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Original paper

A multidimensional approach to rank fuzzy numbers based on the concept of magnitude

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Abstract

Ranking fuzzy numbers have become of growing importance in recent years, especially as decision-making is increasingly performed under greater uncertainty. In this paper, we extend the concept of magnitude to rank fuzzy numbers to a more general definition to increase flexibility and generality. More precisely, we propose a multidimensional approach to rank fuzzy numbers considering alternative magnitude definitions with three novel features: Multidimensionality, normalization, and a ranking based on a parametric distance function. A multidimensional magnitude definition allows us to consider multiple attributes to represent and rank fuzzy numbers. Normalization prevents meaningless comparisons among attributes due to scaling problems, and the use of the parametric Minkowski distance function becomes a more general and flexible ranking approach. The main contribution of our multidimensional approach is the representation of a fuzzy number as a point in a n-dimensional normalized space of attributes in which the distance to the origin is the magnitude value. We illustrate our methodology and provide further insights into different normalization approaches and parameters through several numerical examples. Finally, we describe an application of our ranking approach to a multicriteria decision-making problem within an economic context in which the main goal is to rank a set of credit applicants considering different financial ratios used as evaluation criteria.

Keywords: Ranking, magnitude, multiple dimensions, normalization, fuzzy economics, credit ranking.

1 Introduction

In order to deal with the imprecision of real world, Zadeh [31] proposed the use of fuzzy sets in which there is no sharp separation between complete membership and complete non-membership. Fuzzy sets are characterized by a membership function expressing the degree of membership of an element to the set. This degree of membership usually ranges between zero and one representing, respectively, the lowest and highest degrees of membership. Different extensions of fuzzy sets have been developed and applied in different areas, such as interval-valued fuzzy sets [23, 32], intuitionistic fuzzy sets [3, 4], and neutrosophic fuzzy sets [20]. Other recent approaches are hesitant fuzzy sets [22], and picture fuzzy sets [10].

A fuzzy number is a special case of fuzzy sets on the real line with a normal, fuzzy convex and continuous membership function of bounded support. Triangular and trapezoidal fuzzy numbers are examples of generalized fuzzy numbers whose membership functions present a special set of features [7, 13]. When dealing with generalized fuzzy numbers, we usually need to compare them in order to establish a ranking or an ordering. Ranking fuzzy numbers is considered as one of the key problems of fuzzy set theory [30]. Many ranking methods map fuzzy numbers into a real value and then ranking two fuzzy numbers reduces to comparing the values resulting from this mapping. In this context, Yager [29] proposed the defuzzification by means of the centroid of fuzzy numbers; Chu and Tsao [9] suggested finding the area between the centroid point and the origin to ranking fuzzy numbers; and Wang and Lee [28] proposed a two-step

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procedure based first on the degree of representative location and then on the average height of fuzzy numbers. Later on, Veeramani et al. [25] compared ranking methods and proposed a direct formula for ordering the generalized fuzzy number based on weighted expected value.

Abbasbandy and Hajjari [2] introduced the method of the magnitude for ranking trapezoidal fuzzy numbers, which was later extended to non-normal trapezoidal fuzzy numbers and generalized fuzzy numbers in [14]. Later on, Hajjari [15] also relied on the concept of magnitude to solve some of the limitations of the previous ranking approaches when dealing with symmetric trapezoidal fuzzy numbers with identical mode or with identical centroid points. On the other hand, the concept of possibilistic mean value and variance for fuzzy numbers was derived from possibility theory by Carlsson and Fuller [6]. These concepts were further developed and applied to triangular fuzzy numbers in Wan et al. [26]. Gu and Xuan [13] also proposed a new method to rank fuzzy numbers based on the concept of magnitude. The authors defined the concept of magnitude as the sum of the mean value and the standard deviation as a synthetic measure for ranking purposes. Their proposal was illustrated by comparing the ranking obtained for different data sets of fuzzy numbers proposed by Veeramani et al. [25]. The notion of magnitude has been recently generalized by Reig-Mullor et al. [18] to consider not only the sum of the mean and the standard deviation using an additive function, but possibly many other attributes of fuzzy numbers such as the height, and other type of functions such as the product of attributes. Finally, Stocklasa et al. [21] studied the relationship between possibilistic and other high order standard moments of fuzzy numbers.

Taking the notion of the generalized magnitude proposed by Reig-Mullor et al. [18] and distance-based approaches for ranking purposes by Abbasbandy and Asady [1] as a starting point, we here move one step forward in the theoretical and practical development of the concept of magnitude. From a theoretical perspective, one may hypothesize that alternative magnitude definitions may result in a different ranking of fuzzy numbers. Furthermore, to allow comparisons among magnitudes in practice, there is a need for a methodology to rank fuzzy numbers when using multiple attributes to build a particular definition. In this paper, we propose a methodology to deal with magnitudes that integrate multiple attributes to rank fuzzy numbers. Our approach is based on three main steps:

1. A multidimensional representation of fuzzy numbers.
2. Normalization to avoid bias towards any attribute.
3. The use of a parametric distance function to rank fuzzy numbers.

The first step in our methodology is descriptive and proposes a new way to represent fuzzy numbers in a n-dimensional space when n crisp attributes are used to summarize them. For instance, a fuzzy number can be represented in a bidimensional space when the possibilistic mean is measured by the horizontal axis and the possibilistic standard deviation is measured by the vertical axis. A key point addressed by our approach is the possible bias towards one of the attributes that may be present when the scale of an attribute's measure is very large when compared to the others. We here propose to avoid meaningless comparisons among attributes with different scales by means of normalization. More precisely, we consider max-min normalization, max normalization and percentage normalization. Finally, we rely on the use of the Minkowski parametric distance function to rank fuzzy numbers as a way of generalization. Indeed, the proposal by Gu and Xuan [13] is a particular case of our general approach when only the possibilistic mean and standard deviation are considered, when no normalization method is applied, when a key parameter of the Minkowski distance function is set to one, and when no weighting scheme is used.

The main result of this paper is a multidimensional methodology to rank fuzzy numbers derived from the concept of magnitude. In order to illustrate our approach, we develop several numerical examples. More precisely, we compare alternative magnitude definitions to highlight the ability to represent fuzzy numbers by the set of options considered. We also evaluate the performance of different normalization methods such as the max-min normalization, max normalization, and percentage normalization. We use four different data sets of fuzzy numbers extracted from Veeramani et al. [25] and Gu and Xuan [13], and Hajjari [15]. We also compare the impact of setting a key parameter of the Minkowski distance function to the most widely used values. Finally, we illustrate how our multidimensional approach solves the bias problem and introduces flexibility by describing an application to a multicriteria decision-making problem in which the main goal is to rank a set of credit applicants considering different financial ratios used as evaluation criteria.

Summarizing, this paper contributes to extend the one-dimensional concept of magnitude to a multidimensional space in which multiple attributes can be used to represent fuzzy numbers. In a similar way to the alternative magnitude definitions proposed in this paper, researchers have now the possibility to devise new definitions that may prove to provide advantages in some contexts or other types of fuzzy numbers. In addition, researchers can use the methodology proposed in this paper to evaluate these alternative magnitude definitions or different normalization methods. Although the main application of our approach is ranking fuzzy numbers, we argue that additional practical and theoretical results may derive from our multidimensional approach.

This paper is organized as follows. In Section 2, we provide useful background about fuzzy numbers and the notion of magnitude. In Section 3, we propose a methodology to rank fuzzy numbers when using multiple attributes. In Section 4, we numerically compare alternative magnitude definitions, normalization methods and different parameter values to rank fuzzy numbers in several data sets. Finally, Section 5 concludes this paper.

2 Preliminaries

The concept of magnitude for a given fuzzy number $\tilde{A}$, denoted by $Mag(\tilde{A})$, was proposed by Abbasbandy and Hajjari [2], and later developed by other authors as a crisp representation of fuzzy numbers for ranking purposes. To formally define the concept of magnitude, we need to consider the following concepts.

Definition 2.1. (Fuzzy set [31]). *Let X be a non-empty set, the fuzzy set A is expressed as:*

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\}, \tag{1}$$

where $\mu_A(x)$ is the degree of membership of element x to A, defined as a function $\mu_A : X \to [0, 1]$, with 0 and 1 representing the lowest and highest degrees of membership.

Definition 2.2. (Fuzzy number [6]). *A fuzzy number $\tilde{A}$ is a fuzzy set of the real line $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support.*

Definition 2.3. (Generalized fuzzy number [28]) *A generalized fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4, w)$ is a fuzzy subset of universe of discourse X with membership function $\mu_{\tilde{A}}(x)$ defined as follows:*

- *$\mu_{\tilde{A}}(x)$ is a continuous mapping from X to the closed interval $[0, w]$ with $0 < w \leq 1$;*
- *$\mu_{\tilde{A}}(x) = 0$, where $-\infty < x \leq a_1$;*
- *$\mu_{\tilde{A}}(x)$ is strictly increasing on $[a_1, a_2]$;*
- *$\mu_{\tilde{A}}(x) = w$, where $a_2 \leq x \leq a_3$, with $0 < w \leq 1$ constant;*
- *$\mu_{\tilde{A}}(x)$ is strictly decreasing on $[a_3, a_4]$;*
- *$\mu_{\tilde{A}}(x) = 0$, where $a_4 \leq x < +\infty$;*

Definition 2.4. (Arithmetic operations [8]). *Given two generalized trapezoidal fuzzy numbers $\tilde{A} = (a_1, a_2, a_3, a_4, w_{\tilde{A}})$ and $\tilde{B} = (b_1, b_2, b_3, b_4, w_{\tilde{B}})$, the addition operation is defined as follows:*

$$\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, min(w_{\tilde{A}}, w_{\tilde{B}})), \tag{2}$$

and the multiplication operation as:

$$\tilde{A}\tilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4, min(w_{\tilde{A}}, w_{\tilde{B}})), \tag{3}$$

when $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are all positive real numbers.

