UNIVERSIDAD POLITÉCNICA DE VALENCIA

Departamento de Matemática Aplicada

Asymmetric Norms and the Dual Complexity Spaces

Memoria presentada por

Luis Miguel García Raffi
para optar al Grado de Doctor en

CIENCIAS MATEMÁTICAS

Dirigida por los doctores

Salvador Romaguera Bonilla

Enrique Alfonso Sánchez Pérez

Valencia, Enero de 2003

D. SALVADOR ROMAGUERA BONILLA, Catedrático del Departamento de Matemática Aplicada de la Universidad Politécnica de Valencia y **D. ENRIQUE ALFONSO SÁNCHEZ PÉREZ**, Profesor Titular de Universidad del Departamento de Matemática Aplicada de la Universidad Politécnica de Valencia,

CERTIFICAN:

Que la presente memoria "Asymmetric Norms and the Dual Complexity Spaces " ha sido realizada bajo su dirección por **D. Luis Miguel García Raffi** en el Departamento de Matemática Aplicada de la Universidad Politécnica de Valencia, y constituye su tesis para optar al grado de Doctor en Ciencias Matemáticas.

Y para que así conste, presentan la referida tesis, firmando el presente certificado.

Valencia, 21 de Noviembre de 2002

Fdo. Salvador Romaguera Bonila   Enrique Alfonso Sánchez Pérez

AGRADECIMIENTOS

Quiero utilizar estas líneas para dar las gracias a aquellas personas que de una forma u otra han contribuido a hacer esta tesis. En primer lugar quiero agradecer a mis directores, Salvador y Enrique, la atención prestada. Decir que ha sido realmente fácil trabajar con ellos se queda muy corto. Con ellos no sólo he aprendido muchas matemáticas sino que lo he hecho de la forma más fácil posible, desde la amistad. Además, en mi caso, tuvieron especial atención, tratándome como a un igual en todo el trabajo que hemos realizado en esta tesis. Muchisimas gracias a los dos.

En segundo lugar, y continuando aquí en la Universidad Politécnica de Valencia quiero destacar el apoyo que he recibido de mis compañeros: María José, Félix, Paz, Mila, José María, Alfredo, David, Alberto, Miguel, Juan Vicente y por supuesto, aunque ya no están en nuestro departamento, Pilar y Dolo.

Ya fuera de la Universidad, pero en primer lugar en importancia quiero agradecer a mi mujer su paciencia y comprensión, especialmente por las ausencias mentales, que no físicas, a las que te induce siempre el trabajo de una tesis. También quiero mencionar "als meus xiquets", Anna la "xicoteta"i y Albertet el "incansable", por que ellos han padecido como nadie el trabajo de su padre y, sin embargo siempre me han dado su cariño y comprensión. Sin ellos, jamás hubiera sido posible este trabajo. Esta tesis va dedicada a ellos. También quiero agradecer a mi familia el apoyo que me han brindado en todo momento y muy especialmente a mi madre, que ha sufrido como sólo una madre sabe hacerlo, mi tránsito por la universidad española desde aquellos primeros días de becario. También quiero acordarme de aquellos miembros de mi familia que ya no están; a mi padre y a mis hermanos Gonzalo y Rosa María. Estoy seguro de que estarían muy felices.

También quiero en estas líneas acordarme de otras personas que mucho tienen que ver en mi carrera científica, aunque sea fuera de las matemáticas. Doy las gracias a Berta y José Luis, pues ellos me iniciaron como investigador y me enseñaron muchas cosas que me han sido muy útiles. También quiero acordarme de Gadeoto, Javier, Pepito, Dani y Trino, que durante algunos años formaron parte de mi universo particular.

¡Va por todos!

A Ana, Albert y Anna

Esta tesis se ha realizado en el marco del proyecto **BFM2000-1111** del Ministerio de Ciencia y Tecnología.

Normas Asimétricas y los Espacios de Complejidad Dual

Desde el punto de vista de la Ciencia de la Computación, un avance reciente lo ha constituido el establecimiento de un modelo matemático que da cuenta de la distancia entre algoritmos y programas, cuando estos son analizados desde la óptica de la complejidad computacional, entendiendo por complejidad, por ejemplo, la medida del tiempo de computación.

En la última década se han llevado a cabo notables esfuerzos para elaborar una teoría matemática robusta que goce, en cierta medida, de buenas propiedades y constituya una herramienta que, en este contexto, juegue un papel análogo al que los espacios vectoriales normados han desempeñado en diversos ámbitos de la ciencia y la tecnología.

En el caso de la complejidad computacional, se demuestra que un modelo muy satisfactorio lo constituye el de los espacios vectoriales dotados de una norma asimétrica. En esta tesis, realizamos un estudio general de las propiedades de estos espacios, en analogía con las propiedades que clásicamente se estudian en los espacios vectoriales normados. Así, hemos estudiado las propiedades de separación de los espacios vectoriales de norma asimétrica, obteniendo una caracterización de aquellos espacios que son Hausdorff; hemos obtenido una teoría satisfactoria de la bicompletación de dichos espacios; también hemos realizado un estudio de la compacidad cuando el espacio vectorial tiene dimensión finita; hemos determinado condiciones bajo las cuales una norma asimétrica definida en un conjunto algebraicamente cerrado de un espacio vectorial puede ser extendida a todo el espacio y hemos analizado la estructura del espacio dual y las topologías débiles asociadas. Por último, hemos aplicado los resultados obtenidos al campo de la Ciencia de la Computación, más concretamente a los Espacios de Complejidad Dual.

Asymmetric Norms and the Dual Complexity Spaces

One of the recent advances in Computer Science was due to the possibility of establishing a mathematical model that account the distance between algorithms and programs when they are analyzed in terms of their computational complexity (complexity distance), where computational complexity is interpreted in terms of running time, for example.

In the last decade, several authors have done a big effort in obtaining a robust mathematical theory, which was a useful tool that played, in this context, a similar role that normed linear spaces have played in different scientific areas.

In the context of Computational Complexity, it is shown that Asymmetric Normed Linear Spaces constitute a very satisfactory model. This thesis is focused in the study of the properties of these spaces, similarly to the classical properties that are studied in the case of normed linear spaces. Thus, we have studied separation properties of asymmetric normed linear spaces, obtaining in particular a characterization of Hausdorffness; we have obtained a satisfactory theory of bicompletion for these spaces; we have analyzed compactness on finite dimensional asymmetric normed linear spaces; we have studied conditions under which an asymmetric norm defined on an algebraically closed subset of a linear space can be extended to the whole space and we have analyzed the structure of the dual space and the weak topologies associated to it. Finally, we have applied our theory to Computer Science, specifically to the so-called Dual Complexity Spaces.

Normes Asimètriques i els Espais de Complexitat Dual

Des del punt de vista de la Ciència de la Computació, un avanç recent ho ha constituït l'establiment d'un model matemàtic que done compte de la distància entre algoritmes i programes, quan són analitzats des de l'òptica de la complexitat computacional, entenent per complexitat, per exemple, la mesura del temps de computació.

En l'última dècada s'han dut a terme notables esforços per a elaborar una teoria matemàtica robusta que gaudisca, en certa mesura, de bones propietats i constituïsca una eina que, en aquest context, jugue un paper anàleg al què els espais vectorials normats han jugat en diversos àmbits de la ciència i la tecnologia.

En el cas de la complexitat computacional, es demostra que un model molt satisfactori ho constitueix el dels espais vectorials de norma asimètrica. En la tesi que es presenta, realitzem un estudi general de les propietats dels esmentats espais, en analogia amb les propietats que clàssicament s'estudien en els espais vectorials normats. Així, hem estudiat les propietats de separació dels espais vectorials de norma asimètrica, obtenint una caracterització d'aquells espais que són Hausdorff; hem obtingut una teoria satisfactòria de la bicompletació de dits espais; també hem realitzat un estudi de la compacitat en els espais vectorials de dimensió finita; hem determinat condicions baix les quals una norma asimètrica definida en un conjunt algebraicament tancat d'un espai vectorial pot ser estesa a tot l'espai i hem analitzat l'estructura de l'espai dual i les topologies dèbils associades. Finalment, hem aplicat els resultats obtinguts al camp de la Ciència de la Computació, més concretament als Espais de Complexitat Dual.

Contents

Chapter 1

Introduction

1.1 Asymmetric topology

In the last decade several authors have successfully applied some asymmetric structures of Topological Algebra and Functional Analysis to problems in Approximation Theory and Theoretical Computer Science.

In particular, locally convex cones and quasi-norms (or asymmetric norms, in our context) on cones and linear spaces provide efficient tools to study several questions in sign-sensitive approximation theory ([13]), in obtaining general theorems of Hahn-Banach type ([4], [27], [56], [57]), to characterize the structure of (semi-)Chebyshev sets ([6], [42]), and to measure complexity distances between programs or algorithms ([46]). Furthermore, the notions of fractal semigroup, partial metric monoid and weightable invariant quasi-metric semigroup provide useful frameworks to construct theoretical models for some computational processes that appear in a natural way in programming languages (see, for instance, [7], [16], [17], [45], etc).

These facts have motivated, in part, an increasing interest in the research of such kind of structures and the applications of the asymmetric basic notions (quasi-uniformities and quasi-metrics) on which are supported, to various classical mathematical theories: hyperspaces (e.g. [26], [31], [58]), function spaces (e.g. [10], [40], [55]), fixed point theory (e.g. [20], [51]), topological algebra, of course (e.g. [4], [2],

[3] [5], [30], [35], [41]), etc.

In this direction, the important review by H.P.A. Künzi "Nonsymmetric distances and their associated topologies: About the origin of basic ideas in the area of asymmetric topology"([29]), provides an exhaustive list of references related to these topics.

Talking about Nonsymmetric or Asymmetric Topology, there are two basic references: the book of Murdeshwar and Naimpally ([37]) and the book of Fletcher and Lindgren ([19]).

Let us recall that Smyth studied in [53] some concepts of the theory of quasi-uniform spaces in connection with problems from Theoretical Computer Science and proposed quasi-metric and quasi-uniform spaces as a generalization of *cpo's* and metric spaces as used in denotational semantics. One of the important things was his idea about reworking the basic notions involving limits and completeness in order to accommodate the theory to examples in Computer Science. He introduced the concepts of *S-Cauchy filter* and *S-completability* in quasi-uniform spaces that have been very useful in the applications of asymmetric topology.

In [51], M. Schellekens introduced the notion of "complexity distance ". He defined the complexity space in order to develop a topological foundation for the complexity analysis of programs and algorithms. His complexity spaces are weightable and thus, belong to the class of S-completable quasi-uniform spaces. In this seminal paper, he illustrated the applicability of his theory via the complexity analysis of "Divide and Conquer"algorithms and presented a new proof, based on the Banach fixed point theorem, of the fact that mergesort has asymptotic average running time.

But probably, one of the most influencing papers that has inspired our research is due to Romaguera and Schellekens [44]. In it, the authors introduce the notion of Dual Complexity Space and study its quasi-metric properties. The main results obtained are the Smyth-completeness of the complexity space and the compactness of closed complexity subspaces which possesses a complexity lower bound. In [47], Romaguera an Schellekens show that the structure of quasi-normed semilinear space provides a suitable setting to carry out an analysis of the dual complexity space.

Our aim in this thesis is to develop a systematized theory of asymmetric normed linear spaces, applying our methods and results to obtain a mathematical model for the dual complexity space in the framework of Theoretical Computer Science. This kind of structures will provide a robust mathematical model in the sense that we can obtain several properties following the classical scheme on normed linear spaces. In this sense, our work extends the theory of normed linear spaces to the case of lack of symmetry.

Thus, in Chapter 2 we present a characterization of those asymmetric normed linear spaces which are Hausdorff and show that it is possible, under reasonable conditions, to obtain a procedure in describing an symmetric normed linear space as a direct sum of a Hausdorff subspace and a "purely non Hausdorff"subspace. In Chapter 3, we present a satisfactory theory of bicompletion for asymmetric normed linear spaces obtaining that each asymmetric normed linear space has a unique bicompletion up to isometric isomorphism. In Chapter 4 we extend the classical results about compact sets on finite dimensional normed spaces to the asymmetric case. We prove the equivalence between T_1 separation axiom and normability in the finite dimensional case and thus between T_1 and T_2 separation axioms; we also prove that the Heine-Borel Theorem characterizes finite dimensional asymmetric normed linear spaces that satisfies T_2 axiom. Chapter 5 is devoted to study conditions under which we can extend an asymmetric norm which has been defined on an algebraically closed subset of a linear space (the notion of algebraically closed set is defined below) to the corresponding linear span. In Chapter 6 we define the dual space of an asymmetric normed linear space and in Chapter 7 we present some different weak and weak* topologies that can naturally be defined because of the lack of symmetry. In particular we give an asymmetric version of the celebrated Alaoglu Theorem. Finally in Chapter 8 we make use of this mathematical background to the applied context of the dual complexity space ([44]) and extend our study to algorithms and programs that have exponential running time.

A precedent of our study, in the realm of (para)topological linear spaces may be found in "Estructuras Topológicas no Simétricas y Espacios Bitopológicos ", by Carmen Alegre (Thesis, Universidad Politécnica de Valencia, 1994).

1.2 Preliminaries and basic notions

1.2.1 Quasi-Uniformities and quasi-metrics

Our basic reference for quasi-uniformities and quasi-pseudo-metric is [19].

A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ which satisfies:

i) $\Delta \subset U$, for all $U \in \mathcal{U}$.

ii) Given $U \in \mathcal{U}$ exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$,

where $\Delta = \{(x,x) : x \in X\}$ and $V^2 = \{(x,z) \in X \times X :$ exists $y \in X$ such that $(x,y) \in V, (y,z) \in V\}$ The members of $\mathcal{U}$ are called *entourages*.

The filter $\mathcal{U}^{-1}$, formed for all sets of the form $U^{-1} = \{(x,y) \in X \times X : (y,x) \in U\}$, where $U \in \mathcal{U}$, is a quasi-uniformity on X called the *conjugate quasi-uniformity* of $\mathcal{U}$.

A *quasi-uniform space* is a pair $(X,\mathcal{U})$ such that X is a (nonempty) set and $\mathcal{U}$ is a quasi-uniformity on X.

If $\mathcal{U}$ is a quasi-uniformity on a set X, the coarsest uniformity on X finer than $\mathcal{U}$ will be denoted by $\mathcal{U}^s$, i.e. $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$. This uniformity is called the supremum of the quasi-uniformities $\mathcal{U}$ and $\mathcal{U}^{-1}$.

Every quasi-uniformity $\mathcal{U}$ on X generates a topology $T(\mathcal{U})$ on this set. A neigborhood base for each point $x \in X$ is given by $\{U(x) : U \in \mathcal{U}\}$ where $U(x) = \{y \in X : (x,y) \in U\}$.

A quasi-uniformity $\mathcal{U}$ on X is called *bicomplete* if $\mathcal{U}^s$ is a complete uniformity on X. In this case we say that $(X,\mathcal{U})$ is a bicomplete quasi-uniform space.

A *bicompletion* of a quasi-uniform space $(X,\mathcal{U})$ is a bicomplete quasi-uniform space $(Y,\mathcal{V})$ such that $(X,\mathcal{U})$ is quasi-isomorphic to a $T(\mathcal{V}^s)$-dense subset of Y. It was proved in [11] and in [50] (see also [19]) that every T_0 quasi-uniform space

$(X,\mathcal{U})$ has a unique (up quasi-uniform isomorphism) T_0 bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$. Then $(\widetilde{X},\widetilde{\mathcal{U}})$ is called the bicompletion of $(X,\mathcal{U})$. Moreover $(\widetilde{X},\widetilde{\mathcal{U}^{-1}}) = (\widetilde{X},(\widetilde{\mathcal{U}})^{-1})$ and $(\widetilde{X},\widetilde{\mathcal{U}^s}) = (\widetilde{X},\widetilde{\mathcal{U}} \vee \widetilde{\mathcal{U}^{-1}})$.

In our context a *quasi-metric* on a set X is a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$, and

(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A *quasi-metric space* is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X.

Each quasi-metric d on X generates a T_0 topology $T(d)$ on X, which has as a basic open sets the d-balls $B_d(x, r) = \{y \in X : d(x, y) < r\}$ where $x \in X$ and $r > 0$.

Each quasi-metric d on X induces a metric d^s on X defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ for all $x, y \in X$, where d^{-1} is the *conjugate quasi-metric* of $d : d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$.

A quasi-metric d on X induces a quasi-uniformity $\mathcal{U}_d$ on X with basic entourages of the form $\{(x, y) : d(x, y) < 2^{-n}\}$, for every $x \in X$ and $n = 1, 2, 3, \ldots$.

A quasi-metric d on X is called *bicomplete* if the quasi-uniformity $\mathcal{U}_d$ is bicomplete, i.e. if d^s is a complete metric.

A. Di Concilio ([12]) and S. Salbany ([50]) have independently proved that, similarly to the quasi-uniform case, each quasi-metric space (X, d) has a unique (up to isometry) quasi-metric bicompletion.

Results on bicompletion of some interesting asymmetric structures in topological algebra may be found in [34], [35] and [30].

1.2.2 Asymmetric norms

In the sequel the letters $\mathbb{R}$, $\mathbb{R}^+$, ω and $\mathbb{N}$ will denote the set of real numbers, of nonnegative real numbers, of nonnegative integer numbers and positive integer numbers, respectively.

Let X be a linear space. We say that a function $q : X \to \mathbb{R}^+$ is an *asymmetric norm* on X if for all $x, y \in X$ and $a \in \mathbb{R}^+$:

(i) $q(x) = q(-x) = 0$ if and only if $x = 0$.

(ii) $q(ax) = aq(x)$.

(iii) $q(x + y) \leq q(x) + q(y)$.

The pair (X, q) is called an *asymmetric normed linear space*. Asymmetric norms are called quasi-norms in [18], [4] and [42].

The function $q^{-1} : X \to \mathbb{R}^+$ defined by $q^{-1}(x) := q(-x)$ is also an asymmetric norm. The function $q^s : X \to \mathbb{R}^+$ given by the formula $q^s(x) := max\{q(x), q^{-1}(x)\}$ is a norm on X.

An asymmetric norm q induces a quasi-metric d_q by mean of the formula:

$$d_q(x, y) = q(y - x), \qquad x, y \in X.$$

Hence, if q is an asymmetric norm on X, the sets

$$V_\epsilon(0) := \{x \in X : q(x) < \epsilon\}, \qquad \epsilon > 0,$$

form a fundamental system of neighbourhoods of zero for the topology $T(d_q)$ generated by d_q. In the same way the translated sets $V_\epsilon(y) = y + V_\epsilon(0)$, form a fundamental system of neighbourhoods of y for all $y \in X$. It follows from the definition that $V_\epsilon(y) = B_{d_q}(y, \epsilon)$. In case of q is a norm, the sets

$$B_\epsilon(0) := \{x \in X : q(x) < \epsilon\}, \qquad \epsilon > 0.$$

form a fundamental system of neighbourhoods of zero for the topology generated by d_q.

We denote by $V_{\epsilon,\leq}(0)$, the set:

$$V_{\epsilon,\leq}(0) := \{x \in X : q(x) \leq \epsilon\}, \qquad \epsilon > 0,$$

and in the same way, in case of we are working with a norm, we will use the notation

$$B_{\epsilon,\leq}(0) := \{x \in X : q(x) \leq \epsilon\}, \qquad \epsilon > 0.$$

Of course, the set $B_{\epsilon,\leq}(0)$ is usually denoted by $\bar{B}_\epsilon(0)$ or simply B_X. We will indicate the (asymmetric) norm on the space under consideration by a superscript if necessary. It is not easy to choose a satisfactory notation due to the nature of different subjects involved in this work. Of course, some other different notations to the one selected here could be more appropriated.

In the sequel we will also refer to $T(d_q)$ as the topology generated by q.

A *seminorm* on a (real) linear space X is a nonnegative real valued function p on X that satisfies

i) $p(x+y) \leq p(x) + p(y)$, $x, y \in X$,

and

ii) $p(ax) = |a|p(x)$, $x \in X$ and $a \in \mathbb{R}$.

The seminorm p is a norm if $p(x) = 0$ implies $x = 0$.

Now, let us introduce the notion of algebraically closed space. An *algebraically closed* space M (ac-space for short) is a subset of a (real) linear space X which is closed with respect to the sum on X and with respect to the product by non negative scalars, i.e.

$$x + y \in M, \qquad for\ every\ \ x, y \in M$$

and

$$ax \in M \qquad for\ every\ \ x \in M\ \ and\ \ a \in \mathbb{R}^+.$$

In particular, $0 \in M$.

Clearly every linear space can be considered as an ac-space.

An *asymmetric seminorm* on an ac-space M is a function $q : M \to \mathbb{R}^+$ such that for all $x, y \in M$ and $a \in \mathbb{R}^+$:

1) $q(ax) = aq(x)$.
2) $q(x + y) \leq q(x) + q(y)$.

then, we say that the couple (M, q) is an *asymmetric seminormed ac-space*; moreover, if the function q satisfies the following property,

3) for every $x \in M$ such that $-x \in M$, then $q(x) = q(-x) = 0$ if and only if $x = 0$,

then it is called an *asymmetric norm* on M. In this case, we say that the couple (M, q) is an *asymmetric normed ac-space*. This definition is the reasonable restriction to ac-spaces of the notion of an asymmetric norm on a linear space.

In our context a *semilinear space* on $\mathbb{R}^+$ will be an ordered triple $(E, +, \cdot)$ such that $(E, +)$ is an Abelian monoid (i.e. an Abelian semigroup with neutral element) and $\cdot$ is a function from $\mathbb{R}^+ \times E$ to E such that for all $x, y \in E$ and $a, b \in \mathbb{R}^+$: $a \cdot (b \cdot x) = (ab) \cdot x$, $(a + b) \cdot x = (a \cdot x) + (b \cdot x)$, $a \cdot (x + y) = (a \cdot x) + (a \cdot y)$, and $1 \cdot x = x$.

Observe that every semilinear space is a cone in the sense of Keimel and Roth [27].It is clear that every ac-space is a semilinear space.

An *asymmetric normed semilinear space* is a pair (F, q_F) such that F is a (nonempty) subset of an asymmetric normed linear space (E, q), where q_F denotes the restriction of the asymmetric norm q to F, and $(F, + |_F, \cdot |_F)$ is a semilinear space, i.e. an ac-space in this context (compare [43], [46]).

Chapter 2

Separation properties in asymmetric normed linear spaces

2.1 Introduction

Our aim in this chapter is to study the separation properties in asymmetric normed linear spaces. We say that an asymmetric normed linear space (X, q) is Hausdoff if the topology $T(d_q)$, generated by the quasi-metric d_q, is Hausdorff. It is well known that each quasi-metric generates a T_0 topology, and then every asymmetric normed linear space is T_0. However, asymmetric normed linear spaces are not Hausdorff in general. Of course, if q is a norm the space satisfies this property. However, the most common (non trivial) example of asymmetric norm -the one that is defined in a normed linear lattice $(E, \|.\|, \leq)$ as $q(x) = \|x \vee 0\|$- does not generate a Hausdorff topology (see [4]). In Section 2.2 we will show an easy procedure to construct examples of asymmetric normed linear spaces which are Hausdorff. Another example, of a different nature to the one given here can be found in Example 4.7 of [2]. We also characterize those asymmetric normed linear spaces which are Hausdorff. This characterization motivates the notion of a *purely non Hausdorff asymmetric normed linear space* which is introduced here. As an application we show in Section 2.3 that each asymmetric normed linear space can be written, under reasonable conditions, as a direct sum of a Hausdorff asymmetric normed linear space and a “purely non

Hausdorff"asymmetric normed linear space.

Definitions and results on general topology used in this chapter can be found in [11]. The reader can find the basic properties about Banach lattices that are needed in [33]. The main results of this chapter have been published in [22].

We denote by $\wedge$ and $\vee$ the usual operations in a lattice. If $1 \leq p < \infty$, we write $\|x\|_p$ for the p-norm of a sequence of real numbers $x = (x_i)_{i\in\mathbb{N}}$,

$$\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}.$$

2.2 Hausdorff asymmetric normed linear spaces

It is well known that the norm of a normed space obviously defines a Hausdorff topology. This is not the case when for instance we consider a normed linear lattice $(E, \|.\|, \leq)$ and the asymmetric normed linear space (E, q) defined by mean of the asymmetric norm $q(x) = \|x \vee 0\|$. As we indicated in Section 2.1, the space (E, q) is not Hausdorff. Thus, the first question that appears in a natural way is if each Hausdorff asymmetric normed linear space -or at least each Hausdorff asymmetric normed linear lattice- is isomorphic to a normed space. The following example shows that this is not the case. We construct an asymmetric normed linear space that is Hausdorff but is not isomorphic to a normed space.

Example 2.1 *Consider the linear lattice $(E_0, \leq)$ defined by all sequences of real numbers that are different from zero only in a finite set of indexes endowed with its natural order, and let $q_0 : E_0 \to \mathbb{R}^+$ be the function defined by*

$$q_0(x) = \|x \vee 0\|_1 + \|x \wedge 0\|_2.$$

We prove that this function is in fact an asymmetric norm. Since $x = x \vee 0 + x \wedge 0$ for all $x \in E_0$, we have that $q_0(x) = q_0(-x) = 0$ if and only if $x = 0$. Obviously it is also positively homogeneous. We just need to show that it satisfies the triangle inequality.

First note that for a pair of elements $x, y \in E_0$, $(x+y) \vee 0 \leq x \vee 0 + y \vee 0$. For each $1 \leq p < \infty$, the norm properties related to the order operations of the normed lattices $(E_0, \|.\|_p, \leq)$ (see Chapter I, Vol.II, in [33]) leads to the inequality

$$\|(x+y) \vee 0\|_p \leq \|x \vee 0\|_p + \|y \vee 0\|_p.$$

The following equalities are also satisfied for every $x \in E_0$ and each $1 \leq p < \infty$ (Chapter I, Vol.II, in [33]),

$$\|x \wedge 0\|_p = \| - (x \wedge 0)\|_p = \| - (-((-x) \vee 0))\|_p = \| - x \vee 0\|_p.$$

Then

$$q_0(x+y) = \|(x+y) \vee 0\|_1 + \|(x+y) \wedge 0\|_2 = \|(x+y) \vee 0\|_1 + \|(-x-y) \vee 0\|_2 \leq$$

$$= \|x \vee 0\|_1 + \|y \vee 0\|_1 + \| - x \vee 0\|_2 + \| - y \vee 0\|_2 = q_0(x) + q_0(y).$$

We have shown that (E_0, q_0) is an asymmetric normed linear space.

Note that (E_0, q_0) is a Hausdorff space since for every $x \in E_0$, the norm $|\|.\||$, given by $|\|x\|| := \|x \vee 0\|_2 + \|x \wedge 0\|_2$ is equivalent to $\|.\|_2$, and $|\|x\|| \leq q_0(x)$. This means that the open balls defined by q_0 are contained in the open balls defined by the norm $|\|.\||$ on E_0, and then (E_0, q_0) is a Hausdorff space. The proof of the fact that (E_0, q_0) is not isomorphic to any normed space will be shown as a consequence of the last result of this section.

Before to study the T_2 separation axiom in asymmetric normed linear spaces, we are going to give a simple characterization of the T_1 separation axiom:

Proposition 2.1 *Let (X, q) be an asymmetric normed linear space and $T(d_q)$ the topology generated by the quasi-metric d_q. Then $T(d_q)$ is T_1 if and only if $q(y) \neq 0$, for each $y \in X \backslash \{0\}$*

Proof. Suppose that $q(y) \neq 0$ for every $y \in X \backslash \{0\}$. Let $x, y \in X$ such that $d_q(x, y) = 0$. Then $q(x - y) = 0$, so $x = y$. Conversely, suppose that $T(d_q)$ is T_1 and let $y \in X \backslash \{0\}$. Then $d_q(0, y) > 0$, i.e. $q(y) > 0$. ■

Definition 2.1 *Let (X, q) be an asymmetric normed linear space. We define the function $\|.\|_q : X \to \mathbb{R}^+$ by the formula*

$$\|x\|_q := inf_{x_1 \in X}\{q(x_1) + q(x_1 - x)\}, \qquad x \in X.$$

Lemma 2.1 *$\|.\|_q$ is a seminorm on X. Moreover, it is the supremum of all seminorms p that satisfy*

$$p(x) \leq q(x), \qquad x \in X.$$

Proof. First we define on X the function $\phi_0(x) = min\{q(x), q(-x)\}$. $\phi_0(.)$ is homogeneous, since for every $x \in X$ and $a \in \mathbb{R}$,

$$\phi_0(ax) = min\{q(ax), q(-ax)\} = min\{aq(x), aq(-x)\} = a\phi_0(x),$$

if a is nonnegative, and

$$\phi_0(ax) = min\{q((-a)(-x)), q((-a)x)\} = (-a)\phi_0(x),$$

if a is negative. In particular, $\phi_0(x) = \phi_0(-x)$ for every $x \in X$.

Now let us consider the convexification ϕ of the function ϕ_0 in X, which is defined by

$$\phi(x) = inf\{\sum_{i=1}^{n} \phi_0(x_i) : x = \sum_{i=1}^{n} x_i\}.$$

It can be easily checked that ϕ is a seminorm on X. Moreover, a direct consequence of its definition is that ϕ is the supremum seminorm p satisfying $p(x) \leq q(x)$ for every $x \in X$. Therefore we just need to prove that $\phi = \|.\|_q$.

