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Probabilistic analysis of the steady state of weakly perturbed linear oscillators subject to a class of Gaussian inputs

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ABSTRACT

This paper aims to probabilistically study a class of nonlinear oscillator subject to weak perturbations and driven by stationary zero-mean Gaussian stochastic processes. For the sake of generality in the analysis, we assume that the perturbed term is a polynomial of arbitrary degree in the spatial position, that contains, as a particular case, the important case of the Duffing equation. We then take advantage of the so-called stochastic equivalent linearization technique to construct an *equivalent* linear model so that its behavior consistently approximates, in the mean-square sense, that of the nonlinear oscillator. This approximation allows us to take extensive advantage of the probabilistic properties of the solution of the linear model and its first mean-square derivative to construct reliable approximations of the main statistical moments of the steady state. From this key information, we then apply the principle of maximum entropy to construct approximations of the probability density function of the steady state. We illustrate the superiority of the equivalent linearization technique over the perturbation method through some examples.

1. Introduction and motivation

The study of nonlinear dynamical systems with stochastic perturbations has been the subject of very substantial research in Applied Mathematics [1,2], and a very large number of contributions can, indeed, be found in the technical literature discussing a wide variety of applications, as for example in Control [3], Economy [4] and Nonlinear Vibratory Systems [5], to cite a few.

It is interesting to mention that many types of systems subject to vibrations have been discussed in Physics and Engineering to analyze, for example, several classes of linear and nonlinear oscillators in the deterministic case [6,7], using different approaches, such as energy balance method [8] and its improved modifications [9], removing noise in the oscillation term [10], etc.

Many important vibratory systems, in particular of nonlinear oscillators, are described by scalar differential equations of the following form:

$$\ddot{X}(t) + g(X(t), \dot{X}(t)) = Y(t), \quad t > 0, \tag{1}$$

where $X(t)$ denotes the position of the oscillatory system at the time instant $t > 0$, g is a nonlinear function that depends on both the position and velocity, $\dot{X}(t)$, and the input $Y(t)$ represents an external source/forcing term driven the system. An important class of nonlinear oscillators that belongs to model (1) is that where

$$g(X(t), \dot{X}(t)) = 2\beta\dot{X}(t) + \omega_0^2[X(t) + \epsilon f(X(t))]. \tag{2}$$

Here, the parameter $\beta > 0$ denotes the damping constant, $\omega_0 > 0$ is the undamped angular frequency and ϵ is a small parameter that represents the intensity of a perturbation that affects a nonlinear function of the position, $f(X(t))$. For example, when $f(X(t)) = (X(t))^3$, model (1)–(2) corresponds to the familiar case of the Duffing oscillator subject to the input $Y(t)$ [11]. As happens with the Duffing oscillator, the family of nonlinear oscillators (1)–(2) does not admit, in general, closed-form solutions when f is nonlinear. This fact motivates that the main objective is the achievement of conditions on the perturbation parameter ϵ so that the oscillator reaches the steady state. When $Y(t)$ is a deterministic function, many techniques have been devised to achieve the aforementioned goal [12–15]. However, the external forces producing vibrations when acting on an oscillator are often not deterministically known but involve uncertainties due to complex factors that may include the incomplete knowledge of the properties of

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the surrounding medium, the intrinsic nature of the external forces, etc. So, it is more realistic to consider $Y(t)$ as a stochastic process rather than a deterministic function. These facts have motivated an increasing trend in the research community to incorporate randomness when analyzing vibratory systems, particularly nonlinear oscillators, then using random/stochastic techniques [16,17]. In this context, studying the probabilistic properties of vibratory systems in the steady state, such as the model (1)–(2), is crucial for understanding their long-term behavior. It is important to note that, within this framework, the steady-state behavior, being a random variable, must be characterized by statistics such as the mean and variance (or standard deviation), or by the stationary probability density function. The mean provides an estimate of the expected position of the oscillators, while the standard deviation quantifies the variability around this expected position. The probability density function, in turn, allows for the calculation of key information, such as the probability that the position of the oscillator will, in the long term, fall within any specific interval of interest for the vibratory system. Some contributions illustrating the relevance of studying the probabilistic properties of the steady state of specific vibratory systems can be found, for example, in [18–20].

In the particular case that $\epsilon = 0$ and $Y(t)$ is a stochastic process, Eqs. (1)–(2) describe the dynamics of a random linear oscillator. This class of stochastic oscillators has been studied in [21] under very general assumptions. In that paper, the authors determine the so-called first probability density function of the position, $X(t)$, in different scenarios with respect to the form of the stochastic forcing term $Y(t)$. First, they considered that $Y(t)$ is a Gaussian stochastic process, encompassing the important case of a damped pendulum modeled by an Itô-type stochastic differential equation [22]. Second, they analyzed the case that $Y(t)$ can be approximated by a stochastic process belonging to the space $\mathrm{L}^2([0,T]\times\Omega)$ of Lebesgue square-integrable stochastic processes defined in a complete probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$ [22, ch.2]. This latter scenario includes the relevant cases where $Y(t)$ can be approximated by means of a Karhunen-Loève expansion [23] or represented via a random power series [24]. The success of the aforementioned contribution relied on the fact that a semi-explicit (in terms of an integral) solution can be obtained in the linear case ($\epsilon = 0$). However, as previously indicated, in the nonlinear case a solution to the corresponding differential equation is generally unavailable. Then, the main goal is to obtain reliable probabilistic information about the steady state. It includes the calculation of the expectation, the variance, the correlation, the probability density function, etc., in terms of the size of the perturbation $\epsilon > 0$.

The so-called stochastic perturbation technique is a popular approach to analyzing mechanical systems with small nonlinearities that are subject to external random vibrations. In the setting of vibratory systems, this method was first introduced in the pioneering contributions [25,26] by Crandall. Later, the method was continued with many studies (see [27–29]). More recently, some of the authors have applied this technique to analyze a type of nonlinear oscillators that fall within the class (1)–(2), [18]. In that contribution, one takes $f(X(t)) = \sin(X(t))$ as the nonlinear function affected by the perturbation, and the input or source term $Y(t)$ is assumed to be a mean-square differentiable and stationary Gaussian stochastic process with zero-mean whose correlation function is known. In the present paper, we will extend this study to a wider class for the input term and more general oscillators that those analyzed there using the stochastic equivalent linearization method.

If the perturbation size ϵ is not small enough, the stochastic approximations obtained by the perturbation technique may deteriorate and become unrealistic. An alternative to overcome this drawback in practice is the so-called stochastic equivalent linearization technique. This method is inspired by its deterministic counterpart introduced by Krylov and Bogoliubov [30]. The advantage of the equivalent linearization technique over the perturbation method is that it is not restricted to problems with very small nonlinearities [24, p. 209], which extends its application for a certain class of nonlinear problems appearing in Control and Mechanical Vibrations. Since its inception, the equivalent linearization method has proven to be a powerful approach with enough flexibility to be successfully adapted to analyze different classes of random vibratory systems. Indeed, in [31], one proposes a higher-order equivalent linearization method for analyzing non-linear random vibration problems, where the non-linear terms are replaced by unknown linear terms that are described by extra non-linear differential equations, with the objective to obtain a higher degree-of-freedom equation for the original system. In [32], the dynamics of a non-linear oscillator with multiple static equilibrium excited by a white noise is approximated by means of an equivalent locally linear oscillator. In [33], the authors propose a two-step linearization procedure to improve the accuracy of oscillators' mean-square response estimation with quadratic damping excited by a white noise random process. In [33], one develops a method based on a two-step linearization procedure to improve the accuracy of the mean-square response estimation of oscillators with quadratic damping excited by white noise random process. In [34], one proposes a new mean-square criterion of error sample function to determine the coefficients of the linearized equivalent equation, with the objective to obtain the stationary response of nonlinear systems under zero-mean Gaussian random excitation for both weak and strong nonlinearity. Also, in [35] has developed an efficient analysis procedure for the random vibration analysis of nonlinear structures subjected to nonstationary random excitations based on the equivalent linearization method and the time-domain explicit formulation method. In [36–38], one design weighted averaging equivalent linearization methods to study different classes of the nonlinear Duffing oscillator. Recently, in [39], one has applied this same technique to study systems of two coupled strong nonlinear differential equations, where nonlinear differential equations are transferred into a single equation by using some intermediate variables. For an overview of the equivalent linearization technique, including many of its applications over the years, see [40]. Finally, it is interesting to point out that slight variations of this technique have been proposed and discussed in the scientific community [41].

This paper performs a probabilistic study of random nonlinear oscillators formulated by (1)–(2) by considering that the nonlinear function $f(X(t))$ is a polynomial of arbitrary degree, so containing the relevant case of the Duffing oscillator, for which $f(X(t)) = (X(t))^3$. As it shall be commented later, the foregoing choice also permits approximating the case that $f(X(t))$ is an analytic function of $X(t)$ by truncating at an arbitrary order the corresponding Taylor expansion, so extending the analysis performed in [18], where, as previously indicated, $f(X(t)) = \sin(X(t))$. It is important to point out that our analysis is addressed to approximate the probability density function (PDF) of the steady state. To achieve this goal, we shall approximate the first statistical moments of the steady state via the equivalent linearization technique, and then we will take advantage of the principle of maximum entropy (PME) to approximate the PDF [42]. To show the flexibility of the proposed approach, we shall illustrate the theoretical results with different relevant stochastic processes playing the role of the input $Y(t)$.

The paper is organized as follows. In Section 2, we introduce the main results that will be required throughout the paper. This section is certainly quite lengthy because one requires a large number of previous results so that the interested reader can follow in detail the full development of the paper, which is rather technical in some of its developments. Section 2 contains abundant results about mean-square stochastic calculus and its relationship with key deterministic functions associated with second-order stochastic processes such as the mean, the variance, the (cross-)correlation, and the (cross-)power spectral density function. Section 3 is devoted to explaining how the equivalent linearized technique can be applied to probabilistically analyze the steady state of the general Duffing oscillators we are interested in when they are subject to a class of Gaussian inputs. To address this goal, we first will introduce some statistical properties of the solution and its mean-square derivative in Sections 3.1 and 3.2, respectively. The properties developed in Section 3.1 will play a key role in our subsequent study, particularly the variance of the solution as well as other higher moments. Whereas, those exhibited in Section 3.1 have been included so that the reader can understand a thorough

discussion about alternative approaches to those followed in our development. In Section 3.3, we explain in detail how the stochastic equivalent linearization technique works when applied to study the class of nonlinear oscillators given in (1) which contains the family of oscillators studied in this paper. In Section 4, we focus on the analysis of the steady state of a class of random oscillators where nonlinearity affects the spatial position. This family of oscillators contains, as a particular case, the Duffing oscillator. We then obtain the stochastic equivalent linear system of the aforementioned class of nonlinear oscillators. Obtaining this equivalent system involves determining a reliable approximation of the variance of the solution. This approximation is explained in Section 4.1. In Section 5, we explain how the principle of maximum entropy can be applied to obtain reliable approximations of the probability density function of the steady state using the higher moments of the solution, which have been previously calculated. Section 6 contains several examples illustrating our main theoretical findings. In these examples, we have obtained the main probabilistic information of the steady state solution, including the mean, the variance, and the correlation functions as well as the PDF. We show that the numerical results obtained via the stochastic linearized equivalent technique are consistent with those provided by simulations calculated by stochastic numerical schemes and better than those calculated using the stochastic perturbation method. To better illustrate the applicability of our approach, we have considered a variety of scenarios in the examples that consider as inputs the white noise Gaussian and the Ornstein–Uhlenbeck processes when dealing with different nonlinear oscillators including the Duffing oscillator. Finally, in Section 7, we summarize the main conclusions drawn from our study, including its limitations.

2. Stochastic preliminaries

In this section, and for the sake of completeness, we introduce a number of definitions and technical results that will be required throughout the paper. Many of them belong to the so-called mean-square random calculus for random variables and stochastic processes having finite variance (or equivalently second-order moments) [24,43,44].

We will work with an underlying complete probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$, where Ω denotes the sample space, which is the set of all possible outcomes $\omega \in \Omega$; $\mathcal{F}_\Omega$ is the σ-algebra of Ω, whose elements are called events, an event being a set of outcomes in Ω; and $\mathbb{P}$ is a probability measure. We recall that a real-valued random variable is a function $Z : \Omega \longrightarrow \mathbb{R}$, such that $Z^{-1}(A) \in \mathcal{F}_\Omega$ for all $A \in \mathcal{B}_\mathbb{R}$, where $\mathbb{R}$ denotes the σ-algebra of Borel generated by the collection of all open intervals in the whole real line $\mathbb{R}$. A real-valued stochastic process, $Z(t;\omega)$, is a set of real-valued random variables indexed by $t \in \mathcal{T}$, where $\mathcal{T}$ is a subset. In this paper, we shall take $\mathcal{T} = [0, \infty)$. A random variable can be regarded as a time-constant stochastic process, in other words, given a stochastic process, $Z(t)$, for each t fixed, $Z(t)$ is a random variable. Hereinafter, when convenient, we will alleviate the notation by omitting the ω-dependence, and we will simply write $Z = Z(\omega)$ and $Z(t) = Z(t;\omega)$ for random variables and stochastic processes, respectively. Note that in our setting of the differential equation (1)–(2), the source term, $Y(t)$, and the solution, $X(t)$, are stochastic processes where the ω-dependence has been omitted.

Given a stochastic process, $\{Z(t) : t \in \mathcal{T}\}$, if for every finite set of values $\{t_1, \dots, t_n\} \in \mathcal{T}$, there corresponds a set of random variables, $Z_1 = Z(t_1), \dots, Z_n = Z(t_n)$, with a well-defined joint probability distribution function

$$F_{Z_1,\dots,Z_n}(z_1, t_1; \dots; z_n, t_n) = \mathbb{P}\{\{Z_1 \le z_1\} \cap \cdots \cap \{Z_n \le z_n\}\}, \ z_n \in \mathbb{R}, \ n = 1, 2, \dots,$$

where $\{Z_i \le z_i\} = \{\omega \in \Omega : Z_i(\omega) \le z_i\}$, then this family of joint distributions defines the n-distribution function of the stochastic process $Z(t)$. This deterministic function is often denoted by $F_n(z_1, t_1; \dots; z_n, t_n)$ instead of $F_{Z_1,\dots,Z_n}(z_1, t_1; \dots; z_n, t_n)$ to shorten the notation.

Throughout this paper, we will work with stochastic processes satisfying

$$\mathbb{E}\{(Z(t))^2\} = \int_\Omega (Z(t;\omega))^2 \mathrm{d}\mathbb{P}(\omega) < \infty,$$

usually referred to as second-order stochastic processes. Here $\mathbb{E}\{\cdot\}$ stands for the expectation operator. In the particular case that $Z(t)$ does not depend on t, and it satisfies $\mathbb{E}\{(Z(t))^2\} < \infty$, $Z(t)$ is called a second-order stochastic processes. The set of all second-order stochastic processes (and in particular, second-order random variables) endowed with the following inner product

$$\langle Z_1(t), Z_2(t)\rangle = \mathbb{E}\{Z_1(t)Z_2(t)\} = \int_\Omega Z_1(t;\omega)Z_2(t;\omega)\mathrm{d}\mathbb{P}(\omega) < \infty,$$

is a Hilbert space, denoted as $(\mathrm{L}^2(\Omega), \langle\cdot,\cdot\rangle)$. From the above inner product, one infers the so-called 2-norm: $\|Z(t)\|_2 = \left(\mathbb{E}\{(Z(t))^2\}\right)^{1/2} < \infty$. This space is made up of all the stochastic processes with finite second-order moment, $\mathbb{E}\{(Z(t))^2\} < \infty$ (and so, with finite mean, $\mathbb{E}\{Z(t)\} < \infty$, and variance, $\mathbb{V}\{Z(t)\} = \mathbb{E}\{(Z(t))^2\} - (\mathbb{E}\{Z(t)\})^2 < \infty$) for all $t \in \mathcal{T}$. The 2-norm permits defining the concepts of mean-square continuity, differentiability, and integrability of a second-order stochastic process [24, ch.4], [43]. Given a second-order stochastic process, $Z(t)$, $t \in \mathcal{T}$, we can define a deterministic function, called the covariance function $\mathbb{K}_Z(t_1, t_2) = \mathbb{E}\{Z(t_1)Z(t_2)\} - \mathbb{E}\{Z(t_1)\}\mathbb{E}\{Z(t_2)\}$, $t_1, t_2 \in \mathcal{T}$. For second-order stochastic processes, it can be assumed, without loss of generality, that $\mathbb{E}\{Z(t)\} = 0$ by defining $\hat{Z}(t) = Z(t) - \mathbb{E}\{Z(t)\}$, that obviously satisfies $\mathbb{E}\{\hat{Z}(t)\} = 0$. In such a case, the stochastic process is called centered at its mean, and the auto-correlation function simplifies as $\Gamma_Z(t_1, t_2) = \mathbb{E}\{Z(t_1)Z(t_2)\}$, $t_1, t_2 \in \mathcal{T}$, which is referred to as the correlation function. Throughout this paper, we will work zero-mean stochastic processes for which the covariance and the correlation functions coincide, i.e., $\mathbb{K}_Z(t_1, t_2) = \Gamma_Z(t_1, t_2)$. Because of Jensen and Cauchy–Schwarz inequalities [43], it is clear that the correlation function is well-defined

$$|\Gamma_Z(t_1, t_2)| \le \mathbb{E}\{|Z(t_1)Z(t_2)|\} \le \|Z(t_1)\|_2 \, \|Z(t_2)\|_2 < \infty.$$

Moreover, the correlation function is nonnegative definite on $\mathcal{T} \times \mathcal{T}$, i.e., it satisfies $\sum_{j=1}^n \sum_{k=1}^n \Gamma_Z(t_j, t_k)g(t_j)g(t_k) \ge 0$, for every n and $t_1, \dots, t_n \in \mathcal{T}$, and for an arbitrary function $g(t)$ defined on $\mathcal{T}$.

The correlation function can be extended for two second-order stochastic processes, $Z_1(t)$ and $Z_2(t)$, and then it is referred to as the cross-correlation function $\Gamma_{Z_1,Z_2}(t_1, t_2) = \mathbb{E}\{Z_1(t_1)Z_2(t_2)\}$ (obviously, for a single stochastic process $Z(t)$, one gets $\Gamma_{Z,Z}(t_1, t_2) = \Gamma_Z(t_1, t_2)$). If $\Gamma_{Z_1,Z_2}(t_1, t_2) = \mathbb{E}\{Z_1(t_1)Z_2(t_2)\} = 0$ for all $t_1, t_2 \in \mathcal{T}$, the zero-mean (or centered) processes $Z_1(t)$ and $Z_2(t)$ are termed uncorrelated stochastic processes. For non-centered processes, this definition is given in terms of the cross-covariance function, $\mathbb{K}_{Z_1,Z_2}(t_1, t_2) = \mathbb{E}\{Z_1(t_1)Z_2(t_2)\} - \mathbb{E}\{Z_1(t_1)\}\mathbb{E}\{Z_2(t_2)\}$; clearly, $\mathbb{K}_{Z_1,Z_2}(t_1, t_2) = \mathbb{K}_{Z_2,Z_1}(t_2, t_1)$. Observe that for zero-mean stochastic processes, the cross-covariance and the cross-correlation functions match, i.e., $\mathbb{K}_{Z_1,Z_2}(t_1, t_2) = \Gamma_{Z_1,Z_2}(t_1, t_2)$. As it shall be seen later, throughout this paper will utilize the cross-correlation function for a stochastic process and its mean-square derivative, as well as the cross-correlation function of the input $Y(t)$ and the solution $X(t)$ for a class of nonlinear oscillators of the form (1).