Definition 2.5. (Support [15]). *The support $supp(\tilde{A})$ of fuzzy number $\tilde{A}$ is defined as the set of all elements that have a nonzero degree of membership in $\tilde{A}$:*

$$supp(\tilde{A}) = \{x \in \mathbb{R} | \mu_{\tilde{A}}(x) > 0\}. \tag{4}$$

Definition 2.6. (Height [19]). *The height $h(\tilde{A})$ of fuzzy number $\tilde{A}$ is defined as the largest membership grade obtained by any element of that set:*

$$h(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x). \tag{5}$$

Definition 2.7. (α-level set [6]). The α-level set of fuzzy number $\tilde{A}$ is defined for $\alpha \in [0, 1]$ by:

$$\left[\tilde{A}\right]_\alpha = \{x \in \mathbb{R} | \mu_{\tilde{A}}(x) \geq \alpha\}. \tag{6}$$

Let $\tilde{A}$ be a fuzzy number with $\left[\tilde{A}\right]_\alpha = [a_L(\alpha), a_R(\alpha)]$, where $a_L(\alpha)$ and $a_R(\alpha)$ are, respectively, the lower (left) and upper (right) bounds of $\left[\tilde{A}\right]_\alpha$, the possibilistic mean and standard deviation are defined as follows:

Definition 2.8. (Possibilistic mean [13]). *The possibilistic mean of $\tilde{A}$ is the level-weighted average of the arithmetic means of all α-level sets*

$$M(\tilde{A}) = \frac{1}{w^2}\int_0^w \alpha(a_L(\alpha) + a_R(\alpha))d\alpha. \tag{7}$$

Definition 2.9. (Possibilistic standard deviation [6]). *The possibilistic standard deviation of $\tilde{A}$ is the square root of the expected value of the squared deviations between the arithmetic mean and the end points of its level α-level sets.*

$$\sigma(\tilde{A}) = \left[\frac{1}{2}\int_0^w \alpha(a_R(\alpha) - a_L(\alpha))^2 d\alpha\right]^{1/2}. \tag{8}$$

For a generalized triangular fuzzy number, denoted by $\tilde{A} = (a_1, a_2, a_3, w)$, equations (7) and (8) reduce to the following expressions [13]:

$$M(\tilde{A}) = \frac{(a_1 + 4a_2 + a_3)}{6}, \tag{9}$$

$$\sigma(\tilde{A}) = \frac{w(a_3 - a_1)}{\sqrt{24}}. \tag{10}$$

For a generalized trapezoidal fuzzy number, denoted by $\tilde{A} = (a_1, a_2, a_3, a_4, w)$, equations (7) and (8) reduce to the following expressions [13]:

$$M(\tilde{A}) = \frac{(a_1 + 2a_2 + 2a_3 + a_4)}{6}, \tag{11}$$

$$\sigma(\tilde{A}) = \frac{w}{\sqrt{24}}\left[(a_4 - a_1)^2 + 2(a_4 - a_1)(a_3 - a_2) + 3(a_3 - a_2)^2\right]^{1/2}. \tag{12}$$

Definition 2.10. (The magnitude method to rank fuzzy numbers [2]). *Given an arbitrary trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$, its magnitude is defined as:*

$$Mag(\tilde{A}) = \frac{1}{2}\int_0^1 f(\alpha)(a_R(\alpha) + a_L(\alpha) + a_2 + a_3)d\alpha, \tag{13}$$

where $f(\alpha)$ is a non-negative and increasing function on $[0,1]$, with $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(\alpha)d\alpha = 1/2$. Function $f(\alpha)$ can be considered a weighting function usually set to $f(\alpha) = \alpha$. The resulting scalar value is used to rank fuzzy numbers $\tilde{A}$ and $\tilde{B}$ according to:

1. $Mag(\tilde{A}) > Mag(\tilde{B}) \iff \tilde{A} \succ \tilde{B}$,
2. $Mag(\tilde{A}) < Mag(\tilde{B}) \iff \tilde{A} \prec \tilde{B}$,
3. $Mag(\tilde{A}) = Mag(\tilde{B}) \iff \tilde{A} \sim \tilde{B}$.

The magnitude method was extended by Hajjari [14] to non-normal trapezoidal fuzzy numbers such that the magnitude for fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4, w)$ reduces to:

$$Mag(\tilde{A}) = \frac{(3w^2 + 2)(a_2 + a_3)}{12w} + \frac{(3w - 2)(a_1 + a_4)}{12w}. \tag{14}$$

In order to solve some of the limitations of the initial magnitude method by [2] when ranking crisp and symmetric fuzzy numbers, Hajjari [15] proposed a new magnitude definition denoted by $Mag_N(\tilde{A}_i)$:

Definition 2.11. (New magnitude definition [15]). *The new magnitude definition is given by the following expression:*

$$Mag_N(\tilde{A}_i) = \gamma(\tilde{A}_i) \cdot D(\tilde{A}_i), \tag{15}$$

where function $\gamma(\tilde{A}_i)$ *is a sign function* $\gamma : S \to \{-1, 1\}$*:*

$$\gamma(\tilde{A}_i) = sign\left[\int_0^1 (a_{Li}(\alpha) + a_{Ri}(\alpha))d\alpha\right], \tag{16}$$

that returns value 1 if $\int_0^1 (a_{Li}(\alpha) + a_{Ri}(\alpha))d\alpha \geq 0$*, and returns value -1 otherwise, and function* $D(\tilde{A}_i)$ *is defined for a group of n fuzzy numbers* $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$*, with* $\tilde{A}_i = (a_{1i}, a_{2i}, a_{3i}, a_{4i})$*, as follows:*

$$D(\tilde{A}_i) = (a_{1i} - x_{min}) + |\frac{1}{2}(a_{2i} + a_{3i}) + val(\tilde{A}_i)| + (x_{max} - a_{4i}), \tag{17}$$

such that $x_{min} = inf(S)$*,* $x_{max} = sup(S)$*,* $S = \cup_{i=1}^n S_i$*,* $S_i = \{x|\tilde{A}_i > 0\}$*, and where* $val(\tilde{A}_i)$ *is the value function of a fuzzy number defined as:*

$$val(\tilde{A}_i) = \int_0^1 f(\alpha)(a_{Li}(\alpha) + a_{Ri}(\alpha))d\alpha. \tag{18}$$

From the definitions of possibilistic mean and standard deviation, Gu and Xuan [13] proposed an alternative magnitude definition for a fuzzy number.

Definition 2.12. (Alternative magnitude definition [13]). *The magnitude of fuzzy number* $\tilde{A}$*, denoted by* $\text{Mag}_A(\tilde{A})$*, is the sum of its possibilistic mean and standard deviation:*

$$\text{Mag}_A(\tilde{A}) = M(\tilde{A}) + \sigma(\tilde{A}). \tag{19}$$

The authors claimed that the concept of magnitude presents some advantages such as the integrity of ordering relation, transitivity of ordering relation and independence of irrelevant fuzzy numbers [13]. Finally, the notion of magnitude has been generalized by Reig-Mullor et al. [18] to extend its scope as a multidimensional concept.

Definition 2.13. (General magnitude [18]). *Given a fuzzy number* $\tilde{A}$*, its general magnitude* $GM(\tilde{A})$ *is expressed by:*

$$GM(\tilde{A}) = f(c_1(\tilde{A}), c_2(\tilde{A}), \ldots, c_n(\tilde{A})), \tag{20}$$

where $f(c_1(\tilde{A}), c_2(\tilde{A}), \ldots, c_n(\tilde{A}))$ *is a function* $f : \mathbb{R}^n \to \mathbb{R}$ *mapping an* n*-dimensional set of attributes or characteristics of* $\tilde{A}$*, denoted by* $F(\tilde{A}) = \{c_1(\tilde{A}), c_2(\tilde{A}), \ldots, c_n(\tilde{A})\}$*, to a scalar.*

3 A multidimensional approach to rank fuzzy numbers

Our multidimensional approach to rank fuzzy numbers is based on the fact that a magnitude definition is formed by one or more attributes of fuzzy numbers. The initial definition by Abbasbandy and Hajjari [2] in equation (13) can be viewed as based on the attribute of value in equation (18). Similarly, the later definition by Hajjari [15] in equation (15) can be viewed as based on the attribute of value, but also on the maximum and minimum of the supports within a group of fuzzy numbers. Finally, the alternative magnitude definition by Gu and Xuan [13] in equation (19) is based on the possibilistic mean and standard deviation as critical attributes. By considering two attributes separately, we can represent fuzzy numbers in a bidimensional space. For instance, measuring the possibilistic mean in the horizontal axis and the possibilistic standard deviation in the vertical axis, fuzzy numbers are represented as a point in the space as shown in Figure 2. In this way, fuzzy numbers that are close together can be considered as belonging to the same class. After normalization to avoid meaningless comparisons between attributes due to problems of scale, we here propose a new method of ranking fuzzy numbers based on the distance of each point to the origin of coordinates. Summarizing, our methodology to rank a set of fuzzy numbers is based on three main steps as graphically described in Figure 1: 1) represent fuzzy numbers as n-dimensional vectors when considering n different attributes; 2) normalize the scale of attributes; 3) rank fuzzy numbers by comparing the weighted parametric Minkowski distance of all fuzzy numbers to the origin. As a result, the larger the distance, the higher the rank of the fuzzy number. The main advantage of our approach is its generality because it allows us to consider many attributes and many distance functions to rank fuzzy numbers.