Let $x \in X$ and $\epsilon > 0$ and consider a representation $x = \sum_{i=1}^{n} x_i$ of x such that

$$\sum_{i=1}^{n} \phi_0(x_i) \leq \phi(x) + \epsilon.$$

Let us define the sets $S^+ = \{x \in X : q(x) \leq q(-x)\}$ and $S^- = \{x \in X : q(-x) < q(x)\}$. Note that either S^+ or S^- can be an empty set. Then there is a natural number k, $1 \leq k \leq n$ such that, without loss of generality, we can order the elements $\{x_i : i = 1, ..., n\}$ of the above representation of x as follows,

$$x_1, ..., x_k \in S^+, \qquad\qquad x_{k+1}, ..., x_n \in S^-.$$

Then, if we denote by x_ϵ the sum $\sum_{i=1}^{k} x_i$ we obtain

$$\phi(x) + \epsilon \geq \sum_{i=1}^{k} q(x_i) + \sum_{i=k+1}^{n} q(-x_i) \geq q(\sum_{i=1}^{k} x_i) + q(-\sum_{i=k+1}^{n} x_i) \geq$$

$$\geq q(x_\epsilon) + q(x_\epsilon - x) \geq \|x\|_q.$$

This proves that $\phi(x) \geq \|x\|_q$ for every $x \in X$. For the converse take an element $x \in X$. For each $x_1 \in X$ we obtain

$$q(x_1) + q(x_1 - x) \geq \phi_0(x_1) + \phi_0(x - x_1) \geq \phi(x),$$

since x_1 and $x - x_1$ obviously define a particular representation of x. Then $\|x\|_q \geq \phi(x)$ for every $x \in X$. This concludes the proof. ■

Theorem 2.1 *Let (X, q) be an asymmetric normed linear space. The following statements are equivalent:*

1) $\|.\|_q$ is a norm on X.

2) (X, q) is a Hausdorff space.

3) (X, q^{-1}) is a Hausdorf space.

Proof. 2) → 1). By Lemma 2.1, we just need to prove that $\|x\|_q = 0$ implies $x = 0$. If $\|x\|_q = 0$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$q(x_n) + q(x_n - x) < \frac{1}{n}$$

for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} q(x_n) = 0$ and $\lim_{n\to\infty} q(x_n - x) = 0$, which means that x and 0 are limits of the sequence $(x_n)_{n\in\mathbb{N}}$. Since the space (X, q) is Hausdorff, the limit of each sequence is unique, and then $x = 0$.

1) → 2). Let $x, y \in X$, $x \neq y$. Since $\|.\|_q$ is a norm, there exists an $\epsilon > 0$ such that $\epsilon < \|x - y\|_q$. Consider the following basic neighbourhoods of x and y for the topology generated by q on X,

$$V_{\frac{\epsilon}{2}}(x) = \{z \in X : q(z - x) < \frac{\epsilon}{2}\} \quad , \quad V_{\frac{\epsilon}{2}}(y) = \{z \in X : q(z - y) < \frac{\epsilon}{2}\}.$$

Then $V_{\frac{\epsilon}{2}}(x) \subset B^{\|\cdot\|_q}_{\frac{\epsilon}{2}}(x)$ and $V_{\frac{\epsilon}{2}}(y) \subset B^{\|\cdot\|_q}_{\frac{\epsilon}{2}}(y)$. Since $B^{\|\cdot\|_q}_{\frac{\epsilon}{2}}(x)$ and $B^{\|\cdot\|_q}_{\frac{\epsilon}{2}}(y)$ are disjoint sets, $V_{\frac{\epsilon}{2}}(x)$ and $V_{\frac{\epsilon}{2}}(y)$ are disjoint too, and we obtain the result.

The equivalence of 2) and 3) is obvious.

Remark 2.1 *Note that the argument given in the proof of* 1) $\to$ 2)*, actually shows that* (X, q) *is Hausdorff whenever* $\|.\| \leq q$ *for any norm* $\|.\|$ *on* X*.*

In our context, two asymmetric normed linear spaces (X, q_X) and (Y, q_Y) are called isomorphic if there are a linear bijection T, $T : (X, q_X) \to (Y, q_Y)$ and two positive constants K_1 and K_2 such that

$$K_1 q_X(x) \leq q_Y(T(x)) \leq K_2 q_X(x), \ \ x \in X.$$

Theorem 2.2 *An asymmetric normed linear space* (X, q) *is isomorphic to a normed linear space if and only if there is a constant* $K > 0$ *such that* $q(x) \leq K\|x\|_q$ *for every* $x \in X$*.*

Proof. Suppose that there is a constant $K > 0$ that satisfies the above conditions. First note that in this case $q^s(x) \leq K\|x\|_q$ for all $x \in X$, since $\|.\|_q = \|.\|_{q^{-1}}$. The following inequalities hold for each $x \in X$,

$$q(x) \leq q^s(x) \leq K\|x\|_q \leq Kq(x).$$

In particular, this implies that $\|.\|_q$ is a norm since q^s so is. To prove the isomorphy it suffices to compare the neighbourhoods of zero defined by the norms q^s, $\|.\|_q$ and the asymmetric norm q. Let $\epsilon > 0$. The following inclusions are direct consequences of the above inequalities and prove that (X, q), $(X, \|.\|_q)$ and (X, q^s) are isomorphic.

$$V_{\frac{\epsilon}{K}}(0) \subset B^{\|.\|_q}_{\frac{\epsilon}{K}}(0) \subset B^{q^s}_{\epsilon}(0).$$

It remains to show that if (X, q) is isomorphic to a normed space, then there is a constant K such that the inequality $q(x) \leq K\|x\|_q$ holds for every $x \in X$. If (X, q) is isomorphic to the normed space $(Y, \|.\|)$ *via* a linear map $i : Y \to X$, the formula $\|i(x)\|$ induces a norm on X. Thus, it is sufficient to consider the case that there is a norm $\|.\|$ on X such that (X, q) and $(X, \|.\|)$ are isomorphic. In this case, there are constants K_1 and K_2 such that

$$K_1 B_1^{\|.\|}(0) \subset V_1(0) \subset K_2 B_1^{\|.\|}(0).$$

We can directly conclude that for every $x \in X$ the inequalities

$$K_2^{-1}\|x\| \leq \|x\|_q \leq q(x) \leq K_1^{-1}\|x\|$$

hold, since $K_2^{-1}\|.\|$ is a norm and $\|.\|_q$ is the supremum norm satisfying $\|x\| \leq q(x)$ for all $x \in X$. We finally obtain the inequality $q(x) \leq K\|x\|_q$, for $K = K_1^{-1}K_2$ and for all $x \in X$. ■

To finish this section, let us show that the space (E_0, q_0) given in Example 2.1 is not isomorphic to any normed space $(E_0, \|.\|)$. Straightforward calculations show that in this case $\|x\|_{q_0} = \|x \vee 0\|_2 + \|x \wedge 0\|_2$, $\|.\|_{q_0}$ is equivalent to $\|.\|_2$, and $q_0^s(x)$ is exactly $\|.\|_1$. The condition for (E_0, q_0) to be isomorphic to a normed space given in the above theorem would imply that q^s and $\|.\|_q$ are equivalent. But this is not true, since $\|.\|_1$ and $\|.\|_2$ are not equivalent in E_0. Note that the construction provides more examples of the same situation just by replacing the norms $\|.\|_1$ and $\|.\|_2$ by $\|.\|_r$ and $\|.\|_s$ respectively for any $1 \leq r < \infty$ and $1 \leq s < \infty$, $r \neq s$.

2.3 The canonical decomposition of an asymmetric normed linear space

Let (X, q) be an asymmetric normed linear space. In this section we show that it is always possible to find an asymmetric normed linear subspace (X_0, q) of (X, q) which is not Hausdorff and satisfies the following property: if X_1 is a linear subspace of X such that $X_1 \cap X_0 = \{0\}$, then (X_1, q) is Hausdorff. In fact, we obtain a standard procedure to describe -under reasonable conditions- an asymmetric normed linear space as a direct sum of a Hausdorff subspace and a "purely non Hausdorff" subspace.

Definition 2.2 *Let (X, q) an asymmetric normed linear space. We say that (X, q)*

is a purely non Hausdorff space if $X = \{0\}$ *or for every* $x \in X \backslash \{0\}$ *there is a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *that converges simultaneously to* x *and to* 0.

An easy example of a purely non Hausdorff space is $(\mathbb{R}, u)$, where $u(x) = x \vee 0$.

Another one is the linear lattice E_0 given in Example 2.1 endowed with the lattice asymmetric norm $q(x) = \|x \vee 0\|_2$. If $x \in X$, it can be easily checked that the constant sequence $x_n = x \wedge 0$ satisfies that $q(x_n - x) = q(-(x \vee 0)) = 0$ and $q(x_n) = 0$ for every n. Obviously, if (X, q) is a non trivial Hausdorff space it is not a purely non Hausdorff space. Example 2.1 provides then a space which does not satisfy this property.

Definition 2.3 *Let* (X, q) *be a asymmetric normed linear space. The kernel of the seminorm* $\|.\|_q$ *will be called the purely non Hausdorff kernel of* (X, q)*, i.e.*

$$Ker\|.\|_q = \{x \in X : \|x\|_q = 0\}.$$

Obviously, $(Ker\|.\|_q, q)$ is a purely non Hausdorff asymmetric normed linear subspace of (X, q) (see 2) $\to$ 1) in the proof of Theorem 2.1). It is clear that the purely non Hausdorff kernel of a purely non Hausdorff asymmetric normed linear space is the whole space, and is $\{0\}$ when the space is Hausdorff. In other case, following standard techniques of Functional Analysis we can consider the quotient normed space $(X/Ker\|.\|_q, \|.\|_q^0)$, whose elements are the classes $[x] = \{y \in X : \|x - y\|_q = 0\}$ and the norm is defined by $\|[x]\|_q^0 := inf\{\|z\|_q : z \in [x]\}$.

The following lemma can be found without proof in Proposition 4.1 in [18].

Lemma 2.2 *Let* (X, q) *and* (Y, g) *be asymmetric normed linear spaces. A linear map* $f : (X, q) \to (Y, g)$ *is continuous if and only in there is a constant* K *such that* $g(f(x)) \leq Kq(x)$ *for every* $x \in X$.

Proof. Let $x \in X$ and consider the neighbourhood $V_\epsilon(f(x)) = \{y \in Y | g(y - f(x)) < \epsilon\}$. We just need to prove that the neighbourhood $f(V_{\frac{\epsilon}{K}}(x)) = \{f(z) | q(z - x) < \frac{\epsilon}{K}\} \subset V_\epsilon(f(x))$. But for every $y = f(z) \in f(V_{\frac{\epsilon}{K}}(x))$, $g(y - f(x)) = g(f(z - x)) \leq$

$Kq(z-x) < K\frac{\epsilon}{K} = \epsilon$. This proves that the inequality gives a sufficient condition for continuity. A similar argument for the neighbourhood $V_1(f(x))$ proves the converse implication. ■

We can define the linear maps

$$i : (X, q) \to (X, \|.\|_q), \qquad by \qquad i(x) = x,$$

and

$$P : (X, \|.\|_q) \to (X/Ker\|.\|_q, \|.\|_q^0), \qquad by \qquad P(x) = [x].$$

That they are continuous follows from the above lemma.

Proposition 2.2 *Given an asymmetric normed linear space (X, q) we have:*

1) $(Ker\|.\|_q, q)$ is a purely non Hausdorff closed subspace of (X, q).

2) Let X_1 be a linear subspace of (X, q). Suppose that $X_1 \cap Ker\|.\|_q = \{0\}$. Then (X_1, q) is a Hausdorff space.

Proof. 1) $\{[0]\}$ is a closed subset of $(X/Ker\|.\|_q, \|.\|_q^0)$ since it is a normed space. It is easy to check that the linear maps i and P are continuous, and then $(P \circ i)$ is continuous too. Thus $(P \circ i)^{-1}([0]) = Ker\|.\|_q$ is a closed subset of (X, q).

2) Consider two elements $x, y \in X_1$, $x \neq y$. Then $x - y$ does not belong to $Ker\|.\|_q$, and $(P \circ i)(x - y) \neq [0]$. Thus $\|[x] - [y]\|_q^0 > 0$, and we can find two disjoint balls of $(X/Ker\|.\|_q, \|.\|_q^0)$, $B_\epsilon^{\|.\|_q^0}([x])$ and $B_\epsilon^{\|.\|_q^0}([y])$. Since $q(x) \geq \|[x]\|_q^0$ and $q(y) \geq \|[y]\|_q^0$, the basic neighbourhoods $V_\epsilon(x)$ and $V_\epsilon(y)$ are disjoint. ■

Let us define the function $q^0 : X/Ker\|.\|_q \to \mathbb{R}^+$ by $q^0([x]) := inf_{z\in[x]} q(z)$. Since $z \in [x]$ if and only if there is a $t \in Ker\|.\|_q$ such that $x - t = z$, we also have the formula $q^0([x]) = inf_{t\in Ker\|.\|_q} q(x - t)$.

Proposition 2.3 *$(X/Ker\|.\|_q, q^0)$ is a Hausdorff asymmetric normed linear space.*

Proof. It is clear that the function q^0 defines an asymmetric norm on $X/Ker\|.\|_q$. By Remark 2.1 it is sufficient to prove that there is a norm $\|.\|$ such that $q^0([x]) \geq \|[x]\|$. But obviously $q^0([x]) \geq inf_{z\in[x]}\|z\|_q = \|[x]\|_q^0$. ■

Note that for an asymmetric normed linear space (X, q) the set of continuous linear maps $f : (X, q) \to (X, q)$ does not define a linear space in general. For example, let us consider the identity map $I : (X, q) \to (X, q)$. It is obviously a continuous linear map. However, the map $-I$ defined by $(-I)(x) = -x$ is continuous if and only if q is equivalent to a norm, since $q(-x) \leq Kq(x)$ for all $x \in X$ implies $q(-x) \leq Kq(x) \leq K^2q(-x)$. Moreover, if $f : X \to X$ is a linear map, the condition $q(f(x)) \leq K_1q(x)$ does not imply $q((I - f)(x)) \leq K_2q(x)$. An easy counterexample is $f(x) = 2x$.

If X_1 is linear subspace of X, a linear map $Q : X \to X_1$ is called a *projection* if $Q(x) = x$ for each $x \in X_1$.

Definition 2.4 *An asymmetric normed linear subspace (X_1, q) of (X, q) is called complemented if there is a continuous projection $Q : X \to X_1$ such that $(I - Q)$ is continuous too.*

A consequence of Lemma 2.2 is that a subspace (X_1, q) is complemented if and only if there exists a projection $Q : X \to X_1$ and a constant $K > 0$ satisfying $max\{Q(x), (I - Q)(x)\} \leq Kq(x)$ for every $x \in X$.

Theorem 2.3 *Let (X, q) be an asymmetric normed linear space. Then the following statements are equivalent:*

1) (X, q) is isomorphic to a direct sum of the purely non Hausdorff subspace $(Ker\|.\|_q, q)$ and a Hausdorff subspace (X_0, q) which is isomorphic to the asymmetric normed linear space $(X/Ker\|.\|_q, q^0)$.

2) $(Ker\|.\|_q, q)$ is complemented.

Proof. 1) $\rightarrow$ 2) is obvious. Let us show 2) $\rightarrow$ 1). There exist a projection Q and a constant K such that $max\{Q(x),(I-Q)(x)\} \leq Kq(x)$ for every $x \in X$. Thus the map $(I-Q): X \rightarrow X$ is well defined and it is continuous. We will denote by X_0 the subspace $(I-Q)(X)$.

Let us show that we can factorize $(I-Q)$ through the quotient space $(X/Ker\|.\|_q, q^0)$. The quotient map $P:(X,q) \rightarrow (X/Ker\|.\|_q, q^0)$ is continuous, since obviously $q^0([x]) \leq q(x)$ for every $x \in X$.

Now, note that $Ker\|.\|_q = Ker(I-Q)$. If $x \in Ker\|.\|_q$, then $Q(x)=x$. Thus $x-Q(x)=(I-Q)(x)=0$, and $x \in Ker(I-Q)$. On the other hand, if $x-Q(x)=0$, $Q(x)=x$ and then $x \in Ker\|.\|_q$. We define the linear map $S:(X/Ker\|.\|_q, q^0) \rightarrow (X_0,q)$ by $S([x])=(I-Q)(x)$. In fact, it is well defined since, if $y \in [x]$, there is an element $t \in Ker\|.\|_q$ such that $y-x=t$, and then $(I-Q)(y)=(I-Q)(t+x)=t-Q(t)+(I-Q)(x)=(I-Q)(x)$.

To prove that S is continuous, consider an element $[x] \in X/Ker\|.\|_q$. We show that $q(S([x])) \leq Kq^0([x])$. Take $t \in Ker\|.\|_q$. Since $q(x-Q(x)) \leq Kq(x)$ and $Q(t)=t$ the following inequalities hold.

$$q(x-Q(x)) = q(x-t-Q(x)+Q(t))) = q((x-t)-Q(x-t)) \leq Kq(x-t).$$

Therefore, $q(S([x])) = q(x-Q(x)) \leq Kinf_{t \in Ker\|.\|_q} q(x-t) = Kq^0([x])$. Moreover, since $Q(x) \in Ker\|.\|_q$ we get $q^0([x]) \leq q(S([x]))$. S is an injection, since $S([x]) = S([y])$ implies $x-y=Q(x-y) \in Ker\|.\|_q$, and then $[x]=[y]$. Thus, S defines an isomorphism between $(X/Ker\|.\|_q, q^0)$ and an asymmetric normed linear subspace (X_0,q) of (X,q). Then (X_0,q) is a Hausdorff space by Proposition 2.3.

Let us consider the product space $X_0 \times Ker\|.\|_q$ endowed with the asymmetric norm $q_1(x_0,x_1) = q(x_0)+q(x_1)$. We just need to show that the map $f:(X,q) \rightarrow (X_0 \times Ker\|.\|_q, q_1)$ defined by $f(x) = ((I-Q)(x), Q(x))$ is an isomorphism. We have the following inequalities.

$$q(x) \leq q((I-Q)(x)) + q(Q(x)) = q_1(f(x)) \leq 2Kq(x), \qquad x \in X.$$

Moreover, if $f(x) = f(y)$, we obtain that $x - Q(x) = y - Q(y)$ and $Q(x) = Q(y)$. Then $x = y$ and the map f is an injection. Since it is surjective by the definition of Q, we get the result. ■

Let us finish this chapter with an easy example of the above canonical decomposition. Consider the linear lattice $\mathbb{R}^4$. Let $\{e_i\}_{i=1}^4$ denote the set of vectors of the canonical basis of $\mathbb{R}^4$. Consider the asymmetric norm

$$q(\sum_{i=1}^{4} \lambda_i e_i) = \|(\sum_{i=1}^{4} \lambda_i e_i) \vee 0\|_2 + \|(\lambda_1 e_1 + \lambda_2 e_2) \wedge 0\|_1.$$

Theorem 2.3 can be applied to the asymmetric normed linear space $(\mathbb{R}^4, q)$. In this case, $Ker\|.\|_q = span\{e_3, e_4\}$, the purely non Hausdorff asymmetric normed kernel is $(span\{e_3, e_4\}, \|(\lambda_3 e_3 + \lambda_4 e_4) \vee 0\|_2)$ and the projection is $Q(\sum_{i=1}^4 \lambda_i e_i) = \lambda_3 e_3 + \lambda_4 e_4$. The quotient $(\mathbb{R}^4/Ker\|.\|_q, q^0)$ is isomorphic to the asymmetric normed linear space $span\{e_1, e_2\}$ endowed with the asymmetric norm

$$q^*(\lambda_1 e_1 + \lambda_2 e_2) = \|(\lambda_1 e_1 + \lambda_2 e_2) \vee 0\|_2 + \|(\lambda_1 e_1 + \lambda_2 e_2) \wedge 0\|_1.$$

The direct sum of $span\{e_1, e_2\}$ and $span\{e_3, e_4\}$ with the corresponding asymmetric norm

$$\widetilde{q}(\sum_{i=1}^{4} \lambda_i e_i) = q^*(\lambda_1 e_1 + \lambda_2 e_2) + \|(\lambda_3 e_3 + \lambda_4 e_4) \vee 0\|_2,$$

is clearly isomorphic to $(\mathbb{R}^4, q)$.

Chapter 3

The bicompletion of an asymmetric normed linear space

3.1 Introduction

The main purpose of this chapter is to obtain a satisfactory theory of bicompletion for asymmetric normed linear spaces. Although our study should be seen as a new contribution to the development of the theory of asymmetric norms (compare [18], [3], [4], [13] , [38]), actually it is motivated, in great part, for the recent applications of these structures to the analysis of the so-called dual complexity space ([43], [44], [46]). Furthermore, the dual complexity space is a (semilinear) subspace of a certain biBanach space (see Example 3.2 in Section 3.2).

See [12], [19] and [50] for a general theory of bicompletion. The main results of this chapter have been published in [21].

Let (X, q) be an asymmetric normed linear space. Let us recall that the asymmetric norm q induces, in a natural way, a quasi-metric d_q on X, defined by $d_q(x, y) = q(y - x)$ for all $x, y \in X$. If the quasi-metric d_q is bicomplete, we say that (X, q) is a *biBanach space.*

The following is a first simple but useful instance of a biBanach space (see also Example 3.2 in Section 3.2).

Example 3.1 *Let* $(\mathbb{R}, +, \cdot)$ *be the (usual) Euclidean linear space. For each* $x \in \mathbb{R}$ *define* $u(x) = x \vee 0$. *Then* u *is an asymmetric norm on* $\mathbb{R}$ *such that* u^s *is the Euclidean norm. Therefore* $(\mathbb{R}, u)$ *is a biBanach space.*

3.2 The bicompletion

Definition 3.1 *An isometric isomorphism from an asymmetric normed linear space* (X, q_X) *to an asymmetric normed linear space* (Y, q_Y) *is a linear map* $f : X \to Y$ *such that* $q_Y(f(x)) = q_X(x)$ *for all* $x \in X$.

Note that if f is an isometric isomorphism from the asymmetric normed linear space (X, q_X) to the asymmetric normed linear space (Y, q_Y), then f is an isometric isomorphism from the normed linear space (X, q_X^s) to the normed linear space (Y, q_Y^s) and hence f is injective.

Definition 3.2 *Two asymmetric normed linear spaces* (X, q_X) *and* (Y, q_Y) *are said to be isometrically isomorphic if there is an isometric isomorphism from* X *onto* Y.

Definition 3.3 *Let* (X, q) *be an asymmetric normed linear space. We say that a biBanach space* (Y, q_Y) *is a bicompletion of* (X, q) *if* (X, q) *is isometrically isomorphic to a subspace of* (Y, q_Y) *that is dense in the Banach space* (Y, q_Y^s).

We will prove that each asymmetric normed linear space (X, q) has a bicompletion $(\widetilde{X}, \widetilde{q})$ such that any bicompletion of (X, q) is isometrically isomorphic to $(\widetilde{X}, \widetilde{q})$.

Thus the biBanach space $(\widetilde{X}, \widetilde{q})$ will be called *the bicompletion* of (X, q). Furthermore $(\widetilde{X}, \widetilde{q})$ provides the standard completion of (X, q) when (X, q) is a normed linear space.

Let (X, q) be an asymmetric normed linear space. Denote by $\widehat{X}$ the set of all Cauchy sequences in the normed linear space (X, q^s).

Define an equivalence relation R on $\widehat{X}$ as follows: For each $x := (x_n)_{n\in\mathbb{N}}$ and $y := (y_n)_{n\in\mathbb{N}}$ in $\widehat{X}$ put

$$xRy \Leftrightarrow \lim_{n\to\infty} q^s(x_n - y_n) = 0.$$

Denote by $\widetilde{X}$ the quotient $\widehat{X}/R$. Thus $\widetilde{X} = \{[x] : x \in \widehat{X}\}$, where as usual $[x] = \{y \in \widehat{X} : xRy\}$ for all $x \in \widehat{X}$.

For each $x := (x_n)_{n\in\mathbb{N}}$ and $y := (y_n)_{n\in\mathbb{N}}$ in $\widehat{X}$ and each $a \in \mathbb{R}$ define

$$x+y = (x_n+y_n)_{n\in\mathbb{N}}, \quad a\cdot x = (ax_n)_{n\in\mathbb{N}}, \quad [x]+[y] = [x+y] \quad \text{and} \quad a\cdot[x] = [a\cdot x].$$

Then we have the following result whose straightforward and essentially known proof is omitted.

Lemma 3.1 *Let (X, q) be an asymmetric normed linear space. Then $(\widetilde{X}, +, \cdot)$ is a linear space.*

Let (X, q) be an asymmetric normed linear space. For each $x := (x_n)_{n\in\mathbb{N}}$ in $\widehat{X}$, let

$$\widetilde{q}([x]) = \lim_{n\to\infty} q(x_n).$$

We first observe that if $y \in [x]$, then $\widetilde{q}([x]) = \widetilde{q}([y])$. Indeed, $\widetilde{q}([x]) = \lim_{n\to\infty} q(x_n) \leq \lim_{n\to\infty} q(x_n - y_n) + \lim_{n\to\infty} q(y_n)$. Since $\lim_{n\to\infty} q(x_n - y_n) = 0$, it follows that $\widetilde{q}([x]) \leq \widetilde{q}([y])$. Similarly we show that $\widetilde{q}([y]) \leq \widetilde{q}([x])$.

Next we observe that $\widetilde{q}([x])$ is a nonnegative real number. Indeed, since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,q^s), for each $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that $q(x_n-x_m)<\varepsilon$ for all $n,m\geq n_0$, so $q(x_n)-q(x_m)<\varepsilon$ for all $n,m\geq n_0$, and thus $(q(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^+$. Consequently $\lim_{n\to\infty}q(x_n)$ exists and is finite.

Therefore we may define a function $\widetilde{q}:\widetilde{X}\to\mathbb{R}^+$ given by $\widetilde{q}([x])=\lim_{n\to\infty}q(x_n)$ for all $x\in\widehat{X}$. We will show that actually $\widetilde{q}$ is an asymmetric norm on $\widetilde{X}$ such that $(\widetilde{X},\widetilde{q})$ is a biBanach space.

Lemma 3.2 *Let (X,q) be an asymmetric normed linear space. Then the following statements hold:*

(1) $\widetilde{q}$ is an asymmetric norm on $\widetilde{X}$.

(2) $(\widetilde{X},\widetilde{q})$ is a biBanach space.

(3) (X,q) is isometrically isomorphic to a subspace of $(\widetilde{X},\widetilde{q})$ that is dense is the Banach space $(\widetilde{X},(\widetilde{q})^s)$.

Proof. (1): As we have observed above $\widetilde{q}$ is a nonnegative real valued function on $\widetilde{X}$.

Let $x:=(x_n)_{n\in\mathbb{N}}$ be an element of $\widehat{X}$ such that $\widetilde{q}([x])=\widetilde{q}(-[x])=0$. Then $\lim_{n\to\infty}q(x_n)=\lim_{n\to\infty}q(-x_n)=0$, so $\lim_{n\to\infty}q^s(x_n)=0$, and hence $[x]=[\mathbf{0}]$.

Now let $x:=(x_n)_{n\in\mathbb{N}}$ be an element of $\widehat{X}$ and let $a\in\mathbb{R}^+$. Then $\widetilde{q}(a\cdot[x])=\widetilde{q}([a\cdot x])=\lim_{n\to\infty}q(ax_n)=a\lim_{n\to\infty}q(x_n)=a\widetilde{q}([x])$.

Finally let $x:=(x_n)_{n\in\mathbb{N}}$ and $y:=(y_n)_{n\in\mathbb{N}}$ bet two elements of $\widehat{X}$. Then $\widetilde{q}([x]+[y])=\widetilde{q}([x+y])=\lim_{n\to\infty}q(x_n+y_n)\leq\lim_{n\to\infty}q(x_n)+\lim_{n\to\infty}q(y_n)=\widetilde{q}([x])+\widetilde{q}([y])$.

We have shown that $\widetilde{q}$ is an asymmetric norm on $\widetilde{X}$.

(2): It is well known (see [12], [50]) that the bicompletion of the quasi-metric space (X,d_q) is the quasi-metric space (X^b,d_q^b), where $X^b=\{[x]:x$ is a Cauchy sequence in the metric space $(X,(d_q)^s)\}$, $d_q^b([x],[y])=\lim_{n\to\infty}d_q(x_n,y_n)$ for all $[x],[y]\in X^b$, and for each Cauchy sequence $x := (x_n)_{n\in\mathbb{N}}$ in $(X, (d_q)^s)$, $[x] = \{y := (y_n)_{n\in\mathbb{N}} : y$ is a Cauchy sequence in $(X, (d_q)^s)$ and $\lim_{n\to\infty}(d_q)^s(x_n, y_n) = 0\}$.

It immediately follows that $X^b = \widetilde{X}$ and $d_q^b = d_{\widetilde{q}}$ on $\widetilde{X}$. Therefore $(\widetilde{X}, \widetilde{q})$ is a biBanach space.

(3): For each $x \in X$ denote by $\widehat{x}$ the constant sequence $x, x, ..., x, ..$

Since (X^b, d_q^b) is the bicompletion of (X, d_q), $i(X)$ is dense in $(\widetilde{X}, (\widetilde{q})^s)$, where i denotes the map from X into $\widetilde{X}$ defined by $i(x) = [\widehat{x}]$ for all $x \in X$ (recall that for each $x \in X$, $[\widehat{x}]$ consists of all sequences in X which converges to x in the normed linear space (X, q^s)).

Since for each $x \in X$, $\widetilde{q}(i(x)) = \widetilde{q}([x]) = q(x)$, in order to show that (X, q) is isometrically isomorphic to $(i(X), \widetilde{q}\mid_{i(X)})$ it remains to see that i is linear. Indeed, given $x, y \in X$ and $a, b \in \mathbb{R}$, we have $i(ax + by) = [\widehat{ax + by}] = [a \cdot \widehat{x} + b \cdot \widehat{y}] = a \cdot [\widehat{x}] + b \cdot [\widehat{y}] = a \cdot i(x) + b \cdot i(y)$.

