{\omega}(t) \right\| \leq \|0_n x\|,$$

Theorem 2 is optimized by solving (16)-(17) in order to get robustness properties of the closed-loop against the uncertain parameter β. It is found that none set of parameters, $\bar{s}_i$, satisfying (13), can be found if $\beta > 0.54$.



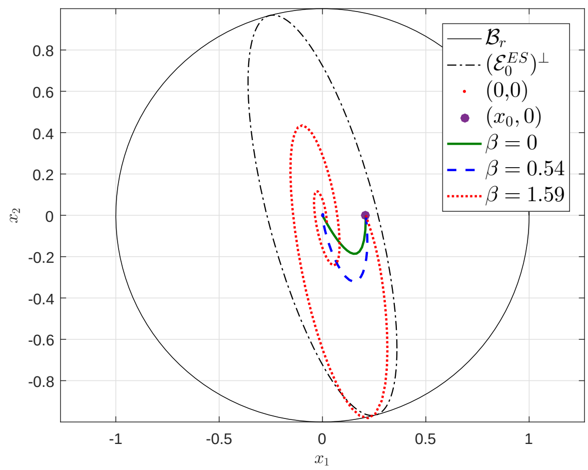


Figure 4: Representation of $\mathcal{B}_r$, $(\mathcal{E}_0^{ES})^\perp$ and state-trajectories for different values of β.

Simulation results are depicted in Figures 3-4. The initial state is set to $x_1(0) = x_0$, $\hat{x}_1(0) = x_0$, $x_2(0) = 0$, $\hat{x}_2(0) = 0$ and $\hat{f}(0) = 0$. It can be checked that, for $x_0 \leq 0.21$, this initial state belongs to the allowable set of initial states for all $\beta < 0.54$; so it is set $x_0 = 0.21$. Figures 3-4 depict simulation results for $\beta = 0$, $\beta = 0.54$, and $\beta = 1.59$, respectively. The trajectories leave $\mathcal{B}_r$ for $\beta > 1.59$ and the closed-loop becomes unstable for $\beta > 2.04$.

6. Conclusions

In this paper, different LMI-based stability conditions for a generalized extended state observer-based control have been developed. The provided stability conditions does not rely on the assumption of global boundedness of the total disturbance derivative. Furthermore, they can be easily optimized by LMI solvers to get numerical properties of the closed-loop behavior. The results presented in this paper are also valid for the conventional Extended State Observer or the Active Disturbance Rejection Control, since both techniques are particular cases of the problem being considered.

Acknowledgments

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Appendix A. Proof of Theorem 1

By Lemma 1. Let us consider the Lyapunov function $V_i(\xi) = \xi^T P_i \xi$, $P_i \succ 0$, and let us set $g_1(x,t) = f(x,\omega(t))$, $g_2(x,t) = \frac{\partial f}{\partial \omega}\big(x,\omega(t)\big)\dot{\omega}(t)$ with $\alpha_1 = \beta_f$ and $\alpha_2 = \beta_{d\omega}\beta_{\dot{\omega}}$. If there exist $\gamma_{1_i} > 0$, $\gamma_{2_i} > 0$, $\delta_i > 0$ such that

$$\dot{V}_i + \delta_i V_i - \gamma_{1_i}\|f(x,\omega(t))\| - \gamma_{2_i}\left\|\frac{\partial f}{\partial \omega}\big(x,\omega(t)\big)\dot{\omega}(t)\right\|^2 \le 0, \quad \text{(A.1)}$$

for all $t \ge 0$, then, the ellipsoid

$$\mathcal{E}_i \triangleq \left\{ \xi \in \mathbb{R}^{2n+q} \,\middle|\, \xi^T P_i \xi \le \frac{\gamma_{1_i}\beta_f^2 + \gamma_{2_i}(\beta_{d\omega}\beta_{\dot{\omega}})^2}{\delta_i} \right\} \quad \text{(A.2)}$$

is positively invariant and exponentially attractive.