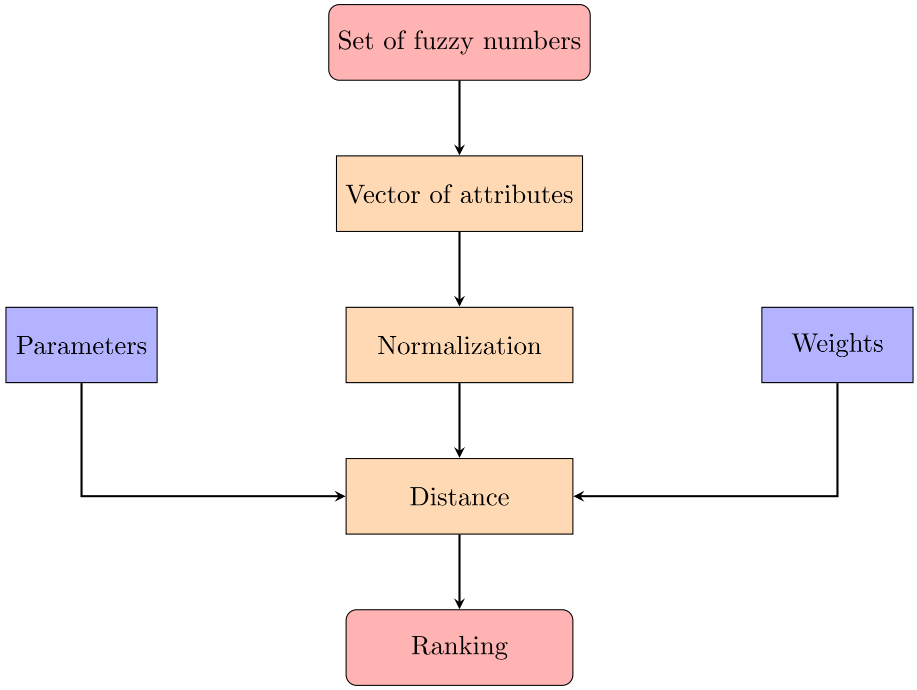

Figure 1: A graphical representation of our multidimensional ranking method.

3.1 Multidimensional representation of fuzzy numbers

In order to derive a multidimensional representation of fuzzy numbers, we rely on the concept of attribute.

Definition 3.1. ***Attribute of a fuzzy number***. *Given fuzzy number $\tilde{A}$, an attribute of $\tilde{A}$, denoted by $c_i(\tilde{A})$, is a mapping $c : \tilde{A} \rightarrow \mathbb{R}$.*

From $\tilde{A} = (a_1, a_2, a_3, a_4, w)$, some examples of attributes that can be extracted from its definition are the possibilistic mean and standard deviation, the height of a fuzzy number given by parameter w, its support given by the interval $[a_1, a_4]$, its skewness and kurtosis that can be computed as described by Stoklasa et al. [21]. Furthermore, parameters a_1, a_2, a_3, a_4 and w are also attributes even though they can be used to derive other attributes as in the case of the possibilistic mean.

As a result, set of attributes $F(\tilde{A}) = \{c_1(\tilde{A}), c_2(\tilde{A}), \ldots, c_n(\tilde{A})\}$ from definition 2.13 is an alternative way of representing fuzzy numbers by means of an arbitrary number of features. Furthermore, $F(\tilde{A})$ forms a space of n dimensions that can be graphically represented when the number of dimensions is three or lower. An example of such a representation for the set of triangular fuzzy numbers in Examples 1 and 2 from Veeramani et al. [25] and Gu and Xuan [13] is shown in Figure 2. We use the horizontal axis to represent the possibilistic mean and the vertical axis to show the possibilistic standard deviation for eight different fuzzy numbers including four triangular fuzzy numbers and four trapezoidal fuzzy numbers. Figure 2 shows an important limitation of the initial definition of magnitude as the sum of the possibilistic mean and standard deviation: magnitudes are biased towards the possibilistic mean, which is the attribute with higher values.

To further illustrate this bias towards the possibilistic mean, Table 1 shows the percentage that both the possibilistic mean and the standard deviation represents with respect to the total magnitude value. On average, the possibilistic mean represents the 84% of the magnitude value. Therefore, we argue that this bias may become an issue when dealing with ranking tasks. Only in the case with fuzzy numbers with equal possibilistic means, the standard deviation will have a fair impact in the ranking. However, in extreme cases with much higher mean values the importance of the standard deviation would be reduced to a negligible value. In order to solve the bias problem due to the different scale of attributes, we next propose the use of a normalization.

3.2 Normalization to solve the bias limitation

When dealing with the aggregation of different attributes for ranking purposes, it is usually convenient to normalize the attributes measures to avoid a meaningless comparison. Very frequently, the scales of the attributes are so different

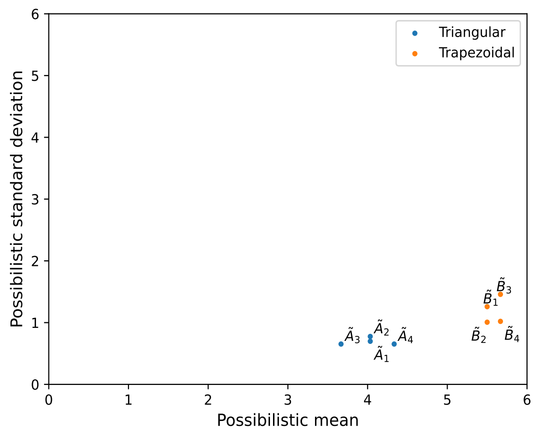

Figure 2: Possibilistic mean-standard deviation space for fuzzy numbers.

Table 1: Possibilistic mean and standard deviation for example fuzzy numbers in [25] and [13].

Set 1 Triangular	Definition	P.Mean	P.Std	Pct Mean	Pct Std
$\tilde{A}_1$	(2.2, 4, 6, 0.9)	4.03	0.70	85%	15%
$\tilde{A}_2$	(2.2, 4, 6, 1)	4.03	0.78	84%	16%
$\tilde{A}_3$	(1, 4, 5, 0.8)	3.67	0.65	85%	15%
$\tilde{A}_4$	(3, 4, 7, 0.8)	4.33	0.65	87%	13%
Set 2 Trapezoidal	**Definition**	**P.Mean**	**P.Std**	**Pct Mean**	**Pct Std**
$\tilde{B}_1$	(3, 5, 6, 8, 1)	5.50	1.26	81%	19%
$\tilde{B}_2$	(3, 5, 6, 8, 0.8)	5.50	1.01	84%	16%
$\tilde{B}_3$	(3, 5, 6, 9, 1)	5.67	1.46	80%	20%
$\tilde{B}_4$	(3, 5, 6, 9, 0.7)	5.67	1.02	85%	15%

that, without a normalization process, the magnitude definition is biased towards the attributes with higher values. Normalization is a data-transformation process that aims to provide a common scale to allow comparisons. A description of the main normalization methods, including vector, linear and non-linear normalization, and logarithmic normalization can be found in Vafaei et al. [24]. Both linear and vector normalization are the most widely used methods.

Given a set of fuzzy numbers $\{\tilde{A}_1, \ldots, \tilde{A}_j, \ldots, \tilde{A}_m\}$ indexed by $j \in [1, m]$, a max-min transformation is achieved by normalizing the i-th attribute value c_{ij} for the j-th fuzzy number to a common scale restricted to the interval $[0, 1]$:

$$\theta_{ij} = \begin{cases} \frac{c_{ij} - \min(c_{ij})}{\max(c_{ij}) - \min(c_{ij})}, & \text{if } \max(c_{ij}) - \min(c_{ij}) > 0 \\ 0, & \text{otherwhise.} \end{cases} \tag{21}$$

Maximum linear normalization is achieved when setting $\min(c_{ij}) = 0$:

$$\theta_{ij} = \begin{cases} \frac{c_{ij}}{\max(c_{ij})}, & \text{if } \max(c_{ij}) \neq 0 \\ 0, & \text{otherwhise.} \end{cases} \tag{22}$$

On the other hand, percentage normalization is achieved by the following transformation:

$$\theta_{ij} = \begin{cases} \frac{c_{ij}}{\sum_{j=1}^{m} c_{ij}}, & \text{if } \sum_{j=1}^{m} c_{ij} \neq 0 \\ 0, & \text{otherwhise.} \end{cases} \tag{23}$$

Note that max-min normalization is restricted to the interval $[0, 1]$ disregarding the sign of the value of attributes. However, maximum and percentage normalization may be affected by the sign of the values of attributes and, because of that, are more appropriate to rank fuzzy numbers with positive attributes.