The proof is complete.■

Lemma 3.3 *Let (X, q_X) be an asymmetric normed linear space and (Y, q_Y) a biBanach space. If there is an isometric isomorphism f from a linear subspace A of X to Y and A is dense in the normed linear space (X, q_X^s), then f has a unique isometric isomorphism extension to X.*

Proof. For each $x \in X\backslash A$ pick a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $\lim_{n\to\infty} q_X^s(x - x_n) = 0$. Since the sequence $(x_n)_{n\in\mathbb{N}}$ associated to $x \in X\backslash A$, is a Cauchy sequence in the normed linear space (X, q_X^s), $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space (Y, q_Y^s), so it converges to a point $x^* \in Y$.

Define $f^* : X \to Y$ by $f^*(x) = f(x)$ for all $x \in A$ and $f^*(x) = x^*$ for all $x \in X\backslash A$.

Observe that the definition of f^* is independent of the choice of sequences $(x_n)_{n\in\mathbb{N}}$. Indeed, if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in A that converge to a point $x \in X\backslash A$ with respect to the norm q_X^s, and denote by x^* and y^* the limit points in (Y, q_Y^s) of

$(f(x_n))_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$, respectively, we deduce that $\lim_{n\to\infty} q^s_Y(f(x_n)-f(y_n)) = 0$, since f is an isometric isomorphism on A and $\lim_{n\to\infty} q^s_X(x_n - y_n) = 0$. So, by the triangle inequality, $x^* = y^*$.

Next we show that f^* is an isometric isomorphism on X. Let $x \in A$. Then $q_Y(f^*(x)) = q_Y(f(x)) = q_X(x)$. Now let $x \in X\backslash A$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A such that $\lim_{n\to\infty} q^s_X(x - x_n) = 0$. Then, for each $\varepsilon > 0$, $q_Y(f^*(x)) = q_Y(x^*) \leq q_Y(f(x_n))+\varepsilon = q_X(x_n)+\varepsilon$ eventually. Therefore, for each $\varepsilon > 0$, $q_Y(f^*(x)) < q_X(x)+2\varepsilon$. Similarly we show that for each $\varepsilon > 0$, $q_X(x) < q_Y(f^*(x)) + 2\varepsilon$. Consequently $q_Y(f^*(x)) = q_X(x)$ for all $x \in X$.

Furthermore f^* is linear on X. Let $x, y \in X$ and $a, b \in \mathbb{R}$. We only consider the case that $x, y \in X\backslash A$ (recall that f is linear on A). Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in A that converge to x and y respectively in the normed linear space (X, q^s_X). Then $(ax_n+by_n)_{n\in\mathbb{N}}$ converges to $ax+by$ with respect to q^s_X, so by definition of f^*, $(f(ax_n+by_n))_{n\in\mathbb{N}}$ converges to $f^*(ax+by)$ with respect to q^s_Y. Since f is linear on A, the sequence $(af(x_n) + bf(y_n))_{n\in\mathbb{N}}$ converges to $f^*(ax + by)$ with respect to q^s_Y. On the other hand, by definition of f^*, $(f(x_n))_{n\in\mathbb{N}}$ converges to $f^*(x)$ and $(f(y_n))_{n\in\mathbb{N}}$ converges to $f^*(y)$ with respect to q^s_Y. So $((af(x_n)+bf(y_n))_{n\in\mathbb{N}}$ converges to $af^*(x)+bf^*(y)$ with respect to q^s_Y. Therefore $f^*(ax+by) = af^*(x)+bf^*(y)$. We conclude that f^* is linear on X.

Finally, suppose that f' is another isometric isomorphism extension of f to X. Let $x \in X\backslash A$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A that converges to x with respect to q^s_X. Then

$$\lim_{n\to\infty}(q^s_Y(f^*(x) - f^*(x_n)) = \lim_{n\to\infty}(q^s_Y(f'(x) - f'(x_n)) = 0.$$

Since $f^*(x_n) = f'(x_n) = f(x_n)$ for all $n \in \mathbb{N}$, it follows that $f^*(x) = f'(x)$. So f^* is unique.■

Lemma 3.4 *Any bicompletion of an asymmetric normed linear space* (X, q) *is isometrically isomorphic to* $(\widetilde{X}, \widetilde{q})$.

Proof. Let (Y, q_Y) be a bicompletion of (X, q). Since X is dense in the Banach space (Y, q_Y^s) and by Lemma 3.2 (3), there is an isometric isomorphism f from (X, q) to $(\widetilde{X}, \widetilde{q})$, it follows from Lemma 3.3 that f has a (unique) isometric isomorphism extension f^* to (Y, q_Y). It remains to show that $f^* : Y \to \widetilde{X}$ is an onto map. Actually, this fact follows from standard arguments. Indeed, let x be an arbitrary point of $\widetilde{X}$. Since $f(X)$ is dense in $(\widetilde{X}, (\widetilde{q})^s)$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(\widetilde{q}(x - f(x_n)))^s \to 0$. Thus $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\widetilde{X}, (\widetilde{q})^s)$. Since f^* is an isometric isomorphism, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (Y, q_Y^s). Let $y \in Y$ such that $q_Y^s(y - x_n) \to 0$. Then $(\widetilde{q}(f^*(y) - f^*(x_n)))^s \to 0$, so $f^*(y) = x$. This completes the proof.■

From the above lemmas we immediately deduce the following

Theorem 3.1 *Each asymmetric normed linear space (X, q) has a unique bicompletion (up to isometric isomorphism). Moreover if (X, q) is a normed linear space, then its bicompletion is the standard completion of (X, q).*

An application of Theorem 2.1 which is related to the bicompletion of a Hausdorff asymmetric normed linear space is given in Proposition 3.1 below.

It must be keep in mind Definition 2.1 as well as the results of Lemma 2.1 and Theorem 2.1. Let us recall that if (X, q) is an asymmetric normed linear space, $\|\cdot\|_q$ is defined by:

$$\|x\|_q := inf_{x_1\in X}\{q(x_1) + q(x_1 - x)\}, \qquad x \in X.$$

Proposition 3.1 *The bicompletion of a Hausdorff asymmetric normed linear space is Hausdorff.*

Proof. Let (X, q) be a Hausdorff asymmetric normed linear space. Denote by $(\widetilde{X}, \widetilde{q})$ the bicompletion of (X, q) and by $(\widetilde{X}_{\|.\|_q}, \|.\|_q^\sim)$ the completion of the normed space $(X, \|.\|_q)$ (see Theorem 2.1).

Since, by Lemma 2.1, $\|x\|_q \leq q(x)$ for all $x \in X$, then every Cauchy sequence in (X, q^s) is a Cauchy sequence in $(X, \|.\|_q)$, so $\widetilde{X} \subseteq \widetilde{X}_{\|.\|_q}$. Therefore $(\widetilde{X}, \|.\|_q^\sim |_{\widetilde{X}})$ is a normed subspace of $(\widetilde{X}_{\|.\|_q}, \|.\|_q^\sim)$.

Since $\|x\|_q^{\sim} = \lim_{n\to\infty} \|x_n\|_q$ for all $x := [(x_n)_{n\in\mathbb{N}}] \in \widetilde{X}_{\|.\|_q}$ and $\widetilde{q}(x) = \lim_{n\to\infty} q(x_n)$ for all $x := [(x_n)_{n\in\mathbb{N}}] \in \widetilde{X}$, it follows that $\|x\|_q^{\sim} \leq \widetilde{q}(x)$ whenever $x \in \widetilde{X}$. Consequently the topology induced on $\widetilde{X}$ by $\widetilde{q}$ is a Hausdorff topology by Remark 2.1. This completes the proof. ■

We finish the chapter by applying Theorem 3.1 to the dual complexity space. The notation and terminology in the following example correspond to the ones used in Chapter 8.

Example 3.2 *The dual complexity space ([44]) is the pair* $(\mathcal{C}^*, d_{\mathcal{C}^*})$, *where*

$$\mathcal{C}^* = \{f \in [0,\infty)^\omega : \textstyle\sum_{n=0}^{\infty} 2^{-n} f(n) < \infty\},$$

and $d_{\mathcal{C}^*}$ *is the quasi-metric defined on* $\mathcal{C}^* \times \mathcal{C}^*$ *by*

$$d_{\mathcal{C}^*}(f,g) = \textstyle\sum_{n=0}^{\infty} 2^{-n}[(g(n) - f(n)) \vee 0].$$

It is shown in [46] (see also [43]) that $\mathcal{C}^*$ *is the positive cone of the biBanach space* $(\mathcal{B}^*, q)$, *where* $\mathcal{B}^* = \{f \in \mathbb{R}^\omega : \sum_{n=0}^{\infty} 2^{-n} \mid f(n) \mid< \infty\}$, *the operations + and* $\cdot$ *(product by a real scalar) are defined in the usual pointwise way, and q is the asymmetric norm defined on* $\mathcal{B}^*$ *by* $q(f) = \sum_{n=0}^{\infty} 2^{-n}(f(n) \vee 0)$ *for all* $f \in \mathcal{B}^*$.

On the other hand, denote, as usual, by l_1 *the set of infinite sequences* $\mathbf{x} := (x_n)_{n\in\omega}$ *of real numbers such that* $\sum_{n=0}^{\infty} \mid x_n \mid< \infty$.

It is well known that $(l_1, \| \,.\, \|_1)$ *is a Banach space, where* $\| \,.\, \|_1$ *is the norm on* l_1 *defined by* $\| \mathbf{x} \|_1 = \sum_{n=0}^{\infty} \mid x_n \mid$ *for all* $\mathbf{x} \in l_1$.

We will split the norm $\| \,.\, \|_1$ *as follows:*

For each $x \in \mathbb{R}$, *let* x^+ *be the nonnegative real number* $x \vee 0$. *For each* $\mathbf{x} := (x_n)_{n\in\omega} \in l_1$ *define* $\mathbf{x}^+ := (x_n^+)_{n\in\omega}$ *and* $q(\mathbf{x})_+ = \|\mathbf{x}^+\|_1$, *i.e.* $q(\mathbf{x})_+ = \sum_{n=0}^{\infty}(x_n^+)$.

It is immediate to show that q_+ *is an asymmetric norm on* l_1 *such that the norm* $(q_+)^s$ *is equivalent to* $\| \, . \, \|_1$. *Furthermore* $(\mathcal{B}^*, q)$ *and* (l_1, q_+) *are isometrically isomorphic.*

Now let $\mathcal{D} = \{f \in \mathcal{B}^* : f \text{ is eventually constant}\}$.

Clearly $(\mathcal{D}, +, \cdot)$ *is a linear subspace of* $(\mathcal{B}^*, +, \cdot)$. *So* $(\mathcal{D}, q \mid \mathcal{D})$ *is an asymmetric normed linear space. Moreover* $\mathcal{D}$ *is dense in* $(\mathcal{B}^*, q^s)$. *Indeed, given* $g \in \mathcal{B}^*$ *and* $\varepsilon > 0$ *there is* $n_\varepsilon \in \mathbb{N}$ *such that* $\sum_{n=n_\varepsilon}^{\infty} 2^{-n} \mid g(n) \mid < \varepsilon/2$. *Let* $f \in \mathcal{D}$ *defined by* $f(n) = g(n)$ *for* $n = 0, ..., n_\varepsilon - 1$, *and* $f(n) = \varepsilon/2$ *for* $n \geq n_\varepsilon$. *Then*

$$\sum_{n=0}^{\infty} 2^{-n} \mid g(n) - f(n) \mid = \sum_{n=n_\varepsilon}^{\infty} 2^{-n} \mid g(n) - f(n) \mid \leq$$

$$\sum_{n=n_\varepsilon}^{\infty} 2^{-n} \mid g(n) \mid + \sum_{n=n_\varepsilon}^{\infty} 2^{-n} \mid f(n) \mid < \varepsilon.$$

We conclude that $\mathcal{D}$ *is dense in the Banach space* $(\mathcal{B}^*, (q)^s)$. *By the above theorem* $(\mathcal{B}^*, q)$ *is the bicompletion of* $(\mathcal{D}, q \mid \mathcal{D})$.

In particular, let $\mathcal{D}^+ = \{f \in \mathcal{D} : f(n) \geq 0 \text{ for all } n \in \omega\}$. *Thus* $\mathcal{D}^+$ *is the positive cone of* $\mathcal{D}$. *Since the dual complexity space* $\mathcal{C}^*$ *is closed in the complete metric space* $(\mathcal{B}^*, (d_q)^s)$ *and* $(\mathcal{B}^*, d_q)$ *is bicomplete,* $(\mathcal{C}^*, d_q \mid_{\mathcal{C}^*})$ *is a bicomplete quasi-metric space ([44]), which is clearly the bicompletion of* $(\mathcal{D}^+, d_q \mid_{\mathcal{D}^*})$.

Chapter 4

Compactness and finite dimension in asymmetric normed linear spaces

4.1 Introduction

The aim of this chapter is to extend the results about compact sets on finite dimensional normed spaces to the case of asymmetric normed linear spaces. In Section 4.2 we introduce the set theoretical arguments that allows to a general description of compact sets of an asymmetric normed linear space. In Section 4.3, we focus our attention in the finite dimensional case to reproduce the classical results of the normed spaces theory. In particular, we prove that a T_1 asymmetric normed linear space is finite dimensional if and only if the unit ball is compact for the topology generated by the asymmetric norm q (Theorem 4.2). Following the terminology given in Chapter 1, we denote by $V_{1,\leq}$ the unit ball in the asymmetric normed linear space (X, q) and $B_{1,\leq}$ the unit ball in the normed linear space (X, q^s), this will be done via the compactness of $V_{1,\leq}$ in the supremum norm q^s. In fact, we will prove the equivalence between T_1 separation axiom and normability in the case of finite dimensional asymmetric normed linear spaces and thus between T_1 and T_2 separation axioms. The T_2 separation axiom in the general case of asymmetric normed linear spaces has been studied in [22]. We also prove that the Heine-Borel Theorem characterizes finite dimensional asymmetric normed linear spaces that satisfies the T_2 axiom (Theorem 4.3). The general situation for nonnecessarily T_1-spaces is also explored.

Basic references about quasi-metrics and asymmetric norms are [4],[18], [42], [44] and [50]. We use standard notation. Definitions and results on general topology can be found in [11].

4.2 Compact sets in asymmetric normed linear spaces

In this section we describe the compact sets of any asymmetric normed linear space. In particular, given a compact set in (X, q^s), we give a way to construct compact sets in X for the topology generated by q.

Definition 4.1 *Let (X, q) be an asymmetric normed linear space and $x \in X$. We define the set $\theta(x)$ as:*

$$\theta(x) = \{z \in X : d_q(x, z) = q(z - x) = 0\}.$$

In particular

$$\theta(0) = \{z \in X : d_q(0, z) = q(z) = 0\}.$$

Observe that $\theta(x)$ is the closure of $\{x\}$ in (X, q^{-1}).

Lemma 4.1 *Given a subset A of an asymmetric normed linear space (X, q), we have that*

$$\bigcup_{x \in A} \theta(x) = A + \theta(0),$$

where

$$A + \theta(0) = \{z \in X : z = x + y, x \in A \text{ and } y \in \theta(0)\}.$$

Proof. Let $z \in \bigcup_{x \in A} \theta(x)$. Then there exists an $x \in A$ such that $q(z - x) = 0$. This implies that $z - x = y$, $y \in \theta(0)$ and then we can express z as $z = x + y$. Thus $\bigcup_{x \in A} \theta(x) \subset A + \theta(0)$.

Now, let $w \in A + \theta(0)$. Then there exists an $x \in A$ and an element $y \in \theta(0)$ such that $w = x + y$ and also $w - x = y$. Then $q(w - x) = q(y) = 0$, so $w \in \theta(x)$ and $w \in \bigcup_{x \in A} \theta(x)$. This implies that $A + \theta(0) \subset \bigcup_{x \in A} \theta(x)$. ■

Lemma 4.2 *Let (X, q) be an asymmetric normed linear space and $x \in X$. Then*

$$V_\epsilon(x) = V_\epsilon(x) + \theta(0).$$

Proof. $V_\epsilon(x) \subset V_\epsilon(x) + \theta(0)$ since $0 \in \theta(0)$ and every $x \in V_\epsilon(x)$ can be written as $x = x + 0$.

Let $z \in V_\epsilon(x) + \theta(0)$. Then there exists an $y \in V_\epsilon(x)$ and $w \in \theta(0)$ such that $z = y + w$. Then

$$q(z - x) = q(y + w - x) \leq q(y - x) + q(w) < \epsilon + 0 = \epsilon.$$

As a consequence, $z \in V_\epsilon(x)$ and $V_\epsilon(x) + \theta(0) \subset V_\epsilon(x)$. ■

Lemma 4.3 *Let (X, q) be an asymmetric normed linear space and $A \subset X$ an open set. Then*

$$A = A + \theta(0).$$

Proof. It is obvious that $A \subset A + \theta(0)$.

Let $z \in A+\theta(0)$. Then we can express z as $z = x+y$ where x is in A and y is an element of $\theta(0)$. Since A is an open set, there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subset A$. Taking into account that by Lemma 4.2 $V_\epsilon(x) = V_\epsilon(x) + \theta(0)$, we conclude that z is in A. ■

Lemma 4.4 *Given a family $\{A_i : i \in I\}$ of sets in (X, q), then*

$$\bigcup_{i\in I}(A_i + \theta(0)) = \left(\bigcup_{i\in I} A_i\right) + \theta(0).$$

Proof. If $x \in \bigcup_{i\in I}(A_i+\theta(0))$, there exists some $i \in I$ satisfying that $x \in A_i+\theta(0)$, then $x = x_i + z$ with $x_i \in A_i$ and $z \in \theta(0)$. Thus $x_i \in \bigcup_{i\in I} A_i$ and x is in $\left(\bigcup_{i\in I} A_i\right) + \theta(0)$.

If $x \in \left(\bigcup_{i\in I} A_i\right) + \theta(0)$ there exists an $x_i \in A_i$ and $z \in \theta(0)$ such that $x = x_i + z$ and then x is in $\bigcup_{i\in I}(A_i + \theta(0))$. ■

Let (X, q) an asymmetric normed linear space endowed with the topology $T(d_q)$ generated by q. A subset $M \subset X$ is said to be compact if it is compact considered as a subspace of X with the induced topology, that is, M is compact with respect to the topology $T(d_q)|_M$.

Proposition 4.1 *Let (X, q) be an asymmetric normed linear space and $K \subset X$. Then K is compact respect to the topology generated by q if and only if $K + \theta(0)$ is compact for the same topology.*

Proof. We first prove the part "if". Let be $\{A_i : i \in I\}$ an open cover of K. By Lemma 4.3 we have that

$$A_i = A_i + \theta(0).$$

Then by Lemma 4.4

$$K + \theta(0) \subset \bigcup_{i \in I} A_i + \theta(0).$$

Since K is compact, there exists a finite subcover of K, $\{A_j : j \in J \subset I,\ J \text{ finite}\}$ such that $K \subset \bigcup_{j \in J} A_j$. Then applying Lemma 4.4 we obtain that $K + \theta(0) \subset \bigcup_{j \in J}(A_j + \theta(0))$. This implies that $K + \theta(0)$ admits a finite subcover $\{A_j + \theta(0) : j \in J \subset I,\ J \text{ finite}\}$ and thus $K + \theta(0)$ is a compact set.

Conversely, if $K + \theta(0)$ is compact, given an open cover of the set K, $\{A_i : i \in I\}$, the family $\{A_i + \theta(0) : i \in I\}$ is an open cover of $K + \theta(0)$ and this set admits a finite subcover $\{A_j + \theta(0) : j \in J \subset I,\ J \text{ finite}\}$. Then by Lemma 4.4, $K + \theta(0) \subset \bigcup_{j \in J} A_j + \theta(0)$ that implies $K \subset \bigcup_{j \in J} A_j$ and thus $\{A_j : j \in J \subset I,\ J \text{ finite}\}$ is a subcover of K obtained from the open cover $\{A_i : i \in I\}$. Hence, K is compact. ■

Corollary 4.1 *Given a subset K_0 such that $K_0 \subset K + \theta(0)$, if $K + \theta(0)$ is a compact set and $K_0 + \theta(0) = K + \theta(0)$ then K_0 is also compact.*

Note that if K is a compact set in (X, q^s), then $K + \theta(0)$ is a compact set in (X, q).

4.3 Compactness and finite dimension

Let us recall the following well known result.

Lemma 4.5 *Let $(X, \|\cdot\|)$ be a finite dimensional normed linear space, with base $\{e_1, e_2, \ldots, e_n\}$. Then, a sequence $(x_k)_{k \in \mathbb{N}}$ in X converges to $x = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n$ if and only if the i-co-ordinate sequence of $(x_k)_{k \in \mathbb{N}}$ converges to λ_i, with respect to the Euclidean norm, $i = 1, \ldots, n$.*

We generalize this classical result to asymmetric normed linear spaces as follows.

Theorem 4.1 *Let (X, q) be a finite dimensional T_1 asymmetric normed linear space, with base $\{e_1, e_2, \ldots, e_n\}$. Then, a sequence $(x_k)_{k\in\mathbb{N}}$ in X converges to $x = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n$ with respect to q if and only if the i-co-ordinate sequence of $(x_k)_{k\in\mathbb{N}}$ converges to λ_i, with respect to the Euclidean norm, $i = 1, \ldots, n$.*

Proof. First suppose that the i-co-ordinate sequence of $(x_k)_{k\in\mathbb{N}}$ converges to λ_i, with respect to the Euclidean norm, $i = 1, \ldots, n$. Given a positive real number $M > 0$ and an $\epsilon > 0$ there is a k_0^i such that when $k \geq k_0^i$ then

$$|(x_k)_i - \lambda_i| < \frac{\epsilon}{nM}.$$

Let $k_0 = \max\{k_0^i : i = 1, \ldots, n\}$. Then, if $k \geq k_0$,

$$q(x_k - x) \leq \sum_{i=1}^{n} q((x_k)_i - \lambda_i) \leq \sum_{i=1}^{n} q^s((x_k)_i - \lambda_i) \leq \sum_{i=1}^{n} M|(x_k)_i - \lambda_i| \leq \epsilon.$$

where we have used the fact that q^s is a norm equivalent to the Euclidean norm with constant M.

Suppose now that $(x_k)_{k\in\mathbb{N}}$ is a sequence in X that converges to 0 with respect to q (if $(x_k)_{k\in\mathbb{N}}$ converges to x respect to q, the sequence $(x_k - x)_{k\in\mathbb{N}}$ converges to 0), but for some $n_0 \in 1, \ldots, n$ the co-ordinate sequence $((\lambda_k)_{n_0})_{k\in\mathbb{N}}$ is not convergent to 0 with respect to the Euclidean norm, where

$$x_k = (\lambda_k)_1 e_1 + (\lambda_k)_2 e_2 + \ldots + (\lambda_k)_n e_n.$$

for each $k \in \mathbb{N}$.

We may assume that there is a constant $r > 0$ such that $|(\lambda_k)_{n_0}| > r$ for all $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$ put $M_k = \max\{|(\lambda_k)_i : i = 1, \ldots, n\}$. Define a sequence $(y_k)_{k\in\mathbb{N}}$ by $y_k = x_k/M_k$ for all $k \in \mathbb{N}$. Then

$$q(y_k) = \frac{q(x_k)}{M_k} < \frac{q(x_k)}{r},$$

for all $k \in \mathbb{N}$, so $(y_k)_{k\in\mathbb{N}}$ converges to 0 with respect to q.

Now observe that there exists a co-ordinate sequence of $(y_k)_{k\in\mathbb{N}}$ that has a co-ordinate subsequence which consists only of terms -1 or 1. Denote this subsequence by $((\lambda_{k_j})_m)_{j\in\mathbb{N}}$ where $m \in \{1, \ldots, n\}$. Consider the corresponding subsequence $(y_{k_j})_{j\in\mathbb{N}}$ of $(y_k)_{k\in\mathbb{N}}$ and its first co-ordinate sequence $((\lambda_{k_j})_1)_{j\in\mathbb{N}}$. Then $((\lambda_{k_j})_1)_{j\in\mathbb{N}}$ has a convergent subsequence. Continuing this process to the n-th co-ordinate sequence, we obtain a subsequence $(y_{k_l})_{l\in\mathbb{N}}$ of $(y_k)_{k\in\mathbb{N}}$ which has each co-ordinate sequence convergent since the m-th co-ordinate subsequence consists only of terms -1 or 1. So by the preceding lemma $(y_{k_l})_{l\in\mathbb{N}}$ converges to a point $y \neq 0$ with respect to the norm q^s. Since $q(y) \leq q(y - y_{k_l}) + q(y_{k_l})$ for all $l \in \mathbb{N}$, it follows that $q(y) = 0$ so $y = 0$, a contradiction.

We conclude that each co-ordinate sequence $((\lambda_k)_i)_{k\in\mathbb{N}}$ converges for $i = 1, \ldots, n$.

Finally, if the sequence $(x_k)_{k\in\mathbb{N}}$ converges to x with respect to q, then the sequence $(x_k - x)_{k\in\mathbb{N}}$ converges to 0 with respect to q. So the i-co-ordinate sequence $((x_k)_i - (x)_i)_{k\in\mathbb{N}}$ converges to 0. Hence the i-co-ordinate sequence $((x_k)_i)_{k\in\mathbb{N}}$ converges to the i-co-ordinate $(x)_i$. This concludes the proof. ■

Definition 4.2 *An asymmetric normed linear space (X, q) is called normable if there is a norm $\|.\|$ on the linear space X such that the topologies $T(d_q)$ and $T(d_{\|.\|})$ coincide on X.*

Corollary 4.2 *Let (X, q) be a finite dimensional T_1 asymmetric normed linear space. Then (X, q) is normable by the norm q^s.*

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X that converges to a point x with respect to q. By Theorem 4.1 and Lemma 4.5, $(x_k)_{k\in\mathbb{N}}$ converges to x with respect to the norm q^s. ■

In particular observe that, because of Corollary 4.2, the T_1 separation axiom implies the T_2 separation axiom in the finite dimensional case.

Theorem 4.2 *The unit ball of a T_1 asymmetric normed linear space (X,q) is compact if and only if (X,q) is finite dimensional.*

Proof. Suppose firstly that $V_{1,\leq}$ is a compact set of (X,q). Then $V_{1,\leq}$ is compact in (X,q^s) by the preceding corollary. Since $B_{1,\leq} \subset V_{1,\leq}$ and $B_{1,\leq}$ is closed in (X,q^s) it follows that $B_{1,\leq}$ is compact in (X,q^s). Hence (X,q^s), and thus (X,q), are finite dimensional.

Conversely, let $\{e_1,e_2,\ldots,e_n\}$ be a base of (X,q). For each $x \in X$ set

$$x = \lambda_1(x)e_1 + \lambda_2(x)e_2 + \ldots + \lambda_n(x)e_n.$$

Thus we have defined n functions $\lambda_i : X \longrightarrow \mathbb{R}$, which are clearly linear functions on X.

By Theorem 4.1, each λ_i is continuous from (X,q) to $\mathbb{R}$ endowed with the Euclidean norm, so there exist n constants $M_i > 0, M_i \in \mathbb{R},\ i = 1,\ldots,n$ such that

$$|\lambda_i(x)| \leq M_i q(x),\ i = 1,\ldots,n,\ \text{for all } x \in X.$$

Now let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $V_{1,\leq}$. Then $|\lambda_i(x_k)| \leq M_i\ i = 1,\ldots,n,\ k \in \mathbb{N}$. Hence, the first co-ordinate sequence $(\lambda_1(x_k))_{k\in\mathbb{N}}$ has a convergent subsequence. The corresponding co-ordinate sequence $(\lambda_2(x_k))_{k\in\mathbb{N}}$ has also a convergent subsequence. Continuing this process, we obtain a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$, which has each co-ordinate sequence convergent. Therefore $(x_{k_j})_{j\in\mathbb{N}}$ converges to some $y \in X$ with respect to the norm q^s by Theorem 4.1. Since $q(x_{k_j}) \leq 1$ and $q(y) - q(x_{k_j}) \leq q(y - x_{k_j})$ for all $j \in \mathbb{N}$, it follows that $q(y) \leq 1$. We conclude that $V_{1,\leq}$ is a compact set of the normed space (X,q^s) and by the preceding corollary it is a compact set of (X,q). ■

Remark 4.1 *The above proof is doing following the customary scheme but there is an straightforward argument to deduce the result from classical theorems. This comes from the observation that all asymmetric norms on a T_1 finite dimensional linear space are equivalent. It was shown in Chapter 2, Proposition 2.1 that an asymmetric normed linear space is T_1 if and only if $q(x) \neq 0$ for all $x \in X \backslash \{0\}$. Let (X, q) be a finite dimensional asymmetric normed linear space and q^s the supremum norm as usual. Then the restriction of q to the unit sphere of (X, q^s) does not attain zero because q is a continuous function in (X, q^s). Thus, it is bounded below, and so q and q^s are equivalent.*

Theorem 4.3 *Let (X, q) be a finite dimensional asymmetric normed linear space. Then (X, q) is normable if and only if each compact set is closed.*

Proof. Suppose that (X, q) is not normable. Then it is not Hausdorff by Corollary 4.2, so there exist a sequence $(x_n)_{n\in\mathbb{N}}$ in X and two points $x, y \in X$ with $x \neq y$ such that $x_n \to x$ and $x_n \to y$ with respect to the topology $T(d_q)$. Since $K = \{x\} \bigcup \{x_n : n \in \mathbb{N}\}$ is compact in (X, q) and $y \in \overline{K} - K$, K cannot be closed.

The converse is well-known ■

Note that, in a finite dimensional linear space, every compact set is bounded and hence this theorem provides a version of the Heine-Borel Theorem for asymmetric normed linear spaces.