Hence, the proof is reduced to show how (9)-(10) imply (A.1). The derivative of V_i, with $\dot{\xi}$ substituted by (8) is

$$\begin{aligned}\dot{V}_i =& \xi^T(P_i\Phi_c + \Phi_c^T P_i)\xi + 2\xi^T P_i \Gamma_1 f + 2\xi^T P_i \Gamma_2 df_x \\ &+ 2\xi^T P_i \Gamma_2 df_t,\end{aligned} \quad \text{(A.3)}$$

where, for simplicity, $f \triangleq f(x,\omega(t))$, $df_x \triangleq \frac{\partial f}{\partial x}\big(x,\omega(t)\big)\dot{x}$, and $df_t \triangleq \frac{\partial f}{\partial \omega}\big(x,\omega(t)\big)\dot{\omega}(t)$.

Substituting (A.3) into (A.1) leads to

$$\begin{aligned}&\xi^T(P_i\Phi_c + \Phi_c^T P_i)\xi + 2\xi^T P_i\Gamma_1 f + 2\xi^T P_i \Gamma_2 df_x + \\ &\qquad + 2\xi^T P_i \Gamma_2 df_t + \delta_i \xi^T P_i \xi - \gamma_{1_i} f^T f - \gamma_{2_i} df_t^T df_t \le 0,\end{aligned}$$

being expressed as $\phi^T(\Psi^i_{iss,0})\phi \le 0$, $\phi \triangleq [\xi^T, f, df_x, df_t]^T$,

$$\Psi^i_{iss,0} \triangleq \begin{bmatrix} P_i\Phi_c + \Phi_c^T P_i + \delta_i P_i & P_i\Gamma_1 & P_i\Gamma_2 & P_i\Gamma_2 \\ (*) & -\gamma_{1_i} & 0 & 0 \\ (*) & (*) & 0 & 0 \\ (*) & (*) & (*) & -\gamma_{2_i} \end{bmatrix}$$

So, if $\Psi^i_{iss,0} \preceq 0$, then (A.1) is satisfied. The next step follows by the application of the S-procedure in the term

df_x. It is known that

$$\begin{aligned} df_x &= \frac{\partial f}{\partial x}\big(x,\omega(t)\big)\,[Ax + B_u u + B_f f] \\ &= \frac{\partial f}{\partial x}\big(x,\omega(t)\big)[Ax + B_u(K_x(x - e_{o,x}) + K_f(f - e_{o,f})) + B_f f] \\ &= \frac{\partial f}{\partial x}\big(x,\omega(t)\big)[\Delta_\xi \xi + \Delta_f f], \end{aligned}$$

with $\Delta_\xi \triangleq [(A + B_u K_x),\, B_u E_o]$, $\Delta_f \triangleq (B_u K_f + B_f)$.

Hence, by Assumption 4 and for all $x \in \mathcal{B}_r$, the following upper bound can be established:

$$df_x^T df_x \leq \beta_{dx}^2 \left[\xi^T \Delta_\xi^T \Delta_\xi \xi + 2\xi^T \Delta_\xi^T \Delta_f f + f^T \Delta_f^T \Delta_f f\right], \tag{A.4}$$

which can be written as $\phi^T(\Psi^i_{iss,1})\phi \leq 0$, with

$$\Psi^i_{iss,1} \triangleq \begin{bmatrix} -\beta_{dx}^2 \Delta_\xi^T \Delta_\xi & -\beta_{dx}^2 \Delta_\xi^T \Delta_f & 0 & 0 \\ (*) & -\beta_{dx}^2 \Delta_f^T \Delta_f & 0 & 0 \\ (*) & (*) & 1 & 0 \\ (*) & (*) & (*) & 0 \end{bmatrix} \tag{A.5}$$

The knowledge of $\Psi^i_{iss,1} \preceq 0$ implies that $\Psi^i_{iss,0} \preceq 0$ if there exist $\tau_i > 0$ such that $\phi^T \Psi^i_{iss,0}\phi \leq \tau_i \phi^T \Psi^i_{iss,1}\phi \leq 0$. This holds if $\phi^T(\Psi^i_{iss,0} - \Psi^i_{iss,1})\phi \leq 0$, leading to (9).

So, if (9) holds, then the ellipsoid (A.2) is attractive and positively invariant. Condition (10) follows from a short analysis of the sets that are being considered in this problem. Lemma 1 considers that $g_1(x,t) = f(x,\omega(t))$ and $g_2(x,t) = \frac{\partial f}{\partial \omega}(x,\omega(t))\dot{\omega}(t)$ are bounded. However, by Assumption 4, this bounds can be only established in $\mathcal{B}_r$. So it must be required that $x(t)$ lies inside $\mathcal{B}_r$ for all $t \geq 0$.