As a graphical example, Figure 3 depicts the location of possibilistic mean and standard deviation after the max-min normalization method described in equation (21) for the triangular and trapezoidal fuzzy numbers in Figure 2 and

Table 1. From the comparison of the figures, we observe that the bias towards the possibilistic mean shown in Figure 2 is solved by a common normalized scale for the two attributes. Now, we are in a position to meaningfully rank fuzzy numbers by means of a parametric distance function.

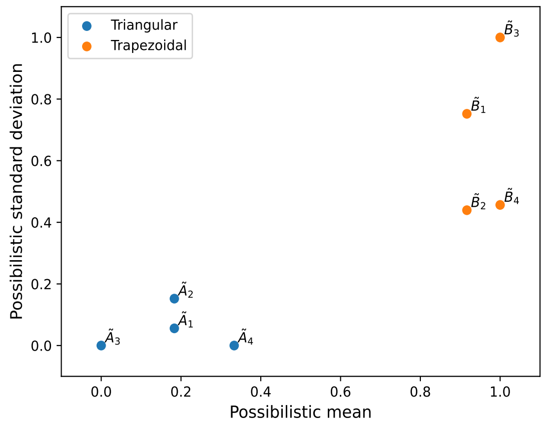

Figure 3: Possibilistic mean-standard deviation normalized space for fuzzy numbers.

3.3 Ranking fuzzy numbers using the Minkowski distance function

Given a set of fuzzy numbers $S = \{\tilde{A}_1, \ldots, \tilde{A}_j, \ldots, \tilde{A}_m\}$, with cardinality m and indexed by $j \in [1, m]$, and a vector of n normalized attributes $\boldsymbol{u}_j(\tilde{A}_j) = \{\theta_{ij}, \ldots, \theta_{nj}\}$ for all $\tilde{A}_j \in S$, we propose to rank fuzzy numbers according to the value the parametric Minkowski distance function as follows:

$$d(\tilde{A}_j, p) = \left[\sum_{i=1}^{n} |\beta_i^p \theta_{ij}^p| \right]^{1/p}, \tag{24}$$

where β_i is the weight attached to the i-th attribute, and parameter p is an integer in the closed interval $[0, \infty]$.

Note that we are implicitly assuming that the longer the distance from any fuzzy number to the origin in a normalized n-dimensional space, the higher the value of the ranking score function encoded in equation (24). Then, we propose a new definition of the concept of magnitude based on parametric distance functions:

Definition 3.2. ***Distance-based magnitude definition.*** *Given set of fuzzy numbers S and distance function $d(\tilde{A}_j, p)$, a distance-based magnitude $Mag_D : S \to \mathbb{R}_{\geq 0}$ is defined for every $\tilde{A}_j \in S$ as follows:*

$$Mag_D(\tilde{A}_j, p) = d(\tilde{A}_j, p). \tag{25}$$

As a result, a ranking for set S is derived from the application of the next preference relations for positive number:

$$\tilde{A}_j \succ \tilde{A}_k \iff Mag_D(\tilde{A}_j, p) > Mag_D(\tilde{A}_k, p), \ \forall \tilde{A}_j, \tilde{A}_k \in S, \tag{26}$$

$$\tilde{A}_k \sim \tilde{A}_j \iff Mag_D(\tilde{A}_j, p)) = Mag_D(\tilde{A}_k, p), \ \forall \tilde{A}_j, \tilde{A}_k \in S. \tag{27}$$

The value of parameter p is a topological metric that results in different distance functions. Let us consider the most relevant cases. For $p = 1$, we are dealing with Manhattan distances to the origin in a normalized space (the sum of the coordinates). For $p = 2$, we are dealing with Euclidean distances (the square root of the sum of the squares of the coordinates). For $p = \infty$, we are dealing Chebyshev distances (the maximum among the coordinates), and for $p = 0$ we are dealing with geometric distances (the product of the coordinates). As a result, if we:

- consider two attributes such as the possibilistic mean and the possibilistic standard deviation ($n = 2$);
- apply no normalization method ($\theta_1 = M(\tilde{A}_j)$ and $\theta_2 = \sigma(\tilde{A}_j)$);

- set $p = 1$ in equation (24); and
- use equal weights for both attributes ($\beta_1, \beta_2 = 1$);

then we obtain the initial magnitude definition by Gu and Xuan [13] as a particular case because we are indeed computing the sum of the equally weighted non-normalized values of the possibilistic mean and standard deviation.

Considering a generalized trapezoidal fuzzy number, its opposite image $-\tilde{A}$, and $Mag_D(\tilde{A}, p)$, we derive the following result:

Proposition 3.3. *Given set $S = \{\tilde{A}, -\tilde{A}\}$, populated with a generalized trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$, with $0 < a_1 < a_2 \le a_3 \le a_4$, and its opposite denoted by $-\tilde{A} = (-a_1, -a_2, -a_3, -a_4)$. Using attributes $M(\tilde{A})$ and $\sigma(\tilde{A})$, and max-min normalization, we have that $Mag_D(\tilde{A}, p) > 0$, and $Mag_D(-\tilde{A}, p) = 0, \forall p \in [0, \infty]$.*

Proof. Let $\theta_1(\tilde{A})$ and $\theta_2(\tilde{A})$ be the normalized attributes for $M(\tilde{A})$ and $\sigma(\tilde{A})$. First, we have that $M(\tilde{A}) > M(-\tilde{A})$ because $M(\tilde{A}) = -M(-\tilde{A})$. Then, $max\{M(\tilde{A}), M(-\tilde{A})\} = M(\tilde{A})$ and $min\{M(\tilde{A}), M(-\tilde{A})\} = M(-\tilde{A})$. According to equation (21), $\theta_1(A) = 1$ and $\theta_1(-A) = 0$. Following a similar reasoning, $\theta_2(A) = 1$ and $\theta_2(-A) = 0$. As a result, $Mag_D(\tilde{A}, p)$ is strictly positive and $Mag_D(-\tilde{A}, p) = 0$ for all integer values of p in the interval $[0, \infty]$ because $d(\tilde{A}, p) = [\beta_1^p \theta_1^p(A) + \beta_2^p \theta_2^p(A)]^{1/p} = [\beta_1^p + \beta_2^p]^{1/p}$ is strictly positive when weights β_1 and β_2 are positive. □

The weighting system $\{\beta_1, \beta_2, \ldots, \beta_n\}$ allows to control for the importance that each attribute has in the ranking function described in equation (24). Even though we are computing distances in a normalized space, there is no limitation to give more weight to attributes that are considered more important for a particular application. It must be noted that using a weighting system different to equal weights introduces bias in the ranking function. However, in contrast to the approach by Gu and Xuan [13], the bias is controlled and applied after the normalization process.

3.4 Reasonable properties

The discussion about the rationality of an ordering procedure was formalized by Wang and Kerre [27] in a set of reasonable properties for ordering method $\mathcal{M}$ applied to a set of fuzzy numbers S, being Γ a finite subset of S:

- P_1: For an arbitrary finite subset Γ of S and $\tilde{A} \in \Gamma$, $\tilde{A} \succeq \tilde{A}$ by $\mathcal{M}$ on Γ.
- P_2: For an arbitrary finite subset Γ of S and $(\tilde{A}, \tilde{B}) \in \Gamma^2$, $\tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{A}$ by $\mathcal{M}$ on Γ, we should have $\tilde{A} \sim \tilde{B}$ by $\mathcal{M}$ on Γ.
- P_3: For an arbitrary finite subset Γ of S and $(\tilde{A}, \tilde{B}, \tilde{C}) \in \Gamma^3$, $\tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{C}$ by $\mathcal{M}$ on Γ, we should have $\tilde{A} \succeq \tilde{C}$ by $\mathcal{M}$ on Γ.
- P_4: For an arbitrary finite subset Γ of S and $(\tilde{A}, \tilde{B}) \in \Gamma^2$, $\inf supp(\tilde{A}) > \sup supp(\tilde{B})$, we should have $\tilde{A} \succeq \tilde{B}$ by $\mathcal{M}$ on Γ.
- P'_4: For an arbitrary finite subset Γ of S and $(\tilde{A}, \tilde{B}) \in \Gamma^2$, $\inf supp(\tilde{A}) > \sup supp(\tilde{B})$, we should have $\tilde{A} \succ \tilde{B}$ by $\mathcal{M}$ on Γ.
- P_5: Let S and S' be two arbitrary finite sets of fuzzy quantities in which $\mathcal{M}$ can be applied and $\tilde{A}$ and $\tilde{B}$ are in $S \cap S'$. We obtain the ranking order $\tilde{A} \succ \tilde{B}$ by $\mathcal{M}$ on S' iff $\tilde{A} \succ \tilde{B}$ by $\mathcal{M}$ on S.
- P_6: Let $\tilde{A}$, $\tilde{B}$, $\tilde{A} + \tilde{C}$, and $\tilde{B} + \tilde{C}$ be elements of S. If $\tilde{A} \succeq \tilde{B}$ by $\mathcal{M}$ on $\{\tilde{A}, \tilde{B}\}$, then $\tilde{A} + \tilde{C} \succeq \tilde{B} + \tilde{C}$ by $\mathcal{M}$ on $\{\tilde{A} + \tilde{C}, \tilde{B} + \tilde{C}\}$.
- P'_6: If $\tilde{A} \succ \tilde{B}$ by $\mathcal{M}$ on $\{\tilde{A}, \tilde{B}\}$, then $\tilde{A} + \tilde{C} \succ \tilde{B} + \tilde{C}$ by $\mathcal{M}$ on $\{\tilde{A} + \tilde{C}, \tilde{B} + \tilde{C}\}$ when $\tilde{C} \neq \emptyset$.
- P_7: Let $\tilde{A}$, $\tilde{B}$, $\tilde{A}\tilde{C}$, and $\tilde{B}\tilde{C}$ be elements of S and $\tilde{C} \ge 0$. $\tilde{A} \succeq \tilde{B}$ by $\mathcal{M}$ on $\{\tilde{A}, \tilde{B}\}$, implies that $\tilde{A}\tilde{C} \succeq \tilde{B}\tilde{C}$ by $\mathcal{M}$ on $\{\tilde{A}\tilde{C}, \tilde{B}\tilde{C}\}$.