Independence between two stochastic processes is an important probability concept with useful operational properties and is closely related to uncorrelation, particularly when both are Gaussian processes. The following results are well-known [45].

Proposition 1. *Let $Z_1(t)$ and $Z_2(t)$ be independent stochastic processes. Then*

(i) $Z_1(t)$ *and* $Z_2(t)$ *are uncorrelated.*
(ii) $\mathbb{E}\{Z_1(t)Z_2(t)\} = \mathbb{E}\{Z_1(t)\}\mathbb{E}\{Z_2(t)\}$. *More generally, for* f *and* g *Borel measurable functions,* $f(Z_1(t))$ *and* $g(Z_2(t))$ *are also independent stochastic processes and*

$$\mathbb{E}\{f(Z_1(t))g(Z_2(t))\} = \mathbb{E}\{f(Z_1(t))\}\mathbb{E}\{g(Z_2(t))\}.$$

In general, the reciprocal of the statement (i) of Proposition 1 is false, unless when the processes are Gaussian

Proposition 2. *Let $Z_1(t)$ and $Z_2(t)$ be Gaussian stochastic processes. Then, the following statements are equivalent:*

(i) $Z_1(t)$ *and* $Z_2(t)$ *are uncorrelated.*
(ii) $Z_1(t)$ *and* $Z_2(t)$ *are independent.*

As we shall see later, some stochastic processes with specific properties will be handled throughout this paper. A stochastic process, say $Z(t) \equiv \{Z(t) : t \in \mathcal{T}\}$, is said to be stationary (or strictly stationary) if all its finite probability distributions, F_n, are invariant under an arbitrary translation of t, i.e.,

$$F_n(z_1, t_1; \ldots; z_n, t_n) = F_n(z_1, t_1 + \tau; \ldots; z_n, t_n + \tau), \quad t_j, t_j + \tau \in \mathcal{T},\ j = 1, \ldots, n. \tag{3}$$

By putting $\tau = -t_1$, it is clear that the probability functions depend upon the time parameters only via the differences, i.e., the statistical properties of a stationary stochastic process are independent of the absolute time origin [24, p. 43]. As a consequence, the mean function is constant, $\mathbb{E}\{Z(t)\} = \mu_Z$ with μ_Z constant for all $t \in \mathcal{T}$; the correlation function depends on a single variable $\Gamma_Z(t_1, t_2) = \Gamma_Z(t_2 - t_1)$ and is an even function $\Gamma_Z(\tau) = \Gamma_Z(-\tau)$. Similar properties for higher-order moments can be easily obtained. However, in general, when dealing with real-world physical problems, it is difficult to check that condition (3) holds for all n in order to ascertain whether the process $Z(t)$ is stationary. This fact motivates the definition of wide-sense (or weakly or covariance) stationary stochastic processes as those satisfying that $|\mathbb{E}\{Z(t)\}| = \mu_Z$, with μ_Z constant; $\mathbb{E}\left\{(Z(t))^2\right\} < \infty$ and $\Gamma_Z(t_1, t_2) = \Gamma_Z(t_2 - t_1)$. Clearly, every weak stationary process is a stationary process provided its second-order moment is finite, but in general, the converse is not true. An important exception is a Gaussian process which is completely determined by its mean and covariance functions, so any wide-sense stationary Gaussian process is also stationary.

The correlation function of a second-order is very important because many relevant properties, such as mean-square continuity, differentiability, and integrability, can be characterized in terms of this two-variable deterministic function. This characterization is particularly readily applicable when the process is also wide stationary. The following result about the mean-square continuity will used later

Proposition 3 (*Th. 4.3.3 [24]*). *Let $Z(t)$, $t \in \mathcal{T}$, be a second-order stochastic process whose correlation function is $\Gamma_Z(t_1, t_2)$. Then, the following statements are equivalent:*

(i) $Z(t)$ *is mean-square continuous in* $\mathcal{T}$.
(ii) $\Gamma_Z(t_1, t_2)$ *is continuous at* $(t, t) \in \mathcal{T} \times \mathcal{T}$.

If $Z(t)$ is wide-sense stationary with correlation function $\Gamma_Z(\tau)$, then, the following statements are equivalent:

(iii) $Z(t)$ *is mean-square continuous in* $\mathcal{T}$.
(iv) $\Gamma_Z(\tau)$ *is continuous at* $\tau = 0$.

Now we recall the concept and main properties of power spectral density of a wide-sense stationary process that will be required later. Let $Z(t)$ be a mean-square continuous wide-sense stationary stochastic process with correlation function $\Gamma_Z(\tau)$. From Proposition 3, we know that $\Gamma_Z(\tau)$ is continuous at $\tau = 0$. As the correlation function of a stochastic process is also real and nonnegative definite, by Bochner theorem [46], it can be represented in the form $\Gamma_Z(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} \exp(\mathrm{i}w\tau)\mathrm{d}\lambda(w)$, $\mathrm{i} = \sqrt{-1}$, where $\lambda(w)$ is real, non-decreasing and bounded. If $\lambda(w)$ is absolutely continuous, i.e., there exists $S_Z(w)$ such that $S_Z(w) = \frac{\mathrm{d}\lambda(w)}{\mathrm{d}w}$, then

$$\Gamma_Z(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} \exp(\mathrm{i}w\tau)S_Z(w)\mathrm{d}w. \tag{4}$$

Because of the properties of $\lambda(w)$, clearly $S_Z(w)$ is real and nonnegative. Furthermore, from (4), one deduces that $(\Gamma_Z(\tau), S_Z(w))$ form a Fourier pair and then

$$S_Z(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp(-\mathrm{i}w\tau)\Gamma_Z(\tau)\mathrm{d}\tau. \tag{5}$$

This function is called the power spectral density function associated with the wide-sense stationary process $Z(t)$. Using that the correlation function is even, i.e., $\Gamma_Z(\tau) = \Gamma_Z(-\tau)$, the Euler's identity $\exp(-\mathrm{i}x) = \cos(x) + \mathrm{i}\sin(x)$, and that $\Gamma_Z(\tau)\cos(x)$ is even and $\Gamma_Z(\tau)\sin(x)$ is odd, one gets

$$S_Z(w) = \frac{2}{\pi}\int_{0}^{\infty} \cos(w\tau)\Gamma_Z(\tau)\mathrm{d}\tau. \tag{6}$$

So, $S_Z(w)$ is an even function. Furthermore, using (4) and that as $\Gamma_Z(\tau)$ is an even function, one deduces

$$2\Gamma_Z(\tau) = \Gamma_Z(\tau) + \Gamma_Z(-\tau) = \frac{1}{2}\int_{-\infty}^{\infty} \left(\exp(\mathrm{i}w\tau) + \exp(-\mathrm{i}w\tau)\right) S_Z(w)\mathrm{d}w = \int_{-\infty}^{\infty} \cos(w\tau)S_Z(w)\mathrm{d}w.$$

As the integrand $\cos(w\tau)S_Z(w)$ is an even function (observe that it is the product of two even functions), one deduces that $\int_{-\infty}^{\infty} \cos(w\tau)S_Z(w)\mathrm{d}w = 2\int_{0}^{\infty} \cos(w\tau)S_Z(w)\mathrm{d}w$. Then substituting this in the last expression, one obtains the following representation for the correlation function

$$\Gamma_Z(\tau) = \int_{0}^{\infty} \cos(w\tau)S_Z(w)\mathrm{d}w. \tag{7}$$

Putting here $\tau = 0$, one gets the following expression of the second-order moment of $Z(t)$ in terms of its power spectral density function

$$\Gamma_Z(0) = \mathbb{E}\left\{(Z(t))^2\right\} = \int_0^\infty S_Z(w)\mathrm{d}w. \tag{8}$$

Since $Z(t)$ is a wide-sense stationary process, observe that this moment does not depend on t, as expected.

The definition of wide-sense stationary can be extended to two stochastic processes. Indeed, $Z_1(t)$ and $Z_2(t)$ are called jointly wide-sense stationary stochastic processes if their means are constant and the cross-covariance function only depends of the time lag, i.e., $\mathbb{E}\left\{Z_1(t)\right\} = \mu_{Z_1}$ with μ_{Z_1} constant, $\mathbb{E}\left\{Z_2(t)\right\} = \mu_{Z_2}$ with μ_{Z_2} constant and $\mathbb{K}_{Z_1,Z_2}(t_1,t_2) = \mathbb{K}_{Z_1,Z_2}(t_2 - t_1)$. Clearly, for two jointly wide-sense stationary processes $Z_1(t)$ and $Z_2(t)$, one satisfies the following property for the cross-covariance function $\mathbb{K}_{Z_1,Z_2}(\tau) = \mathbb{K}_{Z_2,Z_1}(-\tau)$ and for the cross-correlation function $\Gamma_{Z_1,Z_2}(\tau) = \Gamma_{Z_2,Z_1}(-\tau)$. The definition of power spectral density also naturally extends for two jointly wide-sense stationary stochastic processes, $Z_1(t)$ and $Z_2(t)$, whose cross-correlation function, $\Gamma_{Z_1,Z_2}(t_1,t_2)$, is continuous, forming both a Fourier pair

$$S_{Z_1,Z_2}(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp(-\mathrm{i}w\tau)\Gamma_{Z_1,Z_2}(\tau)\mathrm{d}\tau, \quad \Gamma_{Z_1,Z_2}(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} \exp(\mathrm{i}w\tau)S_{Z_1,Z_2}(w)\mathrm{d}w. \tag{9}$$

The function $S_{Z_1,Z_2}(w)$ is termed cross-power spectral density. In contrast to $S_Z(t)$, which is always a real-valued function, in general, $S_{Z_1,Z_2}(w)$ is a complex function. Using the property $\Gamma_{Z_1,Z_2}(\tau) = \Gamma_{Z_2,Z_1}(-\tau)$, previously seen, and denoting $\overline{z}$ the complex conjugate, one gets

$$\begin{aligned}
\overline{S_{Z_2,Z_1}(w)} &= \overline{\frac{1}{\pi}\int_{-\infty}^{\infty} \exp(-\mathrm{i}w\tau)\Gamma_{Z_2,Z_1}(\tau)\mathrm{d}\tau} = \frac{1}{\pi}\int_{-\infty}^{\infty} \overline{\exp(-\mathrm{i}w\tau)}\Gamma_{Z_2,Z_1}(\tau)\mathrm{d}\tau \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \exp(\mathrm{i}w\tau)\Gamma_{Z_2,Z_1}(\tau)\mathrm{d}\tau = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp(\mathrm{i}w\tau)\Gamma_{Z_1,Z_2}(-\tau)\mathrm{d}\tau \\
&= -\frac{1}{\pi}\int_{\infty}^{-\infty} \exp(-\mathrm{i}wu)\Gamma_{Z_1,Z_2}(u)\mathrm{d}u = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp(-\mathrm{i}w\tau)\Gamma_{Z_1,Z_2}(u)\mathrm{d}\tau \\
&= S_{Z_1,Z_2}(w).
\end{aligned}$$

This property will be utilized later.

As previously indicated, we are interested in calculating the statistical moments of the steady state solution of model (1)–(2), where f is a nonlinear function and $Y(t)$ is a stochastic process whose mathematical properties will be detailed later. Now, we introduce some technical results that will be very useful in achieving this goal. It is important to point out that some of these auxiliary results apply when the corresponding stochastic process is Gaussian satisfying additional hypotheses.

The following result permits expressing the moments of the product of an arbitrary number of zero-mean Gaussian random variables in terms of their correlations, and in particular, it gives an explicit expression for the moments of a zero-mean Gaussian random variable:

Proposition 4 ([24, p. 28], [47, p. 148]). *Let $Z_1, \ldots, Z_n$ be zero-mean random variables with a joint Gaussian distribution, i.e., $\mathbb{E}\left\{Z_i\right\} = 0$, $i = 1, \ldots, n$. Then, all odd-order moments of these random variables vanish and, for n even,*

$$\mathbb{E}\left\{Z_1 \cdots Z_n\right\} = \sum_{m_1,m_2,\ldots,m_n} \mathbb{E}\left\{Z_{m_1}Y_{m_2}\right\}\mathbb{E}\left\{Z_{m_3}Z_{m_4}\right\}\cdots\mathbb{E}\left\{Z_{m_{n-1}}Z_{m_n}\right\}, \tag{10}$$

where the sum above is taken over all possible combinations of $n/2$ pairs of n random variables. The number of terms in the summation is $(n-1)!! = 1 \cdot 3 \cdot 5 \cdots (n-3)\cdot(n-1)$.

In the particular case that $Z_1 = \cdots = Z_n = Z$, one gets

$$\mathbb{E}\{Z^n\} = \begin{cases} 0 & \text{if} \quad n \text{ odd}, \\ (n-1)!!\,(\sigma_Z)^n & \text{if} \quad n \text{ even}, \end{cases} \tag{11}$$

where σ_Z denotes the standard deviation of Z.

The following result establishes that the mean-square derivative of a Gaussian stochastic process is also a Gaussian process:

Proposition 5 ([24, p. 110]). *Let $Z(t)$ be a Gaussian process Then, $\dot{Z}(t)$ is also a Gaussian process.*

The following result provides sufficient conditions so that a Gaussian process and its mean-square derivative are independent:

Proposition 6 ([24, p. 114]). *Let $Z(t)$ be a mean-square differentiable stationary zero-mean Gaussian process. Then, $Z(t)$ and $\dot{Z}(t)$ are independent.*

The following proposition permits calculating the expectation and correlation function of the mean-square integral of a stochastic process by commuting the expectation operator.

Proposition 7 ([24, p. 104]). *Let $Y(t) \equiv \{Y(t) : t \in (a,b)\}$ be a mean-square integrable stochastic process with mean and correlation functions $\mathbb{E}\{Y(t)\}$ and $\Gamma_Y(t_1,t_2)$, respectively. Let $f(t,s)$ be a Riemann integrable deterministic function on $t,s \in (a,b)$, $-\infty \le a < b \le \infty$ and consider the integral stochastic process*

$$Z(t) = \int_a^t f(t,s)Y(s)\mathrm{d}s, \quad a \le t \le b.$$

Then,

$$\mathbb{E}\{Z(t)\} = \int_a^t f(t,s)\mathbb{E}\{Y(s)\}\,\mathrm{d}s, \quad a \le t \le b, \tag{12}$$

and

$$\Gamma_Z(t_1,t_2)=\int_a^{t_1}\int_a^{t_2} f(t_1,s_1)f(t_2,s_2)\Gamma_Y(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2,\quad a\le t_1,t_2\le b. \tag{13}$$

The following result is similar to Proposition 5, but changing the mean-square derivative by the mean-square integral of a Gaussian process.

Proposition 8 (*[24, p. 112]*). *Let $Y(t)\equiv\{Y(t):a\le t\le b\}$ be a Gaussian process and let $f(t,s)$ be a Riemann integrable deterministic function on $t,s\in(a,b)$, $-\infty\le a<b\le\infty$ such that the following mean-square integral*

$$Z(t)=\int_a^t f(t,s)Y(s)\mathrm{d}s,\quad a\le t\le b, \tag{14}$$

exists. Then, $Z(t)\equiv\{Z(t):a\le t\le a\}$ is a Gaussian process.

We finish this section stating the mean-square Leibniz rule for differentiating, in the mean-square sense, an integral process:

Proposition 9 (*[24, p. 104]*). *Let $Y(t)\equiv\{Y(t):a\le t\le b\}$ be a mean-square integrable stochastic process. Let $f(t,s)$ be a continuous deterministic function on $t,s\in(a,b)$, $-\infty\le a<b\le\infty$ with finite first partial derivative $\frac{\partial f(t,s)}{\partial t}$. Then, the mean-square derivative of*

$$Z(t)=\int_a^t f(t,s)Y(s)\mathrm{d}s,\quad a\le t\le b, \tag{15}$$

exists for all $t\in(a,b)$ and is given by

$$\dot{Z}(t)=f(t,t)Y(t)+\int_a^t\frac{\partial f(t,s)}{\partial t}Y(s)\mathrm{d}s,\quad a\le t\le b. \tag{16}$$

3. Probabilistic analysis via the equivalent linearization technique

The equivalent linearization technique consists in determining parameters, β_e and k_e^2, such that the linear differential equation

$$\ddot{X}(t)+2\beta_e\dot{X}(t)+k_e^2X(t)=Y(t), \tag{17}$$

approximates the nonlinear differential Eq. (1), using some error measure (for example, the mean-square error). Once this approximation has been constructed, the properties of the nonlinear Eq. (1) are approximated via the corresponding properties for the linear Eq. (17) provided the aforementioned approximation is accurate enough. In the following Sections 3.1 and 3.2, a number of statistical properties for the solution, $X(t)$, of the linear Eq. (17) and its mean-square derivative, $\dot{X}(t)$, are deduced, respectively. As will be seen later, these properties will play a crucial role in our subsequent deductions. In Section 3.3, we detail how the equivalent linearization technique operates in dealing with random nonlinear oscillators of the form (1) without particularizing neither the form of the function $g(X(t),\dot{X}(t))$ nor the function $f(X(t))$ that is affected by the perturbation parameter ϵ [24,48].