The case in which (X, q) is only a T_0 finite dimensional asymmetric normed linear space is actually more complex. Let us now give a characterization for this situation.

Definition 4.3 *Let (X, q) be an asymmetric normed linear space. We say that $V_{1,\leq}$ is right-bounded if there exists a real constant $r > 0$, such that*

$$rV_{1,\leq} \subset B_{1,\leq} + \theta(0)$$

Proposition 4.2 *Let (X, q) be a T_0 finite dimensional asymmetric normed linear space such that $V_{1,\leq}$ is right-bounded . Then $V_{1,\leq}$ is compact.*

Proof. $B_{1,\leq}$ is the unit ball of the normed space (X, q^s). Since X is finite dimensional, $B_{1,\leq}$ is compact. Let $\{A_i,\ i \in I\}$ be an open cover of $V_{1,\leq}$ in $T(d_q)$. Since $B_{1,\leq} \subset B_{1,\leq} + \theta(0) \subset V_{1,\leq}$, then $\{B_{1,\leq} \bigcap A_i,\ i \in I\}$ is an open cover of $B_{1,\leq}$ in $T(d_{q^s})|_{B_{1,\leq}}$. There exists a finite subcover $\{B_{1,\leq} \bigcap A_j,\ j = 1, \cdots, n\}$ of $B_{1,\leq}$ in $T(d_{q^s})|_{B_{1,\leq}}$. Then $B_{1,\leq} + \theta(0) \subset \bigcup_{j=1}^n (B_{1,\leq} \bigcap A_j) + \theta(0) \subset \bigcup_{j=1}^n A_j + \theta(0)$. But $V_{1,\leq}$ is right-bounded, so $rV_{1,\leq} \subset \bigcup_{j=1}^n A_j + \theta(0) \subset \bigcup_{j=1}^n A_j$ by Lemma 4.3. Then $rV_{1,\leq}$ is compact. Taking into account that the function $f(x) = rx$ is continuous for the topology $T(d_q)$, it is obvious that $V_{1,\leq}$ is compact. ■

Chapter 5

Extensions of asymmetric norms to linear spaces

5.1 Introduction

In Chapter 1 we introduced the notion of *algebraically closed* space.

An easy example of an ac-space is the positive cone C_n of the finite dimensional space $\mathbb{R}^n$. For instance,

$$C_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$$

defines an ac-space.

The aim of the present chapter is to obtain conditions under which it is possible to extend an asymmetric norm defined on an ac-space M to the corresponding linear span $span\{M\}$. Our motivation is that a great part of asymmetric normed linear spaces that appear in applied contexts are in fact extensions of asymmetric norms defined on ac-spaces (see [24] and [44]). For example, the natural definition of the dual of an asymmetric normed linear space X (see Chapter 6) provides an asymmetric normed ac-space. In Section 5.2 we characterize those asymmetric seminorms defined on an ac-space M that can be extended at least to an asymmetric seminorm

q on $span\{M\}$. However, note that in general such an extension does not lead to an asymmetric norm on $span\{M\}$, since we cannot assure that the separation axiom ($q(x) = q(-x) = 0$ if and only if $x = 0$) is satisfied. For example, the asymmetric seminorm q_2 defined on C_2 as $q_2((x_1, x_2)) = x_1$ can be extended to the function $\overline{q}_2$,

$$\overline{q}_2((x_1, x_2)) = x_1 \quad \textit{if} \quad x_1 > 0,$$

and $\overline{q}_2((x_1, x_2)) = 0$ otherwise. It is clear that $\overline{q}_2$ does not satisfy the separation axiom of the definition of the asymmetric norm, although q_2 is an asymmetric norm on C_2.

This motivates the study of extensions satisfying the separation axiom. In Section 5.3 we characterize when this condition is also satisfied, under the assumption that such an extension exists. Section 5.4 is devoted to the application of these results to the particular case of the increasing asymmetric seminorms that appear in several interesting applied frameworks.

The main results of this chapter have been published in [23].

5.2 Extensions of asymmetric seminorms defined on ac-spaces

Let M be an ac-space and let $X = span\{M\}$. In this section we develop a constructive technique to obtain extensions of an asymmetric seminorm q from M to X. Two basic functions are needed in order to construct the extension. The first one is q. The second function that is needed is another asymmetric seminorm p_0 on M. It is clear that the inversion map $i(x) = -x$ defines a linear isomorphism $i : X \to X$ such that $i(M) = -M = \{-x \in X : x \in M\}$ and then $-M$ is also an ac-space. Thus we can use p_0 in order to define an asymmetric seminorm p on $-M$ as $p(x) := p_0(-x)$ for every $x \in -M$.

The following definition gives the canonical construction of an asymmetric seminorm from q and p. Note that each element $x \in X$ can be decomposed as a sum $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in -M$.

Definition 5.1 *Let q and p be asymmetric seminorms on the ac-spaces M and $-M$, respectively. We define the function $q^*_{q,p}$ induced by the couple (q,p) by mean of the expression*

$$q^*_{q,p}(x) = inf\{q(x_1) + p(x_2) : x_1 \in M, x_2 \in -M, x = x_1 + x_2\}$$

for every $x \in X$.

It is easy to prove that $q^*_{q,p}$ defines an asymmetric seminorm on X.

Definition 5.2 *Let q be an asymmetric seminorm on the ac-space M. We say that an asymmetric seminorm q^* defined on X is an extension of q if the restriction of q^* to M coincides with q, i.e. $q^*|_M = q$.*

The asymmetric seminorm $q^*_{q,p}$ is closely related to the possible extensions of q to X. For instance, consider the positive cone C_+ of a Köthe function space $(E, \| \|, <)$. A Köthe function space is a Banach lattice of functions with its natural order (see [33]). If (Ω, Σ, μ) is a complete σ-finite measure, a Banach space E consisting of equivalence classes, modulo equality almost everywhere of locally integrable real valued functions is called a Köthe function space if the following conditions hold.

1) If $|f(\omega)| \leq |g(\omega)|$ a.e. on Ω, with f measurable and $g \in E$, then $f \in E$ and $\|f\| \leq \|g\|$.

2) For every $\sigma \in \Sigma$ with $\mu(\sigma) < \infty$, the characteristic function χ_σ of σ belongs to E.

An easy example of such a space is a (real) Hilbert space of integrable functions $L_2(\nu)$, where ν is a finite measure.

If E is a Köthe function space, it is easy to see that the function $r(x) := \|x \vee 0\|$ defines an asymmetric norm. In fact, the definition of r is given by the evaluation of the norm of the positive part of the function. This construction provides a broad class of examples of asymmetric normed linear spaces of the type (E, r). The reader can find information about related examples in [4] and [18].

It is easy to see that the positive cone (C_+, r) is an asymmetric normed ac-space. Now consider the trivial seminorm $p_1(x) = 0$ defined on $-C_+$. A direct calculation shows that $q^*_{r,p_1}|_{C_+} = r$. Another extension of r to E is the norm $\| \ \|$. It is also easy to prove that $\| \ \|$ is equivalent to q^*_{r,p_2}, where $p_2(x) := \|x \wedge 0\| = \|x\|$ for every $x \in -C_+$. Moreover, $q^*_{r,p_2}|_{C_+} = r$.

The example above shows that we can find different extensions of an asymmetric seminorm defined on an ac-space M to the linear space X. In fact, the asymmetric normed linear spaces (E, q^*_{r,p_1}) and (E, q^*_{r,p_2}) are different from a topological point of view. (E, q^*_{r,p_2}) is a Hausdorff space (it is in fact a biBanach space). However, it can be easily proved that q^*_{r,p_1} does not define a Hausdorff topology on E ([22]). Anyway, the existence of such an extension cannot be assured in general. The following theorem characterizes the asymmetric seminorms defined on ac-spaces M which can be extended to $span\{M\}$, in terms of their moduli of asymmetry.

Definition 5.3 *Let q be an asymmetric seminorm on the ac-space M. We define the modulus of asymmetry of q as the real function $\Phi_q : M \to \mathbb{R}$ given by the formula*

$$\Phi_q(x) := sup\{q(y) - q(y + x) : y \in M\}$$

for every $x \in M$.

Note that $\Phi_q(x) = q(-x)$ if q is a norm on X.

Theorem 5.1 *Let q be an asymmetric seminorm on the ac-space M. Then:*

1) There exists an extension of q to X if and only if there is an asymmetric seminorm p on $-M$ such that

$$\Phi_q(x) \le p(-x) \qquad for\ every\ \ x \in M.$$

2) Such an extension can be obtained as the asymmetric seminorm $q^*_{q,p}$ induced by the couple (q, p).

Proof. The proof is a direct consequence of the properties of the asymmetric seminorm $q^*_{q,p}$. It is defined on the whole linear space $span\{M\}$. Then we just need to show that its restriction to M is exactly q. It is clear that $q^*_{q,p}(x) \le q(x)$ for every $x \in M$, since

$$inf\{q(x_1) + p(x_2) : x_1 \in M, x_2 \in -M, x = x_1 + x_2\} \le q(x) + p(0) = q(x).$$

On the other hand, consider an element $x \in M$, an $\epsilon > 0$ and a decomposition $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in -M$, that satisfies

$$q(x_1) + p(x_2) < q^*_{q,p}(x) + \epsilon.$$

Then we obtain the following inequalities using the condition given in 1) for Φ_q.

$$q^*_{q,p}(x) + \epsilon > q(x_1) + p(x_2) = q(x - x_2) + p(x_2) \ge$$

$$\ge q(x - x_2) + sup\{q(y) - q(y - x_2) : y \in M\} \ge q(x - x_2) + q(x) - q(x - x_2) = q(x).$$

Thus, $q^*_{q,p}(x) = q(x)$ for every $x \in M$, since the above inequalities hold for each $\epsilon > 0$.

For the converse, consider an extension q^* of q to $span\{M\}$. Then for every $x, y \in M$,

$$q(x + y) + q^*(-x) = q^*(x + y) + q^*(-x) \ge q^*(y) = q(y),$$

since $x + y \in M$. Now let us define on $-M$ the asymmetric seminorm $p = q^*|_{-M}$ and fix $x \in M$. We obtain for every $y \in M$ the inequality

$$p(-x) \ge q(y) - q(x + y).$$

Then

$$p(-x) \ge \Phi_q(x) \qquad for\ every\ \ x \in M.$$

2) is a direct consequence of the constructive procedure used in the proof of 1). ■

The next example shows that it is possible to find asymmetric seminorms defined on ac-spaces that cannot be extended to the corresponding linear span. According to Theorem 5.1 we just need to show that there is not any seminorm satisfying the required property. In fact, it is enough to find an element $x \in M$ such that $\Phi_q(x) = \infty$.

Example 5.1 *Consider the positive cone S_+ of the lattice $\mathbb{R}_0^{\mathbb{N}}$ whose elements are the sequences of real numbers $(x_n)_{n\in\mathbb{N}}$ that are non zero only for a finite set of indexes, with the usual order. S_+ is obviously an ac-space. Let us define the asymmetric norm q_+ on S_+ as follows. Consider the canonical basis of $\mathbb{R}_0^{\mathbb{N}}$, $\{e_n : n \in \mathbb{N}\}$. Then for every $\overline{x} = (x_n)_{n\in\mathbb{N}}$, if there is no $\lambda \in \mathbb{R}^+$ such that $\overline{x} = \lambda e_n$ for any $n \in \mathbb{N}$, we define*

$$q_+(\overline{x}) := \sum_{n=1}^{\infty} x_n,$$

and $q_+(\lambda e_n) := \lambda n$ otherwise.

It is easy to prove that q_+ is an asymmetric norm on S_+. However, the element e_1 satisfies that $\Phi_{q_+}(e_1) = \infty$ since

$$\Phi_{q_+}(e_1) = sup\{q(\overline{y}) - q(e_1 + \overline{y}) : \overline{y} \in S_+\} \geq sup\{q(e_n) - q(e_1 + e_n) : n \in \mathbb{N}\} =$$

$$= sup\{n - 2 : n \in \mathbb{N}\} = \infty.$$

Then there is no asymmetric seminorm p on $-M$ satisfying $p(-e_1) \geq \Phi_{q_+}(e_1)$. Moreover, note that this conclusion does not depend on the separation properties of the space (C_+, q_+). It is easy to see that $q_+(\overline{x}) = 0$ implies $\overline{x} = 0$ in the above example. However, an easy change of the definition of q_+ would lead to an asymmetric seminorm which does not satisfy this separation property but does not admit an extension yet. The conditions required for the characterization of extensions that are asymmetric norms are different that the ones that assures the existence of the extension. The next section is devoted to study these conditions.

5.3 Extensions of asymmetric norms

Definition 5.4 *Two asymmetric seminorms q and p given on the ac-spaces M and $-M$ respectively, define a compatible couple (q, p) if the extension $q^*_{q,p}$ exists and satisfies that $q^*_{q,p}|_M = q$ and $q^*_{q,p}|_{-M} = p$.*

Note that any extension $q^*_{q,p}$ of an asymmetric seminorm q can be obtained by means of a compatible couple. It is enough to replace the seminorm p by $p_0 = q^*_{q,p}|_{-M}$. A direct computation shows that $q^*_{q,p} = q^*_{q,p_0}$. Thus we can use compatible couples without loss of generality.

Definition 5.5 *Consider an asymmetric seminormed ac-space (M,q) that admits an extension by means of the compatible couple (q,p). We define the set $\overline{M}_{q,p}$ as the closure of M on the seminormed space $(span\{M\}, (q^*_{q,p})^s)$. Moreover, we say that the ac-space M is closed if $M = \overline{M}_{q,p}$.*

For each element $y \in \overline{M}_{q,p}$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $y \in \lim_{n\to\infty} x_n$, where the limit is computed with respect to the seminorm $(q^*_{q,p})^s$. Then we can extend the asymmetric seminorm q to $\overline{M}_{q,p}$ in the following way. Note that for each $n \in N$

$$(q^*_{q,p})^s(x_n - y) \geq q^*_{q,p}(x_n - y) \geq q^*_{q,p}(x_n) - q^*_{q,p}(y)$$

and

$$(q^*_{q,p})^s(x_n - y) \geq q^*_{q,p}(y - x_n) \geq q^*_{q,p}(y) - q^*_{q,p}(x_n).$$

Then it is clear that $\lim_{n\to\infty} q^*_{q,p}(x_n) = q^*_{q,p}(y)$. Taking into account that $q^*_{q,p}|_M = q$, we obtain that the following (topological) extension of q is well defined.

Definition 5.6 *Let (M,q) be an asymmetric seminormed ac-space and let (q,p) be a compatible couple. Then we define the (topological) extension $\overline{q}$ for each $y \in \overline{M}_{q,p}$ by means of the formula*

$$\overline{q}(y) := \lim_{n\to\infty} q(x_n),$$

where $(x_n)_{n\in\mathbb{N}} \subset M$ satisfies that $y \in \lim_{n\to\infty} x_n$.

Lemma 5.1 *Let (M,q) be an asymmetric seminormed ac-space and let (q,p) be a compatible couple. Then $(\overline{M}_{q,p}, \overline{q})$ is an asymmetric seminormed ac-space.*

Proof. Consider two elements $\overline{x}, \overline{y} \in \overline{M}_{q,p}$. Then there are sequences $(x_n)_{n\in\mathbb{N}} \subset M$

and $(y_n)_{n\in\mathbb{N}} \subset M$ such that

$$\lim_{n\to\infty}(q^*_{q,p})^s(x_n - \overline{x}) = 0,$$
$$\lim_{n\to\infty} q(x_n) = \overline{q}(\overline{x}),$$
$$\lim_{n\to\infty}(q^*_{q,p})^s(y_n - \overline{y}) = 0, \text{and}$$
$$\lim_{n\to\infty} q(y_n) = \overline{q}(\overline{y}).$$

Then

$$\lim_{n\to\infty} q^*_{q,p}(x_n + y_n - \overline{x} - \overline{y}) \leq \lim_{n\to\infty} q^*_{q,p}(x_n - \overline{x}) + \lim_{n\to\infty} q^*_{q,p}(y_n - \overline{y}) = 0.$$

This means that $\overline{x} + \overline{y} \in \overline{M}_{q,p}$, since $x_n + y_n \in M$ for every $n \in \mathbb{N}$. It is also possible to prove that $\lim_{n\to\infty} q(x_n + y_n) = q^*_{q,p}(\overline{x} + \overline{y})$ in the same way. Finally,

$$\overline{q}(\overline{x} + \overline{y}) \leq \lim_{n\to\infty} q(x_n) + \lim_{n\to\infty} q(y_n) = \overline{q}(\overline{x}) + \overline{q}(\overline{y}).$$

The proof for the products $\lambda\overline{x}$, where $\lambda \in \mathbb{R}^+$ and $\overline{x} \in \overline{M}$, is similar.

Consider a compatible couple (q,p). Then we can define the corresponding closed ac-space $\overline{M}_{q,p}$ endowed with the asymmetric seminorm $\overline{q}$. Since $(q^*_{q,p})^s$ is a seminorm, the ac-space $\overline{(-M)}_{q,p}$ is also closed and $\overline{(-M)}_{q,p} = -\overline{M}_{q,p}$. Thus, we can also consider the closed ac-space $-\overline{M}_{q,p}$ endowed with the asymmetric seminorm $\overline{p}$. Clearly, $X = span\{M\} = span\{\overline{M}_{q,p}\}$. Moreover, the definition of the extension $q^*_{q,p}$ implies $q^*_{q,p} \geq q^*_{\overline{q},\overline{p}}$. This argument shows that the separation properties that are satisfied by $q^*_{\overline{q},\overline{p}}$ are also fulfilled by $q^*_{q,p}$. Therefore, we can suppose that q and p are seminorms defined on the closed ac-spaces M and $-M$ of $(X,(q^*_{q,p})^s)$ in the following theorem. In the general case, the condition that will be required in order to assure that the separation axiom holds for extensions will be obtained as a direct consequence.

Theorem 5.2 *Let (q,p) be a compatible couple of asymmetric norms on the closed ac-spaces M and $-M$ respectively. Then the following are equivalent.*

1) $\psi(x) := max\{q(x), p(-x)\} = 0$ implies $x = 0$ for every $x \in M$.

2) The extension $q^*_{q,p}$ defined by (q,p) is an asymmetric norm.

Proof. Let us show that 1) implies 2). Suppose that for an element $x \in X$ we have $q^*_{q,p}(x) = 0$ and $q^*_{q,p}(-x) = 0$. Then, as a consequence of the definition of the extension $q^*_{q,p}$, there are sequences $(x_n)_{n\in\mathbb{N}} \subset M$ and $(y_n)_{n\in\mathbb{N}} \subset -M$ such that

$$(x - x_n)_{n\in\mathbb{N}} \subset -M, \quad \lim_{n\to\infty} q(x_n) = 0, \quad \lim_{n\to\infty} p(x - x_n) = 0,$$

and

$$(-x - y_n)_{n\in\mathbb{N}} \subset M, \quad \lim_{n\to\infty} q(-x - y_n) = 0, \quad \lim_{n\to\infty} p(y_n) = 0.$$

Let us define the sequence $(z_n)_{n\in\mathbb{N}} \subset M$, $z_n := x_n - y_n$. Since for every $n \in \mathbb{N}$, $-x + z_n = -x - (y_n - x_n) \in M$, we have that

$$q(-x + z_n) \leq q(x_n) + q(-x - y_n)$$

and

$$p(x - z_n)) \leq p(x - x_n) + p(y_n),$$

we deduce that $\lim_{n\to\infty} q(-x + z_n) = 0$ and $\lim_{n\to\infty} p(x - z_n) = 0$. Moreover, since $q^*_{q,p}|_M = q$ and $q^*_{q,p}|_{-M} = p$, we get

$$q(-x + z_n) = q^*_{q,p}(-x + z_n),$$

and

$$p(x - z_n) = q^*_{q,p}(x - z_n).$$

Then

$$\psi(-x + z_n) = (q^*_{q,p})^s(-x + z_n),$$

and

$$\lim_{n\to\infty} (q^*_{q,p})^s(z_n - x) = 0.$$

Therefore $x \in M$ since M is closed, and $\psi(x) = (q^*_{q,p})^s(x) = 0$. Then an application of 1) gives 2). For the converse we just need to note that $\psi = (q^*_{q,p})^s|_M$. ■

Corollary 5.1 *Let (q,p) be a compatible couple of asymmetric norms on the ac-spaces M and $-M$. Then the following conditions are equivalent, and imply that $q^*_{q,p}$ is an asymmetric norm:*

1) For every $x \in \overline{M}_{q,p}$, $(q^*_{\overline{q},\overline{p}})^s(x) = 0$ implies $x = 0$.

2) $q^*_{\overline{q},\overline{p}}$ is an asymmetric norm.

5.4 Applications. Extensions of increasing asymmetric seminorms

To finish this chapter we apply the results of Section 5.2 and Section 5.3 to a particular case. We define a class of asymmetric seminorms that satisfy an increasing condition. Our definition is motivated by the fact that many asymmetric norms that have been used in applied contexts belong to this class.

Definition 5.7 *Let q be an asymmetric seminorm defined on an ac-space M. We say that q is an increasing asymmetric seminorm if for every pair $x, y \in M$, $q(x) \leq q(x+y)$.*

Note that this property implies a strong restriction on the value of $q(x)$ for the elements $x \in M$ that satisfy that x and $-x$ belong to M, since $q(x) \leq q(x+(-x)) = q(0) = 0$. In particular if M is a linear space, $q = 0$. However, we can find a lot of examples of subsets of Banach lattices that satisfy this property. In particular, the restriction of the norm to an ac-space contained on the positive cone of a Köthe function space satisfies this condition (see [33] for the definition of the Köthe function space). Moreover, the dual complexity space introduced in [44] (see also [24], [51]) satisfies this property too.

Corollary 5.2 *Let q be an increasing asymmetric seminorm on an ac-space M. Then the extension $q^*_{q,p}$ exists for each asymmetric seminorm p defined on $-M$.*

Proof. Since q is increasing, it is obvious that $\Phi_q(x) = sup\{q(y) - q(y+x) : y \in M\} \leq 0$ for every $x \in M$. Then it is clear that each asymmetric seminorm p on $-M$ satisfies $p(-x) \geq \Phi_q(x)$. An application of Theorem 5.1 gives the result. ■

Corollary 5.2 is true even in the trivial case $p = 0$. Moreover, consider a normed lattice $(E, \| \|, <)$. Then the canonical asymmetric norm on E is defined as $q_0(y) := \|y \vee 0\|$ for every $y \in E$ (see [4] and [18]). If we define M as the positive cone of E and $q(x) := \|x\|$ for every $x \in M$, it can be easily proved that $q_0(y) = q^*_{q,p}(y)$ for every $y \in E = span\{M\}$, where $p = 0$.

Corollary 5.3 *Let q be an increasing asymmetric seminorm on an ac-space M that satisfies that for every $x \in M$, $q(x) = 0$ implies $x = 0$. Let p be an asymmetric seminorm on $-M$. Then the extension $q^*_{q,p}$ exists and defines an asymmetric norm if M is closed.*

The proof of the above result is a direct consequence of Corollary 5.2 and Theorem 5.2. We can use the last result in order to extend the asymmetric norm q_0 defined on the normed lattice E. For instance, Corollary 5.3 can be applied to each ac-space M contained in the positive cone of a Köthe function space $(E, \|.\|, <)$. The properties of this class of normed lattices imply that the asymmetric seminorm q defined as the restriction of $\|.\|$ to M is increasing (see [33], p. 28). (Since the elements of M are positive functions, we have that $|f| \leq |f + g|$ for every $f, g \in M$, and then $\|f\| \leq \|f + g\|$). Moreover, $x = 0$ if and only if $q(x) = 0$ for every $x \in M$. If p is an asymmetric seminorm defined on $-M$ such that M is an ac-space, the extension $q^*_{q,p}$ is an asymmetric norm. Of course, this is also true if $p = 0$. In this case, the asymmetric norm $q^*_{q,0}$ is the natural extension of q_0.

Chapter 6

The dual space of an asymmetric normed linear space

6.1 Introduction and preliminaries

Our aim in this chapter is to introduce and study the dual space (X^*, q^*) of an asymmetric normed linear space (X, q). We observe that, in contrast to the classical theory, it is not a linear space in general. However, we prove that if X and Y are asymmetric normed linear spaces, then the space $LC(X, Y)$ of all continuous linear maps from X to Y can be endowed with the structure of an asymmetric normed semilinear space. From this result it follows that (X^*, q^*) is a biBanach semilinear space. We also define the bidual space (X^{**}, q^{**}) and prove that (X, q) is isometrically isomorphic to an asymmetric normed linear space that is an algebraically closed subset of X^{**}.

We will give again some basic definitions presented in Chapter 1 because the introduction of the notion of extended asymmetric norm.

If X is a linear space, A an algebraically closed subset of X and B a subset of A that is algebraically closed in X, then we say that B is an *algebraically closed subset of A.*

Here, we will consider *extended asymmetric norms.* They satisfy the usual axioms, except that we allow $q(x) = \infty$.

We will also consider *extended quasi-metrics* (we allow $d(x, y) = \infty$). If d is an extended quasi-metric on a set X, then the function d^{-1} is also an extended quasi-metric on X and the function d^s is an extended metric on X.

The notions of *extended asymmetric normed semilinear space* and *bicomplete* extended quasi-metric are defined in the obvious manner.

It is well known that an extended quasi-metric d on X induces a T_0 topology as in the usual case.

If (X, q) is an extended asymmetric normed linear space such that the induced extended quasi-metric d_q is bicomplete, we will say, as in the usual case, that (X, q) is a *biBanach space* ([21],[24], [46]).

If A is an algebraically closed subset of X (i.e a semilinear space) such that the restriction of d_q to A is bicomplete, we will say that (A, q) is a *biBanach semilinear space.*

As we will see in Chapter 8, asymmetric normed (semi)linear spaces and other related structures provide suitable tools in some fields of Theoretical Computer Science and Approximation Theory, respectively (see [42], [46], [51], [56], etc.).

6.2 Spaces of continuous linear functions

Given two asymmetric normed linear spaces (X, q) and (Y, p), we will denote by $LC^s(X, Y)$ the linear space of all continuous linear maps from the normed linear space (X, q^s) to the normed linear space (Y, p^s).

According to the classical theory, $(LC^s(X, Y), (q^s)^*_p)$ is a normed linear space, where $(q^s)^*_p$ is the norm on $LC^s(X, Y)$ defined by

$$(q^s)^*_p(f) = \sup\{p^s(f(x)) : q^s(x) \leq 1\},$$

for all $f \in LC^s(X,Y)$. Furthermore $(LC^s(X,Y),(q^s)^*_p)$ is a Banach space whenever (Y,p^s) is so.

In order to obtain a satisfactory generalization of the theory of duality to the asymmetric setting, we will denote by $LC(X,Y)$ the set of all continuous linear maps from the asymmetric normed linear space (X,q) to the asymmetric normed linear space (Y,p).

First we will establish that $LC(X,Y)$ is an algebraically closed subset of $LC^s(X,Y)$. This will be done with the help of the following result, that has been introduced in Lema 2.2 (see Proposition 4.1 in [18]) and, using the notation of this chapter, can be written as follows.

Lemma 6.1 *Let (X,q) and (Y,p) be two asymmetric normed linear spaces and let $f : X \to Y$ be a linear map. Then $f \in LC(X,Y)$ if and only if there is a constant $M > 0$ such that $p(f(x)) \leq Mq(x)$ for all $x \in X$.*

Proposition 6.1 *Let (X,q) and (Y,p) be two asymmetric normed linear spaces. Then, every continuous linear map from (X,q) to (Y,p) is continuous from (X,q^{-1}) to (Y,p^{-1}). Hence $LC(X,Y) \subseteq LC^s(X,Y)$.*

Proof. Let $f \in LC(X,Y)$. Then there is $M > 0$ such that $p(f(x)) \leq Mq(x)$ for all $x \in X$. Thus

$$p^{-1}(f(x)) = p(-f(x)) = p(f(-x)) \leq Mq(-x) = Mq^{-1}(x).$$

Therefore f is continuous from (X,q^{-1}) to (Y,p^{-1}) by Lemma 6.1, and, consequently it is continuous from (X,p^s) to (Y,q^s). We conclude that $LC(X,Y) \subseteq LC^s(X,Y)$.■

Corollary 6.1 *Let (X,q) and (Y,p) be two asymmetric normed linear spaces. Then $LC(X,Y)$ is an algebraically closed subset of $LC^s(X,Y)$. Hence $LC(X,Y)$ is a semi-linear space.*

The next simple example shows that in contrast to the classical theory, $LC(X,Y)$ is not a linear space in general, and justifies the importance of considering semilinear spaces in order to construct a satisfactory dual theory in this context.

Example 6.1 *Let I be the identity function on $\mathbb{R}$. Clearly I is a continuous linear map from $(\mathbb{R}, u)$ into itself. However, it is clear that $-I$ is not continuous. It follows that $LC(\mathbb{R},\mathbb{R})$ is not a linear space. Hence $LC(X,Y) \neq LC^s(X,Y)$, in general. We also observe that for $x < 0$, $u(-x) = -x$, so $\sup\{u(-x) : u(x) \leq 1\} = \infty$.*

Theorem 6.1 *Let (X,q) and (Y,p) be two asymmetric normed linear spaces. For each $f \in LC^s(X,Y)$ set*

$$q_p^*(f) = \sup\{p(f(x)) : q(x) \leq 1\}.$$

Then the following assertions hold:

(1) q_p^* is an extended asymmetric norm on $LC^s(X,Y)$, and $(q^s)_p^* \leq (q_p^*)^s$ on $LC^s(X,Y)$.

(2) The restriction of q_p^* to $LC(X,Y)$ is an asymmetric norm.

(3) $LC(X,Y)$ is a closed subset of $(LC^s(X,Y), (q_p^*)^s)$.