The fact that any ranking method $\mathcal{M}$ has properties P_1 to P_7 depends on the definition of magnitude. In this sense, Abbasbandy and Hajjari [2] proved that the initial proposal of magnitude in Definition 2.10 has properties P_1 to P_6. Similarly, Hajjari [15] claimed that the Mag_N function in Definition 2.11 has properties P_1 to P_6. Gu and Xuan [13] stated that the alternative magnitude definition based on the concepts of possibilistic mean and variance in Definition **??** has properties P_2, P_3 and P_5. However, it is not difficult to show that Definition **??** has properties P_1, P_4, P'_4, P_6 and P'_6.

In this paper, we follow a general approach in which we consider alternative definitions of the notion of magnitude based on a set of attributes. However, we provide the following result for one of the definitions considered in Section 4.

Theorem 3.4. *Given set* $S = \{\tilde{A}_1, \ldots, \tilde{A}_j, \ldots, \tilde{A}_m\}$ *of non-normal triangular fuzzy numbers characterized by set of attributes* $\tilde{A}_j = (a_{1j}, a_{2j}, a_{3j}, w)$, *with* $0 \leq a_{1j} \leq a_{2j} \leq a_{3j}$, *and* $0 < w \leq 1$, *linear maximum normalization* $\theta_{ij} = a_{ij}/\max_j(a_{ij})$, *for* $i = 1, 2, 3$, *and* $\theta_{4j} = w, \forall \tilde{A}_j \in S$, *distance-based magnitude definition with* $p = 1$, *computed as* $Mag_D(\tilde{A}_j, 1) = d(\tilde{A}_j, 1) = \sum_{i=1}^{4} |\theta_{ij}|$ *has properties* P_1 *to* P_6.

Proof. $Mag_D(\tilde{A}_j)$ has property P_1 because $d(\tilde{A}_j, 1)$ results in the same value for the same fuzzy number. Property P_2 holds because $\tilde{A}_j \sim \tilde{A}_k \iff d(\tilde{A}_j, 1) = d(\tilde{A}_k, 1)$. Property P_3 holds because if $d(\tilde{A}_j, 1) \geq d(\tilde{A}_k, 1)$, and $d(\tilde{A}_k, 1) \geq d(\tilde{A}_l, 1)$, then it necessarily follows that $d(\tilde{A}_j, 1) \geq d(\tilde{A}_l, 1)$. Property P_4' holds because if $\inf supp(\tilde{A}_j) > \sup supp(\tilde{A}_k)$, then $a_{1j} > b_{1k}$, $a_{2j} > b_{2k}$, and $a_{3j} > b_{3k}$, resulting in $d(\tilde{A}_j, 1) \geq d(\tilde{A}_k, 1)$ because w is assumed to be constant for all fuzzy numbers in S. Since P_4' is stronger than P_4, property P_4' implies that P_4 also holds. P_5 holds because the ranking order of two fuzzy numbers is independent of any other fuzzy number in S. P_6 holds because the addition of fuzzy number $\tilde{A}_l$ to fuzzy numbers $\tilde{A}_j$ and $\tilde{A}_k$ implies that $\tilde{A}_j + \tilde{A}_l = (a_{1j} + a_{1l}, a_{2j} + a_{2l}, a_{3j} + a_{3l}, w)$ and $\tilde{A}_k + \tilde{A}_l = (a_{1k} + a_{1l}, a_{2k} + a_{2l}, a_{3k} + a_{3l}, w)$. Consequently, $d(\tilde{A}_j + \tilde{A}_l, 1) = |a_{1j} + a_{1l}| + |a_{2j} + a_{2l}| + |a_{3j} + a_{3l}| + w$, and $d(\tilde{A}_k + \tilde{A}_l, 1) = |a_{1k} + a_{1l}| + |a_{2k} + a_{2l}| + |a_{3k} + a_{3l}| + w$. As a result, $\tilde{A}_j \succeq \tilde{A}_k$ also implies that $\tilde{A}_j + \tilde{A}_l \succeq \tilde{A}_k + \tilde{A}_l$ because we are adding (or subtracting) the same quantity to each of the components respectively. Note that we are dealing with vectors of four components and their distances to the origin. Then, adding a third vector to a pair of vectors does not change the initial ranking of distances to the origin. Following a similar reasoning, P_6' holds. □

Remark 3.5. P_7 *does not hold in general for* $Mag_D(\tilde{A}_j) = d(\tilde{A}_j, 1) = \sum_{i=1}^{4} |\theta_{ij}|$ *because we can easily find a counter-example. Given* $\tilde{A}_1 = (1, 4, 6, 1)$, $\tilde{A}_2 = (3, 4, 5, 1)$ *and* $\tilde{A}_3 = (1, 2, 3, 1)$, *we obtain, without normalization for ease of understanding,* $d(\tilde{A}_1) = 12$ *and* $d(\tilde{A}_2) = 13$, *hence* $\tilde{A}_2 \succ \tilde{A}_1$, *and* $d(\tilde{A}_1\tilde{A}_3) = 28$ *and* $d(\tilde{A}_2\tilde{A}_3) = 27$, *hence* $\tilde{A}_1\tilde{A}_3 \succ \tilde{A}_2\tilde{A}_3$.

Proposition 3.6. *The ranking method* $Mag_D(\tilde{A}_j, p) = d(\tilde{A}_j, p)$ *defined over set* $S = \{\tilde{A}_1, \ldots, \tilde{A}_j, \ldots, \tilde{A}_m\}$ *of fuzzy numbers is a map from* S *to the set of real numbers* $\mathbb{R}$ *which is closed under real scalar multiplication.*

Proof. $Mag_D(\tilde{A}_j, p)$ maps fuzzy numbers in S to a vector space in $\mathbb{R}^n$. Each element $\boldsymbol{u}_j = (\theta_{1j}, \ldots, \theta_{nj}) \in \mathbb{R}^n$ is mapped to the real line $\mathbb{R}$ through p-distance $d(\tilde{A}_j, p) = \left[\sum_{i=1}^{n} |\beta_i^p \theta_{ij}^p|\right]^{1/p}$. Given scalar $k \in \mathbb{R}$, we have that $d(k\tilde{A}_j, p)$ is also an element of $\mathbb{R}$ because:

$$d(k\tilde{A}_j, p) = \left[\sum_{i=1}^{n} |\beta_i^p k^p \theta_{ij}^p|\right]^{1/p} = \left[k^p \sum_{i=1}^{n} |\beta_i^p \theta_{ij}^p|\right]^{1/p} = kd(\tilde{A}_j, p) \in \mathbb{R}. \quad (28)$$

As a result, the set of individual ranks derived from $Mag_D(\tilde{A}_j, p)$ is closed under real scalar multiplication. □

4 Numerical examples

In this section, we compare alternative magnitude definitions, normalization techniques, and typical values for parameter p to rank fuzzy numbers. We also compare our method with two recent ranking methods based on binary relations. Finally, we describe a real case study in the field of ranking credit applicants.