3.1. Statistical properties of the solution of the linearized oscillator

To study the steady state solution of the linear Eq. (17) in the underdamped case, i.e., when $\beta_e<k_e$, one can take as initial condition $X(0)=\dot{X}(0)=0$, and then it is well-known that the corresponding solution is given by

$$X(t)=\int_{-\infty}^t h(t-u)Y(u)\mathrm{d}u=\int_0^\infty h(u)Y(t-u)\mathrm{d}u, \tag{18}$$

where

$$h(t)=\begin{cases}\frac{1}{\sqrt{k_e^2-\beta_e^2}}\exp(-\beta_e t)\sin\left(\sqrt{k_e^2-\beta_e^2}t\right), & \text{if}\quad t\ge 0,\\ 0, & \text{if}\quad t<0.\end{cases} \tag{19}$$

Notice that, if $\mathbb{E}\{Y(t)\}=0$, then using (18) and Proposition 7, one gets

$$\mathbb{E}\{X(t)\}=\int_0^\infty h(u)\mathbb{E}\{Y(t-u)\}\,\mathrm{d}u=0. \tag{20}$$

Moreover, using (18) again and Proposition 8, it is clear that the solution $X(t)$ is a Gaussian process. Furthermore, utilizing the property (13) of Proposition 7 and assuming that $Y(t)$ is a wide-sense stationary stochastic process, one gets,

$$\begin{aligned}\Gamma_X(t_1,t_2) &= \int_0^\infty\int_0^\infty h(u_1)h(u_2)\Gamma_Y(t_1-u_1,t_2-u_2)\mathrm{d}u_1\mathrm{d}u_2\\ &= \int_0^\infty\int_0^\infty h(u_1)h(u_2)\Gamma_Y(t_2-t_1+u_1-u_2)\mathrm{d}u_1\mathrm{d}u_2.\end{aligned} \tag{21}$$

So, $X(t)$ is also a wide-sense stationary stochastic process since it has finite mean and second-order moment, and moreover, its correlation function calculated at (t_1,t_2) only depends on the difference t_2-t_1. As $X(t)$ is Gaussian and wide-sense stationary, hence it is also stationary. Finally, observe

that its variance function can be straightforwardly obtained from (21)

$$\begin{aligned}\sigma_X^2 &= \mathbb{V}\{X(t)\} = \Gamma_X(t,t)\\ &= \int_0^\infty\int_0^\infty h(u_1)h(u_2)\Gamma_Y(u_1-u_2)\mathrm{d}u_1\mathrm{d}u_2 = \mathbb{E}\left\{(X(t))^2\right\}\end{aligned} \tag{22}$$

Let us first observe that the variance does not depend on t as expected since $X(t)$ is a stationary stochastic process (hence the previous notation σ_X^2 instead of $\sigma_X^2(t)$). Second, notice that in the last step, we have used that $\mathbb{E}\{X(t)\} = 0$, so the variance coincides with the second-order moment. Moreover, we can calculate higher moments of $X(t)$ using Proposition 4,

$$\mathbb{E}\{(X(t))^n\} = \begin{cases} 0 & \text{if} \quad n \text{ odd},\\ (n-1)!!\ \left(\sigma_X\right)^n & \text{if} \quad n \text{ even},\end{cases} \tag{23}$$

where, according to (22), $\sigma_X = \left(\int_0^\infty\int_0^\infty h(u_1)h(u_2)\Gamma_Y(u_1-u_2)\mathrm{d}u_1\mathrm{d}u_2\right)^{1/2}$.

Now, we show some properties of the cross-correlation function of the input $Y(t)$ and the solution $X(t)$ that will be used later. Let us first observe that $X(t)$, given in (18), can be expressed as follows

$$X(t) = \int_{-\infty}^\infty h(u)Y(t-u)\mathrm{d}u, \tag{24}$$

since, according to (19), $h(t) = 0$ for $t < 0$. Then,

$$\begin{aligned}\Gamma_{YX}(s) &= \mathbb{E}\{Y(t)X(t+s)\} = \mathbb{E}\left\{Y(t)\int_{-\infty}^\infty h(u)Y(t+s-u)\mathrm{d}u\right\}\\ &= \int_{-\infty}^\infty h(u)\mathbb{E}\{Y(t)Y(t+s-u)\}\mathrm{d}u = \int_{-\infty}^\infty h(u)\Gamma_{YY}(s-u)\mathrm{d}u\end{aligned} \tag{25}$$

where, first we have applied Proposition 7 to commute the expectation operator and the integral and second that $\mathbb{E}\{Y(t)Y(t+s-u)\} = \Gamma_{YY}(t,t+s-u) = \Gamma_{YY}(s-u)$ since $Y(t)$ is wide-stationary.

Analogously,

$$\begin{aligned}\Gamma_{XX}(s) &= \mathbb{E}\{X(t)X(t+s)\} = \mathbb{E}\left\{\left(\int_{-\infty}^\infty h(u)Y(t-u)\mathrm{d}u\right)X(t+s)\right\}\\ &= \int_{-\infty}^\infty h(u)\mathbb{E}\{Y(t-u)X(t+s)\}\mathrm{d}u = \int_{-\infty}^\infty h(u)\Gamma_{YX}(u+s)\mathrm{d}u,\end{aligned} \tag{26}$$

where now we have utilized that $\mathbb{E}\{Y(t-u)X(t+s)\} = \Gamma_{YX}(t-u,t+s) = \Gamma_{YX}(u+s)$.

Let us calculate the cross-power spectral density of the input, $Y(t)$, and the solution, $X(t)$, by taking the Fourier transform of $\Gamma_{YX}(s)$ given in (25)

$$\begin{aligned}S_{YX}(w) &= \frac{1}{\pi}\int_{-\infty}^\infty \Gamma_{YX}(\tau)\ \exp(-\mathrm{i}w\tau)\mathrm{d}\tau\\ &= \frac{1}{\pi}\int_{-\infty}^\infty\left(\int_{-\infty}^\infty h(u)\Gamma_{YY}(\tau-u)\mathrm{d}u\right)\exp(-\mathrm{i}w\tau)\mathrm{d}\tau\\ &= \frac{1}{\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty h(u)\Gamma_{YY}(\tau-u)\exp(-\mathrm{i}w\tau)\mathrm{d}u\mathrm{d}\tau\\ &= \frac{1}{\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty h(u)\Gamma_{YY}(\tau-u)\exp(-\mathrm{i}w\tau)\mathrm{d}\tau\mathrm{d}u\\ &= \int_{-\infty}^\infty h(u)\left(\frac{1}{\pi}\int_{-\infty}^\infty \Gamma_{YY}(\tau-u)\exp(-\mathrm{i}w\tau)\mathrm{d}\tau\right)\mathrm{d}u\\ &= \int_{-\infty}^\infty h(u)\left(\frac{1}{\pi}\int_{-\infty}^\infty \Gamma_{YY}(\tau-u)\exp(-\mathrm{i}w(\tau-u))\mathrm{d}\tau\right)\exp(-\mathrm{i}wu)\mathrm{d}u,\end{aligned} \tag{27}$$

where we have applied Fubbini's theorem to interchange the integrals. Using the change of variable $v = \tau - u$ in the last integral, one gets

$$\frac{1}{\pi}\int_{-\infty}^\infty \Gamma_{YY}(\tau-u)\exp(-\mathrm{i}w(\tau-u))\mathrm{d}\tau = \frac{1}{\pi}\int_{-\infty}^\infty \Gamma_{YY}(v)\exp(-\mathrm{i}wv)\mathrm{d}v = S_{YY}(w), \tag{28}$$

where in the last step we have used (9). So, plugging (28) in (27), one obtains

$$S_{YX}(w) = \int_{-\infty}^\infty h(u)S_{YY}(w)\exp(-\mathrm{i}wu)\mathrm{d}u = S_{YY}(w)\int_{-\infty}^\infty h(u)\exp(-\mathrm{i}wu)\mathrm{d}u. \tag{29}$$

Similarly, taking the Fourier transform of $\Gamma_{XX}(s)$ given in (26), one gets

$$\begin{aligned} S_{XX}(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\Gamma_{XX}(\tau)\,\exp(-\mathrm{i}w\tau)\mathrm{d}\tau \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}h(u)\Gamma_{YX}(u+\tau)\mathrm{d}u\right)\exp(-\mathrm{i}w\tau)\mathrm{d}\tau \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(u)\Gamma_{YX}(u+\tau)\exp(-\mathrm{i}w\tau)\mathrm{d}u\mathrm{d}\tau \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(u)\Gamma_{YX}(u+\tau)\exp(-\mathrm{i}w\tau)\mathrm{d}\tau\mathrm{d}u \\ &= \int_{-\infty}^{\infty}h(u)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\Gamma_{YX}(u+\tau)\exp(-\mathrm{i}w\tau)\mathrm{d}\tau\right)\mathrm{d}u \\ &= \int_{-\infty}^{\infty}h(u)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\Gamma_{YX}(u+\tau)\exp(-\mathrm{i}w(u+\tau))\mathrm{d}\tau\right)\exp(\mathrm{i}wu)\mathrm{d}u. \end{aligned} \tag{30}$$

Again, the change of variable $v = u + \tau$, the last integral can be expressed as

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\Gamma_{YX}(u+\tau)\exp(-\mathrm{i}w(u+\tau))\mathrm{d}\tau = \frac{1}{\pi}\int_{-\infty}^{\infty}\Gamma_{YX}(v)\exp(-\mathrm{i}wv)\mathrm{d}v = S_{YX}(w). \tag{31}$$

Therefore, substituting (31) in (30), one obtains

$$S_{XX}(w) = \int_{-\infty}^{\infty}h(u)S_{YX}(w)\exp(\mathrm{i}wu)\mathrm{d}u = S_{YX}(w)\int_{-\infty}^{\infty}h(u)\exp(\mathrm{i}wu)\mathrm{d}u. \tag{32}$$

Plugging $S_{YX}(w)$, given in (29), in (32), one finally gets the relation between the power spectral densities of the response $X(t)$ and the input $Y(t)$,

$$S_{XX}(w) = S_{YY}(w)\int_{-\infty}^{\infty}h(u)\exp(-\mathrm{i}wu)\mathrm{d}u\int_{-\infty}^{\infty}h(u)\exp(\mathrm{i}wu)\mathrm{d}u. \tag{33}$$

This relation can be expressed in terms of the so-called frequency response function associated with the linear model (17)

$$\hat{h}(z) = \int_{-\infty}^{\infty}h(t)\exp(-zt)\mathrm{d}t, \quad z\in\mathbb{C}. \tag{34}$$

Indeed, notice that (33) can be written in terms of $\hat{h}(z)$, defined in (34), evaluated at $z = \mathrm{i}w$

$$S_{XX}(w) = S_{YY}(w)\hat{h}(\mathrm{i}w)\hat{h}(-\mathrm{i}w). \tag{35}$$

Moreover, using the Euler identity, $\exp(\mathrm{i}r) = \cos(r) + \mathrm{i}\sin(r)$, $r\in\mathbb{R}$, one gets

$$\hat{h}(-\mathrm{i}w) = \int_{-\infty}^{\infty}h(t)\exp(\mathrm{i}wt)\mathrm{d}t = \int_{-\infty}^{\infty}h(t)\overline{\exp(-\mathrm{i}wt)}\mathrm{d}t = \overline{\int_{-\infty}^{\infty}h(t)\exp(-\mathrm{i}wt)\mathrm{d}t} = \overline{\hat{h}(\mathrm{i}w)}. \tag{36}$$

This permits expressing (35) as follows

$$S_{XX}(w) = S_{YY}(w)\hat{h}(\mathrm{i}w)\overline{\hat{h}(\mathrm{i}w)} = S_{YY}(w)|\hat{h}(\mathrm{i}w)|^2, \tag{37}$$

where $\hat{h}(\mathrm{i}w)$ is defined in (34). This value can be calculated by integration

$$\hat{h}(\mathrm{i}w) = \frac{1}{\hat{H}(w)}, \quad \hat{H}(w) = \frac{1}{k_e^2 - w^2 + 2\mathrm{i}w\beta_e}. \tag{38}$$

Hence, expression (37) can be written as

$$S_{XX}(w) = S_{YY}(w)\hat{h}(\mathrm{i}w)\overline{\hat{h}(\mathrm{i}w)} = S_{YY}(w)|\hat{H}(w)|^{-2}, \tag{39}$$

where $\hat{H}(w)$ is defined in (38).

In summarizing, if $Y(t)$ is a stationary zero-mean Gaussian process with a given correlation function, $\Gamma_Y(\tau)$, then the stochastic process $X(t)$ defined in (18)–(19) is also a stationary zero-mean Gaussian process whose correlation function is given by (21). Furthermore, we have calculated the spectral density function of the response, $S_{XX}(w)$, in terms of the spectral density function of the input, $S_{YY}(w)$, and the system function or frequency-response function associated with the linear system.

3.2. Statistical properties of the mean-square derivative of the solution of the linearized oscillator

In Section 3.1 we have obtained a number of properties of the stochastic process $X(t)$, given by (18)–(19), from the properties of the input $Y(t)$. Now, we shall rigorously obtain analogous properties for $\dot{X}(t)$ by retaining the same hypotheses for $Y(t)$ assumed in Section 3.1, i.e., it is a stationary zero-mean Gaussian process.

On the one hand, applying the mean-square Leibniz's rule stated in Proposition 9 to the first expression of $X(t)$ given in (18), one gets

$$\dot{X}(t) = h(0)Y(t) + \int_{-\infty}^{t}h'(t-u)Y(u)\mathrm{d}u = \int_{-\infty}^{t}h'(t-u)Y(u)\mathrm{d}u, \tag{40}$$

since $h(0) = 0$ (see (19)). Notice that $h(t)$ is not differentiable at $t = 0$ since

$$h'(t) = \begin{cases} \exp(-\beta_e t)\left(\cos\left(\sqrt{k_e^2 - \beta_e^2}\,t\right) - \dfrac{\beta_e \sin\left(\sqrt{k_e^2 - \beta_e^2}\,t\right)}{\sqrt{k_e^2 - \beta_e^2}}\right), & \text{if} \quad t > 0, \\ 0, & \text{if} \quad t < 0. \end{cases} \tag{41}$$

However, Leibniz's rule is also valid when the partial derivative exists almost everywhere and does not require that it be continuous [49]. Then, applying Proposition 6, one deduces the following key property that will be extensively used later

$$\Gamma_{X,\dot{X}}(t,t) = \mathbb{E}\{X(t)\dot{X}(t)\} = 0, \tag{42}$$

in other words, the stochastic processes given by (18) and (40), where $h(t)$ is defined in (19), are orthogonal or uncorrelated. Even more, as $Y(t)$ is Gaussian then, by Proposition 5, $\dot{X}(t)$ is also Gaussian, so by Proposition 2, $X(t)$ and $\dot{X}(t)$ are independent. Then, applying Proposition 8 to expression (40), one deduces that $\dot{X}(t)$ is Gaussian too. Furthermore, as $\mathbb{E}\{Y(t)\} = 0$, then taking in (40) the expectation operator and applying Proposition 7, one obtains

$$\mathbb{E}\{\dot{X}(t)\} = \int_{-\infty}^{t} h'(t-u)\mathbb{E}\{Y(u)\}\,\mathrm{d}u = 0. \tag{43}$$

From the second expression of $\dot{X}(t)$ given in (40) and formula (13) in Proposition 7, the correlation function of $\dot{X}(t)$ can be calculated as follows

$$\begin{aligned} \Gamma_{\dot{X}}(t_1,t_2) &= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} h'(t_1-u_1)h'(t_2-u_2)\Gamma_Y(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2 \\ &= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} h'(t_1-u_1)h'(t_2-u_2)\Gamma_Y(u_1-u_2)\mathrm{d}u_1\mathrm{d}u_2, \end{aligned} \tag{44}$$

where in the last step we have utilized that $Y(t)$ is a wide-sense stationary stochastic process. As a consequence, the variance (and second-order moment of $\dot{X}(t)$ since $\mathbb{E}\{\dot{X}(t)\} = 0$) is given by

$$\sigma_{\dot{X}}^2 = \mathbb{V}\{\dot{X}(t)\} = \Gamma_{\dot{X}}(t,t) = \int_{-\infty}^{t}\int_{-\infty}^{t} h'(t-u_1)h'(t-u_2)\Gamma_Y(u_1-u_2)\mathrm{d}u_1\mathrm{d}u_2 = \mathbb{E}\{(\dot{X}(t))^2\}. \tag{45}$$

Using the change of variable: $v_1 = t - u_1$ and $v_2 = t - u_2$, observe that (45) writes

$$\sigma_{\dot{X}}^2 = \int_{-\infty}^{0}\int_{-\infty}^{0} h'(v_1)h'(v_2)\Gamma_Y(v_2-v_1)(-\mathrm{d}v_1)(-\mathrm{d}v_2) = \int_{0}^{\infty}\int_{0}^{\infty} h'(v_1)h'(v_2)\Gamma_Y(v_2-v_1)\mathrm{d}v_1\mathrm{d}v_2, \tag{46}$$

showing, as expected, that the variance does not depend on t since $\dot{X}$ is a stationary stochastic process.

Summarizing up, if $Y(t)$ is a stationary zero-mean Gaussian process with a given correlation function, $\Gamma_Y(\tau)$, then the stochastic process $\dot{X}(t)$, defined by (40) and (19), is also a stationary zero-mean Gaussian process which is orthogonal (or uncorrelated) to the $X(t)$ defined in (18)–(19). Observe that, even more, $X(t)$ and $\dot{X}(t)$ are independent because both are Gaussian.

3.3. Stochastic equivalent linearization technique

Now, we detail how to perform the above-mentioned linear approximation. For this goal, let us consider the auxiliary equation

$$\ddot{X}(t) + 2\beta_e\dot{X}(t) + k_e^2 X(t) = Y(t) + N(t), \tag{47}$$

where the stochastic process

$$N(t) = 2\beta_e\dot{X}(t) + k_e^2 X(t) - g(X(t),\dot{X}(t)) \tag{48}$$

is the error function of the linear approximation. If we use the usual mean-square error to minimize this function, then β_e and k_e^2 must be chosen such that

$$e(t) := \mathbb{E}\left\{(N(t))^2\right\} = \mathbb{E}\left\{\left(2\beta_e\dot{X}(t) + k_e^2 X(t) - g(X(t),\dot{X}(t))\right)^2\right\} \tag{49}$$

is minimized for $t > 0$. To perform the minimization, we first calculate the first and second-order derivatives of $e(t)$ w.r.t. β_e and k_e^2, and we then impose the well-known conditions for minimizing a function. This way, one obtains

$$\begin{aligned} \frac{\partial e(t)}{\partial \beta_e} &= 4\mathbb{E}\left\{2\beta_e(\dot{X}(t))^2 + k_e^2 X(t)\dot{X}(t) - \dot{X}(t)g(X(t),\dot{X}(t))\right\} = 0, \\ \frac{\partial e(t)}{\partial k_e^2} &= 2\mathbb{E}\left\{k_e^2(X(t))^2 + 2\beta_e X(t)\dot{X}(t) - X(t)g(X(t),\dot{X}(t))\right\} = 0, \\ \frac{\partial^2 e(t)}{\partial \beta_e^2} &= 8\mathbb{E}\{(\dot{X}(t))^2\} > 0, \\ \frac{\partial^2 e(t)}{\partial (k_e^2)^2} &= 2\mathbb{E}\left\{(X(t))^2\right\} > 0, \\ \frac{\partial^2 e(t)}{\partial k_e^2 \partial \beta_e} &= \frac{\partial^2 e(t)}{\partial \beta_e \partial k_e^2} = 4\mathbb{E}\{X(t)\dot{X}(t)\}. \end{aligned} \tag{50}$$

Therefore,

$$\frac{\partial^2 e(t)}{\partial \beta_e^2}\frac{\partial^2 e(t)}{\partial (k_e^2)^2} - \left[\frac{\partial^2 e(t)}{\partial k_e^2 \partial \beta_e}\right]^2 = 16\left[\mathbb{E}\left\{(X(t))^2\right\}\mathbb{E}\{(\dot{X}(t))^2\} - (\mathbb{E}\{X(t)\dot{X}(t)\})^2\right] \geq 0. \tag{51}$$

To justify the last inequality, we utilize that $X(t)$ is zero-mean stationary Gaussian process (see Section 3.1), so by Proposition 6, $X(t)$ and $\dot{X}(t)$ are independent. Now, applying Proposition 1(ii) with $Z_1(t) = X(t)$, $Z_2(t) = \dot{X}(t)$, $f(u) = g(u) = u^2$, we can guarantee that $(X(t))^2$ and $(\dot{X}(t))^2$ are also independent and hence $\mathbb{E}\left\{(X(t))^2\right\}\mathbb{E}\{(\dot{X}(t))^2\} = \mathbb{E}\{(X(t))^2(\dot{X}(t))^2\}$. As a consequence,

$$\begin{aligned}&\mathbb{E}\left\{(X(t))^2\right\}\mathbb{E}\{(\dot{X}(t))^2\} - (\mathbb{E}\{X(t)\dot{X}(t)\})^2\\ &= \mathbb{E}\{(X(t))^2(\dot{X}(t))^2\} - (\mathbb{E}\{X(t)\dot{X}(t)\})^2 = \mathbb{V}\left\{X(t)\dot{X}(t)\right\} \geq 0,\end{aligned}$$

where $\mathbb{V}\{\cdot\}$ denotes the variance, which by definition is non-negative.