(4) If (Y,p) is a biBanach space, then $(LC^s(X,Y), q_p^*)$ is a biBanach space and $(LC(X,Y), q_p^*)$ is a biBanach semilinear space.

Proof. (1) Clearly $q_p^*(0) = 0$.

Let $f \in LC^s(X,Y)$ be such that $q_p^*(f) = q_p^*(-f) = 0$. Then $p(f(x)) = p(-f(x)) = 0$ whenever $q(x) \leq 1$. Hence $f(x) = 0$ whenever $q(x) \leq 1$. Now, if $x \in X$ verifies $q(x) > 1$ we obtain

$$\frac{1}{q(x)} f(x) = f(\frac{x}{q(x)}) = 0.$$

Therefore $f(x) = 0$ for all $x \in X$.

It is easy to see that for $f, g \in LC^s(X, Y)$ and $r \in \mathbb{R}^+$ we have

$$q_p^*(rf) = rq_p^*(f) \quad \text{and} \quad q_p^*(f+g) \leq q_p^*(f) + q_p^*(g).$$

We conclude that q_p^* is an extended asymmetric norm on $LC^s(X, Y)$ (Example 6.1 above shows that "extended" cannot be omitted in our assertion).

Next we show that $(q^s)_p^* \leq (q_p^*)^s$ on $LC^s(X, Y)$.

Let $f \in LC^s(X, Y)$. Given $\varepsilon > 0$ there is $x \in X$ such that $q^s(x) \leq 1$ and $(q^s)_p^*(f) < p^s(f(x)) + \varepsilon$. Assume without loss of generality that $q^s(x) = q(x)$. Then, if $p^s(f(x)) = p(f(x))$, we obtain

$$(q^s)_p^*(f) < p(f(x)) + \varepsilon \leq q_p^*(f) + \varepsilon.$$

Otherwise, $p^s(f(x)) = p(-f(x))$, so

$$(q^s)_p^*(f) < p(-f(x)) + \varepsilon \leq q_p^*(-f) + \varepsilon.$$

Consequently $(q^s)_p^* < (q_p^*)^s + \varepsilon$. Thus $(q^s)_p^* \leq (q_p^*)^s$ on $LC^s(X, Y)$.

(2) By virtue of statement (1) it suffices to show that for each $f \in LC(X, Y)$, $q_p^*(f) < \infty$. But this is clear because, by Lemma 6.1, for each $f \in LC(X, Y)$ there is $M > 0$ such that $p(f(x)) \leq Mq(x)$ for all $x \in X$, and hence $q_p^*(f) \leq M$.

(3) Let $f \in LC^s(X, Y)$ be such that there is a sequence $(f_n)_{n \in \mathbb{N}}$ in $LC(X, Y)$ which converges to f in $(LC^s(X, Y), (q_p^*)^s)$. We will show that there is $M > 0$ such that $p(f(x)) \leq (M+1)q(x)$ for all $x \in X$, and thus $f \in LC(X, Y)$.

Choose $n_0 \in \mathbb{N}$ such that $q_p^*(f - f_{n_0}) < 1$. Since $f_{n_0} \in LC(X, Y)$, there is $M > 0$ such that $p(f_{n_0}(x)) \leq Mq(x)$ for all $x \in X$.

Let $x \in X$. If $q(x) \neq 0$. Then

$$p\left(\frac{f(x) - f_{n_0}(x)}{q(x)}\right) = p\left((f - f_{n_0})(\frac{x}{q(x)})\right) \leq q_p^*(f - f_{n_0}).$$

Hence

$$p(f(x)) - p(f_{n_0}(x)) \leq q_p^*(f - f_{n_0})q(x) < q(x).$$

So

$$p(f(x)) < p(f_{n_0}(x)) + q(x) \leq (M+1)q(x).$$

If $q(x) = 0$, for each $\varepsilon > 0$ choose $n_\varepsilon \in \mathbb{N}$ such that $q_p^*(f - f_{n_\varepsilon}) < \varepsilon$. Since $p(f_{n_\varepsilon}(x)) = 0$ we obtain

$$p(f(x)) = p(f(x)) - p(f_{n_\varepsilon}(x)) \leq p((f - f_{n_\varepsilon})(x)) \leq q_p^*(f - f_{n_\varepsilon}) < \varepsilon.$$

Therefore $p(f(x)) = 0$.

We have shown that $f \in LC(X,Y)$, and consequently $LC(X,Y)$ is closed in $(LC^s(X,Y), (q_p^*)^s)$. (Note that actually we have proved the more general fact that $LC(X,Y)$ is closed in $(LC^s(X,Y), (q_p^*)^{-1})$).

(4) Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the extended normed linear space $(LC^s(X,Y), (q_p^*)^s)$. It immediately follows that for each $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space (Y, p^s). Thus we can construct a map $f : X \to Y$, where for each $x \in X$, $f(x)$ is the limit point of the sequence $(f_n(x))_{n\in\mathbb{N}}$ in the Banach space (Y, p^s).

On the other hand, since by (1), $(q^s)_p^* \leq (q_p^*)^s$, it follows that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(LC^s(X,Y), (q^s)_p^*)$, so $(f_n)_{n\in\mathbb{N}}$ converges to some $g \in LC^s(X,Y)$. Hence $(f_n(x))_{n\in\mathbb{N}}$ converges to $g(x)$ in the Banach space (Y, p^s) for all $x \in X$. Consequently $g = f$.

Next we show that actually $(f_n)_{n\in\mathbb{N}}$ converges to f in $(LC^s(X,Y), (q_p^*)^s)$.

Indeed, let $\varepsilon > 0$. Then there is $n_0 \in \mathbb{N}$ such that $(q_p^*)^s(f_n - f_m) < \varepsilon/2$ for all $n, m \geq n_0$. Choose an arbitrary point $x \in X$ such that $q(x) \leq 1$. There is $m \geq n_0$ such that $p^s(f(x) - f_m(x)) < \varepsilon/2$. Therefore for each $n \geq n_0$ we have

$$\begin{aligned} p^s((f - f_n)(x)) &= p^s(f(x) - f_n(x)) \leq p^s(f(x) - f_m(x)) + p^s(f_m(x) - f_n(x)) \\ &< \frac{\varepsilon}{2} + (q_p^*)^s(f_n - f_m) < \varepsilon. \end{aligned}$$

We deduce that for each $n \geq n_0$, $(q_p^*)^s(f - f_n) \leq \varepsilon$, and thus $(LC^s(X,Y), q_p^*)$ is a biBanach space.

Finally, since by (3), $LC(X,Y)$ is closed in the extended Banach space $(LC^s(XY), (q_p^*)^s)$, it follows that $(LC(X,Y), q_p^*)$ is a biBanach semilinear space.■

Remark 6.1 *Let us observe that if in the above theorem (Y,p) is a normed linear space, then q_p^* is an extended norm on $LC^s(X,Y)$ and thus $(q^s)_p^* \leq q_p^*$. Hence the topology induced by q_p^* is finer than the topology induced by $(q^s)_p^*$ on $LC^s(X,Y)$. We show that actually these topologies do not coincide in general. Indeed, for each $n \in \mathbb{N}$ let $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = -x/n$. Then f_n is linear and continuous with respect to the Euclidean norm. Furthermore $f_n \to 0$ with respect to the usual norm of uniform convergence because* $\sup\{|-x/n| : |x| \leq 1\} = 1/n$ *for all $n \in \mathbb{N}$. Nevertheless we have* $\sup\{|-x/n| : u(x) \leq 1\} = \infty$ *for all $n \in \mathbb{N}$.*

Next we discuss the preservation by $(LC(X,Y), q_p^*)$ of properties as Hausdorffness, complete regularity and normability.

We say that an extended asymmetric normed linear space (X,q) is Hausdorff (resp. completely regular) if the topology induced by q is Hausdorff (resp. completely regular).

Proposition 6.2 *Let (X,q) and (Y,p) be two asymmetric normed linear spaces. If (Y,p) is Hausdorff, then $(LC^s(X,Y), q_p^*)$ is Hausdorff.*

Proof. Let $f, g \in LC^s(X,Y)$ such that $f \neq g$. Then there is $x_0 \in X$ with $f(x_0) \neq g(x_0)$, and we may assume without loss of generality that $q(x_0) \leq 1$. Let $\varepsilon > 0$ such that $B_{d_p}(f(x_0), \varepsilon) \cap B_{d_p}(g(x_0), \varepsilon) = \emptyset$. It follows that $B_{d_{q_p^*}}(f, \varepsilon) \cap B_{d_{q_p^*}}(g, \varepsilon) = \emptyset$. We conclude that $(LC^s(X,Y), q_p^*)$ is a Hausdorff space.■

Since Hausdorffness is a hereditary property we obtain the following.

Corollary 6.2 *Let (X,q) and (Y,p) be two asymmetric normed linear spaces. If (Y,p) is Hausdorff, then $(LC(X,Y),q_p^*)$ is Hausdorff.*

We do not know if the preceding corollary remains valid when "Hausdorff" is replaced by "completely regular". However, for normable asymmetric normed linear spaces we will obtain a positive result.

The natural extension of normability to the class of extended asymmetric normed linear spaces is the following. An (extended) asymmetric normed linear space (X,q) is called *normable* if there is a norm $\|.\|$ on the linear space X such that the topologies $T(d_q)$ and $T(d_{\|.\|})$ coincide on X.

In Chapter 2 were obtained examples of Hausdorff asymmetric normed linear spaces that are not normable. Next we give an easy example of an asymmetric normed linear space which is not a normed linear space but is normable.

Example 6.2 *Let k be a positive real number different from 1 and let q be the function defined on $\mathbb{R}$ by*

$$q(x) = x \text{ if } x \geq 0 \quad \text{and} \quad q(x) = k(-x) \text{ if } x < 0.$$

It is routine to check that q is an asymmetric norm on the Euclidean linear space $\mathbb{R}$. Clearly q is not a norm.

Furthermore the ball $B_{d_q}(0,\varepsilon)$ is the open interval $]-\varepsilon/k,\varepsilon[$. We deduce that the topology $T(d_q)$ coincides with the Euclidean topology on $\mathbb{R}$.

Note that if $k < 1$, q^s is exactly the Euclidean norm on $\mathbb{R}$, and if $k > 1$, $q^s(x) = k \mid x \mid$ for all $x \in \mathbb{R}$.

Lemma 6.2 *Let (X,q) be an extended asymmetric normed linear space. If $(X,T(d_q))$ is a topological group, then (X,q) is normable.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$ such that $x_n \to x$ with respect to $T(d_q)$. Then $-x_n \to -x$ with respect to $T(d_q)$. Hence $q(x_n - x) \to 0$ and

$q(-x_n + x) \to 0,$ so $q^s(x_n - x) \to 0$. We have shown that $T(d_q) = T(d_{q^s})$ on X. It immediately follows that $(X, T(d_q))$ is a topological linear space that admits a bounded and convex neighborhood of 0. Hence (X, q) is normable ([49]).■

Proposition 6.3 *Let (X, q) and (Y, p) be two asymmetric normed linear spaces. If (Y, p) is normable, then $(LC^s(X, Y), q_p^*)$ is normable.*

Proof. By Lemma 6.2 it suffices to show that $(LC^s(X, Y), T(d_{q_p^*}))$ is a topological group. Indeed, choose an arbitrary $\varepsilon > 0$. Since (Y, p) is normable, we have $T(d_p) = T(d_{p^{-1}}) = T(d_{p^s})$ and thus there exists $\delta > 0$ such that $B_{d_p^{-1}}(0, \delta) \subseteq B_{d_p}(0, \varepsilon/2)$. Then, an easy computation shows that $B_{d_{(q_p^*)^{-1}}}(0, \delta) \subseteq B_{d_{q_{p^*}}}(0, \varepsilon)$. Therefore $T(d_{q_p^*}) \subseteq T(d_{(q_{p^*})^{-1}})$. Similarly we prove that $T(d_{(q_p^*)^{-1}}) \subseteq T(d_{q_{p^*}})$. Hence $(LC^s(X, Y), T(d_{q_{p^*}}))$ is a topological group. We conclude that $(LC^s(X, Y), q_p^*)$ is normable.■

6.3 The dual space of an asymmetric normed linear space

Given an asymmetric normed linear space (X, q) let

$$X^{s*} = \{f : (X, q^s) \to (\mathbb{R}, |.|) : f \text{ is linear and continuous}\},$$

and let

$$X^* = \{f : (X, q) \to (\mathbb{R}, u) : f \text{ is linear and continuous}\}.$$

Then X^{s*} is a linear space.

Note that $f \in X^*$ if and only if it is a linear and upper semicontinuous real-valued function on (X, q).

By Corollary 6.1, X^* is an algebraically closed subset of X^{s*}, and thus it is a semilinear space. Moreover, by Theorem 6.1, q_u^* is an extended asymmetric norm on X^{s*} such that the restriction of q_u^* to X^* is an asymmetric norm, where

$$q_u^*(f) = \sup\{f(x) \vee 0 : q(x) \leq 1\},$$

for all $f \in X^{s*}$. In the following q_u^* will be simply denoted by q^*.

Observe that (X^{s*}, q^*) is a biBanach space and that (X^*, q^*) is a biBanach semilinear space, by Theorem 6.1.

If (X, q) is an asymmetric normed linear space, then the pair (X^*, q^*) is called the *dual space* of (X, q).

It is interesting to observe that actually we have

$$q^*(f) = \sup\{f(x) : q(x) \leq 1\},$$

for all $f \in X^{s*}$.

Example 6.1 above shows that X^* is not a linear space in general. Since the space of this example is finite dimensional, we next present an example of an infinite dimensional asymmetric normed linear space (X, q) for which X^* is not a linear space. Proposition 3.4 in [18] provides more examples.

Example 6.3 *Consider the asymmetric normed linear space (l_2, q_2^+) defined by the sequences $(\lambda_i)_{i=1}^\infty$ that belong to the Hilbert space l_2 and the asymmetric norm*

$$q_2^+((\lambda_i)) = \|(\lambda_i \vee 0)\|_2, \qquad (\lambda_i)_{i=1}^\infty \in l_2,$$

where $\|(\lambda_i)\|_2 := (\sum_{i=1}^\infty |\lambda_i|^2)^{1/2}$. An easy computation applying the definition leads to the representation of the dual space l_2^* as the set

$$l_2^* := \{(\mu_i)_{i=1}^\infty : \mu_i \geq 0,\ \|(\mu_i)\|_2 < \infty\}.$$

Each element $\mu = (\mu_i)_{i=1}^\infty \in l_2^*$ defines a linear and upper semicontinuous function f_μ by the formula

$$f_\mu((\lambda_i)) := \sum_{i=1}^{\infty} \mu_i \lambda_i, \qquad (\lambda_i)_{i=1}^{\infty} \in l_2.$$

Thus, we can identify the function f_μ with the element μ. Moreover, a straightforward calculation shows that in this case the restriction of the asymmetric norm $(q_2^+)^*$ to the algebraically closed subset l_2^* of l_2^{s*} is given by the expression

$$(q_2^+)^*((\mu_i)) = (\sum_{i=1}^{\infty} \mu_i^2)^{1/2}, \qquad (\mu_i)_{i=1}^{\infty} \in l_2^*.$$

Furthermore, $(l_2^{s*}, (q_2^+)^*)$ is a biBanach space and $(l_2^*, (q_2^+)^*)$ is a biBanach semilinear space by Theorem 6.1.

Given an asymmetric normed linear space (X, q), we denote by $V_{1,\le}^{X^*}$ the unit ball of the dual space (X^*, q^*), i.e. $V_{1,\le}^{X^*} = \{f \in X^* : q^*(f) \le 1\}$.

The following somewhat surprising identification of $V_{1,\le}^{X^*}$ will be useful at the last part of this section.

Proposition 6.4 *Let (X, q) be an asymmetric normed linear space. Then*

$$V_{1,\le}^{X^*} = \{f \in X^{s*} : q^*(f) \le 1\} \quad \text{and} \quad X^* = \{f \in X^{s*} : q^*(f) < \infty\}.$$

Proof. Let $f \in V_{1,\le}^{X^*}$. Then $q^*(f) \le 1$ and $f \in X^{s*}$ because $X^* \subseteq X^{s*}$.

Now let $f \in X^{s*}$ such that $q^*(f) \le 1$. We want to show that $f(x) \le q(x)$ for all $x \in X$. Indeed, fix $x \in X$. We will distinguish two cases.

Case 1. $q(x) = 0$.

Suppose $f(x) > 0$. Choose $r > 0$ such that $rf(x) > 1$. Put $y = rx$. Then $q(y) = 0$. Since $q^*(f) \le 1$ it follows that $f(y) \le 1$. However $f(y) = rf(x) > 1$, a contradiction. Therefore $f(x) \le 0$.

Case 2. $q(x) > 0$.

Then we obtain
$$\frac{1}{q(x)}f(x) = f(\frac{x}{q(x)}) \leq q^*(f) \leq 1,$$

and thus $f(x) \leq q(x)$.

From Lemma 6.1 it follows that $f \in X^*$, so $f \in V_{1,\leq}^{X^*}$.

Now let $f \in X^{s*}$ such that $q^*(f) < \infty$. Then the function $g = f/q^*(f)$ is in X^{s*} and $q^*(g) = 1$. Therefore $g \in V_{1,\leq}^{X^*}$. We conclude that $f \in X^*$. The proof is complete.■

Lemma 6.3 *([4], [18]). Let (X, q) be an asymmetric normed linear space, let A be an algebraic closed subset of X and let g be a linear and upper semicontinuous real-valued function on A. Then there exists a linear and upper semicontinuous real-valued function f on X such that $f\mid_A = g$ and $q^*(f) = q^*\mid_A(g)$.*

Lemma 6.4 *Let (X, q) be an asymmetric normed linear space. Then for each $x_0 \in X$ there is $f \in V_{1,\leq}^{X^*}$ such that $f(x_0) = q(x_0)$.*

Proof. If $q(x_0) = 0$, the function $f \equiv 0$, satisfies obviously the requirements.

Suppose then that $q(x_0) > 0$. Consider the algebraically closed subspace of X, $span\{x_0\}$. Let $g : span\{x_0\} \to \mathbb{R}$ given by $g(ax_0) = aq(x_0)$ for all $a \in \mathbb{R}$. Clearly g is linear. Furthermore g is upper semicontinuous on $(span\{x_0\}, q\mid_{span\{x_0\}})$. Indeed, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and $a \in \mathbb{R}$ such that $q(a_nx_0 - ax_0) \to 0$. If $a_n \leq a$ eventually, we have $g(a_nx_0) - g(ax_0) = (a_n - a)q(x_0) \leq 0$ eventually, and hence $g(a_nx_0) \to g(ax_0)$, in $(\mathbb{R}, u)$, obviously. Otherwise, we may assume without loss of generality that $a_n > a$ eventually. Then, we have $q(a_nx_0 - ax_0) = (a_n - a)q(x_0)$ eventually, so $(a_n - a)q(x_0) \to 0$, and hence $g(a_nx_0) \to g(ax_0)$ with respect to the Euclidean norm on $\mathbb{R}$.

Therefore, we may apply Lemma 6.3, and thus g has an extension to a linear and upper semicontinuous real-valued function f such that

$$q^*(f) = \sup\{g(x) : x \in span\{x_0\} \text{ and } q(x) \leq 1\}.$$

Hence $f(x_0) = g(x_0)$, so $f(x_0) = q(x_0)$. Furthermore, it is clear for the definition of g that

$$q^*(f) = \sup\{aq(x_0) : a > 0 \text{ and } aq(x_0) \leq 1\}.$$

Thus $q^*(f) \leq 1$. This completes the proof.■

Theorem 6.2 *Let (X, q) be an asymmetric normed linear space. Then for each $x \in X$,*

$$q(x) = \sup\{f(x) : f \in V_{1,\leq}^{X^*}\}.$$

Proof. Fix $x \in X$. If $q(x) = 0$, then $f(x) \leq 0$ for all $f \in V_{1,\leq}^{X^*}$ because $f(x) \leq Mq(x)$ for some $M > 0$. Consequently $0 = \sup\{f(x) : f \in V_{1,\leq}^{X^*}\} = q(x)$.

If $q(x) > 0$, then for each $f \in V_{1,\leq}^{X^*}$, we obtain

$$\frac{1}{q(x)} f(x) = f(\frac{x}{q(x)}) \leq q^*(f) \leq 1,$$

and thus $f(x) \leq q(x)$. Therefore

$$\sup\{f(x) : f \in V_{1,\leq}^{X^*}\} \leq q(x).$$

On the other hand, by Lemma 6.4 there exists $f_0 \in V_{1,\leq}^{X^*}$ such that $f_0(x) = q(x)$. Hence

$$q(x) \leq \sup\{f(x) : f \in V_{1,\leq}^{X^*}\}.$$

This completes the proof.■

It is a classical result that each normed linear space X is isometrically isomorphic to a closed linear subspace of its bidual X^{**}. In the rest of this section we discuss the corresponding situation for asymmetric normed linear spaces.

Let (X, q) be an asymmetric normed linear space. By analogy with the notion of the dual X^* of X, introduced above, we define the following sets.

$$X^{s**} = \{\varphi : (X^{s*}, (q^*)^s) \to (\mathbb{R}, |.|) : \varphi \text{ is linear and continuous}\},$$

and

$$X^{**} = \{\varphi : (X^{s*}, q^*) \to (\mathbb{R}, u) : \varphi \text{ is linear and continuous}\}.$$

Then X^{s**} is a linear space and X^{**} is an algebraically closed subset of X^{s**}.

Now for each $\varphi \in X^{**}$ set

$$q^{**}(\varphi) = \sup\{\varphi(f) : q^*(f) \leq 1\}.$$

Then (X^{**}, q^{**}) is an asymmetric normed semilinear space, which will be called *the bidual space of* (X, q).

Given two asymmetric normed semilinear spaces (X, q) and (Y, p), a linear map $f : X \to Y$ such that $p(f(x)) = q(x)$ for all $x \in X$, is called an *isometric isomorphism* from (X, q) to (Y, p).

Observe that every isometric isomorphism is a one-to-one map.

Two asymmetric normed linear spaces (X, q) and (Y, p) are called *isometrically isomorphic* if there exists an isometric isomorphism from (X, q) onto (Y, p).

Theorem 6.3 *Let (X, q) be an asymmetric normed linear space. For each $x \in X$ let $\varphi_x : X^{s*} \to \mathbb{R}$ defined by*

$$\varphi_x(f) = f(x), \qquad f \in X^{s*},$$

and let $\varphi(X) = \{\varphi_x : x \in X\}$. *Then the following statements hold.*

(1) $\varphi(X)$ *is a linear space which is algebraically closed in* X^{**}.

(2) $(\varphi(X), q^{**})$ *is an asymmetric normed linear space isometrically isomorphic to* (X, q).

(3) $(\varphi(X), q^{**})$ *is a biBanach space if* (X, q) *is so.*

Proof. (1) We first prove that $\varphi(X)$ is a subset of X^{**}. Fix $x_0 \in X$. Let $f, g \in X^{s*}$ and $a, b \in \mathbb{R}$. Then

$$\varphi_{x_0}(af + bg) = (af + bg)(x_0) = af(x_0) + bg(x_0) = a\varphi_{x_0}(f) + b\varphi_{x_0}(g).$$

Hence φ_{x_0} is a linear function.

Now let $(f_n)_{n\in\mathbb{N}}$ be a sequence in X^{s*} that converges to a function f in (X^{s*}, q^*). Given $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $q^*(f_n - f) < \varepsilon$ for all $n \geq n_\varepsilon$.

If $q(x_0) = 0$ it follows that $f_n(x_0) - f(x_0) < \varepsilon$ for all $n \geq n_\varepsilon$.

If $q(x_0) > 0$ it follows that

$$f_n(\frac{x_0}{q(x_0)}) - f(\frac{x_0}{q(x_0)}) < \varepsilon$$

for all $n \geq n_\varepsilon$. So $f_n(x_0) - f(x_0) < \varepsilon q(x_0)$.

We deduce that $(f_n(x_0))_{n\in\mathbb{N}}$ converges to $f(x_0)$ in $(\mathbb{R}, u)$, and thus φ_{x_0} is continuous from (X^{s*}, q^*) to $(\mathbb{R}, u)$. Consequently $\varphi_{x_0} \in X^{**}$. Hence $\varphi(X) \subseteq X^{**}$.

Next we show that $\varphi(X)$ is a linear space. Let $x, y \in X$ and $a, b \in \mathbb{R}$. Then, for each $f \in X^{s*}$,

$$(a\varphi_x + b\varphi_y)(f) = a\varphi_x(f) + b\varphi_y(f) = af(x) + bf(y) = f(ax + by) = \varphi_{ax+by}(f).$$

So $a\varphi_x + b\varphi_y \in \varphi(X)$. It immediately follows that $\varphi(X)$ is a linear space and it is algebraically closed in X^{**}.

(2) From (1) we obtain that $(\varphi(X), q^{**})$ is an asymmetric normed linear space.

Next we prove that (X, q) and $(\varphi(X), q^{**})$ are isometrically isomorphic. Define a map $\Psi : X \to \varphi(X)$ by $\Psi(x) = \varphi_x$ for all $x \in X$. Then Ψ is linear because for $x, y \in X$ and $a, b \in \mathbb{R}$ we obtain

$$\Psi(ax + by) = \varphi_{ax+by} = a\varphi_x + b\varphi_y = a\Psi(x) + b\Psi(y).$$

Clearly, Ψ is an onto map.

Given $x \in X$, we have by Proposition 6.4 and Theorem 6.2,

$$q(x) = \sup\{f(x) : f \in X^{s*} \text{ and } q^*(f) \leq 1\}$$

$$= \sup\{\varphi_x(f) : f \in X^{s*} \text{ and } q^*(f) \leq 1\} = q^{**}(\varphi_x) = q^{**}(\Psi(x)).$$

(3) If (X, q) is a biBanach space, it is obvious by (2) that $(\varphi(X), q^{**})$ is also a biBanach space.■

Remark 6.2 *Although it does not follow the idea proposed above to constructing the bidual of an asymmetric normed linear space (X, q), there is a temptation to give an alternative and apparently more simple notion of biduality, working directly on the set of all linear and upper semicontinuous real-valued functions defined on (X^*, q^*). Thus we define*

$$(X^*)^* = \{\varphi : (X^*, q^*) \to (\mathbb{R}, u) : \varphi \textit{ is linear and continuous}\}.$$

Since span$\{(X^*)^*\}$ is clearly a linear space and $(X^*)^*$ is an algebraically closed subset of it, we deduce that $((X^*)^*, (q^*)^*)$ is an asymmetric normed semilinear space, where

$$(q^*)^*(\varphi) = \sup\{\varphi(f) : q^*(f) \leq 1\},$$

for all $\varphi \in (X^*)^*$.

Now, for each $x \in X$ *let* $\varphi_{x|X^*}$ *be the restriction to* X^* *of the function* φ_x *constructed in Theorem 6.3 (thus* $\varphi_{x|X^*}(f) = f(x)$ *for all* $f \in X^*$*), and let* $\varphi(X)_{|X^*} = \{\varphi_{x|X^*} : x \in X\}$*. Then, as in the proof of Theorem 6.3, we obtain that* $(\varphi(X)_{|X^*}, (q^*)^*)$ *is an asymmetric normed linear space isometrically isomorphic to* (X, q)*. Hence* $(\varphi(X), q^{**})$ *and* $(\varphi(X)_{|X^*}, (q^*)^*)$ *are isometrically isomorphic.*

Chapter 7

Weak topologies on asymmetric normed linear spaces

7.1 Introduction

The use of the structure of the dual space is one of the main tools of the theory of the locally convex spaces, since it leads to the definition of the weak topologies for the spaces as a consequence of the properties of the space of the (real) continuous linear maps. In this chapter we show how we can construct weak topologies in the context of the asymmetric normed linear spaces, and we present several results related to the basic properties of these topologies.

7.2 Preliminary results

The proof of the following result was given essentially in Chapter 6.

Proposition 7.1 *Let (X, q) be an asymmetric normed linear space. The dual space X^* is an ac-closed subset of X^{s*} and $(q^s)^*(f) \leq q^*(f)$ for every $f \in X^*$. Moreover, q^* is an asymmetric norm on X^*. In particular, this means that if $f \in X^*$ and $-f \in X^*$, $q^*(f) = q^*(-f) = 0$ implies $f = 0$.*

The following pointwise boundedness property can be obtained directly. In fact, it is the main idea in the proof of Proposition 7.1. For every $f \in X^*$ and every $x \in X$, we have:

$$-q(-x)q^*(f) \leq f(x) \leq q(x)q^*(f).$$

Proposition 7.2 gives a representation of the linear span of X^* which will be used in the following section.

Proposition 7.2 *Let (X, q) be an asymmetric normed linear space and X^* its dual space. Then*

$$span\{X^*\} = \{f \in X^{s*} : f = f_1 - f_2,\ f_1, f_2 \in X^*\} = X^* - X^*.$$

Proof. Let $f \in span\{X^*\}$. Then we can write f as

$$f = \sum_{i=1}^{n} \alpha_i g_i - \sum_{i=n+1}^{m} \alpha_i g_i,$$

where $n, m \in \mathbb{N}$, $m \geq n$, and for every $i = 1, ..., m$ α_i is a non negative real number and $g_i \in X^*$.

Since X^* is algebraically closed, the functions $f_1 = \sum_{i=1}^{n} \alpha_i g_i$ and $f_2 = \sum_{i=n+1}^{m} \alpha_i g_i$ are in fact elements of X^*. ■

Therefore, $X^* - X^*$ is a linear subspace of X^{s*}. Moreover, in the following section we show that the equality between these linear spaces gives conditions for the coincidence of several weak topologies.

7.3 Weak topologies on X

The first definition of weak topology that we give for the asymmetric normed linear space (X, q) is induced when the linear functionals of X^* are considered as elements of X^{s*}.