4.1 Ranking for alternative magnitude definitions

The first two data sets A and B taken from Gu and Xuan [13] are populated with non-normal triangular and trapezoidal fuzzy numbers as summarized in Table 1. Furthermore, we consider data sets C and D used by Hajjari [15]. Data set C is populated with four normal triangular fuzzy numbers $\tilde{C}_1 = (-4, 0, 4)$, $\tilde{C}_2 = (-2, 0, 2)$, $\tilde{C}_3 = (0, 1, 1.9)$ and $\tilde{C}_4 = (0, 2, 2.9)$. In addition, data set D includes crisp fuzzy number $\tilde{D}_1 = (1, 1, 1)$, and two symmetric fuzzy numbers $\tilde{D}_2 = (0, 1, 2)$ and $\tilde{D}_3 = (-1, 1, 3)$. All fuzzy numbers in the four data sets are shown in Figure 4. To compare the ranking obtained by different magnitude definitions, we consider the following alternatives:

1. The initial magnitude definition proposed by Abbasbandy and Hajjari [2], denoted by Mag.
2. The new magnitude definition proposed by Hajjari [15], denoted by Mag_N.
3. The alternative magnitude definition proposed by Gu and Xuan [13], denoted by Mag_A.
4. Several distance-based magnitude definitions proposed in this work, denoted by Mag_D, combining different attributes, normalization methods, and usual parameters in the Minkowski distance function.

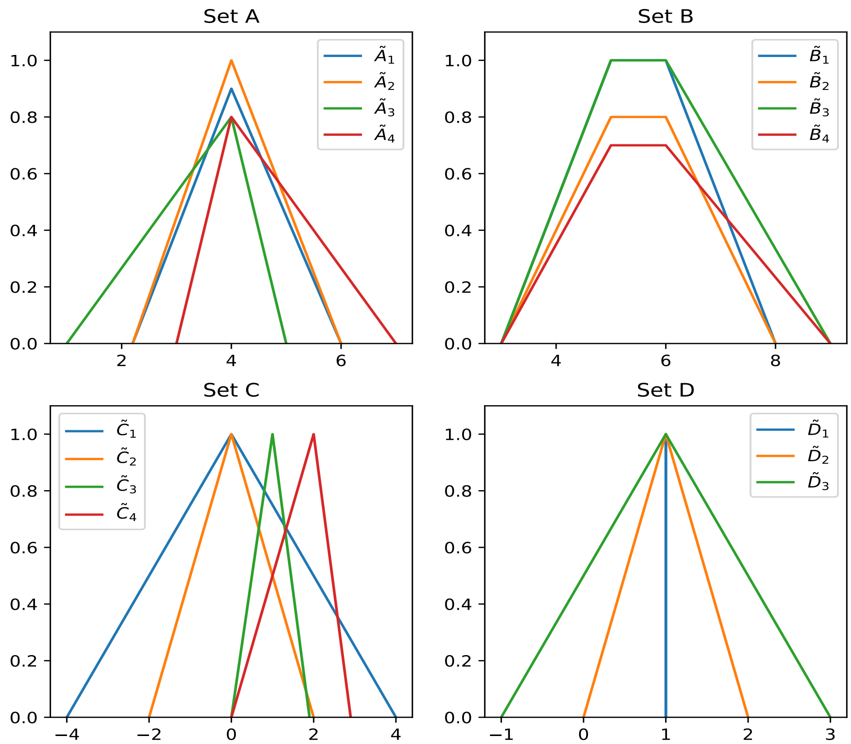

Figure 4: Data sets 1 to 4 of fuzzy numbers with their intuitive ranking.

In addition to magnitude definitions Mag, Mag_N, and Mag_A, in data set A we use four different distance-based magnitude definitions. More precisely, we use the defining parameters of fuzzy numbers as the initial set of attributes $\{a_{j1}, a_{j2}, a_{j3}\}$, apply max-min normalization, set $p = 1$ in the Minkowski distance function, and consider equal weights $\beta_1, \beta_2, \beta_3 = 1$ to compute Mag_{D1}. In order to explore the differences of using Euclidean distances instead of Manhattan distances, we use the same set of attributes, equal weights and max-min normalization as in Mag_{D1}, but we set $p = 2$ to compute Mag_{D2}. For illustrative purposes, we also use the possibilistic mean, the possibilistic standard deviation, and the maximum support value as a new set of attributes $\{M(\tilde{B}_j), \sigma(\tilde{B}_j), \sup supp(\tilde{B}_j)\}$, percentage normalization, $p = 1$, and equal weights $\beta_1, \beta_2, \beta_3 = 1$ to compute Mag_{D3}. Finally, we use the same configuration as in Mag_{D3}, but setting $p = 2$ to compute Mag_{D4}. In what follows, we always use equal weights in the Minkowski distance function for all distance-based magnitude definitions.

The results of the first comparison of magnitudes for data set A are summarized in Table 2. The first point that is worth mentioning is that using the set of defining parameters in Mag_{D1} and Mag_{D2} as a set of attributes is usually an option as good as other attributes in providing useful information to rank fuzzy numbers. Indeed, the central value of a fuzzy number, which is an attribute that usually provides critical information, is a function of the defining parameters. The main advantage of using the set of defining parameters is twofold: first, there is no need to design complex score functions for ranking purposes; and second, a particular set of weights can be attached to each of the parameters according to the preference of the decision-maker. In addition, the use of the height of a fuzzy number in Mag_{D3} and Mag_{D4} is justified because of the presence of non-normal triangular fuzzy numbers with different height values. The use of this attribute will not make any difference in the case of either all normal fuzzy numbers or non-normal fuzzy numbers with equal height. As a result, we argue that our method enhances the flexibility of the ranking procedure while other approaches are restricted to a fixed score function. We observe a difference in the ranking produced by Mag_{D1} and Mag_{D2} compared to Mag_{D3} and Mag_{D4} due to the presence of different attributes. On the contrary, we observe no difference in the rankings when parameter p is changed from 1 to 2. Finally, the use of max-min normalization and max normalization is indifferent because there is no attribute with negative values.

Table 2: Ranking for Set A derived from alternative magnitude definitions

Ranking approach	Definition	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	Ranking
Abbasbandy and Hajjari [2]	Mag	3.81	4.02	3.52	3.68	$\tilde{A}_2 \succ \tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3$
Hajjari [15]	Mag_N	16.37	17.03	15.55	15.97	$\tilde{A}_2 \succ \tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3$
Gu and Xuan [13]	Mag_A	4.73	4.81	4.32	4.99	$\tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_1 \succ \tilde{A}_3$
$\{a_{j1}, a_{j2}, a_{j3}, w_j\}$, max-min, $p = 1$	Mag_{D1}	1.60	2.10	0.00	2.00	$\tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_1 \succ \tilde{A}_3$
$\{a_{j1}, a_{j2}, a_{j3}, w_j\}$, max-min, $p = 2$	Mag_{D2}	0.93	1.27	0.00	1.41	$\tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_1 \succ \tilde{A}_3$
$\{M, \sigma, h\}$, max, $p = 1$	Mag_{D3}	2.73	2.93	2.49	2.64	$\tilde{A}_2 \succ \tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3$
$\{M, \sigma, h\}$, max, $p = 2$	Mag_{D4}	1.58	1.69	1.44	1.53	$\tilde{A}_2 \succ \tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3$

The results of the comparison for data set B are summarized in Table 3. In this case, we use again the set of defining parameters in Mag_{D1} and Mag_{D2}, and the possibilistic mean, the possibilistic standard deviation, and the height of a fuzzy number in Mag_{D3} and Mag_{D4} as a set of attributes. In this case, it is important to highlight how the introduction of the height in Mag_{D3} and Mag_{D4}, results in a change of ranking with respect to Mag_A that considers the sum of the possibilistic mean and standard deviation. Similarly to data set B, we observe no difference in the rankings when parameter p is changed from 1 to 2.

Table 3: Ranking for Set B derived from alternative magnitude definitions

Ranking approach	Definition	$\tilde{B}_1$	$\tilde{B}_2$	$\tilde{B}_3$	$\tilde{B}_4$	Ranking
Abbasbandy and Hajjari [2]	Mag	5.50	4.95	5.58	4.69	$\tilde{B}_3 \succ \tilde{B}_1 \succ \tilde{B}_2 \succ \tilde{B}_4$
Hajjari [15]	Mag_N	14.00	12.02	13.17	10.28	$\tilde{B}_1 \succ \tilde{B}_3 \succ \tilde{B}_2 \succ \tilde{B}_4$
Gu and Xuan [13]	Mag_A	6.76	6.51	7.12	6.69	$\tilde{B}_3 \succ \tilde{B}_1 \succ \tilde{B}_4 \succ \tilde{B}_2$
$\{b_{j1}, b_{j2}, b_{j3}, b_{j4}, w_j\}$, max-min, $p = 1$	Mag_{D1}	1.00	0.33	2.00	1.00	$\tilde{B}_3 \succ \tilde{B}_1 \sim \tilde{B}_4 \succ \tilde{B}_2$
$\{b_{j1}, b_{j2}, b_{j3}, b_{j4}, w_j\}$, max-min, $p = 2$	Mag_{D2}	1.00	0.33	1.41	1.00	$\tilde{B}_3 \succ \tilde{B}_1 \sim \tilde{B}_4 \succ \tilde{B}_2$
$\{M, \sigma, h\}$, max, $p = 1$	Mag_{D3}	2.80	2.44	3.00	2.40	$\tilde{B}_3 \succ \tilde{B}_1 \succ \tilde{B}_2 \succ \tilde{B}_4$
$\{M, \sigma, h\}$, max, $p = 2$	Mag_{D4}	1.62	1.42	1.73	1.41	$\tilde{B}_3 \succ \tilde{B}_1 \succ \tilde{B}_2 \succ \tilde{B}_4$