Then, according to the first two equations above and using the linearity of the expectation operator $\mathbb{E}\{\cdot\}$, it is sufficient that β_e and k_e^2 solve the following system of algebraic equations

$$\begin{cases} 2\beta_e\mathbb{E}\{(\dot{X}(t))^2\} + k_e^2\mathbb{E}\{X(t)\dot{X}(t)\} - \mathbb{E}\{\dot{X}(t)g(X(t),\dot{X}(t))\} = 0,\\ k_e^2\mathbb{E}\{(X(t))^2\} + 2\beta_e\mathbb{E}\{X(t)\dot{X}(t)\} - \mathbb{E}\{X(t)g(X(t),\dot{X}(t))\} = 0,\end{cases} \tag{52}$$

in order to the error function $e(t)$, defined in (49), be minimized. Notice that in the particular case that $X(t)$ is given by (18)–(19), where $Y(t)$ is a mean-square differentiable zero-Gaussian stationary Gaussian process, these two conditions simplify as

$$\begin{cases} 2\beta_e\mathbb{E}\{(\dot{X}(t))^2\} - \mathbb{E}\{\dot{X}(t)g(X(t),\dot{X}(t))\} = 0,\\ k_e^2\mathbb{E}\{(X(t))^2\} - \mathbb{E}\{X(t)g(X(t),\dot{X}(t))\} = 0,\end{cases} \tag{53}$$

because of (42).

Remark 1. Observe that in Sections 3.1 and 3.2 we have deduced that $X(t)$ and $\dot{X}(t)$ are stationary zero-mean independent Gaussian process, so its joint probability density function is given by

$$f_{X(t),\dot{X}(t)}(x,\dot{x}) = \frac{1}{2\pi\sigma_X^2\sigma_{\dot{X}}^2}\exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma_X^2} + \frac{\dot{x}^2}{\sigma_{\dot{X}}^2}\right)\right) = f_{X(t)}(x)f_{\dot{X}(t)}(\dot{x}) \tag{54}$$

where

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-\frac{x^2}{2\sigma_X^2}\right), \quad f_{\dot{X}(t)}(\dot{x}) = \frac{1}{\sqrt{2\pi\sigma_{\dot{X}}^2}}\exp\left(-\frac{\dot{x}^2}{2\sigma_{\dot{X}}^2}\right). \tag{55}$$

Then, the expectations involved in the Eq. (53) can be calculated in terms of $f_{X(t)}(x)$, $f_{\dot{X}(t)}(\dot{x})$ and $f_{X(t),\dot{X}(t)}(x,\dot{x})$:

$$\mathbb{E}\{(X(t))^2\} = \int_{-\infty}^{\infty} x^2 f_{X(t)}(x)\,\mathrm{d}x, \quad \mathbb{E}\{(\dot{X}(t))^2\} = \int_{-\infty}^{\infty} \dot{x}^2 f_{\dot{X}(t)}(\dot{x})\,\mathrm{d}\dot{x},$$

$$\mathbb{E}\{X(t)g(X(t),\dot{X}(t))\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xg(x,\dot{x})f_{X(t),\dot{X}(t)}(x,\dot{x})\,\mathrm{d}x\mathrm{d}\dot{x},$$

and

$$\mathbb{E}\{\dot{X}(t)g(X(t),\dot{X}(t))\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot{x}g(x,\dot{x})f_{X(t),\dot{X}(t)}(x,\dot{x})\,\mathrm{d}x\mathrm{d}\dot{x}.$$

However, observe that these expectations depend ultimately on β_e and k_e^2. Indeed, $f_{X(t)}(x)$ and $f_{\dot{X}(t)}(\dot{x})$ depend on σ_X^2 and $\sigma_{\dot{X}}^2$ (see (55)) that in turn depend on $h(t)$ (see expressions (22) and (46), respectively), and, $h(t)$ is a function of β_e and k_e^2 (see (19)). Therefore, the parameters β_e and k_e^2 could be theoretically calculated using iterative methods to the system of Eq. (53). However, it seems to be an unfeasible approach in practice since the resulting system is highly nonlinear in both unknowns, β_e and k_e^2, which makes extremely difficult their computation. In the next section, we develop an alternative approach to circumvent this drawback.

4. Random oscillators where nonlinearity affects the spatial position

In this section, we study the class of random nonlinear oscillators formulated in (1)–(2) for the case that

$$f(X(t)) = \sum_{n=1}^{N} a_n(X(t))^n. \tag{56}$$

Notice that when we take $a_3 \neq 0$ and $a_n = 0$, $n \neq 3$, one gets the Duffing oscillator. We will apply the results obtained in the foregoing section to study the class of random oscillators formulated in (1)–(2), where $f(X(t))$ is given in (56). So, according to (2),

$$g(X(t),\dot{X}(t)) = 2\beta\dot{X}(t) + \omega_0^2\left[X(t) + \epsilon\sum_{n=1}^{N} a_n(X(t))^n\right]. \tag{57}$$

First, we calculate the constants β_e and k_e^2, with $\beta_e < k_e^2$ (underdamped case), to determine the equivalent linear oscillator (17). We know, from our previous analysis, that β_e and k_e^2 must solve the algebraic system of Eq. (53) with $g(X(t),\dot{X}(t))$ given in (57). So, from the first equation of (53) using the linearity of the expectation operator, one obtains

$$2\beta_e\mathbb{E}\{(\dot{X}(t))^2\} - 2\beta\mathbb{E}\{(\dot{X}(t))^2\} - \epsilon\omega_0^2\sum_{n=0}^{N} a_n\mathbb{E}\{(X(t))^n\dot{X}(t)\} = 0. \tag{58}$$

In Section 3.2 we have proven that $X(t)$ and $\dot{X}(t)$ are independent stochastic processes, so by Proposition 1(ii) with $f(u) = u^n$ and $g(u) = u$, it is guaranteed that $(X(t))^n$ and $\dot{X}(t)$ are also independent and $\mathbb{E}\{(X(t))^n\dot{X}(t)\} = \mathbb{E}\{(X(t))^n\}\mathbb{E}\{\dot{X}(t)\}$. As $X(t)$ is Gaussian (see Section 3.1), then all its moments are finite, i.e., $\mathbb{E}\{(X(t))^n\} < \infty$, and then using that by Section 3.2, $\mathbb{E}\{\dot{X}(t)\} = 0$, one gets $\mathbb{E}\{(X(t))^n\dot{X}(t)\} = 0$. Using this fact, Eq. (58) writes

$$2(\beta_e - \beta)\mathbb{E}\{(\dot{X}(t))^2\} = 0, \tag{59}$$

hence

$$\beta_e = \beta. \tag{60}$$

This result is consistent (and intuitive) since $\dot{X}(t)$ appears linearly in (57) and its coefficient matches the corresponding to the linear model (17) when $\beta_e = \beta$.

Now, we substitute (57) in the second equation of (53) and apply the linearity of the expectation operator. This leads to

$$k_e^2\mathbb{E}\{(X(t))^2\} - 2\beta\mathbb{E}\{X(t)\dot{X}(t)\} - \omega_0^2\mathbb{E}\{(X(t))^2\} - \epsilon\omega_0^2\sum_{n=1}^{N} a_n\mathbb{E}\{(X(t))^{n+1}\} = 0.$$

The orthogonality of $X(t)$ and $\dot{X}(t)$, deduced in (42), simplifies the previous expression as

$$(k_e^2 - \omega_0^2)\mathbb{E}\{(X(t))^2\} - \epsilon\omega_0^2\sum_{n=1}^{N} a_n\mathbb{E}\{(X(t))^{n+1}\} = 0.$$

Now, using the fact that the odd moments of a zero-mean Gaussian random variable are zero (see Proposition 4), the foregoing relation writes

$$(k_e^2 - \omega_0^2 - \epsilon\omega_0^2 a_1)\mathbb{E}\{(X(t))^2\} - \epsilon\omega_0^2\sum_{n=2}^{\lfloor\frac{N+1}{2}\rfloor} a_{2n-1}\mathbb{E}\{(X(t))^{2n}\} = 0.$$

We now substitute the explicit values of the even moments of a zero-mean Gaussian random variable (see Proposition 4) and utilize the time-independent notation $\mathbb{E}\{(X(t))^2\} = \sigma_X^2$ (since $X(t)$ is stationary and $\mathbb{E}\{X(t)\} = 0$),

$$(k_e^2 - \omega_0^2 - \epsilon\omega_0^2 a_1)\sigma_X^2 - \epsilon\omega_0^2\sum_{n=2}^{\lfloor\frac{N+1}{2}\rfloor} a_{2n-1}(2n-1)!!(\sigma_X^2)^n = 0.$$

Finally, solving for k_e^2, one obtains the value of the second parameter of the linearized model (47)

$$k_e^2 = (1 + \epsilon a_1)\omega_0^2 + \epsilon\omega_0^2\sum_{n=2}^{\lfloor\frac{N+1}{2}\rfloor} a_{2n-1}(2n-1)!!(\sigma_X^2)^{n-1}, \tag{61}$$

which can be rewritten

$$k_e^2 = \omega_0^2\left[1 + \epsilon\sum_{n=1}^{\lfloor\frac{N+1}{2}\rfloor} a_{2n-1}(2n-1)!!(\sigma_X^2)^{n-1}\right], \tag{62}$$

where σ_X^2 is given in (22), which in turns is defined in terms of the function h defined in (19). The expression (62) for k_e^2 admits the following intuitive interpretation: k_e^2 is the undamped angular frequency, ω_0^2, plus a deviation whose size is proportional to the perturbation parameter ϵ. Such deviation depends on ultimately the correlation of the input $Y(t)$.

Remark 2. We here discuss the form chosen in (56) for the function f. It might be natural to start the summation in (56) from $n = 0$ instead of $n = 1$; however, in such a case, the input would violate the condition of having a null expectation. Indeed, if $f(X(t)) = \sum_{n=0}^{N} a_n(X(t))^n$, then the nonlinear oscillator (1)–(2) writes

$$\ddot{X}(t) + 2\beta\dot{X}(t) + \omega_0^2\left[X(t) + \epsilon\sum_{n=0}^{N} a_n(X(t))^n\right] = Y(t), \tag{63}$$

or equivalently,

$$\ddot{X}(t) + 2\beta\dot{X}(t) + \omega_0^2\left[X(t) + \epsilon\sum_{n=1}^{N} a_n(X(t))^n\right] = \hat{Y}(t)$$

where

$$\hat{Y}(t) := Y(t) - \epsilon\omega_0^2 a_0. \tag{64}$$

Then,

$$\mathbb{E}\{\hat{Y}(t)\} = \mathbb{E}\{Y(t)\} - \epsilon\omega_0^2 a_0 = -\epsilon\omega_0^2 a_0 \neq 0,$$

since $\mathbb{E}\{Y(t)\} = 0$. So, the previous development cannot be applied unless $a_0 = 0$ which leads to $\mathbb{E}\{\hat{Y}(t)\} = 0$. More broadly, the previous development can be applied to any analytic function $f(z)$, such that $f(0) = 0$, by taking (56) as the truncation, at an arbitrary order N, of its Taylor expansion, so $a_n = \frac{d^n f(z)}{dz^n}|_{z=0}$.

Example 1. We determine the coefficients β_e and k_e^2 of the linear model (17) that approximates the nonlinear oscillator (1)–(2) in the case that $f(x) = x^3$ that corresponds to the Duffing oscillator. Therefore, in accordance with (60) and (62),

$$\beta_e = \beta, \qquad k_e^2 = \omega_0^2 \left[1 + 3\epsilon \left(\sigma_X^2\right)\right]. \tag{65}$$

These values for β_e and k_e^2 will be utilized later in the first part of Example 3.

Example 2. We determine the coefficients β_e and k_e^2 of the linear model (17) that approximates the nonlinear oscillator (1)–(2) in the case that $f(x) = \sin(x)$. For it, we approximate $f(x)$ via its Taylor expansion truncated at order $N = 5$. Then, observe that this approximation corresponds to expression (56) with $N = 5$:

$$f(x) = \sum_{n=1}^{N} a_n x^n = \sum_{n=1}^{\lfloor \frac{N+1}{2} \rfloor} a_{2n-1} x^{2n-1}, \quad a_{2n-1} = \frac{(-1)^n}{(2n-1)!}.$$

Therefore, in accordance with (60) and (62),

$$\begin{aligned} \beta_e &= \beta, \\ k_e^2 &= \omega_0^2 \left[1 + \epsilon \sum_{n=1}^{3} a_{2n-1}(2n-1)!!(\sigma_X^2)^{n-1}\right] = \omega_0^2 \left[1 + \epsilon \left(1 - \frac{1}{2}\sigma_X^2 + \frac{1}{8}(\sigma_X^2)^2\right)\right]. \end{aligned} \tag{66}$$

These values for β_e and k_e^2 will be utilized later in the second part of Example 3.

4.1. Approximating the variance

In Remark 1, we explained that σ_X^2 depends on β_e and k_e^2, so the expression given in (62) for k_e^2 is not completely useful in practice since in many cases it may result in an equation very difficult to solve since k_e^2 could appear in a highly nonlinear way. In this section, we present an alternative to calculate reliable approximations of (62). To this end, we will take advantage of the fact that if $Y(t)$ is wide-sense stationary then $X(t)$ does (see Section 3.1), and the relationship between the variance of $X(t)$, σ_X^2, and the spectral density function of $X(t)$ that ultimately depends on the spectral density function of the stochastic input $Y(t)$.

Applying the general result (8) to the wide-sense stationary process $Z = X(t)$ (for which $\mathbb{E}\{X(t)\} = 0$, so its variance coincides with its second-order moment), and taking into account (39), one gets

$$\sigma_X^2 = \mathbb{E}\left\{(X(t))^2\right\} = \int_0^\infty S_{XX}(w)\mathrm{d}w = \int_0^\infty |\hat{H}(w)|^{-2} S_{YY}(w)\mathrm{d}w, \tag{67}$$

where $\hat{H}(w)$ is given in (38). As $\beta_e = \beta$ (see (60)), $\hat{H}(w)$ only depends on k_e^2,

$$\hat{H}(w) = \frac{1}{k_e^2 - w^2 + 2\mathrm{i}w\beta}, \tag{68}$$

and (67) writes

$$\sigma_X^2 = \int_0^\infty \frac{1}{|k_e^2 - w^2 + 2\mathrm{i}w\beta|^2} S_{YY}(w)\mathrm{d}w. \tag{69}$$

Substituting this value in (62), one arrives at the following nonlinear equation in k_e^2

$$k_e^2 = \omega_0^2 \left[1 + \epsilon \sum_{n=1}^{\lfloor \frac{N+1}{2} \rfloor} a_{2n-1}(2n-1)!! \left(\int_0^\infty \frac{S_{YY}(w)}{|k_e^2 - w^2 + 2\mathrm{i}w\beta|^2}\mathrm{d}w\right)^{n-1}\right], \tag{70}$$

which is completely defined once a spectral density function, $S_{YY}(w)$, of the input $Y(t)$, has been given. As solving Eq. (70) for k_e^2 may become challenging, one can check whether the simplest approximation consisting of approximating $\hat{H}(w)$ given in (68) by

$$\hat{H}_0(w) = \frac{1}{w_0^2 - w^2 + 2\mathrm{i}w\beta}, \tag{71}$$

that is the inverse of the frequency response function of the linear system resulting from (1)–(2) when $\epsilon = 0$. In such a case, the variance given in (69) is approximated by the following constant (recall that both w_0^2 and β are given constants)

$$\sigma_{X_0}^2 = \int_0^\infty \frac{1}{|w_0^2 - w^2 + 2\mathrm{i}w\beta|^2} S_{YY}(w)\mathrm{d}w = \int_0^\infty \frac{1}{(w_0^2 - w^2)^2 + 4w^2\beta^2} S_{YY}(w)\mathrm{d}w. \tag{72}$$

Example 3. In the setting of Examples 1 and 2, the equivalent linear equations using the approximation (72) are given by

$$\ddot{X}(t) + 2\beta\dot{X}(t) + \omega_0^2 \left[1 + 3\epsilon \left(\sigma_{X_0}^2\right)\right] X(t) = Y(t). \tag{73}$$

and

$$\ddot{X}(t) + 2\beta\dot{X}(t) + \omega_0^2 \left[1 + \epsilon \left(1 - \frac{1}{2}\sigma_{X_0}^2 + \frac{1}{8}(\sigma_{X_0}^2)^2\right)\right] X(t) = Y(t), \tag{74}$$

respectively. The Eq. (73) will used later in Example 4 while Eq. (74) will used in Examples 5 and 6.

5. Approximating the PDF via the maximum entropy principle

In this section, we explain how we can take advantage of the approximations of the first statistical moments of the steady state to compute approximations of its PDF. To address this goal, we will apply the principle of maximum entropy (PME) based on the Shannon entropy [42]. It is clear that the knowledge of the PDF, $f_Z(z)$, of a random variable, say Z, is very advantageous since it permits calculating any statistical moments $\mathbb{E}\{Z^m\} = \int_{-\infty}^{\infty} z^m f_Z(z)\,\mathrm{d}z$, $m = 1, 2, \ldots$, provided they exist as well as calculating confidence intervals at any specific confidence level, say $\alpha \in (0, 1)$, via $1-\alpha = \mathbb{P}\left\{\mu_Z - k\sigma_Z \le Z \le \mu_Z + k\sigma_Z\right\} = \int_{\mu_Z-k\sigma_Z}^{\mu_Z+k\sigma_Z} f_Z(z)\,\mathrm{d}z$, where μ_Z and σ_Z denote the expectation and standard deviation of Z, respectively. Furthermore, the PDF permits calculating the probability that Z lies within any interval of interest, $\mathbb{P}\{a \le Z \le b\} = \int_a^b f_Z(z)\mathrm{d}z$.