Definition 7.1 *We define the weak topology for X, denoted by τ_{weakq}, as the one that has as a basis of neighborhoods of 0 the following subsets. For every natural number n, each finite sequence $f_1, ..., f_n \in X^*$ and each $\epsilon > 0$, we define*

$$W_{\epsilon,f_1,...,f_n}(0) := \{x \in X : |f_1(x)| < \epsilon, ..., |f_n(x)| < \epsilon\}.$$

A basis of neighborhoods for an element $y \in X$ is obtained by translations of these neighborhoods, i.e.

$$W_{\epsilon,f_1,...,f_n}(y) := y + W_{\epsilon,f_1,...,f_n}(0).$$

Note that each neighborhood of y can be written as

$$W_{\epsilon,f_1,...,f_n}(y) = \{x \in X : |f_1(x-y)| < \epsilon, ..., |f_n(x-y)| < \epsilon\}.$$

We can consider the asymmetry of the norm on the original space (X, q) to define two different topologies that are coarser than τ_{weakq}.

Definition 7.2 *The weak positive topology for X (weak+ topology for short), denoted by τ_{weak+}, is the one that has as a basis of neighborhoods of 0 the following subsets. For every natural number n, each finite sequence $f_1, ..., f_n \in X^*$ and each $\epsilon > 0$, we define*

$$W^+_{\epsilon,f_1,...,f_n}(0) := \{x \in X : f_1(x) < \epsilon, ..., f_n(x) < \epsilon\}.$$

As in the case of the weak topology, a basis of neighborhoods for an element $y \in X$ is obtained by

$$W^+_{\epsilon,f_1,...,f_n}(y) := y + W^+_{\epsilon,f_1,...,f_n}(0).$$

In this case, each neighborhood of y can be written as

$$W^+_{\epsilon,f_1,...,f_n}(y) = \{x \in X : f_1(x-y) < \epsilon, ..., f_n(x-y) < \epsilon\}.$$

Definition 7.3 *The weak negative topology for X (weak$-$ topology for short), denoted by τ_{weak-}, is the one that is defined by the basis of neighborhoods of 0 given by the following subsets. For every natural number n, each finite sequence $f_1, ..., f_n \in X^*$ and each $\epsilon > 0$, we define*

$$W^-_{\epsilon,f_1,...,f_n}(0) := \{x \in X : -f_1(x) < \epsilon, ..., -f_n(x) < \epsilon\}.$$

A basis of neighborhoods for an element $y \in X$ is given by

$$W^-_{\epsilon,f_1,...,f_n}(y) := y + W^-_{\epsilon,f_1,...,f_n}(0),$$

which can also be defined as

$$W^-_{\epsilon,f_1,...,f_n}(y) = \{x \in X : -f_1(x-y) < \epsilon, ..., -f_n(x-y) < \epsilon\}.$$

Note that the continuity properties of a function $f \in X^*$ with respect to the topologies defined above can be characterized by mean of the study of the continuity in 0, since they are invariant by translations.

We can also consider the weak topology on X induced by the elements of the dual of the normed space (X, q^s). We will denote it by τ_{weakq^s}.

Theorem 7.1 *The following relations between the topologies defined on X by the asymmetric norm q and the dual space X^* hold.*

1) τ_{weakq} is coarser than τ_{weakq^s}.

2) τ_{weak+} is coarser than the topology $T(d_q)$ generated by q .

3) τ_{weak+} and τ_{weak-} are coarser than τ_{weakq}.

4) $\tau_{weakq} = \tau_{weak+} \vee \tau_{weak-}$.

5) If $X^{s*} = X^* - X^*$, then $\tau_{weakq} = \tau_{weakq^s}$.

Proof. The statement 1) is obvious. To prove, 2), consider a neighborhood $W^+_{\epsilon,f_1,...,f_n}(0)$ of τ_{weak+}. Then there are positive constants $M_1, ..., M_n$ such that

$f_i(x) \leq M_i q(x)$ for every $x \in X$ and each f_i, $i = 1, ..., n$. We just need to consider the ball $V_{\frac{\epsilon}{M}}(0)$, where M is the maximum of the constants M_i, $i = 1, ..., n$, since for every $x \in V_{\frac{\epsilon}{M}}(0)$ and $i = 1, ..., n$

$$f_i(x) \leq M_i q(x) \leq M q(x) < \epsilon.$$

The statement 3) is just a consequence of the definitions, since for every $x \in X$ and each function $f \in X^*$ we have $f(x) \leq |f(x)|$ and $-f(x) \leq |f(x)|$. The same inequalities show 4), since

$$W^+_{\epsilon,f}(0) \cap W^-_{\epsilon,f}(0) = \{x \in X : -f(x) < \epsilon, f(x) < \epsilon\} = W_{\epsilon,f}(0)$$

for every $\epsilon > 0$ and $f \in X^*$, and the same argument can be used to construct the neighborhoods $W_{\epsilon,f_1,...,f_n}(0)$ as $W^+_{\epsilon,f_1,...,f_n}(0) \cap W^-_{\epsilon,f_1,...,f_n}(0)$ for finite sets of functions $f_1, ..., f_n \in X^*$.

To prove 5), consider the neighborhood of 0 for the topology τ_{weakq^s} $W^s_{\epsilon,f_1,...,f_n}(0) = \{x \in X : |f(x)| < \epsilon\}$, where $f_i \in X^{s*}$ for every $i = 1, ..., n$. Then we can find functions $f_{i,1}, f_{i,2} \in X^*$ for each f_i, $i = 1, ..., n$, such that $f_i = f_{i,1} - f_{i,2}$. Since $|f_i(x)| \leq |f_{i,1}(x)| + |f_{i,2}(x)|$ for each $i = 1, ..., n$ and every $x \in X$, we obtain that

$$W_{\epsilon,f_{1,1},...,f_{n,1}}(0) \cap W_{\epsilon,f_{1,2},...,f_{n,2}}(0) \subset W^s_{\epsilon,f_1,...,f_n}(0),$$

which gives the result. ■

Let us show that, contrarily to the classical case, $T(d_q)$ can be weaker than τ_{weakq}. Consider the space $(\mathbb{R}^2, q_2^+)$, where $q_2^+((x, y)) = \sqrt{(x \vee 0)^2 + (y \vee 0)^2}$. It is clear that $(\mathbb{R}^2)^*$ is defined by the positive cone of $\mathbb{R}^2$ (with respect to the usual order) with the Euclidean norm, and each element of the dual space of $(\mathbb{R}^2, \|.\|_2)$ can be written as a difference of two elements of $(\mathbb{R}^2)^*$. Thus, we have that the $\tau_{weakq} = \tau_{weakq^s}$. Since the weak and the norm topology coincide for finite dimensional normed spaces, we obtain $T(d_{q^s}) = \tau_{weakq}$. However, it is clear that there is no ball of $(\mathbb{R}^2, q_2^+)$ contained in the unit ball of $\mathbb{R}^2$ (endowed with the Euclidean norm), since the set $\{\lambda(-1, -1) : \lambda \in \mathbb{R}^+\}$ is contained into every ball $V_\epsilon(0)$, $\epsilon > 0$ for the topology generated by q_2^+. Therefore, in this case $T(d_q)$ is weaker than τ_{weakq}.

Proposition 7.3 *Let (X, q) be an asymmetric normed linear space. Then*

1) The weak topology is the coarsest that makes continuous all linear functionals in X^*.

2) The weak+ topology is the coarsest that makes upper semicontinuous all linear functionals of X^*.

3) The weak− topology is the coarsest that makes lower semicontinuous all linear functionals of X^*.

Proof. 1) If $f \in X^*$ and $\epsilon > 0$, we just need to consider the neighborhood of 0, $W_{\epsilon,f}$ to show that f is continuous, since $x \in W_{\epsilon,f}$, $|f(x)| \leq \epsilon$, implies $f(x) \in (-\epsilon, \epsilon)$. To see that it is the coarsest, it is enough to take into account that $W_{\epsilon,f}(0) = \{x \in X : |f(x)| < \epsilon\} = f^{-1}((-\epsilon, \epsilon))$, and then these sets must be contained in every topology such that all the functions $f \in X^*$ are continuous. Since the single neighborhoods as $W_{\epsilon,f}(0)$ define a subbases for τ_{weakq}, we obtain the result.

2) The proof is similar for the topology τ_{weak+}. In this case, we just need to consider an upper semicontinuous function $f \in X^*$ as a continuous function $f : X \to (\mathbb{R}, u)$. A basic neighborhood of 0 in $(\mathbb{R}, u)$ is $(-\infty, \epsilon)$ for an $\epsilon > 0$. It is clear that $f(W^+_{\epsilon,f}) \subset (-\infty, \epsilon)$, and then we obtain the upper semicontinuity of f. A similar argument that the one of 1) gives that τ_{weak+} is the coarsest topology that satisfies this condition for every function $f \in X^*$. The proof of 3) follows the same lines. ■

7.4 Weak topologies on X^*

As in the case of the asymmetric normed linear space (X, q), we can give several definitions for the space (X^*, q^*). In this section we present these notions and we show that the definitions of Section 7.3 lead to the same topology when we adapt them to the dual space X^*. We restrict our attention to the case of the weak* topologies, i.e. the topologies induced by X on X^*. Therefore, in this section we are interested in the (pointwise) topologies generated by the elements of X when we consider them as functions acting on X^*.

Definition 7.4 *We define the weak* topology for X^*, denoted by τ_{weak*}, as the one that has as a basis of neighborhoods of* 0 *the following subsets. For every natural number n, each finite sequence $x_1, ..., x_n \in X$ and each $\epsilon > 0$, we define*

$$W^*_{\epsilon,x_1,...,x_n}(0) := \{f \in X^* : |f(x_1)| < \epsilon, ..., |f(x_n)| < \epsilon\}.$$

A basis of neighborhoods for a function $g \in X^*$ is obtained by translations of these neighborhoods, i.e.

$$W^*_{\epsilon,x_1,...,x_n}(g) := g + W^*_{\epsilon,x_1,...,x_n}(0).$$

It is obvious that we get in this way a translation invariant topology. In the same way, we can define the weak* positive topology, denoted by τ_{weak*+}, on the space X^* (the weak*+ topology for short) as in the case of the weak topologies for X. In this case, this would be the one that has the following neighborhoods of 0. For each finite subset of elements $x_1, ..., x_n \in X$ and each $\epsilon > 0$, we define

$$W^{*+}_{\epsilon,x_1,...,x_n}(0) := \{f \in X^* : f(x_1) < \epsilon, ..., f(x_n) < \epsilon\}.$$

The translations of these sets $W^{*+}_{\epsilon,x_1,...,x_n}(g) = g + W^{*+}_{\epsilon,x_1,...,x_n}(0)$, define a fundamental system of neighborhoods of g for every $g \in X^*$.

We can also define the weak* negative topology, denoted by τ_{weak*-}, with the following neighborhoods of 0. If $x_1, ..., x_n$ are elements of X and ϵ is a positive real number, we define $W^{*-}_{\epsilon,x_1,...,x_n}(0)$ as

$$W^{*-}_{\epsilon,x_1,...,x_n}(0) := \{f \in X^* : -f(x_1) < \epsilon, ..., -f(x_n) < \epsilon\}.$$

Although the definitions above seems to give different topologies, it is easy to prove that these topologies are in fact the same. Hence,

$$\tau_{weak*} = \tau_{weak*+} = \tau_{weak*-}$$

on every dual X^* of an asymmetric normed linear space (X, q).

To see this, it is enough to take a neighborhood of 0 for the weak* topology as $W^*_{\epsilon,x}(0)$. Then we can consider $W^{*+}_{\epsilon,x,-x}(0)$, and it is clear that $W^*_{\epsilon,x}(0) = W^{*+}_{\epsilon,x,-x}(0)$.

Since these sets define a subbases of the weak* topology, the equivalence is proved. Therefore, this is a consequence of the linearity of X.

In the same way, we can consider the linearization $X^* - X^*$ of the dual space X^* and extend the definition of the topologies to this new space. This leads to the following definition.

Definition 7.5 *We define the pc-weak* topology, denoted by $\tau_{pc-weak*}$, to the topology induced by X on $X^* - X^*$, i.e. the one that has as neighborhoods of $g \in X^* - X^*$ the sets*

$$W^{pc*}_{\epsilon,x_1,...,x_n}(g) := \{f \in X^* - X^* : |(f-g)(x_1)| < \epsilon, ..., |(f-g)(x_n)| < \epsilon\},$$

for every finite set of elements $x_1, ..., x_n$ of X.

The notation pc-weak* is due to the obvious fact that this topology is exactly the topology of the pointwise convergence ("pc "for short).

Note that, in this case, we do not define the neighborhoods of an element $g \in X^* - X^*$ as translations of the neighborhoods of 0. However, we have also a translation invariant topology, since

$$W^{pc*}_{\epsilon,x_1,...,x_n}(g) = g + W^{pc*}_{\epsilon,x_1,...,x_n}(0).$$

for every g and every neighborhood of 0.

We could give other definitions of pointwise topologies on $X^* - X^*$ that are related to the topology $\tau_{pc-weak*}$ but taking into account the asymmetry of the space X^*, following the definition of the dual topologies for X. As in the case of τ_{weak*}, the definition of the corresponding positive topology by mean of neighborhoods of g as

$$W^{pc*+}_{\epsilon,x_1,...,x_n}(g) := \{f \in X^* - X^* : (f-g)(x_1) < \epsilon, ..., (f-g)(x_n) < \epsilon\}$$

for every finite set of elements $x_1, ..., x_n$ of X leads to the topology $\tau_{pc-weak*}$.

Remark 7.1 *It is interesting to note that the topology τ_{weak*} does not coincide on X^* with the topology $\tau_{pc-weak*}$ when we restrict it to the space X^*. It is easy to* *prove that τ_{weak*} is finer than the restriction $\tau_{pc-weak*}|X^*$, but the converse is not true in general. The neighborhoods of an element $g \in X^*$ for τ_{weak*} are translations of neighborhoods of 0 in X^*. This means that*

$$W^*_{\epsilon,x_1,...,x_n}(g) = g + W^*_{\epsilon,x_1,...,x_n}(0) =$$

$$= \{f \in X^* : f - g \in X^*, |(f-g)(x_1)| < \epsilon, ..., |(f-g)(x_n)| < \epsilon\}.$$

However, the restriction to X^* of the corresponding neighborhood of $g \in X^*$ for $\tau_{pc-weak*}$ is

$$W^{pc*}_{\epsilon,x_1,...,x_n}(g) \bigcap X^* = (g + W^{pc*}_{\epsilon,x_1,...,x_n}(0)) \bigcap X^* =$$

$$= \{f \in X^* : |(f-g)(x_1)| < \epsilon, ..., |(f-g)(x_n)| < \epsilon\}.$$

which are not in general the same sets, since X^* is not in general a linear space. In fact, τ_{weak*} is translation invariant, but this is not the case for $\tau_{pc-weak*}|X^*$. The following example illustrates this fact.

Example 7.1 *Consider the linear space $\mathbb{R}_0^{\mathbb{N}}$ of sequences of real numbers (λ_i) that are different of 0 only for a finite subset of co-ordinates. We define the asymmetric normed linear space l_1^+ as the pair $(\mathbb{R}_0^{\mathbb{N}}, q_1)$, where q_1 is the asymmetric norm defined by*

$$q_1((\lambda_i)) := \|(\lambda_i \vee 0)\|_1,$$

where $\|.\|_1$ is the usual 1-norm, i.e., for every $(\lambda_i) \in \mathbb{R}_0^{\mathbb{N}}$,

$$\|(\lambda_i)\|_1 = \sum_{i=1}^{\infty} |\lambda_i|.$$

The dual of this space can be directly computed by using the lattice properties of the space $\mathbb{R}_0^{\mathbb{N}}$ with the usual order and the well-known duality between the space of summable sequences l_1 and the space of bounded sequences l_∞. Since for this kind of asymmetric norms, the continuous functions are exactly the positive continuous functions for the original norm in the lattice (in this case $\|.\|_1$) (see [4]), we obtain that $(l_1^+)^*$ is exactly the positive cone of l_∞. Now, consider the constant sequence $(1,1,1,...) \in (l_1^+)^*$ and the corresponding neighborhood defined by the element $(1,0,0,0...) \in l_1^+$,

$$W^*_{\epsilon,(1,0,0...)}((1,1,...) = \{(\lambda_i) \in l_\infty : \lambda_i \geq 1, i \in \mathbb{N}, \lambda_1 - 1 < \epsilon\} =$$

$$= (1,1,1,...) + W^{*}_{\epsilon,(1,0,0...)}((0,0,0,...).$$

However, the restriction to $(l_1^+)^*$ of the neighborhood of $(1,1,1,...)$ for $\tau_{pc-weak*}$ is

$$W^{pc*}_{\epsilon,(1,0,0...)}((1,1,...) = \{(\lambda_i) \in l_\infty : \lambda_i \geq 0, i \in \mathbb{N}, |\lambda_1 - 1| < \epsilon\}.$$

which can not be written as the translation of any neighborhood of 0 of $\tau_{pc-weak*}|(l_1^+)^*$. Then, note that every neighborhood of $(1,1,1,...)$ for the weak* topology have all its co-ordinates greater or equal to 1. But every neighborhood for $\tau_{pc-weak*}$ contains an element that have all its co-ordinates equal to 0 after a finite number of non-zero co-ordinates, since the neighborhoods are defined by finite sets of sequences that has finitely many non-zero co-ordinates.

We are interested in the pc-weak* topology, since it leads to a good weak reflexivity relation for asymmetric normed linear spaces as we will show in the next section. However, $\tau_{pc-weak*}$ is the topology that really acts as a weak* topology for the linearization of the dual space X^*, as the following proposition shows.

Proposition 7.4 *The pc-weak* topology is the coarsest topology which makes continuous the functionals $x : X^* - X^* \to \mathbb{R}$, defined by $x(f) := f(x)$ for every $x \in X$.*

Proof. First we show that every functional defined by an element $x \in X$ on $X^* - X^*$ is continuous for $\tau_{pc-weak*}$. Let $\epsilon > 0$ and consider the neighborhood of zero in $\mathbb{R}$ given by $(-\epsilon, \epsilon)$. Take the neighborhood $W^{pc*}_{\epsilon,x}(0) := \{f \in X^* - X^* : |f(x)| < \epsilon\}$. It is clear that $x(W^{pc*}_{\epsilon,x}(0)) \subset (-\epsilon, \epsilon)$, and then the function is continuous. On the other hand, each topology τ that makes continuous the map $x(f) := f(x)$ satisfies that $x^{-1}((-\epsilon, \epsilon)) \in \tau$. Since $x^{-1}((-\epsilon, \epsilon)) = W^{pc*}_{\epsilon,x}(0)$ and the pc-weak* topology is the coarsest topology generated by the neighborhoods $\{W^{pc*}_{\epsilon,x}(0) : x \in X\}$, we obtain the result. ■

7.5 The Alaoglu theorem for asymmetric normed linear spaces

Let (X, q) be an asymmetric normed linear space and let $B^{X^{s*}}_{1,\leq} = \{f \in X^{s*} : (q^s)^*(f) \leq 1\}$. Then, the celebrated Alaoglu theorem states that $B^{X^{s*}}_{1,\leq}$ is compact for the weak* topology on X^{s*}. Here we will show that the unit ball $V_{1,\leq}^{X^*}$ is compact for the pc-weak* topology on X^*. In fact, we present two proofs of this result. The proof of the following lemma can be found in Chapter 6 (see Proposition 6.4).

Lemma 7.1 *Let (X,q) be an asymmetric normed linear space. Then $V_{1,\leq}^{X^*} \subseteq B_{1,\leq}^{X^{s*}}$.*

Theorem 7.2 *Let (X,q) be an asymmetric normed linear space. Then $V_{1,\leq}^{X^*}$ is compact in X^* with respect to $\tau_{\text{pc}-\text{weak}*}|_{X^*}$.*

Proof. Let $(f_\alpha)_{\alpha\in\Delta}$ be a net in $V_{1,\leq}^{X^*}$. Since $V_{1,\leq}^{X^*} \subseteq B_{1,\leq}^{X^{s*}}$ and $B_{1,\leq}^{X^{s*}}$ is compact for the weak* topology by Alaoglu Theorem, there is a subnet $(f_{\alpha_\lambda})_{\lambda\in\Lambda}$ of $(f_\alpha)_{\alpha\in\Delta}$, which converges to a function $f \in B_{X^{s*}}$ with respect to the weak* topology on X^{s*}. Thus f is linear. Moreover, for $x \in X$ and $\varepsilon > 0$ there is λ_0 such that for $\lambda \geq \lambda_0$, $|f(x) - f_{\alpha_\lambda}(x)| < \varepsilon$. Since by Theorem 6.2, $f_{\alpha_\lambda}(x) \leq q(x)$ for all $\lambda \in \Lambda$, we obtain

$$f(x) < \varepsilon + f_{\alpha_\lambda}(x) \leq \varepsilon + q(x),$$

for all $\lambda \geq \lambda_0$. Hence, $f(x) \leq q(x)$. Consequently f is continuous from (X,q) to $(\mathbb{R},u)$ and $q^*(f) \leq 1$. We conclude that $f \in V_{1,\leq}^{X^*}$. Therefore $V_{1,\leq}^{X^*}$ is compact with respect to $\tau_{\text{pc}-\text{weak}*}|_{X^*}$.■

Remark 7.2 *Theorem 7.2 admits a direct proof without using explicitly Alaoglu's Theorem as we show in the following.*

Indeed, let $x \in X$. The interval $[-q(-x), q(x)]$ is a compact subset of $(\mathbb{R}, |.|)$. For each function $f \in V_{1,\leq}^{X^*}$ we have, by Theorem 6.2, that $f(x) \in [-q(-x), q(x)]$ for every $x \in X$.

Now, consider the product space $H := \Pi_{x\in X}[-q(-x), q(x)]$ endowed with the product topology. We can identify each function $f \in V_{1,\leq}^{X^*}$ with its range $(f(x))_{x\in X} \in H$. Moreover, a direct argument shows that the restriction of the product topology to the following subset of H,

$$\{(f(x))_{x\in X} : f \in V_{1,\leq}^{X^*}\},$$

coincides with the pc-weak topology of* $V_{1,\leq}^{X^*}$ *(see the classical proof of Alaoglu Theorem in [9] or [59]).*

As a consequence of Tychonov's theorem, the product space H *endowed with its product topology is compact. Now we just need to prove that* $\{(f(x))_{x\in X} : f \in V_{1,\leq}^{X^*}\}$ *is a compact subset of the product space. In order to do this, we will prove that it is closed. Fix the elements* $x, y \in X$*. Let us define the function* $\Psi_{x,y} : H \to \mathbb{R}$ *as*

$$\Psi_{x,y}(f) := f(x) + f(y) - f(x+y), \qquad f \in H.$$

This function is obviously continuous for the product topology, since its definition only depends on a finite subset of elements of X *-in fact, two-. If* $a \in \mathbb{R}$ *and* $x \in X$*, we can define in the same way the function*

$$\Phi_{a,x}(f) := af(x) - f(ax), \qquad f \in H,$$

that is also continuous. Now we define the set

$$A := (\cap_{x,y\in X}\Psi_{x,y}^{-1}(\{0\})) \cap (\cap_{a\in\mathbb{R},x\in X}\Phi_{a,x}^{-1}(\{0\})).$$

It is a closed subset, since it is the intersection of a family of closed subsets. Moreover, A *is clearly the representation of the unit ball* $V_{1,\leq}^{X^*}$ *via the range* $(f(x))_{x\in X}$ *of each function* f*. Therefore* $V_{1,\leq}^{X^*}$ *is compact.*

7.6 Applications. Reflexivity of Hausdorff asymmetric normed linear spaces

In this section we prove that the dual of the topological space $X^* - X^*$ endowed with the pc-weak* topology is the original space X when the asymmetric normed linear space X is Hausdorff. We generalize in this way the classical result for the weak* topology of normed spaces. We have shown that each element $x \in X$ defines a continuous map $x : X^* - X^* \to \mathbb{R}$ when we consider the pc-weak* topology on $X^* - X^*$. Thus, we just need to show that the functionals defined in this way are the only ones.

First, it is interesting to declare that there are asymmetric normed linear spaces that are Hausdorff and are not normable (see Chapter 2 or [22]). For normed spaces, Theorem 7.1 is already known (see for example Ch.II of [9]). An easy example of an asymmetric normed linear space that is Hausdorff but not normable is Example 2.1.

Theorem 7.3 *Let (X, q) be a Hausdorff asymmetric normed linear space. Let ϕ be a linear functional on $X^* - X^*$ which is continuous with respect to the pc-weak* topology of X^*. Then it can be identified with an element $x \in X$, i.e. there is an element $x \in X$ such that $\phi(f) = f(x)$ for every $f \in X^* - X^*$.*

Proof. By Proposition 7.4, we know that all the functionals defined by mean of the elements of X are continuous with respect to the pc-weak* topology of $X^* - X^*$. Thus, we just need to prove that we can find an element $x \in X$ for every functional $\phi : X^* - X^* \to \mathbb{R}$ satisfying the required property. Since ϕ is continuous, for every $\epsilon > 0$ there is a $\delta > 0$, and elements $x_1, ..., x_n$ such that, if $f \in W^{pc*}_{\delta, x_1, ..., x_n}(0)$, then $|\phi(f)| < \epsilon$. This means that the conditions $|f(x_1)| < \delta$,..., $|f(x_n)| < \delta$ implies $|\phi(f)| < \epsilon$.

Now suppose that the first r elements, $r \leq n$, of $x_1, ..., x_n$ define a basis. Let us show that, if $f \in X^* - X^*$ and all the images $f(x_1)$,...,$f(x_r)$ are equal to 0, then $\phi(f) = 0$. Suppose that f is a non zero element of $X^* - X^*$ and satisfies the above condition for the elements $x_1, ..., x_r$. The linearity of f shows that $f(x_i) = 0$ also for the rest of the elements of the sequence $x_1, ..., x_n$. Then, it belongs to the neighborhood $W^{pc*}_{\delta, x_1, ..., x_n}(0)$, and for each $\lambda \in \mathbb{R}$ the element $\lambda \phi$ also belongs to this subset. But then, if $\phi(f) \neq 0$, there is a λ such that $|\phi(\lambda f)| = |\lambda \phi(f)| > \epsilon$, which contradicts the continuity of ϕ.

Consider the finite dimensional subspace S of X generated by the elements $x_1, ..., x_r$. Since X is Hausdorff, Corollary 4.2 shows that S is isomorphic to a normed space, and then we can find linear functionals f_i on S such that $f_i(x_j) = 1$ if $i = j$, and 0 otherwise, which are continuous with respect to the topology induced by q. By the Hahn-Banach Theorem for asymmetric normed linear spaces obtained in [4], we can extend these functionals to the whole space X. We denote them by $\overline{f}_i$, $i = 1, ..., r$.

Take an element $g \in X^* - X^*$, and define the linear functional

$$g' = g - \sum_{i=1}^{r} g(x_i)\overline{f}_i \in X^* - X^*.$$

It is equal to 0 in S, since for each x_j, $j = 1, ..., r$, we have

$$g'(x_j) = g(x_j) - \sum_{i=1}^{r} g(x_i)\overline{f}_i(x_j) = g(x_j) - g(x_j) = 0.$$

Then, $\phi(g') = 0$, and

$$\phi(g) = \sum_{i=1}^{r} g(x_i)\phi(\overline{f}_i) = g(\sum_{i=1}^{r} \alpha_i x_i),$$

where $\alpha_i = \phi(\overline{f}_i)$, $i = 1, ..., r$. This gives the result, since ϕ is determined by the element $\sum_{i=1}^{r} \alpha_i x_i$. ■

Chapter 8

Sequence spaces and asymmetric norms in the theory of computational complexity

8.1 Introduction and preliminaries

Our purpose in this chapter consists in developing a robust mathematical model for the theory of computational complexity of algorithms and programs in the context of Theoretical Computer Science, by using the mathematical background of the preceding chapters

In [51] M. Schellekens introduced the complexity (quasi-metric) space as a part of the development of a topological foundation for the complexity analysis of programs and algorithms. In particular, he presented some applications of this theory to the complexity analysis of Divide&Conquer algorithms.

The complexity space ([51]) is the pair $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \{f \in (0, \infty]^{\omega} : \textstyle\sum_{n=0}^{\infty} 2^{-n}(1/f(n)) < \infty\},$$

and $d_{\mathcal{C}}$ is the quasi-metric defined on $\mathcal{C} \times \mathcal{C}$ by

$$d_{\mathcal{C}}(f,g) = \sum_{n=0}^{\infty} 2^{-n}[(\tfrac{1}{g(n)} - \tfrac{1}{f(n)}) \vee 0].$$

The dual complexity space was introduced in [44], where several quasi-metric properties of the complexity space which are interesting from a computational point of view are obtained via the analysis of its dual. Some motivations for the use of the dual space instead of the original complexity space are given in [44] p. 313. In particular, the structure of an asymmetric normed semilinear space provides a suitable setting to develop a consistent theory for the analysis of the dual complexity space ([46]) and, by other hand, the dual has a definite appeal, since in this context, it has a minimum $\perp$ which corresponds directly to the minimum of semantics domains. Moreover the dual complexity space can be directly used for the complexity analysis of algorithms where the running time of computing is the complexity measure (compare [51] Section 4, and [44] page 313).

The dual complexity space ([44]) is the pair $(\mathcal{C}^*, d_{\mathcal{C}^*})$, where

$$\mathcal{C}^* = \{f \in [0,\infty)^\omega : \sum_{n=0}^{\infty} 2^{-n} f(n) < \infty\},$$

and $d_{\mathcal{C}^*}$ is the quasi-metric defined on $\mathcal{C}^* \times \mathcal{C}^*$ by

$$d_{\mathcal{C}^*}(f,g) = \sum_{n=0}^{\infty} 2^{-n}[(g(n) - f(n)) \vee 0].$$

As is noted in [44], the inversion map $\Psi : \mathcal{C}^* \to \mathcal{C}$ is an isometry from $(\mathcal{C}^*, d_{\mathcal{C}^*})$ to $(\mathcal{C}, d_{\mathcal{C}})$.