As mentioned in Section 1.1, the ellipsoid (A.2) has a projection onto $\mathbb{R}^n$ given by

$$\mathcal{E}_i^\perp \triangleq \left\{ x \in \mathbb{R}^n \,\middle|\, x^T P_i^\perp x \leq \frac{\gamma_{1_i}\beta_f^2 + \gamma_{2_i}(\beta_{d\omega}\beta_{\dot{\omega}})^2}{\delta_i} \right\}.$$

Therefore, it must be imposed that $\mathcal{E}_i^\perp \subseteq \mathcal{B}_r$, which is satisfied if (10) holds. Finally, the theorem follows by defining two independent solutions, given by $i \triangleq \{0, \infty\}$, such that both satisfy (9)-(10). In this case, two attractive ellipsoids, i.e $\mathcal{E}_0$, $\mathcal{E}_\infty$; are obtained. If the initial state is restricted to be inside $\mathcal{E}_0$, then $x(t) \in \mathcal{B}_r$ for all $t \geq 0$, and $x(t)$ approaches to $\mathcal{E}_\infty$ with a exponential rate $\delta_\infty/2$.

Appendix B. Proof of Theorem 2

By Lemma 2. Let us consider the Lyapunov function $V_i = \xi^T P_i \xi$, where the derivative $\dot{V}_i$, which is given by (A.3), is substituted into (12) and it is expressed as $\phi^T(\Psi^i_{es,0})\phi \leq 0$, with $\phi \triangleq [\xi^T, f, df_x, df_t]^T$ and

$$\Psi^i_{es,0} = \begin{bmatrix} P\Phi_c + \Phi_c^T P + \delta P & P\Gamma_1 & P\Gamma_2 & P\Gamma_2 \\ (*) & 0 & 0 & 0 \\ (*) & (*) & 0 & 0 \\ (*) & (*) & (*) & 0 \end{bmatrix}$$

The term df_x satisfies the inequality (A.4), which leads to $\phi^T(\Psi^i_{iss,1})\phi \leq 0$, being $\Psi^i_{iss,1}$ defined in (A.5). The terms f and df_t satisfy the inequalities in Assumption 5, which lead to $\phi^T(\Psi^i_{es,1})\phi \leq 0$, $\phi^T(\Psi^i_{es,2})\phi \leq 0$, with

$$\Psi^i_{es,1} = \begin{bmatrix} -H^T \Pi_1^T \Pi_1 H & 0 & 0 & 0 \\ (*) & 1 & 0 & 0 \\ (*) & (*) & 0 & 0 \\ (*) & (*) & (*) & 0 \end{bmatrix}$$

$$\Psi^i_{es,2} = \begin{bmatrix} -\beta_{\dot{\omega}}^2 H^T \Pi_2^T \Pi_2 H & 0 & 0 & 0 \\ (*) & 0 & 0 & 0 \\ (*) & (*) & 0 & 0 \\ (*) & (*) & (*) & 1 \end{bmatrix}$$

where $H \triangleq [I_n,\, 0_{n\times(n+q)}]^T$ is defined so that $x = H\xi$.

By the $\mathcal{S}$-procedure, the knowledge of $\Psi^i_{iss,1} \preceq 0$, $\Psi^i_{es,1} \preceq 0$, $\Psi^i_{es,2} \preceq 0$, implies that $\Psi^i_{es,0} \preceq 0$ if there exist $\tau_1 > 0$, $\tau_2 > 0$ and $\tau_3 > 0$ such that

$$\phi^T \Psi^i_{es,0}\phi \leq \tau_1 \phi^T \Psi^i_{iss,1}\phi + \tau_2 \phi^T \Psi^i_{es,1}\phi + \tau_3 \phi^T \Psi^i_{es,2}\phi \leq 0,$$

leading to (13).

Finally, similarly to Theorem 1, the proof follows by defining two independent solutions that satisfy (13); while the set of allowable initial states is defined so that it strictly fits inside $\mathcal{B}_r$.