The principle of maximum entropy (PME) is a technique for assigning a probability density function (PDF) to a random variable in a way that remains consistent with the available information without introducing unwarranted assumptions. This information typically includes values like the domain, mean, variance, etc., which are gathered from sampling the random variable. In simple terms, PME seeks a PDF that represents the highest level of uncertainty and the minimal amount of information. A PDF with lower entropy, even if it satisfies the same constraints, would imply more information and thus make stronger claims than intended. Therefore, the PDF with the highest entropy, given the constraints, is the most unbiased in terms of its predictions [50,51].

To determine the PDF, $f_Z(z)$, of a random variable with domain (a, b) one maximizes the Shannon's entropy defined by the functional

$$\mathcal{S}\left\{f_Z(z)\right\} = -\int_a^b f_Z(z)\log\left(f_Z(z)\right)\mathrm{d}z, \tag{75}$$

subject to the following $M+1$ restrictions:

$$\int_a^b f_Z(z)\mathrm{d}z = 1, \tag{76}$$

$$\mathbb{E}\{Z^n\} = \int_a^b z^n f_Z(z)\mathrm{d}z = m_n \quad n = 1, \ldots, M. \tag{77}$$

Condition (76) guarantees $f_Z(z)$ is a PDF, and the next M conditions given in (77) impose that the sampled moments, m_n, match the theoretical moments, $\mathbb{E}\{Z^n\}$. In our setting, the values m_n are obtained by the stochastic equivalent linearization method. To maximize $\mathcal{S}\left\{f_Z(z)\right\}$ subject to (76)–(77), one applies the Lagrange multipliers method by defining the auxiliary function

$$\mathcal{L}\left\{f_Z, \lambda_0, \ldots, \lambda_M\right\} = \mathcal{S}\left\{f_Z(z)\right\} + \sum_{n=0}^{M}\lambda_n\left[m_n - \int_a^b z^n f_Z(z)\mathrm{d}z\right], \tag{78}$$

where $m_0 = 1$. Then, using Variational Calculus, it can be seen that [50]

$$f_Z(z) = \mathbb{1}_{[a,b]}\exp\left(-\sum_{n=0}^{M}\lambda_i z^n\right), \tag{79}$$

where $\mathbb{1}_{[a,b]}$ denotes the characteristic function of the interval $[a, b]$. In practice, the Lagrange multipliers λ_n, $n = 0, 1, \ldots, M$, can be calculated solving numerically the system of nonlinear equations (76)–(77). In Examples 4–6, the Mathematica© `FindRoot` function will be used to solve numerically this integro-algebraic system (76)–(77) [52].

As shall be specified in the examples, once the values of β_e and k_e^2 have been calculated, we will apply the PME taking $[a, b] = [\mu_Z - k\sigma_Z, \mu_Z + k\sigma_Z]$ with $k = 10$ and different values of M until we observe that two consecutive approximations are very close. Observe that the application of the PME for different values of M is feasible since we have previously obtained the explicit information about the moments of $Z \equiv X(t)$ (see (23)). Finally, notice that, according to the Bienaymé-Chebyshev inequality [53], taking $k = 10$ for the domain of the PDF, we guarantee the probability of values lying within the interval $[\mu_Z - 10\sigma_Z, \mu_Z + 10\sigma_Z]$, which is called the *coverage*, is at least 99% regardless the distribution of Z.

6. Numerical examples

This section is devoted to illustrating the theoretical findings obtained in previous sections through three examples. In all the examples we consider Eqs. (1)–(2) with parameters $\beta = \frac{1}{20} = 0.05$ and $\omega_0^2 = 1$, i.e.,

$$\ddot{X}(t) + \frac{1}{10}\dot{X}(t) + X(t) + \epsilon f(X(t)) = Y(t), \quad t > 0, \tag{80}$$

where $f(X(t))$ is given (or can be approximated) by (56) and the excitation $Y(t)$ is a stationary zero-mean Gaussian stochastic process that will be defined in each one of the examples. Table 1 shows the choice of $f(X(t))$ and its approximation, if any, together with $Y(t)$ in each example. Specifically, Example 4 is devoted to studying a Duffing oscillator that corresponds to the choice $f(x) = x^3$. As an extension of this model, in Examples 5 and 6, the function involving nonlinear terms will be $f(x) = \sin(x)$, that we approximate (as in Example 2) via its Taylor expansion truncated at order $N = 5$, $f(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$, in order to use the previous developments when the nonlinear term is given by (56).

Finally, the excitation term $Y(t)$ has been chosen for Examples 4 and 5 as $Y(t) = \xi(t)$, a Gaussian white-noise (WN) process with mean zero. The correlation function of $Y(t)$ is

$$\Gamma_{YY}(\tau) = \pi S_0\delta(\tau), \tag{81}$$

where $\delta(\tau)$ is the Dirac delta function and $S_0 = \frac{1}{200\pi}$ is the noise power. This type of random noise has been extensively used in the literature since the earliest contributions [26]. The power spectral density of $Y(t)$ is obtained from Eqs. (28) and (81), and it is given by

$$S_{YY}(\omega) = S_0, \quad \omega \in (-\infty, \infty). \tag{82}$$

We have considered the Ornstein–Uhlenbeck (OU) process to play the role of the external source, $Y(t)$, in Example 6. This process is defined as the stationary solution of the Langevin equation

$$\frac{dY(t)}{dt} + \alpha_1 Y(t) = \sigma_2\frac{dW(t)}{dt}, \quad \alpha_1 > 0, \tag{83}$$

Table 1
Specification of the data $f(X(t))$, its polynomial, representation according to (56), and the excitation term $Y(t)$ to Eq. (80) in the Examples 4–6 WN and OU stand for white noise and Ornstein–Uhlenbeck processes, respectively.

	$f(X(t))$	$\sum_{n=1}^{N} a_n(X(t))^n$	$Y(t)$
Example 4	$(X(t))^3$	$(X(t))^3$	WN
Example 5	$\sin(X(t))$	$X(t) - \frac{(X(t))^3}{3!} + \frac{(X(t))^5}{5!}$	WN
Example 6	$\sin(X(t))$	$X(t) - \frac{(X(t))^3}{3!} + \frac{(X(t))^5}{5!}$	OU

where $W(t)$ is the Wiener process [3]. Notice that $\alpha_1 > 0$ is a necessary and sufficient condition to have a stationary solution. In particular, we consider the following values for the parameters $\sigma_2 = 0.01$ and $\alpha_1 = 0.5$, so the existence of the steady state solution is ensured. The correlation function of $Y(t)$ is

$$\Gamma_{YY}(\tau) = \sigma_2^2 \exp(-\alpha_1|\tau|). \tag{84}$$

The power spectral density of $Y(t)$ is obtained from Eqs. (28) and (84), and it is given by

$$S_{YY}(\omega) = \frac{2\sigma_2^2\alpha_1}{\pi(\omega^2 + \alpha_1^2)}. \tag{85}$$

Notice that the two types of excitations, WN and OU, playing the role of $Y(t)$ are well-known wide-sense stationary zero-mean Gaussian stochastic processes [24, p. 33–71]. So, the stochastic perturbation method and the equivalent linearization technique can be applied to study the probabilistic properties of the steady state of model (80) using the approach exhibited in previous sections.

The organization of the examples is as follows. We compute the mean and standard deviation at the steady state using the perturbation method for different values of the perturbation parameter ϵ. It is well known that the perturbation method is only applicable for small values of ϵ. We provide the maximum value of ϵ where the method can be used. Solutions deteriorate drastically when we are near this maximum value.

In each one of the examples, we shall take advantage of the stochastic equivalent linearization technique to obtain reliable approximations of the aforementioned statistics of the steady state for greater values of the ϵ. These results are compared with those calculated by the Kloeden–Platen–Schurz numerical method, an explicit order 1.5 strong numerical scheme for solving stochastic differential equations [54, Sec. 4.2 B]. This numerical method is implemented in Mathematica©'s `ItoProcess` function with the `Method ''KloedenPlatenSchurz''` option. Notice that we are interested in obtaining the probabilistic properties of the steady state in each example. As the foreign numerical method constructs oscillatory solutions, we obtain the solution in a time interval in which the solution is stabilized. An interval where the solution has stabilized is determined by calculating the mean with the smallest number of simulations, 10^3, which is the worst case. It is also checked that the standard deviation has stabilized in that interval. When a larger number of simulations are performed, it is found that in all cases the working interval considered is adequate. Therefore, in addition to calculating the mean and standard deviation in that interval, it is also important to provide an estimate of the error by calculating the standard deviation. To check the efficiency of the numerical method, solutions have been obtained with 10^3, 10^4, 10^5 and 10^6 simulations. In Examples 5 and 6, we consider the exact nonlinear term $f(x) = \sin(x)$. Results obtained applying the different methods are collected in Tables 2–3 (Example 4), Tables 8–9 (Example 5) and Tables 14–15 (Example 6).

Once we have verified that good results are obtained with the stochastic equivalent linearization technique, we proceed to calculate the correlation function for different values of the perturbation parameter ϵ. In each example, we provide their explicit expressions, and we also show their graphical representation in Fig. 1 (Example 4), Fig. 5 (Example 5) and Fig. 9 (Example 6).

In addition, as shown in the previous theoretical study, we can calculate the moments of order M for each one of the three examples. With this key information, we construct approximations to the PDF using the PME taking $M = 3, 5$ and for different values of ϵ, including those where perturbation cannot be used to better illustrate the limitations of this method. Results corresponding to the application of the PME technique, for $M = 3, 5$, are given in Tables 5–6 (Example 4), Tables 11–12 (Example 5) and Tables 16–17 (Example 6). The corresponding PDFs, for $M = 5$, are shown in Figs. 3–4 (Example 4), Figs. 7–8 (Example 5) and Figs. 10–11 (Example 6). The PDFs corresponding to $M = 3$ are very similar, as can be seen when we measure the error using the L_1-norm by comparing the PDFs calculated from different moments. The results of the corresponding errors are given in Table 7 (Example 4), Table 13 (Example 5) and Table 18 (Example 6). We have also compared the results obtained for the approximation of the PDF using the PME with $M = 7$ for the first value of ϵ computed in each example where the stochastic perturbation method cannot be used. The results are shown in Table 4 (Example 4), Table 10 (Example 5) and Table 19 (Example 6), and Fig. 2 (Example 4), Fig. 6 (Example 5) and Fig. 12 (Example 6). Finally, as indicated at the end of Section 5, in all the examples we have taken $[x_1, x_2] = [\mu_{X(t)} - 10\sigma_{X(t)}, \mu_{X(t)} + 10\sigma_{X(t)}]$ as the domain of the PDF when applying the PME to ensure at least 99% coverage.

Example 4. Taking the data from Table 1 corresponding to Duffing oscillators, we are going to apply the stochastic equivalent linearization method to study model (80). First, we must determine the coefficients β_e and k_e^2 of the equivalent Eq. (17). From Example 1, these are given by

$$\beta_e = \beta = \frac{1}{20}, \qquad k_e^2 = \omega_0^2\left[1 + 3\epsilon\sigma_X^2\right] = 1 + 3\epsilon\sigma_X^2.$$

Notice that

$$\beta_e = \frac{1}{20} < 1 \leq 1 + 3\epsilon\sigma_X^2 = k_e^2, \quad \forall \epsilon \geq 0,$$

so the condition corresponding to the underdamped case is fulfilled as was assumed to develop the theoretical study.

Since $\mathbb{E}\{X(t)\} = 0$, the following step is to compute the variance given by (69) that is approximated by (72), obtaining $\sigma_{X_0}^2 = \frac{1}{40}$. Now, it is easy to check that, using expression (22), the second-order moment is given by

$$\mathbb{E}\left\{(X(t))^2\right\} = \mathbb{V}\{X(t)\} = \frac{1}{40 + 3\epsilon}. \tag{86}$$

Table 2
Comparison of the mean (μ_X) and the standard deviation (σ_X) between stochastic perturbation method, stochastic equivalent linearization method and Kloeden–Platen–Schurz scheme with 10^6 simulations (in the latter method, we also include an estimate of the error by calculating the standard deviation of the corresponding approximations obtained via the simulations). We have carried out for different $\epsilon \in \{0,\ 0.01,\ 0.05,\ 0.1,\ 0.3, 0.5, 1, 2, 3\}$. The largest value for which the perturbation method works is $\epsilon < 0.33333$, so from that value on, no approximation is included. It is indicated by –. Example 4.

ϵ		Perturbation method	Equivalent linearization method	Kloeden–Platen–Schurz 10^6 simulations
0	μ_X	0	0	0.000012 ± 0.000150
	σ_X	0.158114	0.158114	0.158193 ± 0.000088
0.01	μ_X	0	0	0.000013 ± 0.000210
	σ_X	0.155724	0.158055	0.158194 ± 0.000120
0.05	μ_X	0	0	$0,000000 \pm 0.00014$
	σ_X	0.145774	0.157818	0.158009 ± 0.000093
0.1	μ_X	0	0	-0.000004 ± 0.00014
	σ_X	0.132288	0.157524	0.157717 ± 0.000110
0.3	μ_X	0	0	-0.000032 ± 0.000260
	σ_X	0.05	0.156365	0.156594 ± 0.000110
0.5	μ_X	0	0	-0.000010 ± 0.000140
	σ_X	–	0.155230	0.155540 ± 0.000099
1.0	μ_X	0	0	0.000002 ± 0.000130
	σ_X	–	0.152499	0.153282 ± 0.000092
2.0	μ_X	0	0	0.000007 ± 0.000200
	σ_X	–	0.147442	0.149321 ± 0.000076
3.0	μ_X	0	0	0.000003 ± 0.000120
	σ_X	–	0.142857	0.146010 ± 0.000077

Table 3
Approximations of the mean (μ_X) and the standard deviation (σ_X) using the Kloeden–Platen–Schurz numerical scheme with 10^3, 10^4 and 10^5 simulations including an estimate of the error by calculating the standard deviation of the corresponding approximations obtained via the simulations. We have carried out for different $\epsilon \in \{0,\ 0.01,\ 0.05,\ 0.1,\ 0.3, 0.5, 1, 2, 3\}$. Example 4.

ϵ		10^3 simulations	10^4 simulations	10^5 simulations
0	μ_X	0.0001 ± 0.0037	0.00006 ± 0.00088	0.00000 ± 0.00060
	σ_X	0.1571 ± 0.0034	0.1581 ± 0.0011	0.15817 ± 0.00027
0.01	μ_X	-0.0001 ± 0.0040	0.0001 ± 0.0013	-0.00002 ± 0.00042
	σ_X	0.1587 ± 0.0030	0.15780 ± 0.00070	0.15798 ± 0.00040
0.05	μ_X	0.0003 ± 0.0053	0.0000 ± 0.0014	-0.00004 ± 0.00072
	σ_X	0.1601 ± 0.0040	0.15700 ± 0.00073	0.15767 ± 0.00036
0.1	μ_X	0.0000 ± 0.0056	0.0000 ± 0.0013	-0.00002 ± 0.00053
	σ_X	0.1586 ± 0.0029	0.15779 ± 0.00086	0.15766 ± 0.00032
0.3	μ_X	0.0000 ± 0.0044	0.0000 ± 0.0015	0.00000 ± 0.00052
	σ_X	0.1558 ± 0.0024	0.15686 ± 0.00076	0.15650 ± 0.00034
0.5	μ_X	0.0002 ± 0.0044	0.0000 ± 0.0018	-0.00002 ± 0.00039
	σ_X	0.1543 ± 0.0030	0.1553 ± 0.0010	0.15543 ± 0.00029
1.0	μ_X	-0.0003 ± 0.0058	-0.0002 ± 0.0017	0.00000 ± 0.00056
	σ_X	0.1544 ± 0.0032	0.1531 ± 0.0014	0.15296 ± 0.00029
2.0	μ_X	-0.0001 ± 0.0052	-0.0000 ± 0.0011	0.00001 ± 0.00041
	σ_X	0.1485 ± 0.0032	0.14924 ± 0.00082	0.14933 ± 0.00032
3.0	μ_X	0.0000 ± 0.0045	0.0000 ± 0.0013	0.00002 ± 0.00050
	σ_X	0.1477 ± 0.0025	0.14585 ± 0.00079	0.14593 ± 0.00029

As mentioned above, to check that the approximations obtained with the stochastic equivalent linearization technique are reliable, we compare the mean and standard deviation of the approximate solution obtained via the equivalent linearization technique, with those calculated applying the stochastic perturbation method and the approximations computed by the Kloeden–Platen–Schurz numerical method using different number of simulations. It is important to mention that the bound for the perturbation parameter, ϵ, when applying the perturbation method at Example 4 is $\epsilon < 0.33333$. The results are collected in Tables 2–3. We can observe that the stochastic perturbation method is valid only for small ϵ. In this example, it is only reliable up to $\epsilon = 0.01$. However, with the stochastic equivalent linearization method, we are able to obtain good approximations even up to $\epsilon = 3$. The higher the number of simulations, the better the approximation obtained at the expense of a higher computational burden. Specifically, the timing for carrying out 10^3 simulations is about 1 min, while 10^6 simulations lasted about 1 day. All computations were performed in a desktop PC with 64 GB of DDR5 RAM, and i9-13900K CPU. On the other hand, once the theoretical expressions developed in the article have been obtained, the calculation of the mean and variance of the steady state of the model by the stochastic equivalent linearization method is virtually immediate.

Now, applying (21), we obtain the following approximation of the correlation function,

$$\Gamma_{XX}(\tau) = \begin{cases} f_1(\tau), & \text{if } \tau \geq 0, \\ f_2(\tau), & \text{if } \tau < 0, \end{cases} \tag{87}$$

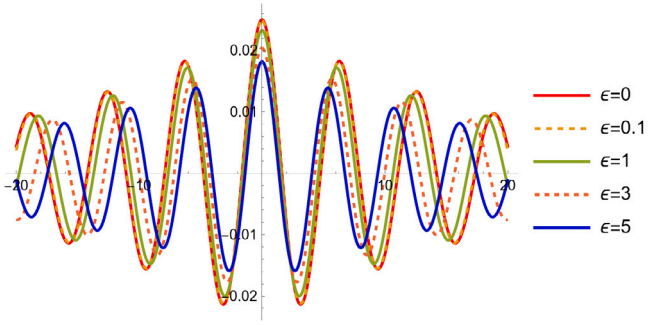

Fig. 1. Comparison of the correlation function $\Gamma_{XX}(\tau)$ of $X(t)$, given in (87)–(89), for different values of ϵ. Example 4.