The results of the comparison for data set C are summarized in Table 4. In this case, we aim to rank four normal triangular fuzzy numbers. This data set was used by Hajjari [15] to highlight the advantages of Mag_N with respect to the Mag definition by Abbasbandy and Hajjari [2]. Indeed, Mag score function considers the central value of the support of a fuzzy number as the most important one. Then $\tilde{C}_1 \sim \tilde{C}_2$ because they are centered on zero. To overcome this shortcoming, Hajjari [15] introduces in magnitude Mag_N the extremes of the supports in a set of fuzzy numbers under the assumption that the larger the support of fuzzy numbers with the same central value the lower the ranking because of the imprecision introduced. As a result, $\tilde{C}_2 \succ \tilde{C}_1$. However, Gu and Xuan [13] defend the opposite view when considering that the larger the possibilistic standard deviation (and support) of fuzzy numbers with the same central value the higher the ranking. Consequently, $\tilde{D}_1 \succ \tilde{D}_2$. In the case of the distance-based magnitude definitions proposed in this work, we use the set of defining parameters in Mag_{D1} and Mag_{D2} with max-min normalization. On the other hand, we use the possibilistic mean, the possibilistic standard deviation, and the maximum value of the support of fuzzy numbers in Mag_{D3} and Mag_{D4} as a set of attributes with percentage normalization because all three attributes are positive. By doing this, we are introducing a third attribute to discriminate fuzzy numbers with the same possibilistic mean and standard deviation. Similarly to data sets A and B, we observe no difference in the rankings when parameter p is changed from 1 to 2.

The results of the last comparison for data set D are summarized in Table 5. Here, we are dealing with two symmetric triangular fuzzy numbers and crisp fuzzy number $\tilde{D}_1 = (1, 1, 1)$. This data set was also used by Hajjari [15] to highlight the advantages of Mag_N with respect to the Mag definition by Abbasbandy and Hajjari [2]. The results derived from the application of Mag lead to $\tilde{D}_1 \sim \tilde{D}_2 \sim \tilde{D}_3$. Following the reasoning by Hajjari [15], the larger the support of fuzzy numbers with the same central value the lower the ranking because of the imprecision introduced. If one of these fuzzy numbers with the same central value is a crisp number such as $\tilde{D}_1 = (1, 1, 1)$, then the highest ranking corresponds to the crisp number. As a result, $\tilde{D}_1 \succ \tilde{D}_2 \succ \tilde{D}_3$. Similarly to the case of data set C, Gu and

Table 4: Ranking for Set C derived from alternative magnitude definitions

Ranking approach	Definition	$\tilde{C}_1$	$\tilde{C}_2$	$\tilde{C}_3$	$\tilde{C}_4$	Ranking
Abbasbandy and Hajjari [2]	Mag	0.00	0.00	0.99	1.91	$\tilde{C}_4 \succ \tilde{C}_3 \succ \tilde{C}_2 \sim \tilde{C}_1$
Hajjari [15]	Mag_N	0.00	4.00	8.08	8.92	$\tilde{C}_4 \succ \tilde{C}_3 \succ \tilde{C}_2 \succ \tilde{C}_1$
Gu and Xuan [13]	Mag_A	1.63	0.82	1.37	2.41	$\tilde{C}_4 \succ \tilde{C}_1 \succ \tilde{C}_3 \succ \tilde{C}_2$
$\{c_{j1}, c_{j2}, c_{j3}\}$, max-min, $p = 1$	Mag_{D1}	1.00	0.55	1.50	2.48	$\tilde{C}_4 \succ \tilde{C}_3 \succ \tilde{C}_1 \succ \tilde{C}_2$
$\{c_{j1}, c_{j2}, c_{j3}\}$, max-min, $p = 2$	Mag_{D2}	1.00	0.50	1.12	1.49	$\tilde{C}_4 \succ \tilde{C}_3 \succ \tilde{C}_1 \succ \tilde{C}_2$
$\{M, \sigma, c_{j3}\}$, percentage, $p = 1$	Mag_{D3}	0.85	0.42	0.64	1.09	$\tilde{C}_4 \succ \tilde{C}_1 \succ \tilde{C}_3 \succ \tilde{C}_2$
$\{M, \sigma, c_{j3}\}$, percentage, $p = 2$	Mag_{D4}	0.60	0.30	0.41	0.72	$\tilde{C}_4 \succ \tilde{C}_1 \succ \tilde{C}_3 \succ \tilde{C}_2$

Xuan [13] follow a different reasoning when considering fuzzy numbers with the same central value. Consequently, the ranking derived from Mag_A is $\tilde{D}_3 \succ \tilde{D}_2 \succ \tilde{D}_1$. Due to the similarity of the values of the defining parameters and the reduced size and variability of the fuzzy numbers in data set D, the max-min normalization used in Mag_{D1} and Mag_{D2} produces almost no discrimination for $p = 1$ and $p = 2$. In this case, it is advisable to use no normalization procedure as we do in Mag_{D3} to produce a ranking. In Mag_{D4}, we use the possibilistic mean, the possibilistic standard deviation, and the maximum value of the support of fuzzy numbers to produce a ranking equal to that derived from Mag_A.

Table 5: Ranking for Set D derived from alternative magnitude definitions

Ranking approach	Definition	$\tilde{D}_1$	$\tilde{D}_2$	$\tilde{D}_3$	Ranking
Abbasbandy and Hajjari [2]	Mag	1.00	1.00	1.00	$\tilde{D}_1 \sim \tilde{D}_2 \sim \tilde{D}_3$
Hajjari [15]	Mag_N	6.00	4.00	2.00	$\tilde{D}_1 \succ \tilde{D}_2 \succ \tilde{D}_3$
Gu and Xuan [13]	Mag_A	1.00	1.41	1.82	$\tilde{D}_3 \succ \tilde{D}_2 \succ \tilde{D}_1$
$\{d_{j1}, d_{j2}, d_{j3}\}$, max-min, $p = 1$	Mag_{D1}	1.00	1.00	1.00	$\tilde{D}_1 \sim \tilde{D}_2 \sim \tilde{D}_3$
$\{d_{j1}, d_{j2}, d_{j3}\}$, max-min, $p = 2$	Mag_{D2}	1.00	0.71	1.00	$\tilde{D}_1 \sim \tilde{D}_3 \succ \tilde{D}_2$
$\{d_{j1}, d_{j2}, d_{j3}\}$, no-norm, $p = 2$	Mag_{D3}	1.73	2.24	3.32	$\tilde{D}_3 \succ \tilde{D}_2 \succ \tilde{D}_1$
$\{M, \sigma, d_{j3}\}$, percentage, $p = 1$	Mag_{D4}	0.37	0.58	0.90	$\tilde{D}_3 \succ \tilde{D}_2 \succ \tilde{D}_1$

Even though all ranking approaches in Tables 2, 3, 4, and 5 have most of the reasonable properties described in Section 3.4, the discussion about what is intuitively the best ranking approach is subject to what is found more important in a particular work or application. Some authors ([2]) consider the central value of the support of a fuzzy number as the most important one. Other authors ([15]) assume that the larger the support of fuzzy numbers with the same central value, the lower the ranking. Some other authors ([13]) defend the opposite view and assume that the larger the possibilistic standard deviation of fuzzy numbers with the same central value, the higher the ranking. In this work, we follow a more general approach in which the attributes for ranking purposes and their respective importance are selected by practitioners according to their preferences and the characteristics of the problem under consideration. This flexibility can be achieved by introducing a set of attributes that are more important for a particular application. In addition, we argue that our method improves the flexibility of existing ranking approaches by not restricting ourselves to a fixed score function.

4.2 Comparative study using pairwise comparison methods

Using pairwise comparison methods such as those recently described in [16, 11] is a different way to rank fuzzy numbers. In this case, the ranking is determined by comparing fuzzy numbers in pairs through a binary relation function. For instance, Adabitabar Firozja et al. [11] use a real function measuring the degree that an arbitrary real number is greater or smaller than a given fuzzy number. Table 6 presents a comparative study of two recent pairwise comparison methods and four versions of our distance-based magnitude definition. More precisely, we use four different data sets described in [11]:

- Set E: $\tilde{E}_1 = (0.4, 0.5, 1)$, $\tilde{E}_2 = (0.4, 0.7, 1)$, $\tilde{E}_3 = (0.4, 0.9, 1)$.

- Set F: $\tilde{F}_1 = (0.3, 0.4, 0.7, 0.9)$, $\tilde{F}_2 = (0.3, 0.7, 0.9)$, $\tilde{F}_3 = (0.5, 0.7, 0.9)$.
- Set G: $\tilde{G}_1 = (0.3, 0.5, 0.7)$, $\tilde{G}_2 = (0.3, 0.5, 0.8, 0.9)$, $\tilde{G}_3 = (0.3, 0.5, 0.9)$.
- Set H: $\tilde{H}_1 = (0, 0.4, 0.7, 0.8)$, $\tilde{H}_2 = (0.2, 0.5, 0.9)$, $\tilde{H}_3 = (0.1, 0.6, 0.8)$.