Following M. Schellekens ([51], Section 4), the intuition behind the complexity distance between two functions $f, g \in \mathcal{C}$ is that $d_{\mathcal{C}}(f,g)$ measures relative progress made in lowering the complexity by replacing any program P with complexity function f by any program Q with complexity function g. Let $f, g \in \mathcal{C}^*$. As $d_{\mathcal{C}^*}(f,g) = d_{\mathcal{C}}(1/f, 1/g)$, we deduce that $d_{\mathcal{C}^*}(f,g)$ measures relative progress made in lowering the complexity by replacing g by f. In particular $d_{\mathcal{C}^*}(f,g) = 0$, with $f \neq g$, can be interpreted as g is "more efficient"than f.

Anyway, there are some algorithms which are time exponential. Several of these problems lead to a complexity analysis that cannot be performed using the dual complexity space. In fact, as the reader can check, an algorithm with running time $\mathcal{O}(\frac{2^n}{\sqrt{n}})$ generates the function f given by $f(n) = 2^n/\sqrt{n}$ for all $n \in \mathbb{N}$, which obviously does not belong to the dual complexity space $\mathcal{C}^*$ ([1] page 312). However, this function belongs to a generalized (p-norm) version of the dual complexity space (see Examples 8.1 and 8.2 below).

Motivated, in part, by this kind of examples, we here define and study several properties of the asymmetric normed linear space $(l_p, \|.\|_{+p})$ and the so-called dual p-complexity space (see Section 8.3 for definitions), which can be used for the complexity analysis of several exponential time algorithms. In particular the asymmetric norms defined on these duals provide a suitable interpretation in terms of running time. We observe that the dual p-complexity space is isometrically isomorphic to the positive cone of $(l_p, \|.\|_{+p})$ and show strong completeness (in the sense of [48]) of the dual p-complexity space. Finally a compactness result for upper bounded subsets of the dual p-complexity space is stated.

On the other hand, there is in the last years a renewed interest in automata of infinite objects due to their intimate relation with temporal and modal logics of programs. Thus, E.A. Emerson and C.S. Jutla ([14]) have successfully applied complexity of tree automata to obtain optimal deterministic exponential time algorithms in some important modal logics of programs, where by an exponential time algorithm we mean an algorithm with running time $\mathcal{O}(2^{P(n)})$, such that $P(n)$ is a polynomial with $P(n) > 0$ for all n. This running time corresponds to the function f given by $f(n) = 2^{P(n)}$ for all n, which does not belong to any dual p-complexity space whenever $P(n) \geq n$.

In Section 8.4 and subsequent we show that the supremum asymmetric norms that one can define in a natural way on certain sequence algebras provide an efficient tool to study those complexity functions that generate exponential time algorithms. In this direction, we construct a very general class of asymmetric normed linear spaces whose positive cones constitute a suitable setting for extending Schellekens' idea of complexity distance to the measure of improvements in complexity of exponential time algorithms. Furthermore, these positive cones are biBanach semialgebras which are isometrically isomorphic to the positive cone of the biBanach space $(l_\infty, \|.\|_{+\infty})$, where $\|\mathbf{x}\|_{+\infty} = \sup\{x_n \vee 0 : n \in \omega\}$ for each $\mathbf{x} := (x_n)_{n\in\omega} \in l_\infty$. Schellekens proved in [51] that Divide & Conquer algorithms induce contraction maps on the complexity space. In the last section, we will show that this fact also follows from our approach.

The main results of this chapter have been published in [24].

8.2 Some asymmetric norms on sequence spaces

It is proved in [46] that the dual complexity space is a semilinear subspace of an asymmetric normed linear space whose induced quasi-metric is bicomplete.

Let us now give some definitions for sequence spaces.

For $1 \leq p < \infty$, we will denote by l_p the set of infinite sequences $\mathbf{x} := (x_n)_{n\in\omega}$ of real numbers such that $\sum_{n=0}^{\infty} \mid x_n \mid^p < \infty$.

It is well known that $(l_p, \|.\|_p)$ is a Banach space, where $\| \,.\, \|_p$ is the norm on l_p defined by $\| \mathbf{x} \|_p = (\sum_{n=0}^{\infty} \mid x_n \mid^p)^{1/p}$ for all $\mathbf{x} \in l_p$.

We will split the norm $\| \,.\, \|_p$ as follows:

For each $x \in \mathbb{R}$, let x^+ be the nonnegative real number $x \vee 0$.

Fix $p \in [1, \infty)$. For each $\mathbf{x} := (x_n)_{n\in\omega} \in l_p$ define $\mathbf{x}^+ := (x_n^+)_{n\in\omega}$ and $\|\mathbf{x}\|_{+p} = \|\mathbf{x}^+\|_p$, i.e.

$$\|\mathbf{x}\|_{+p} = (\sum_{n=0}^{\infty} (x_n^+)^p)^{1/p}.$$

We will show that $\|.\|_{+p}$ is an asymmetric norm on l_p such that the norm $(\|.\|_{+p})^s$ is equivalent to $\|.\|_p$. To this end the following well-known relations will be useful.

Lemma 8.1 *For* $\mathbf{x} := (x_n)_{n\in\omega} \in l_p$, $\mathbf{y} := (y_n)_{n\in\omega} \in l_p$ *and* $a \in \mathbb{R}^+$ *the following statements hold:*

(a) $\mathbf{x} = \mathbf{x}^+ - (-\mathbf{x})^+$;

(b) $(a\mathbf{x})^+ = a\mathbf{x}^+$;

(c) $(x_n + y_n)^+ \leq x_n^+ + y_n^+$ *for all* $n \in \omega$.

Proposition 8.1 *(compare [18] Theorem 3.1).* *For each* $p \in [1, \infty)$, $\|.\|_{+p}$ *is an asymmetric norm on* l_p.

Proof. Fix $p \in [1, \infty)$. Let $\mathbf{x} := (x_n)_{n\in\omega} \in l_p$ such that $\|\mathbf{x}\|_{+p} = \|-\mathbf{x}\|_{+p} = \mathbf{0}$. Then $\mathbf{x}^+ = (-\mathbf{x})^+ = \mathbf{0}$ and by Lemma 8.1 (a), $\mathbf{x} = \mathbf{0}$. On the other hand, it is clear that $\|\mathbf{0}\|_{+p} = \mathbf{0}$.

Now let $a \in \mathbb{R}^+$, $\mathbf{x} := (x_n)_{n\in\omega} \in l_p$ and $\mathbf{y} := (y_n)_{n\in\omega} \in l_p$. Then

$\|(a\mathbf{x})\|_{+p} = \|(a\mathbf{x})^+\|_p = a\,\|\mathbf{x}\|_{+p}$, by Lemma 8.1 (b).

Finally,

$\|\mathbf{x} + \mathbf{y}\|_{+p} = \left\|(\mathbf{x} + \mathbf{y})^+\right\|_p \leq (\sum_{n=0}^{\infty}(x_n^+ + y_n^+)^p)^{1/p}$ by Lemma 8.1 (c), so

$\|\mathbf{x} + \mathbf{y}\|_{+p} \leq \|\mathbf{x}^+ + \mathbf{y}^+\|_p \leq \|\mathbf{x}^+\|_p + \|\mathbf{y}^+\|_p = \|\mathbf{x}\|_{+p} + \|\mathbf{y}\|_{+p}$. ■

Corollary 8.1 *For each* $p \in [1, \infty)$, $(l_p, \|.\|_{+p})$ *is an asymmetric normed linear space.*

Proposition 8.2 *For each* $p \in [1, \infty)$, $(\|\mathbf{x}\|_{+p})^s \leq \| \mathbf{x} \|_p \leq \|\mathbf{x}\|_{+p} + \|-\mathbf{x}\|_{+p}$, *whenever* $\mathbf{x} \in l_p$.

Proof. Fix $p \in [1, \infty)$. Let $\mathbf{x} := (x_n)_{n\in\omega} \in l_p$. Then, it is clear that

$(\|\mathbf{x}\|_{+p})^s = \max\{\|\mathbf{x}\|_{+p}, \|-\mathbf{x}\|_{+p}\} \leq \|\mathbf{x}\|_p$.

Finally, by Lemma 8.1 (a), we obtain

$\|\mathbf{x}\|_p = \left\|\mathbf{x}^+ - (-\mathbf{x})^+\right\|_p \leq \|\mathbf{x}^+\|_p + \left\|(-\mathbf{x})^+\right\|_p = \|\mathbf{x}\|_{+p} + \|-\mathbf{x}\|_{+p}$. ■

Corollary 8.2 *For each* $p \in [1, \infty)$, $(\|.\|_{+p})^s \leq \|.\|_p \leq 2(\|.\|_{+p})^s$. *Therefore* $(\|.\|_{+p})^s$ *and* $\|.\|_p$ *are equivalent norms in* l_p.

Corollary 8.3 *For each* $p \in [1, \infty)$, $(l_p, \|.\|_{+p})$ *is a biBanach space.*

Following [46], set $\mathcal{B}^* = \{f \in \mathbb{R}^\omega : \sum_{n=0}^{\infty} 2^{-n} \mid f(n) \mid < \infty\}$. Note that the dual complexity space $\mathcal{C}^*$ is the positive cone of $\mathcal{B}^*$. Furthermore it is clear that $l_1 \subsetneq \mathcal{B}^*$.

If for each $f, g \in \mathcal{B}^*$ and each $a \in \mathbb{R}$ we define $f + g$ and $a \cdot f$ in the usual pointwise way, then $(\mathcal{B}^*, +, \cdot)$ is a linear space (on $\mathbb{R}$), and we deduce that $(\mathcal{B}^*, q)$ is an asymmetric normed linear space, where $q(f) = \sum_{n=0}^{\infty} 2^{-n} f(n)^+$ for all $f \in \mathcal{B}^*$ ([46]).

It is proved in [46] Theorem 1, that actually $(\mathcal{B}^*, q)$ is a biBanach space, for which $\mathcal{C}^*$ is a semilinear subspace closed in the Banach space $(\mathcal{B}^*, q^s)$.

In order to obtain a general theory which implies the possibility of extending the notion of dual complexity for any $p > 1$, we introduce the following class of spaces.

For each $p \in [1, \infty)$ set

$$\mathcal{B}_p^* = \{f \in \mathbb{R}^\omega : \textstyle\sum_{n=0}^{\infty} (2^{-n} |f(n)|)^p < \infty\}.$$

If for each $f, g \in \mathcal{B}_p^*$ and each $a \in \mathbb{R}$ we define $f + g$ and $a \cdot f$ in the usual pointwise way, then it easily follows that $(\mathcal{B}_p^*, +, \cdot)$ is a linear space.

Now denote by q_p the nonnegative real valued function defined on $\mathcal{B}_p^*$ by

$$q_p(f) = (\textstyle\sum_{n=0}^{\infty} (2^{-n} f(n)^+)^p)^{1/p}.$$

For each $f \in \mathcal{B}_p^*$ let $\mathbf{x}_f := (2^{-n} f(n))_{n \in \omega}$. Then $\mathbf{x}_f \in l_p$ and we have

$$q_p(f) = \left\| \mathbf{x}_f^+ \right\|_{+p}.$$

Since, by Proposition 8.1, $\|.\|_{+p}$ is an asymmetric norm on l_p, then q_p is an asymmetric norm on $\mathcal{B}_p^*$ and consequently $(\mathcal{B}_p^*, q_p)$ is an asymmetric normed linear space.

Observe that, in particular, $(\mathcal{B}_1^*, q_1)$ is exactly the biBanach space $(\mathcal{B}^*, q)$ defined above.

The above simple but useful relationship between q_p and $\|.\|_{+p}$ actually permits us to show that $(\mathcal{B}_p^*, q_p)$ and $(l_p, \|.\|_{+p})$ are isometrically isomorphic as our next result shows. (Let us recall that two (asymmetric) normed linear spaces (X, q_X) and (Y, q_Y) are isometrically isomorphic if there is a linear map F from X onto Y such that $q_Y(F(x)) = q_X(x)$ for all $x \in X$.)

Fix $p \in [1, \infty)$. Define a map $\phi : \mathcal{B}_p^* \to l_p$ by the rule:

$$(\phi(f))(n) = 2^{-n} f(n).$$

for all $f \in \mathcal{B}_p^*$ and $n \in \omega$. Thus $\phi(f) = \mathbf{x}_f$, where $\mathbf{x}_f$ is the element of l_p defined above. We then have the following result.

Proposition 8.3 *ϕ is a linear bijection between $(\mathcal{B}_p^*, q_p)$ and $(l_p, \|.\|_{+p})$ such that $q_p(f) = \|\phi(f)\|_{+p}$ for all $f \in \mathcal{B}_p^*$.*

Proof. We first show that ϕ is onto. Indeed, let $\mathbf{x} = (x_n)_{n\in\omega}$ be an element of l_p. Define $f \in \mathbb{R}^\omega$ by $f(n) = 2^n x_n$ for all $n \in \omega$. Then $f \in \mathcal{B}_p^*$ since

$$(\textstyle\sum_{n=0}^{\infty} \mid 2^{-n} f(n) \mid^p)^{1/p} = (\textstyle\sum_{n=0}^{\infty} \mid x_n \mid^p)^{1/p} = \|\mathbf{x}\|_p .$$

Furthermore $(\phi(f))(n) = x_n$ for all $n \in \omega$, so $\phi(f) = \mathbf{x}$.

Clearly ϕ is one-to-one and hence it is a bijection.

On the other hand, since for each $f, g \in \mathcal{B}_p^*$ and each $a, b \in \mathbb{R}$ we have

$$(\phi(af+bg))(n) = 2^{-n}(af(n)+bg(n)) = a\phi(f(n)) + b\phi(g(n))$$

whenever $n \in \omega$, we deduce that $\phi(af+bg) = a\phi(f) + b\phi(g)$, and thus ϕ is linear.

Finally, for each $f \in \mathcal{B}_p^*$ we have

$$\|\phi(f)\|_{+p} = \|\mathbf{x}_f\|_{+p} = q_p(f).$$

The proof is complete. ■

Corollary 8.4 *$(\mathcal{B}_p^*, q_p)$ and $(l_p, \|.\|_{+p})$ are isometrically isomorphic.*

Corollary 8.5 *$(\mathcal{B}_p^*, q_p)$ is a biBanach space.*

8.3 The dual *p*-complexity space

For each $p \in [1,\infty)$ consider the biBanach space $(\mathcal{B}_p^*, q_p)$ defined in the preceding section and set

$$\mathcal{C}_p^* = \{f \in \mathcal{B}_p^* : f(n) \geq 0 \text{ for all } n \in \omega\}.$$

The restriction of the asymmetric norm q_p to $\mathcal{C}_p^*$ will be also denoted by q_p if no confusion arises. Then, the proof of the following result is straightforward and so is omitted.

Proposition 8.4 *For each $p \in [1,\infty)$, $(\mathcal{C}_p^*, q_p)$ is an asymmetric normed semilinear space which is closed in the Banach space $(\mathcal{B}_p^*, (q_p)^s)$.*

In the following the asymmetric normed semilinear space $(\mathcal{C}_p^*, q_p)$ will be called *the dual p-complexity space.*

As in the case $p = 1$ (see Section 1), the fact that $d_{q_p}(f, g) = 0$, with $f \neq g$, can be interpreted as g is more efficient than f. Furthermore $q_p(f) = d_{q_p}(0, f)$ measures relative progress made in lowering complexity by replacing f by the "optimal" complexity function 0, assuming that the complexity measure is the running time of computing, of course.

Example 8.1 *Consider the World Series Odds problem. Suppose two teams, A and B are playing a match to see who is the first to win n games. Let $P(i, j)$ be the probability that if A needs i games to win, and B needs j games , that A will eventually win the match. To compute $P(i, j)$ it can be used a recursive algorithm in two variables with running time $\mathcal{O}\left(2^n/\sqrt{n}\right)$(see [1] page 312, for more details). As we indicate in Section 1 this running time induces the function $f \in \mathcal{C}_p^*$ for every $p > 2$, given by $f(0) = 0$ and $f(n) = 2^n/\sqrt{n}$ for all $n \in \mathbb{N}$. Obviously $f \notin \mathcal{C}_p^*$ for $p \leq 2$.*

Example 8.2 *Suppose a problem with running time $\mathcal{O}(2^n)$ (see [1]). In case we had always the same number of processors than the size of the instance of such a problem, say n, the running time is reduced to $\mathcal{O}(2^n/n)$ in the ideal case of 100% parallel processing efficiency. As in Example 8.1 the situation leads to a function $f \in \mathcal{C}_p^*$ for every $p > 1$, given by $f(0) = 0$ and $f(n) = 2^n/n$ for all $n \in \mathbb{N}$. Obviously $f \notin \mathcal{C}^*$.*

Note that the natural definition of the asymmetric norm for the case of running time $\mathcal{O}(2^n/n)$ would be the "infinite" version of q_p, i.e. $q_\infty(f) = sup\{2^{-n}f(n) : n \in \omega\}$. This case will discuss later in this chapter.

In Proposition 8.5 below we extend Proposition 8.3 and Corollary 8.4 to the dual p-complexity space and the positive cone of l_p.

For each $p \in [1, \infty)$ denote by l_p^+ the positive cone of l_p. Thus

$$l_p^+ = \{\mathbf{x}^+ : \mathbf{x} \in l_p\}.$$

It is immediate to see that $(l_p^+, \|.\|_{+p})$ is an asymmetric normed semilinear space which is closed in the Banach space $(l_p, (\|.\|_{+p})^s)$, where the restriction of $\|.\|_{+p}$ to l_p^+ is also denoted by $\|.\|_{+p}$.

Furthermore, it is clear that the restriction to $\mathcal{C}_p^*$ of the map $\phi : \mathcal{B}_p^* \to l_p$, defined in Section 2, is a linear bijection between the dual p-complexity space $(\mathcal{C}^*, q_p)$ and the positive cone $(l_p^+, \|.\|_{+p})$ which preserves asymmetric norms.

Hence, considering the notion of an isometric isomorphism between asymmetric normed semilinear spaces, we deduce from the above observations the following result.

Proposition 8.5 *For each $p \in [1, \infty)$, $(\mathcal{C}_p^*, q_p)$ and $(l_p^+, \|.\|_{+p})$ are isometrically isomorphic.*

Remark 8.1 *Although $(\mathcal{C}_p^*, q_p)$ and $(l_p^+, \|.\|_{+p})$ are isometrically isomorphic, the dual p-complexity space has the advantage that it allows us to interpret as convergent, with respect to q_p, for instance programs whose computing-time is constant (or at least it has a polynomial growth). However such programs provide series that are clearly divergent in $(l_p^+, \|.\|_{+p})$. On the other hand, note that the functions constructed in Examples 8.1 and 8.2 are in $\mathcal{C}_p^*$ for $p > 2$ and $p > 1$, respectively, but they are not in l_p.*

S.G. Matthews introduced in [36] the notion of a weightable quasi-metric space as a part of the study of denotational semantics of dataflow networks.

A quasi-metric space (X, d) is called weightable if there is a nonnegative real valued function w on X such that

$$d(x, y) + w(x) = d(y, x) + w(y),$$

for all $x, y \in X$. The function w is called a weighting function for d and the quasi-metric d is called weightable.

Both the complexity space and the dual complexity space are weightable, with weighting functions $w_{\mathcal{C}}$ and $w_{\mathcal{C}^*}$ defined by $w_{\mathcal{C}}(f) = \sum_{n=0}^{\infty} 2^{-n}(1/f(n))$ for all $f \in \mathcal{C}$, and $w_{\mathcal{C}^*}(f) = \sum_{n=0}^{\infty} 2^{-n} f(n)$ for all $f \in \mathcal{C}^*$, respectively ([51], [44]).

Note that the weighting function $w_{\mathcal{C}^*}$ coincides with the asymmetric norm q_1 on $\mathcal{C}_1^*$.

We say that an asymmetric normed (semi)linear space (E, q) is *weightable* if (E, d_q) is a weightable quasi-metric space.

We want to show that the dual p-complexity space is weightable only for $p = 1$. To this end, we will use the following technical lemma.

Lemma 8.2 *The real valued function u defined on $(0, \infty)$ by $u(p) = 3^{1/p} - 2^{1/p}$ is strictly decreasing.*

Proof. It suffices to see that $u'(p) < 0$ for all $p > 0$. Indeed, we have

$$u'(p) = p^{-2}(2^{1/p} \log 2 - 3^{1/p} \log 3).$$

Since for each $p > 0$, $(3/2)^{1/p} > 1 > \log 2/\log 3$, it follows that $3^{1/p} \log 3 > 2^{1/p} \log 2$, so $u'(p) < 0$ for all $p > 0$. ■

Theorem 8.1 *The dual p-complexity space is weightable if and only if $p = 1$.*

Proof. As we have indicate above the dual *(1-)* complexity space is weightable. Conversely, suppose that $(\mathcal{C}_p^*, q_p)$ is weightable via the weighting function w on $\mathcal{C}_p^*$. Then for each $f, g \in \mathcal{C}_p^*$,

$$w(f) + q_p(g - f) = w(g) + q_p(f - g),$$

and

$$w(f) + q_p(-f) = w(\mathbf{0}) + q_p(f) \quad \text{and} \quad w(g) + q_p(-g) = w(\mathbf{0}) + q_p(g).$$

Since $q_p(-f) = q_p(-g) = 0$, it follows that $w(\mathbf{0}) = w(f) - q_p(f) = w(g) - q_p(g)$, and thus

$$q_p(f) + q_p(g - f) = q_p(g) + q_p(f - g).$$

Now define $f, g : \omega \to \mathbb{R}^+$ by $f(n) = 2^{n-(n/p)}$ for all $n \in \omega$, and $g(n) = 0$ for n odd and $g(n) = 2^{n-(n/p)}$ for n even. Clearly $f, g \in \mathcal{C}_p^*$ with $q_p(f) = 2^{1/p}$ and $q_p(g) = (4/3)^{1/p}$. Moreover, $q_p(g - f) = 0$ and $q_p(f - g) = (2/3)^{1/p}$. So we obtain $2^{1/p} = (4/3)^{1/p} + (2/3)^{1/p}$, and, hence, $2^{1/p}3^{1/p} = 2^{1/p}(2^{1/p} + 1)$, i.e. $3^{1/p} - 2^{1/p} = 1$. By Lemma 8.2, this equality only holds when $p = 1$. We conclude that $(\mathcal{C}_p^*, q_p)$ is weightable only for $p = 1$. ■

Remark 8.2 *It is known that $(\mathcal{B}_1^*, q_1)$ is not weightable ([46]). This observation, joint with Theorem 8.1 and the obvious fact that every subspace of a weightable quasi-metric space is weightable, shows that for each $p > 1$, the biBanach space $(\mathcal{B}_p^*, q_p)$ is not weightable.*

The theory of Smyth completable quasi-metric spaces provides an efficient setting to give a topological foundation for many kinds of spaces which arise naturally in several fields of Theoretical Computer Science ([36], [44], [46], [51], [53], [54], etc.).

A quasi-metric space (X, d) is Smyth completable if and only if every left K-Cauchy sequence in (X, d) is a Cauchy sequence in (X, d^s) ([32], [52]). (Let us recall that a sequence $(x_n)_{n \in \mathbb{N}}$ in (X, d) is left K-Cauchy ([39]) provided that for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $k \leq n \leq m$.)

A quasi-metric space (X, d) is Smyth complete if and only if every left K-Cauchy sequence in (X, d) has a $T(d^s)$-limit point ([32], [52]).

It immediately follows that a quasi-metric space is Smyth complete if and only if it is bicomplete and Smyth completable.

It was proved in [28] that every weightable quasi-metric space is Smyth completable, so every weightable bicomplete quasi-metric space is Smyth complete.

We say that an asymmetric normed semilinear space (E, q) is bicomplete (Smyth complete) if (E, d_q) is a bicomplete (Smyth complete) quasi-metric space.

Combining Corollary 3.2 with the second statement of Proposition 8.4, we obtain that the dual p-complexity space is bicomplete. In particular, the dual (*1-*) complexity space is Smyth complete because it is weightable. However, it is possible to prove that for each $p > 1$, the dual p-complexity space is Smyth complete. Actually, we will show that it admits a stronger kind of completeness, namely, strong completeness in the sense of [48].

A filter $\mathcal{F}$ on a quasi-metric space (X, d) is called a *Cauchy filter* if for each $n \in \mathbb{N}$ there is $x \in X$ such that $B_d(x, 2^{-n}) \in \mathcal{F}$ ([19]).

A quasi-metric space (X, d) is called *strongly complete* if every Cauchy filter on (X, d) has a $T(d^s)$-cluster point ([48]).

Several properties of strongly complete quasi-metric spaces were discussed in [48]. In particular, every strongly complete quasi-metric space is Smyth complete, but the converse does not hold.

Let u be the upper quasi-metric on $\mathbb{R}$ defined by $u(r) = r \vee 0$, $r \in \mathbb{R}$. Then we can define the quasi-metric u_P of pointwise convergence as the quasi-metric on $\mathbb{R}^\omega \times \mathbb{R}^\omega$ given by $u_P(f,g) = \sum_{n=0}^{\infty} 2^{-n} \min\{1, u(f(n), g(n))\}$ for all $f, g \in \mathbb{R}^\omega \times \mathbb{R}^\omega$.

Theorem 8.2 *For each $p \in [1, \infty)$, the dual p-complexity space $(\mathcal{C}_p^*, q_p)$ is strongly complete.*

Proof. Fix $p \in [1, \infty)$. Let $\mathcal{F}$ be a Cauchy filter on $(\mathcal{C}_p^*, q_p)$. Then, for each $k \in \mathbb{N}$ there is a $f_k \in \mathcal{C}_p^*$ such that $F_k \in \mathcal{F}$, where $F_k = \{f \in \mathcal{C}_p^* : q_p(f - f_k) < 2^{-3k}\}$.

Therefore, for each $f \in F_1$,

$$\sum_{n=0}^{\infty}(2^{-n}(f(n) - f_1(n))^+)^p < 2^{-3p},$$

so $f(n) < f_1(n) + 2^{n-3}$ for all $f \in F_1$ and $n \in \omega$.

Denote by K the compact space $\prod_{n=0}^{\infty}[0, f_1(n)+2^{n-3}]$, and by $\overline{F \cap K}$ the closure of $F \cap K$ in K for all $F \in \mathcal{F}$. (Note that for each $F \in \mathcal{F}$, $F \cap K \neq \emptyset$ because $F_1 \subseteq K$.)

Next we show that for each $F \in \mathcal{F}$, $(\overline{F \cap K}) \cap (\bigcap_{k=1}^{\infty} \overline{F_k \cap K}) \neq \emptyset$.

Indeed, fix $F \in \mathcal{F}$. For each $k \in \mathbb{N}$ there is $g_k \in F \cap (\bigcap_{j=1}^{k} F_j)$, so $(g_k)_{k\in\mathbb{N}}$ is a sequence in $F_1 \subseteq K$ and, thus, it clusters to some $g \in K$ with respect to $T((u_P)^s)$. Therefore $g \in (\overline{F \cap K}) \cap (\bigcap_{k=1}^{\infty} \overline{F_k \cap K})$.

In particular, it follows from the above observation that $\bigcap_{k=1}^{\infty} \overline{F_k \cap K}$ is a nonempty compact subset of K, so the filter base $\{(\overline{F \cap K}) \cap (\bigcap_{k=1}^{\infty} \overline{F_k \cap K}) : F \in \mathcal{F}\}$ clusters to some $h \in \bigcap_{k=1}^{\infty} \overline{F_k \cap K}$ with respect to $T((u_P)^s)$. (Note that $h(n) \geq 0$ for all $n \in \omega$).

Now we want to show that $h \in \mathcal{C}_p^*$ and that $\mathcal{F}$ clusters to h with respect to the metric induced by the norm $(q_p)^s$. Thus $(\mathcal{C}_p^*, q_p)$ will be strongly complete.

Suppose that $h \notin \mathcal{C}_p^*$. Then, for each $j \in \mathbb{N}$ there is an $m_j \in \omega$ such that $j^p < \sum_{n=0}^{m_j}(2^{-n}h(n))^p$. Since $h \in \overline{F_1 \cap K}$, there exists $g_j \in F_1$ such that $\mid h(n) - g_j(n) \mid < 2^{-j}$ for $n = 0, 1, ..., m_j$. So $h(n) < g_j(n) + 2^{-j}$ for $n = 0, 1, ..., m_j$, and thus

$$j^p < \sum_{n=0}^{m_j}(2^{-n}(g_j(n) + 2^{-j}))^p < \sum_{n=0}^{\infty}(2^{-n}(g_j(n) + v_j(n)))^p = (q_p(g_j + v_j))^p,$$

where v_j is the constant function in $\mathcal{C}_p^*$ defined by $v_j(n) = 2^{-j}$ for all $n \in \omega$. Hence

$$j < q_p(g_j + v_j) \leq q_p(g_j) + q_p(v_j) = q_p(g_j) + 2^{-(j-1)} \leq q_p(g_j) + 1$$

for all $j \in \mathbb{N}$. Since $q_p(g_j - f_1) < 2^{-3}$, it follows that $q_p(g_j) < q_p(f_1) + 2^{-3}$. So, for each $j \in \mathbb{N}$,

$$j < q_p(f_1) + (1 + 2^{-3}),$$

which contradicts the fact that $q_p(f_1) < \infty$. Consequently $h \in \mathcal{C}_p^*$.

Finally, we will prove that $\mathcal{F}$ clusters to h with respect to the metric induced by $(q_p)^s$.