Table 4
Values for λ_i, $i \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ and the domain $[x_1, x_2]$ obtained via the PME method, for $\epsilon = 0.3$. Example 4.

Order	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
$M = 3$	−1.936626498	0.0	20.45	0.0	–	–	–	–
$M = 5$	−1.936626498	0.0	20.449999999	0.0	$1.81925 \cdot 10^{-9}$	0.0	–	–
$M = 7$	−1.936617097	0.0	20.449677961	0.0	$-6.69252 \cdot 10^{-3}$	0.0	$4.77832 \cdot 10^{-2}$	0.0

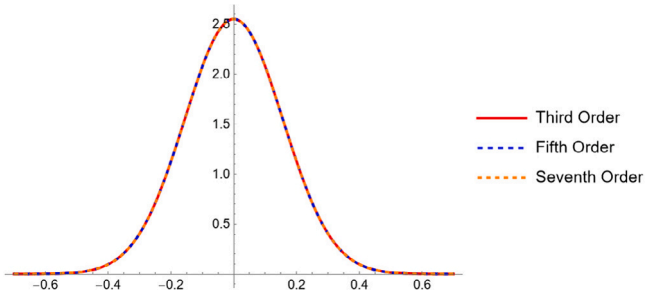

Fig. 2. Approximation of PDF, $f_{X(t)}(x)$, using the PME with $M = 3, 5, 7$ moments for $\epsilon = 0.3$. Example 4.

Table 5
Values of λ_i, $i \in \{0, 1, 2, 3\}$, and the domain $[x_1, x_2]$ obtained via PME based on the three first moments ($M = 3$), for $\epsilon \in \{0, 0.01, 0.05, 0.1, 0.3, 0.5, 1.0, 2.0, 3.0\}$. Example 4.

	$\epsilon = 0$	$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.1$	$\epsilon = 0.3$
λ_0	−1.925501193	−1.925876053	−1.927372686	−1.929237201	−1.936626498
λ_1	0	0	0	0	0
λ_2	20	20.015	20.075	20.15	20.45
λ_3	0	0	0	0	0
$[x_1, x_2]$	[−1.581, 1.581]	[−1.580, 1.580]	[−1.578, 1.578]	[−1.575, 1.575]	[−1.563, 1.563]
	$\epsilon = 0.5$	$\epsilon = 1.0$	$\epsilon = 2.0$	$\epsilon = 3.0$	
λ_0	−1.943908180	−1.961661524	−1.995382165	−2.026971615	
λ_1	0	0	0	0	
λ_2	20.749999999	21.500000000	23.000000000	24.500000000	
λ_3	0	0	0	0	
$[x_1, x_2]$	[−1.552, 1.552]	[−1.524, 1.524]	[−1.474, 1.474]	[−1.428, 1.428]	

$$f_1(\tau) = \frac{\exp\left(-\frac{\tau}{20}\right)}{40 + 3\epsilon}\left[\cos\left(\frac{\tau}{20}\sqrt{399 + 30\epsilon}\right) + \frac{1}{\sqrt{399 + 30\epsilon}}\sin\left(\frac{\tau}{20}\sqrt{399 + 30\epsilon}\right)\right], \tag{88}$$

$$f_2(\tau) = \frac{\exp\left(\frac{\tau}{20}\right)}{40 + 3\epsilon}\left[\cos\left(\frac{\tau}{20}\sqrt{399 + 30\epsilon}\right) - \frac{1}{\sqrt{399 + 30\epsilon}}\sin\left(\frac{\tau}{20}\sqrt{399 + 30\epsilon}\right)\right]. \tag{89}$$

In Fig. 1, we show the graphical representation of the correlation function, $\Gamma_{XX}(\tau)$, given by expressions (87)–(89), for different values of ϵ. We can observe from this plot that the approximations obtained for $\Gamma_{XX}(\tau)$ preserve the symmetry with respect to the vertical axis (even functions) as desirable.

Once we have obtained the standard deviation, $\sigma_{X(t)}$, all the even-order moments can be directly calculated using (23). Notice that the odd-order moments are null. Now, we shall utilize this information to construct approximations of the PDF of $X(t)$, $f_{X(t)}(x)$, applying the PME with different orders of truncation, M, corresponding to the number of statistical moments used.

We first consider the case, $\epsilon = 0.3$, where the stochastic perturbation technique fails. We calculate the PDF using $M = 3, 5, 7$. The values of the parameters λ_i required to construct the PDF according to (79), are given in Table 4. The graphical representation of the corresponding

Table 6
Values of λ_i, $i \in \{0, 1, 2, 3, 4, 5\}$, and the domain $[z_1, z_2]$ obtained via PME based on the five first moments ($M = 5$), for $\epsilon \in \{0, 0.01, 0.05, 0.1, 0.3, 0.5, 1.0, 2.0, 3.0\}$. Example 4.

	$\epsilon = 0$	$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.1$	$\epsilon = 0.3$
λ_0	−1.925501193	−1.925876053	−1.927372686	−1.929237201	−1.936626498
λ_1	0	0	0	0	0
λ_2	19.999999999	20.014999999	20.074999999	20.149999999	20.449999999
λ_3	0	0	0	0	0
λ_4	$1.74018 \cdot 10^{-9}$	$1.74274 \cdot 10^{-9}$	$1.7531 \cdot 10^{-9}$	$1.76642 \cdot 10^{-9}$	$1.81925 \cdot 10^{-9}$
λ_5	0	0	0	0	0
$[x_1, x_2]$	[−1.581, 1.581]	[−1.580, 1.580]	[−1.578, 1.578]	[−1.575, 1.575]	[−1.563, 1.563]
	$\epsilon = 0.5$	$\epsilon = 1.0$	$\epsilon = 2.0$	$\epsilon = 3.0$	
λ_0	−1.943908180	−1.961661524	−1.995382165	−2.026971615	
λ_1	0	0	0	0	
λ_2	20.749999999	21.499999999	22.999999999	24.499999999	
λ_3	0	0	0	0	
λ_4	$1.8731\ 10^{-9}$	$2.011\ 10^{-9}$	$2.30126\ 10^{-9}$	$2.6113\ 10^{-9}$	
λ_5	0	0	0	0	
$[x_1, x_2]$	[−1.552, 1.552]	[−1.524, 1.524]	[−1.474, 1.474]	[−1.428, 1.428]	

Table 7
Values of the error (L_1-norm) between the PDF generated by the PME with $M = 3, 5$, for $\epsilon \in \{0, 0.01, 0.05, 0.1, 0.3, 0.5, 1.0, 2.0, 3.0\}$. Example 4.

	$\epsilon = 0$	$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.1$	$\epsilon = 0.3$
Error	$4.61006 \cdot 10^{-12}$	$4.60995 \cdot 10^{-12}$	$4.60959 \cdot 10^{-12}$	$4.61028 \cdot 10^{-12}$	$4.60978 \cdot 10^{-12}$
$[x_1, x_2]$	[−1.581, 1.581]	[−1.580, 1.580]	[−1.578, 1.578]	[−1.575, 1.575]	[−1.563, 1.563]
	$\epsilon = 0.5$	$\epsilon = 1.0$	$\epsilon = 2.0$	$\epsilon = 3.0$	
Error	$4.61006 \cdot 10^{-12}$	$4.61031 \cdot 10^{-12}$	$4.60966 \cdot 10^{-12}$	$4.61001 \cdot 10^{-12}$	
$[x_1, x_2]$	[−1.552, 1.552]	[−1.524, 1.524]	[−1.474, 1.474]	[−1.428, 1.428]	

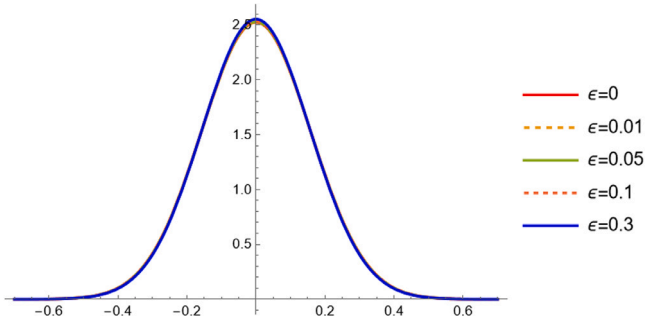

Fig. 3. Approximation of the PDF, $f_{X(t)}(x)$, using until the five order moment ($M = 5$) for $\epsilon \in \{0, 0.01, 0.05, 0.1, 0.3\}$ via the PME. Example 4.

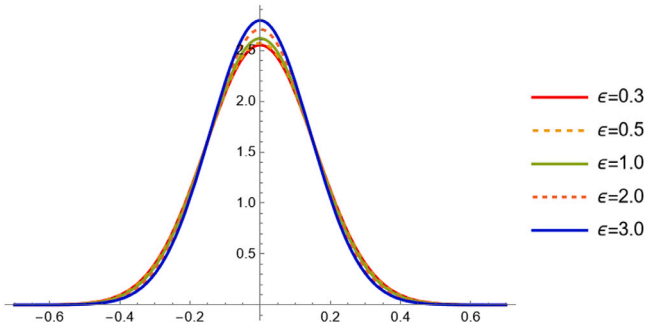

Fig. 4. Approximation of the PDF, $f_{X(t)}(x)$, using until the five order moment ($M = 5$) for $\epsilon \in \{0.3, 0.5, 1.0, 2.0, 3.0\}$ via the PME. Example 4.

PDFs is shown in Fig. 2. We can observe that the three approximations are very close. Then, we have made the decision to only represent the PDF using $M = 5$ when applying the PME method. The graphical results are shown in Fig. 3 (for $\epsilon \in \{0, 0.01, 0.05, 0.1, 0.3\}$) and Fig. 4 (for $\epsilon \in \{0.3, 0.5, 1.0, 2.0, 3.0\}$). We can conclude that the approximations obtained with $M = 5$ are good, as confirmed in Table 7, where it can be checked the small values of the error between the approximations of order $M = 3$ (Table 5) and $M = 5$ (Table 6), measured using the L_1-norm.

Finally, to reinforce that our decision of taking the approximating of the PDF corresponding to $M = 5$ when applying the PME is adequate, in Table 7 we show the error (L_1-norm) obtained when comparing the PDFs calculated with $M = 3$ and $M = 5$ for different values of ϵ. We evidence this error is very small.

Table 8
Comparison of the mean (μ_X) and the standard deviation (σ_X) between stochastic perturbation method, stochastic equivalent linearization method and Kloeden–Platen–Schurz scheme with 10^6 simulations (in the latter method, we also include an estimate of the error by calculating the standard deviation of the corresponding approximations obtained via the simulations). We have carried out for different $\epsilon \in \{0,\ 0.01,\ 0.1,\ 0.5,\ 1, 1.5, 2, 3\}$. The largest value for which the perturbation method works is $\epsilon < 1.01258$, so from that value on, no approximation is included. It is indicated by –. Example 5.

ϵ		Perturbation method	Equivalent linearization method	Kloeden–Platen–Schurz 10^6 simulations
0	μ_X	0	0	0.00000 ± 0.00013
	σ_X	0.158114	0.158114	0.15818 ± 0.00012
0.01	μ_X	0	0	0.00000 ± 0.00025
	σ_X	0.157331	0.157339	0.15738 ± 0.00011
0.1	μ_X	0	0	0.00000 ± 0.00022
	σ_X	0.150103	0.150841	0.15096 ± 0.00011
0.5	μ_X	0	0	0.00000 ± 0.00026
	σ_X	0.112496	0.129368	0.129501 ± 0.000068
1.0	μ_X	0	0	0.000006 ± 0.000095
	σ_X	0.017622	0.112152	0.112366 ± 0.000076
1.5	μ_X	0	0	-0.000003 ± 0.000079
	σ_X	–	0.100375	0.101348 ± 0.000059
2.0	μ_X	0	0	0.00000 ± 0.00010
	σ_X	–	0.091667	0.093091 ± 0.000071
3.0	μ_X	0	0	0.000000 ± 0.000097
	σ_X	–	0.079428	0.081880 ± 0.000065

Table 9
Approximations of the mean (μ_X) and the standard deviation (σ_X) using the Kloeden–Platen–Schurz numerical scheme with 10^3, 10^4 and 10^5 simulations including an estimate of the error by calculating the standard deviation of the corresponding approximations obtained via the simulations. We have carried out for different $\epsilon \in \{0,\ 0.01,\ 0.1,\ 0.5, 1, 1.5, 2, 3\}$. Example 5.

ϵ		10^3 simulations	10^4 simulations	10^5 simulations
0	μ_X	-0.0004 ± 0.0051	0.0001 ± 0.0014	-0.00001 ± 0.00037
	σ_X	0.1588 ± 0.0036	0.15842 ± 0.00095	0.15816 ± 0.00036
0.01	μ_X	0.0001 ± 0.0045	0.0001 ± 0.0010	0.00000 ± 0.00051
	σ_X	0.1567 ± 0.0033	0.1580 ± 0.0011	0.15726 ± 0.00033
0.1	μ_X	0.0003 ± 0.0046	0.0000 ± 0.0012	0.00002 ± 0.00058
	σ_X	0.1517 ± 0.0037	0.1508 ± 0.0011	0.15097 ± 0.00037
0.5	μ_X	0.0002 ± 0.0044	0.00008 ± 0.00064	0.00006 ± 0.00041
	σ_X	0.1283 ± 0.0025	0.12938 ± 0.00081	0.12952 ± 0.00033
1.0	μ_X	0.0001 ± 0.0044	0.0000 ± 0.0011	-0.00001 ± 0.00029
	σ_X	0.1117 ± 0.0022	0.11307 ± 0.00075	0.11239 ± 0.00023
1.5	μ_X	-0.0001 ± 0.0027	0.0000 ± 0.0012	-0.00001 ± 0.00026
	σ_X	0.1020 ± 0.0023	0.10088 ± 0.00064	0.10056 ± 0.00025
2.0	μ_X	0.0001 ± 0.0025	0.0000 ± 0.0010	0.00001 ± 0.00023
	σ_X	0.0916 ± 0.0022	0.09168 ± 0.00070	0.09221 ± 0.00019
3.0	μ_X	0.0000 ± 0.0025	0.00006 ± 0.00094	-0.00006 ± 0.00023
	σ_X	0.0813 ± 0.0014	0.08033 ± 0.00058	0.08050 ± 0.00017

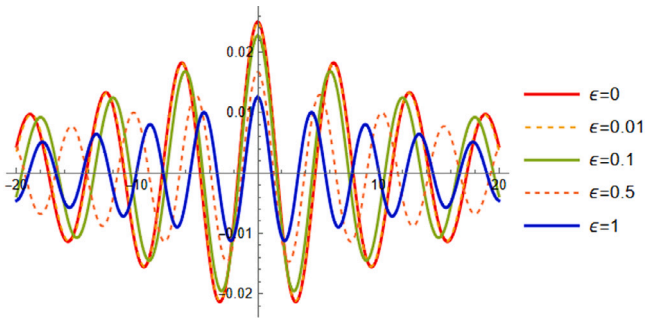

Fig. 5. Comparison of the correlation function $\Gamma_{XX}(\tau)$ of $X(t)$, given in (93)–(95), for different values of ϵ. Example 5.

Example 5. In this example, we will assume the same excitation $Y(t)$ that in previous example, but considering a nonlinear function more general that the corresponding to Duffing oscillator. In particular, we are going to apply the stochastic equivalent linearization method to study model (80) taking the data from Table 1 corresponding to Example 5. The first step is to determine the coefficients β_e and k_e^2 of the equivalent Eq. (17). We

Table 10
Values for λ_i, $i \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ and the domain $[x_1, x_2]$ obtained via the PME method, for $\epsilon = 1.0$. Example 5.

Order	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
$M = 3$	−2.268959631	0.0	39.751562499	0.0	–	–	–	–
$M = 5$	−2.268959631	0.0	39.751562499	0.0	$6.87437 \cdot 10^{-9}$	0.0	–	–
$M = 7$	−2.268959729	0.0	39.751585900	0.0	$-6.20366 \cdot 10^{-4}$	0.0	$3.30217 \cdot 10^{-3}$	0.0

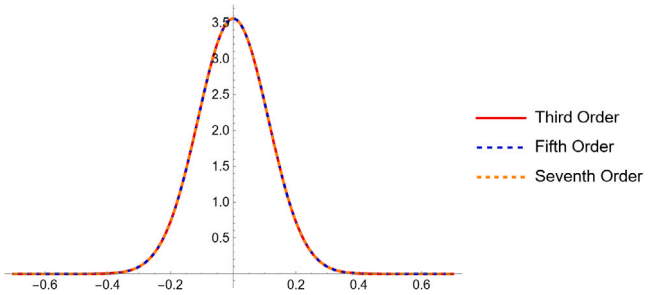

Fig. 6. Approximation of PDF, $f_{X(t)}(x)$, using the PME with $M = 3, 5, 7$ moments for $\epsilon = 1.0$. Example 5.

Table 11
Values of λ_i, $i \in \{0, 1, 2, 3\}$, and the domain $[x_1, x_2]$ obtained via PME based on the three ($M = 3$) first moments, for $\epsilon \in \{0, 0.01, 0.1, 0.5, 1.0, 1.5, 2.0, 3.0\}$. Example 5.

	$\epsilon = 0$	$\epsilon = 0.01$	$\epsilon = 0.1$	$\epsilon = 0.5$
λ_0	−1.925501193	−1.930414861	−1.972591334	−2.126159137
λ_1	0	0	0	0
λ_2	20.0	20.197515624	21.975156249	29.875781249
λ_3	0	0	0	0
$[x_1, x_2]$	[−1.581, 1.581]	[−1.573, 1.573]	[−1.508, 1.508]	[−1.293, 1.293]
	$\epsilon = 1.0$	$\epsilon = 1.5$	$\epsilon = 2.0$	$\epsilon = 3.0$
λ_0	−2.268959631	−2.379906040	−2.470649473	−2.613968336
λ_1	0	0	0	0
λ_2	39.751562499	49.627343749	59.503125000	79.254687499
λ_3	0	0	0	0
$[x_1, x_2]$	[−1.121, 1.121]	[−1.003, 1.003]	[−0.916, 0.916]	[−0.794, 0.794]

Table 12
Values of λ_i, $i \in \{0, 1, 2, 3\}$, and the domain $[x_1, x_2]$ obtained via PME based on the five ($M = 5$) first moments, for $\epsilon \in \{0, 0.01, 0.1, 0.5, 1.0, 1.5, 2.0, 3.0\}$. Example 5.