In this case, we do not use any normalization because the scale of the attributes is comparable. For illustrative purposes, we employ four magnitude definitions. For the first and second, we use the defining parameters of fuzzy numbers as attributes, denoted as θ_{ji}, and we set $p = 1$ and $p = 2$. For the third and fourth, we use the possibilistic mean, the standard deviation, and the maximum support as attributes, and we also consider the cases $p = 1$ and $p = 2$. We observe three different rankings compared to the methods in [16, 11]. Although there is no definitive reason to claim that one ranking method is better than another, the main advantage of our method is the simplicity of computations, especially when considering the sum of defining parameters to rank fuzzy numbers.

Table 6: Ranking for data sets described in [11].

Ranking approach	Set E	Set F	Set G	Set H
Hierro et al. [16]	$\tilde{E}_3 \succ \tilde{E}_2 \succ \tilde{E}_1$	$\tilde{F}_3 \succ \tilde{F}_2 \succ \tilde{F}_1$	$\tilde{G}_2 \succ \tilde{G}_3 \succ \tilde{G}_1$	$\tilde{H}_2 \succ \tilde{H}_3 \succ \tilde{H}_1$
Firozja et al. [11]	$\tilde{E}_3 \succ \tilde{E}_2 \succ \tilde{E}_1$	$\tilde{F}_3 \succ \tilde{F}_2 \succ \tilde{F}_1$	$\tilde{G}_2 \succ \tilde{G}_3 \succ \tilde{G}_1$	$\tilde{H}_2 \succ \tilde{H}_3 \succ \tilde{H}_1$
$\{\theta_{j1}, \theta_{j2}, \theta_{j3}, \theta_{j4}\}$, no-norm, $p = 1$	$\tilde{E}_3 \succ \tilde{E}_2 \succ \tilde{E}_1$	$\tilde{F}_3 \succ \tilde{F}_2 \succ \tilde{F}_1$	$\tilde{G}_2 \succ \tilde{G}_3 \succ \tilde{G}_1$	$\tilde{H}_2 \sim \tilde{H}_3 \succ \tilde{H}_1$
$\{\theta_{j1}, \theta_{j2}, \theta_{j3}, \theta_{j4}\}$, no-norm, $p = 2$	$\tilde{E}_3 \succ \tilde{E}_2 \succ \tilde{E}_1$	$\tilde{F}_3 \succ \tilde{F}_2 \succ \tilde{F}_1$	$\tilde{G}_2 \succ \tilde{G}_3 \succ \tilde{G}_1$	$\tilde{H}_3 \succ \tilde{H}_2 \succ \tilde{H}_1$
$\{M, \sigma, \theta_{j4}\}$, no-norm, $p = 1$	$\tilde{E}_3 \succ \tilde{E}_2 \succ \tilde{E}_1$	$\tilde{F}_2 \succ \tilde{F}_3 \succ \tilde{F}_1$	$\tilde{G}_2 \succ \tilde{G}_3 \succ \tilde{G}_1$	$\tilde{H}_2 \succ \tilde{H}_3 \succ \tilde{H}_1$
$\{M, \sigma, \theta_{j4}\}$, no-norm, $p = 2$	$\tilde{E}_3 \succ \tilde{E}_2 \succ \tilde{E}_1$	$\tilde{F}_3 \succ \tilde{F}_2 \succ \tilde{F}_1$	$\tilde{G}_2 \succ \tilde{G}_3 \succ \tilde{G}_1$	$\tilde{H}_2 \succ \tilde{H}_3 \succ \tilde{H}_1$

4.3 An application to economic decision-making

In this section, we adapt the credit case study described in [17] to a fuzzy context to illustrate an application of our ranking approach to a multicriteria decision-making problem within an economic context. The main goal of the problem is to allocate a financing budget of a bank to a set of applicants. These applicants are evaluated through several financial ratios. With applicants in rows and ratios in columns, we build a decision table with each element $\tilde{V}_{ij}$ set to the evaluation for each applicant and ratio to make the final selection of applicants. A simplified version of the decision table with seven companies and four ratios is shown in Table 7. For illustrative purposes, we here introduce imprecision by considering triangular fuzzy numbers and a random left and right spread from the real mean value of the evaluation for each company and ratio. Each spread is drawn from a uniform distribution between zero and the standard deviation of the ratios for the companies.

Table 7: A fuzzy decision table for financing applicants.

Name	Ratio 1	Ratio 2	Ratio 3	Ratio 4
EDP	(-0.01,0.02,0.03)	(-0.03,0.01,0.01)	(-0.17,0.02,0.30)	(-0.04,0.01,0.15)
CATALANA	(0.10,0.11,0.14)	(0.05,0.10,0.15)	(0.07,0.11,0.23)	(0.04,0.17,0.26)
MAPFRE	(0.05,0.05,0.08)	(-0.01,0.04,0.05)	(0.04,0.05,0.11)	(-0.06,0.05,0.25)
IAG	(0.02,0.04,0.10)	(0.01,0.03,0.06)	(0.68,0.83,1.26)	(-0.03,0.03,0.04)
EBRO FOOD	(0.06,0.07,0.12)	(0.06,0.08,0.08)	(0.22,0.71,1.14)	(-0.04,0.08,0.18)
ACERINOX	(-0.02,0.01,0.04)	(-0.01,0.02,0.06)	(0.74,0.99,1.12)	(-0.11,0.01,0.04)
NH HOTELES	(-0.02,0.00,0.05)	(-0.05,0.00,0.06)	(0.18,0.47,0.77)	(-0.14,-0.03,0.16)

To assess the quality of the applicants in terms of the ratios considered, the authors followed the criterion proposed by Ballestero [5] under a strict uncertainty scenario, meaning the relative importance of each ratio is ignored. This criterion implies the use of the following aggregation fuzzy weight for the j-th ratio:

$$\tilde{w}_j = \frac{1}{\tilde{V}_{j,max} - \tilde{V}_{j,min}}. \tag{29}$$

Consequently, Ballestero's criterion requires the identification of the maximum and minimum fuzzy evaluation of the ratios in Table 7. To this end, we use our multidimensional approach to ranking the set of fuzzy numbers in columns.

Note that fuzzy numbers in Ratio 3 present on average a central value much higher than the deviation from the central value. Consequently, if we use the sum of possibilistic mean and standard deviation as suggested by Gu and Xuan [13], the ranking of applicants in Ratio 3 would be clearly biased to the mean value due to problems of scale. As a result, two ways of action are possible: first, we can normalize the possibilistic mean and standard deviation as proposed in this work to solve the problem of scale; or second, we can select a different set of attributes such as the sum of normalized parameters for ranking purposes. Following the latter procedure, we find that CATALANA and NH HOTELES present respectively the maximum and minimum evaluation for Ratios 1, 2, and 4, and that ACERINOX and EDP present respectively the maximum and minimum evaluation for Ratio 3.

Once we have computed fuzzy weights $\tilde{w}_j$ for each ratio, we are in a position to compute the final ranking of applicants by computing the following fuzzy score function:

$$\tilde{S}_i = \sum_{j=1}^{4} \tilde{w}_j \tilde{V}_j. \tag{30}$$

The higher the fuzzy score, the higher the ranking of the applicants. To obtain this final ranking, we apply our multidimensional ranking approach and compute the sum of normalized parameters derived from the fuzzy score function. Consequently, the ranking of applicants is as follows:

$$\text{CATALANA} \succ \text{EBRO FOODS} \succ \text{IAG} \succ \text{ACERINOX} \succ \text{MAPFRE} \succ \text{NH HOTELS} \succ \text{EDP}. \tag{31}$$

5 Concluding remarks

In this paper, we extend the concept of magnitude to a multidimensional concept in which multiple attributes are used to represent fuzzy numbers. More precisely, we propose a general methodology based on three novel features: multidimensional representation, normalization, and the use of the Minkowski parametric distance function. A multidimensional approach is required to deal with multiple attributes; normalization avoids bias towards any attribute, and a distance function is used to obtain the final ranking. The rationale behind this proposal is to enrich the defuzzification process with alternative attributes and to solve the scale limitations of some magnitude definitions.

When comparing alternative magnitude definitions to illustrate the ability of our approach to rank fuzzy numbers, we find that some authors ([2]) consider the central value of the support of a fuzzy number as the most important one. Other authors ([15]) assume that the larger the support of fuzzy numbers with the same central value, the lower the ranking, while some other authors ([13]) defend the opposite view. However, these ranking approaches have most of the reasonable properties proposed by [27]. We argue that the multidimensional ranking approach proposed is a more general approach than previous approaches in the sense that multiple attributes can be used for ranking purposes and their respective importance are selected by practitioners according to their preferences and also to the characteristics of the problem under consideration. In addition, our method improves the flexibility of existing ranking approaches by not restricting the method to a fixed score function.

Summarizing, this paper contributes to extend the one-dimensional concept of magnitude to a multidimensional space in which multiple attributes can be used to represent fuzzy numbers. Researchers have now the possibility to engineer new attributes and combine them to build novel magnitude definitions. The methodology proposed in this paper is a suitable way to empirically evaluate the advantages and disadvantages of magnitudes, normalization techniques and distance functions. Finally, although the main application our approach is ranking fuzzy numbers, we argue that additional practical and theoretical results may derive from our multidimensional approach.

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