	$\epsilon = 0$	$\epsilon = 0.01$	$\epsilon = 0.1$	$\epsilon = 0.5$
λ_0	−1.925501193	−1.930414861	−1.972591334	−2.126159137
λ_1	0	0	0	0
λ_2	19.999999999	20.197515624	21.975156249	29.875781249
λ_3	0	0	0	0
λ_4	$1.74018 \cdot 10^{-9}$	$1.77476 \cdot 10^{-9}$	$2.10067 \cdot 10^{-9}$	$3.88272 \cdot 10^{-9}$
λ_5	0	0	0	0
$[x_1, x_2]$	[−1.581, 1.581]	[−1.573, 1.573]	[−1.508, 1.508]	[−1.293, 1.293]
	$\epsilon = 1.0$	$\epsilon = 1.5$	$\epsilon = 2.0$	$\epsilon = 3.0$
λ_0	−2.268959631	−2.379906040	−2.470649473	−2.613968336
λ_1	0	0	0	0
λ_2	39.751562499	49.627343749	59.503124998	79.254687498
λ_3	0	0	0	0
λ_4	$6.87437 \cdot 10^{-9}$	$1.07083 \cdot 10^{-8}$	$1.54024 \cdot 10^{-8}$	$2.73267 \cdot 10^{-8}$
λ_5	0	0	0	0
$[x_1, x_2]$	[−1.121, 1.121]	[−1.003, 1.003]	[−0.916, 0.916]	[−0.794, 0.794]

have obtained them in Example 2 (see expression (66)), and they are given by

$$\begin{aligned} \beta_e &= \beta = \tfrac{1}{20}, \\ k_e^2 &= \omega_0^2 \left[1 + \epsilon \left(1 - \tfrac{1}{2}\sigma_X^2 + \tfrac{1}{8}(\sigma_X^2)^2\right)\right] = 1 + \epsilon \left(1 - \tfrac{1}{2}\sigma_X^2 + \tfrac{1}{8}(\sigma_X^2)^2\right). \end{aligned} \tag{90}$$

Notice that the condition corresponding to the underdamped case (assumed in the theoretical development) is fulfilled, since

$$\beta_e = \frac{1}{20} < 1 \leq 1 + \epsilon \left(1 - \frac{1}{2}\sigma_X^2 + \frac{1}{8}(\sigma_X^2)^2\right) = 1 + \frac{\epsilon}{8}\left(\left(\sigma_X^2 - 2\right)^2 + 4\right) = k_e^2, \quad \forall \epsilon \geq 0. \tag{91}$$

As mentioned in the theoretical development, all the moments of odd-order are null, in particular the mean. Next step is to compute the variance, given by (69), that is approximated by (72), obtaining $\sigma_{X_0}^2 = \frac{1}{40}$. With this information, it is easy to check that, using expression (22),

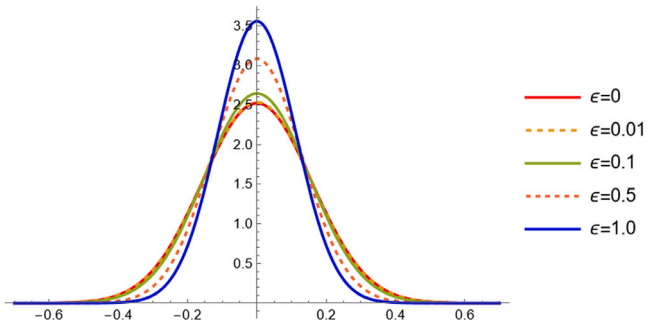

Fig. 7. Approximation of the PDF, $f_{X(t)}(x)$, using until the five order moment ($M = 5$) for $\epsilon \in \{0,\ 0.01,\ 0.1,\ 0.5,\ 1.0\}$ via the PME. Example 5.

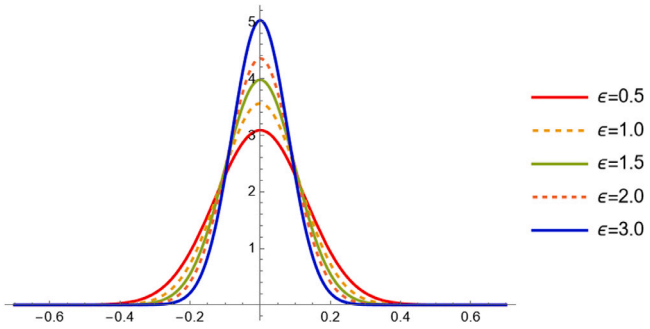

Fig. 8. Approximation of the PDF, $f_{X(t)}(x)$, using until the five order moment ($M = 5$) for $\epsilon \in \{0.5,\ 1.0,\ 1.5,\ 2.0,\ 3.0\}$ via the PME. Example 5.

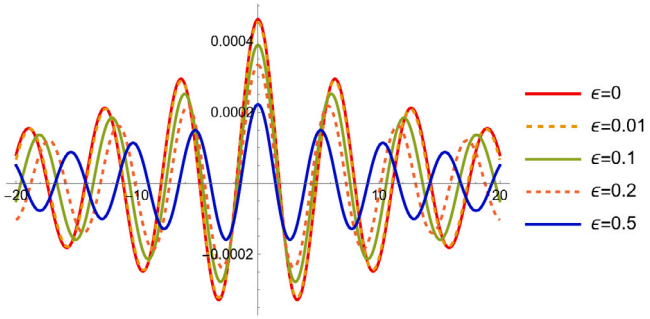

Fig. 9. Comparison of the correlation function $\Gamma_{XX}(\tau)$ of $X(t)$, given in (97)–(99), for different values of ϵ. Example 6.

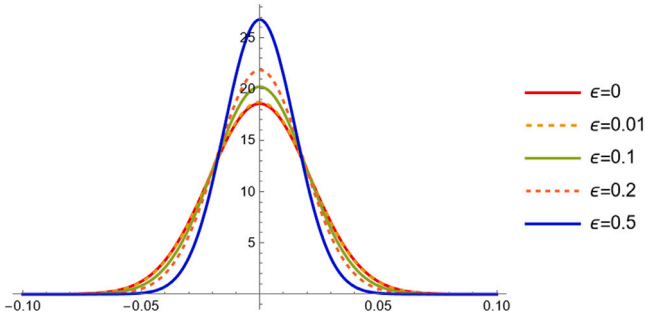

Fig. 10. Approximation of the PDF, $f_{X(t)}(x)$, using until the five ($M = 5$) order moment ($M = 5$) for $\epsilon \in \{0,\ 0.01,\ 0.1,\ 0.2,\ 0.5\}$ via the PME. Example 6.

the second-order moment is determined by

$$\mathbb{E}\left\{(X(t))^2\right\} = \mathbb{V}\left\{X(t)\right\} = \frac{320}{12800 + 12641\epsilon}. \tag{92}$$

In a similar way as in the previous example, the correlation function, $\Gamma_{XX}(\tau)$, can be obtained applying (21),

$$\Gamma_{XX}(\tau) = \begin{cases} f_1(\tau) & \text{if } \tau \geq 0, \\ f_2(\tau) & \text{if } \tau < 0, \end{cases} \tag{93}$$

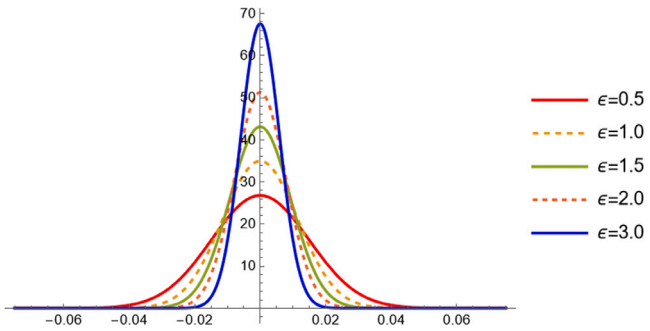

Fig. 11. Approximation of the PDF, $f_{X(t)}(x)$, using until the five ($M = 5$) order moment ($M = 5$) for $\epsilon \in \{0.5,\ 1.0,\ 1.5,\ 2.0,\ 3.0\}$ via the PME. Example 6.

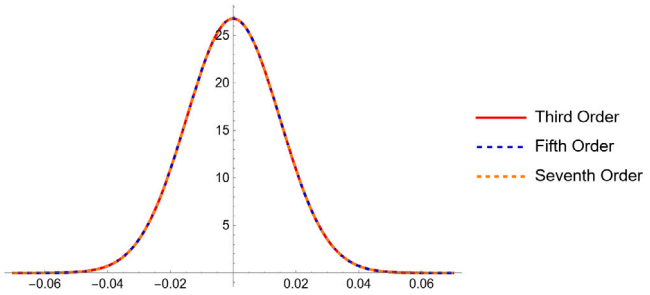

Fig. 12. Approximations of PDF, $f_{X(t)}(x)$, using the PME with $M = 3, 5, 7$ moments for $\epsilon = 0.5$. Example 6.

Table 13
Values of the error (L_1-norm) between the PDF generated by the PME with $M = 3, 5$ for $\epsilon \in \{0,\ 0.01,\ 0.1,\ 0.5,\ 1.0,\ 1.5,\ 2.0,\ 3.0\}$. Example 5.

	$\epsilon = 0$	$\epsilon = 0.01$	$\epsilon = 0.1$	$\epsilon = 0.5$
Error	$4.61006 \cdot 10^{-12}$	$4.61009 \cdot 10^{-12}$	$4.60943 \cdot 10^{-12}$	$4.60960 \cdot 10^{-12}$
$[x_1,\ x_2]$	$[-1.581,\ 1.581]$	$[-1.573,\ 1.573]$	$[-1.508,\ 1.508]$	$[-1.293,\ 1.293]$
	$\epsilon = 1.0$	$\epsilon = 1.5$	$\epsilon = 2.0$	$\epsilon = 3.0$
Error	$4.60905 \cdot 10^{-12}$	$4.60479 \cdot 10^{-12}$	$4.60974 \cdot 10^{-12}$	$4.61009 \cdot 10^{-12}$
$[x_1,\ x_2]$	$[-1.121,\ 1.121]$	$[-1.003,\ 1.003]$	$[-0.916,\ 0.916]$	$[-0.794,\ 0.794]$

$$f_1(\tau) = \frac{64}{(12768 + 12641\epsilon)(12800 + 12641\epsilon)} \exp\left(-\frac{\tau}{20}\right) \left[(63840 + 63205\epsilon) \cos\left(\tau \sqrt{\frac{399}{400} + \frac{12641\epsilon}{12800}}\right) + 20\sqrt{25536 + 25282\epsilon} \sin\left(\tau \sqrt{\frac{399}{400} + \frac{12641\epsilon}{12800}}\right) \right], \tag{94}$$

$$f_2(\tau) = \frac{64}{(12768 + 12641\epsilon)(12800 + 12641\epsilon)} \exp\left(\frac{\tau}{20}\right) \left[(63840 + 63205\epsilon) \cos\left(\tau \sqrt{\frac{399}{400} + \frac{12641\epsilon}{12800}}\right) - 20\sqrt{25536 + 25282\epsilon} \sin\left(\tau \sqrt{\frac{399}{400} + \frac{12641\epsilon}{12800}}\right) \right]. \tag{95}$$

In Fig. 5, we show the graphical representation of the correlation function, $\Gamma_{XX}(\tau)$, given by expressions (93)–(95) for different values of ϵ. Also, in Tables 8–9, we show the comparison between the mean and standard deviation of the stationary approximated solution obtained via the perturbation method, the equivalent linearization method versus the ones calculated by Kloeden–Platen–Schurz scheme with different number or simulations (in Table 8, we show the results with 10^6 simulations, and in Table 9, we show the results with 10^3, 10^4 and 10^5 simulations). It is important to mention that the bound for the perturbative parameter when applying the perturbation method at Example 5 is $\epsilon < 1.01258$.

Applying the results presented in Section 5, we obtain the approximation of the PDF, $f_{X(t)}(x)$, of the steady state, using the PME based on the three ($M = 3$) and five ($M = 5$) first moments. Tables 11 and 12 show the values of λ_i, $i \in \{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$, and the corresponding domain $[x_1,\ x_2]$ for different values of ϵ.

As the procedure followed in the presentation of results is the same as in Example 4, we will proceed directly to comment on the corresponding figures and tables.

As in Example 4, we have obtained the approximation of the PDF, $f_{X(t)}(x)$, for $\epsilon = 1.0$ (where the stochastic perturbation technique is near to the value where it begins to fail) in order to compute the approximation based on the three ($M = 3$), five ($M = 5$) and seven ($M = 7$) first moments, given in Table 10. In Fig. 6, we compare their graphical representations, where it can be observed that the PDFs are very similar to each other with the moments until $M = 3, 5, 7$.

In Figs. 7 and 8, we compare the graphical representations of the PDF, $f_{X(t)}(x)$ for different values of ϵ (for the sake of clarify in the presentation, we split the values of ϵ in both plots as follows: $\epsilon \in \{0, 0.01, 0.1, 0.5, 1\}$ and $\epsilon \in \{0.5, 1, 1.5, 2, 3\}$, respectively) in order to compute the approximation

Table 14
Comparison of the mean (μ_X) and the standard deviation (σ_X) between stochastic perturbation method, stochastic equivalent linearization method and Kloeden–Platen–Schurz scheme with 10^6 simulations (in the latter method, we also include an estimate of the error by calculating the standard deviation of the corresponding approximations obtained via the simulations). We have carried out for different $\epsilon \in \{0,\ 0.01,\ 0.1,\ 0.2,\ 0.5, 1, 1.5, 2, 3\}$. The largest value for which the perturbation method works is $\epsilon < 0.56534$, so from that value on, no approximation is included. It is indicated by −. Example 6.

ϵ		Perturbation method	Equivalent linearization method	Kloeden–Platen–Schurz 10^6 simulations
0	μ_X	0	0	0.000001 ± 0.000024
	σ_X	0.021483	0.021483	0.021497 ± 0.000012
0.01	μ_X	0	0	0.000003 ± 0.000025
	σ_X	0.021293	0.021295	0.021294 ± 0.000016
0.1	μ_X	0	0	-0.000003 ± 0.000026
	σ_X	0.019491	0.019739	0.019747 ± 0.000014
0.2	μ_X	0	0	-0.000001 ± 0.000015
	σ_X	0.017270	0.018258	0.0182797 ± 0.0000091
0.5	μ_X	0	0	0.000002 ± 0.000024
	σ_X	0.007304	0.014908	0.014929 ± 0.000010
1.0	μ_X	0	0	0.000002 ± 0.000018
	σ_X	–	0.011422	0.0114944 ± 0.0000086
1.5	μ_X	0	0	0.000002 ± 0.000035
	σ_X	–	0.009259	0.0093562 ± 0.0000054
2.0	μ_X	0	0	0.000000 ± 0.000045
	σ_X	–	0.007786	0.00790689 ± 0.0000047
3.0	μ_X	0	0	0.000001 ± 0.000047
	σ_X	–	0.005907	0.0060859 ± 0.0000041

Table 15
Approximations of the mean (μ_X) and the standard deviation (σ_X) using the Kloeden–Platen–Schurz numerical scheme with 10^3, 10^4 and 10^5 simulations including an estimate of the error by calculating the standard deviation of the corresponding approximations obtained via the simulations. We have carried out for different $\epsilon \in \{0,\ 0.01,\ 0.1,\ 0.2,\ 0.5, 1, 1.5, 2, 3\}$. Example 6.

ϵ		10^3 simulations	10^4 simulations	10^5 simulations
0	μ_X	-0.00004 ± 0.00073	-0.00001 ± 0.00028	-0.000003 ± 0.000055
	σ_X	0.02181 ± 0.00052	0.02148 ± 0.00014	0.021493 ± 0.000044
0.01	μ_X	0.00003 ± 0.00082	0.00002 ± 0.00018	0.000009 ± 0.000060
	σ_X	0.02118 ± 0.00047	0.02133 ± 0.00014	0.021315 ± 0.000039
0.1	μ_X	0.00002 ± 0.00074	-0.00001 ± 0.00021	-0.000011 ± 0.000053
	σ_X	0.01990 ± 0.00036	0.01980 ± 0.00014	0.019737 ± 0.000040
0.2	μ_X	0.00000 ± 0.00049	0.00001 ± 0.00016	0.000005 ± 0.000063
	σ_X	0.01834 ± 0.00040	0.01831 ± 0.00013	0.018277 ± 0.000033
0.5	μ_X	-0.00004 ± 0.00033	0.00001 ± 0.00013	-0.000006 ± 0.000051
	σ_X	0.01487 ± 0.00030	0.01491 ± 0.00010	0.014906 ± 0.000034
1.0	μ_X	-0.00003 ± 0.00038	0.00000 ± 0.00012	0.000006 ± 0.000029
	σ_X	0.01140 ± 0.00022	0.011454 ± 0.000074	0.011457 ± 0.000029
1.5	μ_X	0.00003 ± 0.00028	0.000024 ± 0.000054	0.000001 ± 0.000028
	σ_X	0.00940 ± 0.00020	0.009302 ± 0.000057	0.009303 ± 0.000023
2.0	μ_X	0.00000 ± 0.00031	0.000002 ± 0.000060	-0.000002 ± 0.000022
	σ_X	0.00790 ± 0.00017	0.007853 ± 0.000053	0.007845 ± 0.000018
3.0	μ_X	0.00000 ± 0.00023	0.000001 ± 0.000075	-0.000001 ± 0.000022
	σ_X	0.00602 ± 0.00012	0.006005 ± 0.000042	0.005994 ± 0.000013

based on the five ($M = 5$) first moments. We can confirm that the approximations obtained for the different values of ϵ are good, taking into account the error given by the L_1-norm presented in Table 13 with respect to the PDFs with $M = 3$ (Table 11) and $M = 5$ (Table 12) for different values of ϵ.

Example 6. In this example, we will consider the same nonlinear function as in Example 5, but considering now $Y(t)$ an Ornstein–Uhlenbeck (OU) stochastic process to play the role of the external source. In particular, we are going to apply the stochastic equivalent linearization method to study model (80) taking the data from Table 1 corresponding to Example 6. When applying the stochastic equivalent linearization technique, the first step is to determine the coefficients β_e and k_e^2 of the equivalent Eq. (17). As in Example 5, they are given by (90). In this case, the condition corresponding to the underdamped case assumed in the theoretical development is fulfilled for all $\epsilon \geq 0$ (see (91)).

From Eq. (23), one can ensure that all the odd-order moments are null, and all the even-order moments can be directly computed from the standard deviation, $\sigma_{X(t)}$. So, obtaining the second-order moments is our main goal.

Table 16
Values of λ_i, $i \in \{0, 1, 2, 3\}$, and the domain $[x_1, x_2]$ obtained via PME based on the three ($M = 3$) first moments, for $\epsilon \in \{0, 0.01, 0.1, 0.2, 0.5, 1.0, 1.5, 2.0, 3.0\}$. Example 6.

	$\epsilon = 0$	$\epsilon = 0.01$	$\epsilon = 0.1$	$\epsilon = 0.2$	$\epsilon = 0.5$
λ_0	−3.921534050	−3.930338629	−4.006224397	−4.084210638	−4.286907297
λ_1	0	0	0	0	0
λ_2	1083.333333333	1102.578872314	1283.285262401	1499.896167603	2249.682740012
λ_3	0	0	0	0	0
$[x_1, x_2]$	[−0.214, 0.214]	[−0.212, 0.212]	[−0.197, 0.197]	[−0.182, 0.182]	[−0.149, 0.149]
	$\epsilon = 1.0$	$\epsilon = 1.5$	$\epsilon = 2.0$	$\epsilon = 3.0$	
λ_0	−4.553272216	−4.763175955	−4.936472449	−5.212639576	
λ_1	0	0	0	0	
λ_2	3832.506550039	5831.804763413	8247.577380136	14 328.545823625	
λ_3	0	0	0	0	
$[x_1, x_2]$	[−0.114, 0.114]	[−0.092, 0.092]	[−0.077, 0.077]	[−0.059, 0.059]	

Table 17
Values of λ_i, $i \in \{0, 1, 2, 3\}$, and the domain $[x_1, x_2]$ obtained via PME based on the five ($M = 5$) first moments, for $\epsilon \in \{0, 0.01, 0.1, 0.2, 0.5, 1.0, 1.5, 2.0, 3.0\}$. Example 6.

	$\epsilon = 0$	$\epsilon = 0.01$	$\epsilon = 0.1$	$\epsilon = 0.2$	$\epsilon = 0.5$
λ_0	−3.921534050	−3.930338629	−4.006224397	−4.084210638	−4.286907297
λ_1	0	0	0	0	0
λ_2	1083.333333314	1102.578872295	1283.285262379	1499.896167577	2249.682739973
λ_3	0	0	0	0	0
λ_4	$5.10517 \cdot 10^{-6}$	$5.28847 \cdot 10^{-6}$	$7.16397 \cdot 10^{-6}$	$9.78709 \cdot 10^{-6}$	$2.20165 \cdot 10^{-5}$
λ_5	0	0	0	0	0
$[x_1, x_2]$	[−0.214, 0.214]	[−0.212, 0.212]	[−0.197, 0.197]	[−0.182, 0.182]	[−0.149, 0.149]
	$\epsilon = 1.0$	$\epsilon = 1.5$	$\epsilon = 2.0$	$\epsilon = 3.0$	
λ_0	−4.553272216	−4.763175955	−4.936472449	−5.212639576	
λ_1	0	0	0	0	
λ_2	3832.506549972	5831.804763312	8247.577380009	14 328.545823504	
λ_3	0	0	0	0	
λ_4	$6.38942 \cdot 10^{-5}$	$1.47961 \cdot 10^{-4}$	$2.51535 \cdot 10^{-4}$	$2.89265 \cdot 10^{-4}$	
λ_5	0	0	0	0	
$[x_1, x_2]$	[−0.114, 0.114]	[−0.092, 0.092]	[−0.077, 0.077]	[−0.059, 0.059]	

Table 18
Values of the error (L_1-norm) between the PDF generated by the PME with $M = 3, 5$, for $\epsilon \in \{0, 0.01, 0.1, 0.2, 0.5, 1.0, 1.5, 2.0, 3.0\}$. Example 6.

	$\epsilon = 0$	$\epsilon = 0.01$	$\epsilon = 0.1$	$\epsilon = 0.2$	$\epsilon = 0.5$
error	$4.60967 \cdot 10^{-12}$	$4.60983 \cdot 10^{-12}$	$4.60978 \cdot 10^{-12}$	$4.61020 \cdot 10^{-12}$	$4.60973 \cdot 10^{-12}$
$[x_1, x_2]$	[−0.214, 0.214]	[−0.212, 0.212]	[−0.197, 0.197]	[−0.182, 0.182]	[−0.149, 0.149]
	$\epsilon = 1.0$	$\epsilon = 1.5$	$\epsilon = 2.0$	$\epsilon = 3.0$	
error	$4.60959 \cdot 10^{-12}$	$4.61014 \cdot 10^{-12}$	$4.17807 \cdot 10^{-12}$	$2.74973 \cdot 10^{-12}$	
$[x_1, x_2]$	[−0.114, 0.114]	[−0.092, 0.092]	[−0.077, 0.077]	[−0.059, 0.059]	

Table 19
Values for λ_i, $i \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ obtained via the PME method for different orders M and for $\epsilon = 0.5$. Example 6.

Order	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
$M = 3$	−4.286907297	0.0	2249.682740012	0.0	–	–	–	–
$M = 5$	−4.286907297	0.0	2249.682739973	0.0	$2.20165 \cdot 10^{-5}$	0.0	–	–
$M = 7$	−4.286907621	0.0	2249.685721446	0.0	$1.52476 \cdot 10^{-2}$	0.0	$5.17235 \cdot 10^{-4}$	0.0

The variance is given by (69), that is approximated by (72), obtaining for this example $\sigma^2_{X_0} = \frac{3}{6500}$. Then, using expression (22), the second-order moment is determined by

$$\mathbb{E}\left\{(X(t))^2\right\} = \mathbb{V}\left\{X(t)\right\} = \frac{68546400000000}{(338000000 + 337922009\epsilon)(439400000 + 337922009\epsilon)}. \tag{96}$$

In Fig. 9, we show the graphical representation of the correlation function, $\Gamma_{XX}(\tau)$, obtained from expression (97) for different values of ϵ. The particular expression of the correlation function when $\epsilon = 0.1$ is given by

$$\Gamma_{XX}(\tau) = \begin{cases} f_1(\tau) & \text{if } \tau > 0, \\ f_2(\tau) & \text{if } \tau \leq 0, \end{cases} \tag{97}$$

where

$$f_1(\tau) = \frac{1142440000000000\ \exp(-\frac{\tau}{2})}{28674925332322978191245361476543450856 1}\left[1379152762403054608 1\right.$$
$$+\ 840032565886567304 05\ \exp\left(\frac{9\tau}{20}\right)\cos\left(\frac{1}{26000}\sqrt{\frac{3709472009}{5}}\tau\right) \tag{98}$$
$$\left.+\ 77772279175500\sqrt{18547360045}\ \exp\left(\frac{9\tau}{20}\right)\sin\left(\frac{1}{26000}\sqrt{\frac{3709472009}{5}}\tau\right)\right]$$

and

$$f_2(\tau) = \frac{1142440000000000\ \exp\left(\frac{\tau}{20}\right)}{28674925332322978191245361476543450856 1}\left[1379152762403054608 1\ \exp\left(\frac{9\tau}{20}\right)\right.$$
$$+\ 840032565886567304 05\cos\left(\frac{1}{26000}\sqrt{\frac{3709472009}{5}}\tau\right) \tag{99}$$
$$\left.-\ 77772279175500\sqrt{18547360045}\sin\left(\frac{1}{26000}\sqrt{\frac{3709472009}{5}}\tau\right)\right].$$

Also, to check that our approximations obtained via the stochastic equivalent linearization method are reliable, we compare the mean and standard deviation against those obtained via the stochastic perturbation method and the Kloeden-Platen-Schurz numerical method using different number of simulations. Results are presented in Tables 14–15. As in the previous examples, we observe that with the stochastic equivalent linearization method good results are obtained even with $\epsilon = 3$, while the approximations with the stochastic perturbation method the approximations are only reliable for very small values of ϵ, specifically for $\epsilon < 0.56534$.

Next, we obtain the approximation of the PDF, $f_{X(t)}(x)$, of the steady state, using the PME based on the three ($M = 3$) and five ($M = 5$) first moments. Tables 16 and 17 show, respectively, the values of λ_i, $i \in \{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$, necessary in the computation of the PDF given by expression (79) and the corresponding domain $[x_1,\ x_2]$ for different values of ϵ.

In Figs. 10 and 11, we compare the graphical representations of the PDF, $f_{X(t)}(x)$ for different values of ϵ in order to compute the approximation based on the five ($M = 5$) first moments from Table 17. Notice that for the sake of clarity in the visualization of both plots, we have split the range of ϵ. Fig. 10 shows the range $\epsilon \in \{0,\ 0.01,\ 0.1,\ 0.2,\ 0.5\}$, while Fig. 11 shows the range $\epsilon \in \{0.5,\ 1.0,\ 1.5,\ 2.0,\ 3.0\}$. Table 18 shows the error (L_1-norm) obtained from the comparison between the PDF obtained by the PME with $M = 3, 5$ for each values of $\epsilon \in \{0,\ 0.01,\ 0.1,\ 0.2,\ 0.5,\ 1.0,\ 1.5,\ 2.0,\ 3.0\}$. From the order of the error we can conclude that the approximations of the PDFs depicted in Figs. 10 and 11 are reliable.

As in previous examples, we have obtained the approximation of the PDF, $f_{X(t)}(x)$, of the steady state for $\epsilon = 0.5$ using the PME method wit the $M = 3, 5, 7$ (see Table 19). In Fig. 12, we compare the graphical representations of these approximations of the PDF. As expected from previous results, we observe that plots are virtually identical.

7. Conclusions

The study of the probabilistic properties of weakly perturbed nonlinear oscillators is, in general, a difficult task often limited to analyzing particular cases with regard to both the type of nonlinearity and the stochastic noise affecting the oscillators. Moreover, many contributions usually focus on calculating a few statistics for the steady state such as mean and the variance. In this paper, we have contributed to the advance in the probabilistic analysis of the steady state of a general class of nonlinear oscillators whose nonlinear term is an arbitrary polynomial of the position that is affected by a small perturbation. This family contains, as a particular case, the important case of the Duffing oscillator but also permits approximating oscillators whose nonlinear term is an analytic function of the position by truncating its Taylor series expansion. To conduct our probabilistic analysis, we have taken advantage of the stochastic equivalent linearization technique, assuming that the input or external force is a zero-mean stationary Gaussian stochastic process, which includes many relevant cases such as the white noise or the Ornstein–Uhlenbeck processes. Our approach has enabled us to approximate not only the mean and variance but also the density function of the steady state. Our numerical experiments show full agreement with the approximations obtained using stochastic numerical integrators and better results than the stochastic perturbation method. Nevertheless, the study presented in this contribution has several aspects we would like to undertake in the future. First, improve the approximations of the undamped angular frequency of the linear equivalent equation of the nonlinear oscillator to expand the range of perturbation values, ϵ, for which the steady state statistics are reliable. And secondly, to extend our analysis to oscillators whose nonlinear term depends not only on the position but also on the velocity via very general functions.

CRediT authorship contribution statement

J.-C. Cortés: Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Funding acquisition. **J.-V. Romero:** Writing – review & editing, Writing – original draft, Supervision, Software, Methodology, Investigation. **M.-D. Roselló:** Writing – review & editing, Writing – original draft, Validation, Supervision, Software, Methodology, Investigation. **J.F. Valencia Sullca:** Writing – review & editing, Writing – original draft, Software, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] Caraballo T, Colucci R, López-De-La-Cruz J, Rapaport A. A way to model stochastic perturbations in population dynamics models with bounded realizations. Commun Nonlinear Sci Numer Simul 2019;77:239–57.
[2] de la Cruz H. Stabilized explicit methods for the approximation of stochastic systems driven by small additive noises. Chaos Solitons Fractals 2020;140:110195.
[3] Li X, Song S. Research on synchronization of chaotic delayed neural networks with stochastic perturbation using impulsive control method. Commun Nonlinear Sci Numer Simul 2014;19(10):3892–900.
[4] Shaikhet L. Stability of the zero and positive equilibria of two connected neoclassical growth models under stochastic perturbations. Commun Nonlinear Sci Numer Simul 2019;68:86–93.
[5] Laudani R, Falsone G. An evolutive probability transformation method for the dynamic stochastic analysis of structures. Probab Eng Mech 2022;69:103313.
[6] Andronov AA, Vitt AA, Khaikin SE. Theory of oscillators. Adiwes international series in physics, Pergamon; 1966.
[7] Cveticanin L. Strong nonlinear oscillators: Analytical solutions. Mathematical engineering, Springer; 2018.
[8] Yildirim A, Askari H, Saadatnia Z, KalamiYazdi M, Khan Y. Analysis of nonlinear oscillations of a punctual charge in the electric field of a charged ring via a Hamiltonian approach and the energy balance method. Comput Math Appl 2011;62:486–90.
[9] Khan Y, Mirzabeigy A. Improved accuracy of He's energy balance method for analysis of conservative nonlinear oscillator. Neural Comput Appl 2014;25:889–95.
[10] Khan Y, Vazquez-Leal H, Hernandez-Martinez L. Removal of noise oscillation term appearing in the nonlinear equation solution. J Appl Math 2012;387365.
[11] Cveticanin L. Analysis techniques for the various forms of the Duffing equation. In: Kovacic I, Brennan M, editors. The Duffing equation: nonlinear oscillators and their behaviour. Wiley; 2011, p. 81–135, Ch. 4.
[12] Hinch E. Perturbation methods. Cambridge texts in applied mathematics, New York, NY, USA; 1991.
[13] Nayfeh AH. Perturbation methods. Wiley classics library, New York, NY, USA: Wiley VCH; 2000.
[14] Bellman RE. Perturbation techniques in mathematics, engineering and physics. Dover books on physics, New York, NY, USA: Dover Publications; 2003.
[15] Simmonds JG, Mann Jr JE. A first look at perturbation theory. Dover books on physics, New York, NY, USA: Dover Publications; 2013.
[16] Lutes LD, Sarkani S. Random vibrations: Analysis of structural and mechanical systems. Elsevier; 2004.
[17] Li J, Chen J. Stochastic dynamics of structures. Wiley; 2009.
[18] Cortés J-C, López-Navarro E, Romero J-V, Roselló M-D. Probabilistic analysis of random nonlinear oscillators subject to small perturbations via probability density functions: theory and computing. Eur Phys J Plus 2021;136(7):1–23.
[19] Liu D, Li J, Meng Y. Probabilistic response analysis for a class of nonlinear vibro-impact oscillator with bilateral constraints under colored noise excitation. Chaos Solitons Fractals 2019;122:179–88. http://dx.doi.org/10.1016/j.chaos.2019.03.024.
[20] Li J, Liu D, Li M. Probabilistic response analysis of nonlinear vibro-impact systems with two correlated Gaussian white noises. Int J Non-Linear Mech 2023;151:104370. http://dx.doi.org/10.1016/j.ijnonlinmec.2023.104370.
[21] Calatayud J, Cortés J-C, Jornet M. The damped pendulum random differential equation: A comprehensive stochastic analysis via the computation of the probability density function. Phys A 2018;512:261–79.
[22] Allen E. Modeling with Itô stochastic differential equations. Mathematical modelling: theory and applications, Springer Dordrecht; 2007.
[23] Lord GJ, Powell CE, Shardlow T. An introduction to computational stochastic PDEs. Cambridge University Press; 2014.
[24] Soong TT. Random differential equations in science and engineering. New York, NY, USA: Academic Press; 1973, Volume 103.
[25] Crandall S. Random vibration. New York: Massachusetts Institute of Technology, Wiley and Chapman & Hall; 1958.
[26] Crandall SH. Perturbation techniques for random vibration of nonlinear systems. J Acoust Soc Am 1963;35(11):1700–5.
[27] Gitterman M. The noisy oscillator: Random mass, frequency, damping. Singapore: World Scientific; 2013.
[28] Ibrahim RA. Parametric random vibration. Dover Publications Inc.; 2008.
[29] Newland DE. An introduction to random vibrations, spectral & wavelet analysis. 3rd ed.. Dover; 2005.
[30] Krylov N, Bogoliubov N. Introduction to nonlinear mechanics. Princeton, New Jersey: Princeton Univ. Press; 1950.
[31] Iyengar RN. Higher order linearization in non-linear random vibration. Int J Non-Linear Mech 1988;23(5–6):385–91.
[32] Ismaili MA, Bernard P. Asymptotic analysis and linearization of the randomly perturbed two-wells Duffing oscillator. Probabilistic Eng Mech 1997;12(3):171–8.
[33] Nicoreştianu FE, Sireteanu T. Analysis of nonlinear random vibrations by statistical equivalent methods. UPB Sci Bull Ser A 2014;76(3):19–30.
[34] Anh N, Schiehlen W. A technique for obtaining approximate solutions in Gaussian equivalent linearization. Comput Methods Appl Mech Engrg 1999;168(1–4):113–9.
[35] Su C, Huang H, Ma H. Fast equivalent linearization method for nonlinear structures under nonstationary random excitations. J Eng Mech 2016;142(8):04016049.
[36] Hung D, Hieu D. Solution of nonlinear cubic-quintic duffing oscillators using the equivalent linearization method with a weighted averaging. 2017, p. 137–42, 169.
[37] Anh N, Hai N, Hieu D. The equivalent linearization method with a weighted averaging for analyzing of nonlinear vibrating systems. Lat Am J Solids Struct 2017;14:1723–40.
[38] Hieu D, Hai N, Hung D. The equivalent linearization method with a weighted averaging for solving undamped nonlinear oscillators. J Appl Math 2018;2018.
[39] Van Hieu D. Analysis of nonlinear vibrations of two-degree-of-freedom systems. World J Adv Eng Technol Sci 2023;8(2):293–304.
[40] Elishakoff I, Crandall SH. Sixty years of stochastic linearization technique. Meccanica 2017;52:299–305.
[41] Crandall S. Is stochastic equivalent linearization a subtly flawed procedure? Probab Eng Mech 2001;16(2):169–76.
[42] Michalowicz JV, Nichols JM, Bucholtz F. Handbook of differential entropy. Abingdon, UK: CRC Press; 2018.
[43] Loève M. Probability theory. Vol. I–II, New York, NY, USA: Springer; 1977.
[44] Neckel T, Rupp F. Random differential equations in scientific computing. Walter de Gruyter; 2013.
[45] Grimmett G, Stirzaker D. Probability and random processes. Oxford: Clarendon Press; 2000.
[46] Krylov N. Introduction to the theory of random processes. Graduate studies in mathematics, vol. 42, American Mathematical Society; 2002.
[47] Papoulis A, Pillai U. Probability, random variables and stochastic processes. 4th ed.. McGraw-Hill Education; 2015.
[48] Socha L. Linearization methods for stochastic dynamic systems. Springer; 2008.
[49] Talvila E. Necessary and sufficient conditions for differentiating under the integral sign. Amer Math Monthly 2001;108:544–8.
[50] Michalowicz JV, Nichols JM, Bucholtz F. Handbook of differential entropy. CRC Press; 2013.
[51] Conrad K. Probability distributions and maximum entropy. Entropy 2004;452(6):10.
[52] Inc. WR. Mathematica, version 14.1. Champaign, IL; 2024, URL https://www.wolfram.com/mathematica.
[53] Casella G, Berger R. Statistical inference. Cengage Learning India; 2002.
[54] Kloeden P, Platen E, Schurz H. Numerical solution of SDE through computer experiments. Springer Verlag; 1994, Corrected Third Printing 2003.