Fix $k \in \mathbb{N}$ and $F \in \mathcal{F}$. Since h and f_k are in $\mathcal{C}_p^*$, there is $n_0 \in \mathbb{N}$ such that

$$(1) \quad \textstyle\sum_{n=n_0}^{\infty}(2^{-n}h(n))^p < 2^{-2kp} \quad \text{and} \quad \textstyle\sum_{n=n_0}^{\infty}(2^{-n}f_k(n))^p < 2^{-3kp}.$$

On the other hand, since $h \in \overline{F \cap F_k \cap K}$, there is $f \in F \cap F_k$ such that

$$\textstyle\sum_{n=0}^{n_0-1}(2^{-n}\,|f(n) - h(n)|)^p < 2^{-2kp}.$$

We want to show that $q_p^s(f - h) < 2^{-k}$.

To this end, let h', f_k' and f' be the functions in $\mathcal{C}_p^*$ defined by $h'(n) = h(n)$, $f_k'(n) = f_k(n)$ and $f'(n) = f(n)$ whenever $n \geq n_0$, and $h'(n) = f_k'(n) = f'(n) = 0$ whenever $n < n_0$.

Note that, by the inequalities (1), it follows that $q_p(h') < 2^{-2k}$ and $q_p(f_k') < 2^{-3k}$. Furthermore $q_p(-h') = q_p(-f_k') = 0$ because $h'(n) \geq 0$ and $f_k'(n) \geq 0$ for all $n \in \omega$.

Then, we have

$$(\textstyle\sum_{n=n_0}^{\infty}(2^{-n}(f-h)(n)^+)^p)^{1/p} = q_p(f' - h') \leq q_p(f') + q_p(-h') = q_p(f'),$$

and, on the other hand,

$$q_p(f' - f_k') = (\textstyle\sum_{n=n_0}^{\infty}(2^{-n}(f - f_k)(n)^+)^p)^{1/p} \leq q_p(f - f_k) < 2^{-3k}.$$

So

$$q_p(f') < q_p(f_k') + 2^{-3k} < 2^{-3k} + 2^{-3k} \leq 2^{-2k}.$$

Therefore

$$q_p(f-h) = (\sum_{n=0}^{\infty}(2^{-n}(f-h)(n)^+)^p)^{1/p}$$

$$= (\sum_{n=0}^{n_0-1}(2^{-n}(f-h)(n)^+)^p + \sum_{n=n_0}^{\infty}(2^{-n}(f-h)(n)^+)^p)^{1/p}$$

$$< (2^{-2kp} + (q_p(f'))^p)^{1/p} < (2^{-2kp} + 2^{-2kp})^{1/p} \leq 2^{-k}.$$

It remains to show that $q_p(h-f) < 2^{-k}$.

Observe that $q_p(h'-f') \leq q_p(h') + q_p(-f') = q_p(h') < 2^{-2k}$. Thus

$$q_p(h-f) = (\sum_{n=0}^{n_0-1}(2^{-n}(h-f)(n)^+)^p + \sum_{n=n_0}^{\infty}(2^{-n}(h-f)(n)^+)^p)^{1/p}$$

$$< (2^{-2kp} + (q_p(h'-f'))^p)^{1/p} < (2^{-2kp} + 2^{-2kp})^{1/p} \leq 2^{-k}.$$

We conclude that $(q_p)^s(f-h) < 2^{-k}$. Hence $(\mathcal{C}_p^*, q_p)$ is strongly complete. ■

For an arbitrary Tychonoff topological space X we denote, as usual, by $C_p(X)$ the space of all continuous real valued functions on X with the topology of pointwise convergence.

The celebrated Grothendieck theorem ([25]) states that if X is a Tychonoff countably compact space and A is a subset of $C_p(X)$ such that every infinite subset of A has a limit point in $C_p(X)$, then the closure of A is compact in $C_p(X)$. Asanov and Velichko ([8]) have obtained the following generalization of Grothendieck's theorem: if X is a Tychonoff countably compact space, then the closure in $C_p(X)$ of every bounded subset A of $C_p(X)$ is compact. In our next theorem we extend Asanov-Velichko's theorem to the dual p-complexity space $(\mathcal{C}_p^*, q_p)$.

Following [47], a subset A of a topological space X is called upper bounded if every upper semicontinuous real valued function on X is upper bounded on A.

Lemma 8.3 *([47]). A subset A of a quasi-metrizable space X is upper bounded if and only if every sequence in A has a cluster point in X.*

Theorem 8.3 *Let A be an upper bounded subset of the dual p-complexity space $(\mathcal{C}_p^*, q_p)$. Then the closure of A in $(\mathcal{C}_p^*, (q_p)^{-1})$ is compact in $(\mathcal{C}_p^*, (q_p)^s)$.*

Proof. Denote by $\overline{A}$ the closure of A in $(\mathcal{C}_p^*, (q_p)^{-1})$. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\overline{A}$. Then there is a sequence $(g_k)_{k\in\mathbb{N}}$ in A such that $(q_p)^{-1}(g_k - f_k) \to 0$. By Lemma 8.3, there are a subsequence $(g_{k_j})_{j\in\mathbb{N}}$ of $(g_k)_{k\in\mathbb{N}}$ and a $g \in \mathcal{C}_p^*$ such that $q_p(g_{k_j} - g) \to 0$. Hence $q_p(f_{k_j} - g) \to 0$. Then, the filter generated by $\{\{f_{k_j} : j \geq m\} : m \in \mathbb{N}\}$ is a Cauchy filter on $(\mathcal{C}_p^*, q_p)$, and by Theorem 8.2, there is $f \in \mathcal{C}_p^*$ which is a cluster point of $(f_{kj})_{j\in\mathbb{N}}$, and thus of $(f_k)_{k\in\mathbb{N}}$, in $(\mathcal{C}_p^*, (q_p)^s)$. Obviously $f \in \overline{A}$. We conclude that $\overline{A}$ is compact in $(\mathcal{C}_p^*, (q_p)^s)$. ■

Corollary 8.6 *Let A be an upper bounded subset of the dual p-complexity space $(\mathcal{C}_p^*, q_p)$. Then the closure of A in $(\mathcal{C}_p^*, (q_p)^s)$ is compact in $(\mathcal{C}_p^*, (q_p)^s)$.*

8.4 The supremum asymmetric norm on sequence algebras

In the precedent sections we have seen that the complexity analysis of algorithms with running time $\mathcal{O}(2^n/n^r)$, $0 < r \leq 1$, cannot be performed via the dual complexity space. This is the reason because we have introduced ([24]) the so-called dual p-complexity space ($p \geq 1$), which provides, for $p > 1$, an appropriate framework to discuss complexity functions generating this kind of algorithms. In particular, it has been shown that the dual p-complexity space is an asymmetric normed semilinear space which is isometrically isomorphic to the positive cone of $(l_p, \|.\|_{+p})$.

Here, motivated by the work of E.A. Emerson and C.S. Jutla ([14]) we present the precise context that will be used in order to obtain a robust mathematical model for discussing those complexity functions that generate exponential time algorithms.

We start by recalling some pertinent concepts.

By an algebra we mean a linear space E (on $\mathbb{R}$) with a binary (multiplicative) operation that is commutative, has identity element and satisfies for all $x, y, z \in E$

and $a \in \mathbb{R}$ the following conditions: $x(yz) = (xy)z$, $x(y+z) = xy + xz$, and $a(xy) = (ax)y = (ay)x$.

A (n asymmetric) normed algebra is an algebra E with a (n asymmetric) norm $\|.\|$ satisfying $\|xy\| \leq \|x\| \, \|y\|$ for all $x, y \in E$. By a Banach algebra is meant a normed algebra that is also a Banach space, and by a biBanach algebra is meant an asymmetric normed algebra that is also a biBanach space.

As usual we denote by l_∞ the algebra consisting of all bounded infinite sequences of real numbers.

It is well known that $(l_\infty, \|.\|_\infty)$ is a Banach algebra for the usual multiplication operation on l_∞, where $\|.\|_\infty$ is the supremum norm on l_∞, i.e. $\|\mathbf{x}\|_\infty = \sup\{| x_n |: n \in \omega\}$ for all $\mathbf{x} := (x_n)_{n\in\omega} \in l_\infty$.

As in the l_p-case (see Section 8.2) we may split the norm $\|.\|_\infty$ as follows:

For each $\mathbf{x} := (x_n)_{n\in\omega} \in l_\infty$ define $\|\mathbf{x}\|_{+\infty} = \|\mathbf{x}^+\|_\infty$, i.e. $\|\mathbf{x}\|_{+\infty} = \sup\{x_n \vee 0 : n \in \omega\}$.

It is immediate to see that $\|.\|_{+\infty}$ is an asymmetric norm on l_∞.

In addition, we have the following facts.

Proposition 8.6 $(\|.\|_{+\infty})^s = \|.\|_\infty$ *on* l_∞.

Proof. Let $\mathbf{x} := (x_n)_{n\in\omega} \in l_\infty$. It is clear that $\|\mathbf{x}\|_{+\infty} \leq \|\mathbf{x}\|_\infty$ and $\|-\mathbf{x}\|_{+\infty} \leq \|\mathbf{x}\|_\infty$.

On the other hand, for each $\varepsilon > 0$ there is $k \in \omega$ such that

$$\|\mathbf{x}\|_\infty < \varepsilon + | x_k | = \varepsilon + (x_k \vee (-x_k)) \leq \varepsilon + (\|\mathbf{x}\|_{+\infty} \vee \|-\mathbf{x}\|_{+\infty}).$$

We conclude that $(\|\mathbf{x}\|_{+\infty})^s = \|\mathbf{x}\|_\infty$. ■

Corollary 8.7 *$(l_\infty, \|.\|_{+\infty})$ is a biBanach space.*

Example 8.3 *Note that $(l_\infty, \|.\|_{+\infty})$ is a not an asymmetric normed algebra. Indeed, let $\mathbf{x} := (x_n)_{n\in\omega} \in l_\infty$ with $x_n = -1$ for all n. Clearly $\|\mathbf{xx}\|_{+\infty} = 1$. However $\|\mathbf{x}\|_{+\infty} = 0$.*

For each polynomial $P(n)$, with $P(n) > 0$ for all $n \in \omega$, define

$$\mathcal{B}^*_{P(n),\infty} := \{f \in \mathbb{R}^\omega : \sup\{2^{-P(n)} \mid f(n) \mid : n \in \omega\} < \infty\}.$$

It easily follows that $\mathcal{B}^*_{P(n),\infty}$ is a linear space for the usual pointwise operations.

Observe that, in particular, $\mathcal{B}^*_{n,\infty} = \bigcap_{P(n)>n} \mathcal{B}_{P(n),\infty}$, and $\mathcal{C}^*_p \subsetneq \mathcal{B}^*_p \subsetneq \mathcal{B}^*_{n,\infty}$ for all $p \geq 1$.

Now define a binary operation $\star$ on $\mathcal{B}^*_{P(n),\infty}$ as follows: For each $f, g \in \mathcal{B}^*_{P(n),\infty}$ let $f \star g$ be the element of $\mathcal{B}^*_{P(n),\infty}$ given by the rule

$$(f \star g)(n) = 2^{-P(n)} f(n) g(n).$$

An easy computation shows that, equipped with the operation $\star$, $\mathcal{B}^*_{P(n),\infty}$ is an algebra with identity element the function $e : \omega \to \mathbb{R}$ given by $e(n) = 2^{P(n)}$ for all n.

Next denote by $q_{P(n),\infty}$ the nonnegative real valued function defined on $\mathcal{B}^*_{P(n),\infty}$ by

$$q_{P(n),\infty}(f) = \sup\{2^{-P(n)} f(n)^+ : n \in \omega\}.$$

For each $f \in \mathcal{B}^*_{P(n),\infty}$ let $\mathbf{x}_f := (2^{-P(n)} f(n))_{n\in\omega}$. Then $\mathbf{x}_f \in l_\infty$ and we have

$$q_{P(n),\infty}(f) = \|\mathbf{x}_f\|_{+\infty}.$$

Since $\|.\|_{+\infty}$ is an asymmetric norm on l_∞ it follows that $q_{P(n),\infty}$ is an asymmetric norm on $\mathcal{B}^*_{P(n),\infty}$ and consequently $(\mathcal{B}^*_{P(n),\infty}, q_{P(n),\infty})$ is an asymmetric normed linear space.

We will show that this space is isometrically isomorphic to $(l_\infty, \|.\|_{+\infty})$.

To this end define a map $\phi : \mathcal{B}^*_{P(n),\infty} \to l_\infty$ by the rule:

$$(\phi(f))(n) = 2^{-P(n)} f(n),$$

for all $f \in \mathcal{B}^*_{P(n),\infty}$ and $n \in \omega$. Thus $\phi(f) = \mathbf{x}_f$, where $\mathbf{x}_f$ is the element of l_∞ defined above. We then have the following result. (Let us recall that a map φ from an algebra X to an algebra Y is a homomorphism provided that φ is a linear map such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in X$).

Proposition 8.7 *ϕ is a bijective homomorphism between $(\mathcal{B}^*_{P(n),\infty}, q_{P(n),\infty})$ and $(l_\infty, \|.\|_{+\infty})$ such that $q_{P(n),\infty}(f) = \|\phi(f)\|_{+\infty}$ for all $f \in \mathcal{B}^*_{P(n),\infty}$.*

Proof. We first show that ϕ is bijective.

Suppose that $\phi(f) = \phi(g)$. Then $2^{-P(n)} f(n) = 2^{-P(n)} g(n)$ for all $n \in \omega$, so $f = g$. Thus ϕ is one-to-one.

Now let $\mathbf{x} := (x_n)_{n\in\omega} \in l_\infty$. Then the function f defined by $f(n) = 2^{P(n)} x_n$ for all $n \in \omega$, satisfies $\phi(f) = \mathbf{x}$. Hence ϕ is onto.

We conclude that ϕ is bijective.

In order to see that ϕ is an homomorphism, let $f, g \in \mathcal{B}^*_{P(n),\infty}$ and let $a, b \in \mathbb{R}$. Then

$$\phi(af + bg)(n) = 2^{-P(n)}(af(n) + bg(n)) = a\phi(f)(n) + b\phi(g)(n),$$

for all $n \in \omega$. Therefore ϕ is linear.

Moreover $\phi(f \star g)(n) = 2^{-P(n)}(f \star g)(n) = 2^{-2P(n)} f(n)g(n) = \phi(f)(n)\phi(g)(n)$ for all $n \in \omega$, and thus $\phi(f \star g) = \phi(f)\phi(g)$.

We have shown that ϕ is a homomorphism.

Finally, given $f \in \mathcal{B}^*_{P(n),\infty}$ we obtain

$$\|\phi(f)\|_{+\infty} = \|\mathbf{x}_f\|_{+\infty} = q_{P(n),\infty}(f),$$

which concludes the proof.■

Corollary 8.8 *$(\mathcal{B}^*_{P(n),\infty}, q_{P(n),\infty})$ and $(l_\infty, \|.\|_{+\infty})$ are isometrically isomorphic.*

Corollary 8.9 *$(\mathcal{B}^*_{P(n),\infty}, q_{P(n),\infty})$ is a biBanach space.*

8.5 The $\boldsymbol{sup}_{P(n)}$-complexity space

By a semialgebra we mean a semilinear space E (on $\mathbb{R}^+$) with a binary (multiplicative) operation that is commutative, has identity element and satisfies for all $x, y, z \in E$ and $a \in \mathbb{R}^+$ the following conditions: $x(yz) = (xy)z$, $x(y+z) = xy + xz$, and $a(xy) = (ax)y = (ay)x$.

By an asymmetric normed semialgebra we mean an asymmetric normed semilinear space $(F, \|.\|_F)$ such that F is a semialgebra satisfying $\|xy\|_F \leq \|x\|_F \|y\|_F$ for all $x, y \in F$. If, in addition, $(F, \|.\|_F)$ is a biBanach semilinear space, we say that $(F, \|.\|_F)$ is a biBanach semialgebra.

Two asymmetric normed semialgebras $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are called isometrically isomorphic if there is a map φ from X onto Y such that for all $x, y \in X$ and $a, b \in \mathbb{R}^+$, $\varphi(ax + by) = a\varphi(x) + b\varphi(y)$, $\varphi(xy) = \varphi(x)\varphi(y)$ and $\|x\|_X = \|\varphi(x)\|_Y$.

Next we obtain a simple but crucial example of an asymmetric normed semialgebra.

Denote by l^+_∞ the positive cone of l_∞, i.e. $l^+_\infty = \{\mathbf{x}^+ : \mathbf{x} \in l_\infty\}$.

It is immediate to see that $(l_\infty^+, \|.\|_{+\infty})$ is an asymmetric normed semilinear space which is closed in the Banach space $(l_\infty, (\|.\|_{+\infty})^s)$, where the restriction of $\|.\|_{+\infty}$ to l_∞^+ is also denoted by $\|.\|_{+\infty}$.

Clearly l_∞^+ is a semialgebra and for each $\mathbf{x}, \mathbf{y} \in l_\infty^+$ we have $\|\mathbf{xy}\|_{+\infty} \leq \|\mathbf{x}\|_{+\infty} \|\mathbf{y}\|_{+\infty}$ (compare 8.3).

Consequently, we obtain the following result.

Proposition 8.8 $(l_\infty^+, \|.\|_{+\infty})$ *is a biBanach semialgebra.*

For each polynomial $P(n)$, with $P(n) > 0$ for all $n \in \omega$, consider the biBanach space $(\mathcal{B}^*_{P(n),\infty}, q_{P(n),\infty})$ constructed in the preceding section and let

$$\mathcal{C}^*_{P(n),\infty} := \{f \in \mathcal{B}^*_{P(n),\infty} : f(n) \geq 0 \text{ for all } n \in \omega\}.$$

The restriction of the asymmetric norm $q_{P(n),\infty}$ to $\mathcal{C}^*_{P(n),\infty}$ will be also denoted by $q_{P(n),\infty}$ if no confusion arises. Similarly, the restriction of the multiplication operation $\star$ to $\mathcal{C}^*_{P(n),\infty}$ is also denoted by $\star$. Therefore $\mathcal{C}^*_{P(n),\infty}$ is a semialgebra for the operation $\star$.

It is clear that the restriction to $\mathcal{C}^*_{P(n),\infty}$ of the map $\phi : \mathcal{B}^*_{P(n),\infty} \to l_\infty$, defined before, is a bijective homomorphism between the asymmetric normed semialgebra $(\mathcal{C}^*_{P(n),\infty}, q_{P(n),\infty})$ and the positive cone $(l_\infty^+, \|.\|_{+\infty})$ which preserves asymmetric norms.

As a consequence of these observations and Proposition 8.8 we have the following result.

Proposition 8.9 $(\mathcal{C}^*_{P(n),\infty}, q_{P(n),\infty})$ *and* $(l_\infty^+, \|.\|_{+\infty})$ *are isometrically isomorphic biBanach semialgebras, and hence* $\mathcal{C}^*_{P(n),\infty}$ *is a closed subset of the Banach space* $(\mathcal{B}^*_{P(n),\infty}, (q_{P(n),\infty})^s)$.

In the following the biBanach semialgebra $(\mathcal{C}^*_{P(n),\infty}, q_{P(n),\infty})$ will be called *the $sup_{P(n)}$-complexity space.*

Remark 8.3 *Observe that, in particular, $\mathcal{C}^*_{n,\infty} = \bigcap_{P(n)>n} \mathcal{C}^*_{P(n),\infty}$, and $\mathcal{C}^*_p \subsetneq \mathcal{C}^*_{n,\infty}$ for all $p \geq 1$. Furthermore, if $P(n) \geq n$ for all $n \in \omega$, the identity element e of the semialgebra $\mathcal{C}^*_{P(n),\infty}$ does not belong to any $\mathcal{C}^*_p$, $p \geq 1$, (recall that e is defined by $e(n) = 2^{P(n)}$ for all $n \in \omega$, and we have $q_{P(n),\infty}(e) = 1$.)*

Remark 8.4 *If $P(n) < Q(n)$ for all $n \in \omega$, then $\mathcal{C}^*_{P(n),\infty} \subseteq \mathcal{C}^*_{Q(n),\infty}$ and $q_{Q(n),\infty}(f) \leq q_{P(n),\infty}(f)$ for all $f \in \mathcal{C}^*_{P(n),\infty}$.*

Next we show that the (complexity) quasi-metric induced by the asymmetric norm $q_{P(n),\infty}$ also provides a suitable interpretation of the functions in $sup_{P(n)}$-complexity space.

Let f be a function from ω to $\mathbb{R}^+$. As usual, a function $g : \omega \to \mathbb{R}^+$ is said to be in the class $\mathcal{O}(f(n))$ if there is $c > 0$ such that $g(n) \leq cf(n)$ for all $n \in \omega$.

Let $f \in \mathcal{C}^*_{P(n),\infty}$ and let g be in class $\mathcal{O}(f(n))$. Then $g \leq cf$, for some $c > 0$. Obviously $g \in \mathcal{C}^*_{P(n),\infty}$.

- If $c \leq 1$, we have $g \leq f$, and hence

$$d_{q_{P(n),\infty}}(f, g) = q_{P(n),\infty}(g - f) = 0.$$

Thus, as in the case of the dual p-complexity space, condition $d_{q_{P(n),\infty}}(f, g) = 0$ (with $f \neq g$), agrees with the fact that g is more efficient than f on all inputs. Furthermore $q_{P(n),\infty}(f) = d_{q_{P(n),\infty}}(0, f)$ measures relative progress made in lowering complexity by replacing f by the "optimal" complexity function 0, assuming that the complexity measure is the running time of computing.

- If $c > 1$, then

$$\begin{aligned} q_{P(n),\infty}(g) - q_{P(n),\infty}(f) &\leq q_{P(n),\infty}(g-f) \\ &= \sup\{2^{-P(n)}((g(n)-f(n)) \vee 0)) : n \in \omega\} \\ &\leq \sup\{2^{-P(n)}(c-1)f(n) : n \in \omega\} \\ &= (c-1)q_{P(n),\infty}(f), \end{aligned}$$

and consequently

$$q_{P(n),\infty}(g) \leq c\, q_{P(n),\infty}(f) \quad \text{and} \quad d_{q_{P(n),\infty}}(f,g) \leq (c-1)d_{q_{P(n),\infty}}(0,f).$$

The following example shows that unfortunately the $sup_{P(n)}$-complexity space is not Smyth completable, hence not Smyth complete.

Example 8.4 *Let $P(n)$ be a polynomial (with $P(n) > 0$ for all $n \in \omega$). Define a sequence $(f_k)_{k\in\omega}$ by $f_k(n) = 0$ for $n = 0, 1, ..., k$, and $f_k(n) = 2^{P(n)}$ for $n > k$. Clearly $f_k \in \mathcal{C}^*_{P(n),\infty}$ for all $k \in \omega$ (actually each f_k is in class $\mathcal{O}(2^{P(n)})$).*

Then

$$d_{q_{P(n),\infty}}(f_k, f_{k+1}) = \sup\{2^{-P(n)}((f_{k+1}(n) - f_k(n)) \vee 0)\} = 0,$$

for all $k \in \omega$. Hence $(f_k)_{k\in\omega}$ is a left K-Cauchy sequence in $(\mathcal{C}^*_{P(n),\infty}, d_{q_{P(n),\infty}})$.

However, for each $j, k \in \omega$ with $j > k$, we have

$$d_{q_{P(n),\infty}}(f_j, f_k) = \sup\{2^{-P(n)}((f_k(n) - f_j(n)) \vee 0)\} = 1.$$

Therefore $(\mathcal{C}^*_{P(n),\infty}, q_{P(n),\infty})$ is not Smyth completable.

8.6 Contraction maps

It is known that for applications the complexity space $(\mathcal{C}, d_{\mathcal{C}})$ is typically restricted to functions which range over positive integers which are power of a given integer b (see Section 6 of [51]).

Let $a, b, c \in \mathbb{N}$ with $a, b \geq 2$, let n range over the set $\{b^k : k \in \omega\}$ and let $h \in \mathcal{C}$. A functional Φ corresponding to a Divide & Conquer algorithm in the sense of [51], is typically defined by

$$(\Phi(f))(n) = \begin{cases} c & \text{if } n = 1 \\ af(n/b) + h(n) & \text{if } n \in \{b^k : k \in \mathbb{N}\}. \end{cases}$$

We recall that this functional intuitively corresponds to a Divide & Conquer algorithm which recursively splits a given problem in a subproblems of size n/b and which takes $h(n)$ time to recombine the separately solved problems into the solution of the original problem.

It was proved in Theorem 6.1 of [51], that Φ is a contraction map for $d_{\mathcal{C}}$ with contraction constant $1/a$. This result was extended in Section 4 of [44] to the dual complexity space $(\mathcal{C}_1^*, d_{q_1})$, where the corresponding functional Φ^* is given, for $h \in \mathcal{C}$, by

$$(\Phi^*(f))(n) = \begin{cases} 1/c & \text{if } n = 1 \\ \frac{f(n/b)}{a+f(n/b)h(n)} & \text{if } n \in \{b^k : k \in \mathbb{N}\}. \end{cases}$$

A slight modification of the proof of Theorem 6.1 of [51] shows that such a result also follows in the realm of any dual p-complexity space. We conclude the chapter by obtaining an extension of Theorem 6.1 of [51] to the $\sup_{P(n)}$-complexity space when $P(n^{k+1}) \geq P(n^k)$ for all $n, k \in \omega$.

Under the above assumptions, define

$\mathcal{C}^*_{P(n),\infty} \mid b, c := \{f : f$ is the restriction to arguments n of the form b^k, $k \in \omega$, of $f' \in \mathcal{C}^*_{P(n),\infty}$ such that $f'(1) = 1/c\}$.

Observe that each $f \in \mathcal{C}^*_{P(n),\infty} \mid b, c$ can be considered as an element of $\mathcal{C}^*_{P(n),\infty}$, by defining $f(n) = 0$ whenever $n \notin \{b^k : k \in \omega\}$. Thus, if for each $f \in \mathcal{C}^*_{P(n),\infty} \mid b, c$, $\Phi^*(f)$ is defined as above, we obtain the following.

Proposition 8.10 *Let $f, g \in \mathcal{C}^*_{P(n),\infty} \mid b, c$. Then $\Phi^*(f), \Phi^*(g) \in \mathcal{C}^*_{P(n),\infty} \mid b, c$, and*

$$d_{q_{P(n),\infty}}(\Phi^*(f), \Phi^*(g)) \leq \frac{1}{a} d_{q_{P(n),\infty}}(f, g).$$

Proof. It is easy to check that $\Phi^*(f), \Phi^*(g) \in \mathcal{C}^*_{P(n),\infty} \mid b, c$. Furthermore

$$\begin{aligned}
& d_{q_{P(n),\infty}}(\Phi^*(f), \Phi^*(g)) \\
= & \sup_{n \in \{b^k : k \in \mathbb{N}\}} 2^{-P(n)} \left(\left(\frac{g(n/b)}{a + g(n/b)h(n)} - \frac{f(n/b)}{a + f(n/b)h(n)} \right) \vee 0 \right) \\
\leq & \sup_{n \in \{b^k : k \in \mathbb{N}\}} 2^{-P(n)} \left(\frac{a(g(n/b) - f(n/b))}{a^2} \vee 0 \right) \\
\leq & \frac{1}{a} \sup_{n \in \{b^k : k \in \omega\}} 2^{-P(n)} \left((g(n) - f(n)) \vee 0 \right) = \frac{1}{a} d_{q_{P(n),\infty}}(f, g).
\end{aligned}$$

This completes the proof.■

Bibliography

[1] Aho, A. V., Hopcroft, J.E., Ullman, J.D. *Data Structures and Algorithms.* Addison-Wesley. Massachusetts. 1983.

[2] Alegre, C., Ferrer, J., Gregori, V. *Quasi-uniformities on real vector spaces.* Indian J. Pure and Appl. Math. **28** (1997), 929-937.

[3] Alegre, C., Ferrer, J., Gregori, V. *On a class of real normed lattices,* Czech. Math. J. **48** (1998), 785-792.

[4] Alegre, C., Ferrer, J., Gregori, V. *On the Hahn-Banach theorem in certain linear quasi-uniform structures.* Acta Math. Hungar. **82** (1999), 315-320.

[5] Alegre, C., Romaguera, S. *On paratopological vector spaces,* Acta Math. Hungar., to appear.

[6] Alimov, A. R., *On the structure of the complements of Chebyshev sets.* Funct. Anal. Appl. **35** (2001), 176-182.

[7] Arenas, F.G., Puertas, M.L., Romaguera, S. *Ordered fractal semigroups as a model of computation,* Math. Comput. Model., to appear.

[8] Asanov, M.O., Velichko, N. V. *Compact sets in $C_p(X)$*, Comm. Math. Univ. Carolinae **22** (1981), 255-266.

[9] Beauzamy, B. *Introduction to Banach Spaces and their Geometry.* North Holland Math. Studies. Amsterdam. 1985.

[10] Cao, J., Reilly, I.L., Romaguera, S. *Some properties of quasi-uniform multifunction spaces,* J. Austral. Math. Soc. (Series A), **64** (1998), 169-177.

[11] Császár, Á. *Fondements de la Topologie Générale*. Budapest-Paris. 1960.

[12] Di Concilio, A. *Spazi quasimetrici e topologie ad essi associate*. Rend. Accad. Sci. Fis. Mat. Napoli **38** (1971), 113-130.

[13] Dolzhenko, E. P., Sevast'yanov, E.A. *Sign-sensitive approximations, the space of sign-sensitive weights. The rigidity and the freedom of a system,* Russian Acad. Sci. Dokl. Math. **48** (1994), 397-401.

[14] Emerson, A.E., Jutla, C.S. *The complexity of tree automata and logic of programs,* Siam J. Comput. **29** (1999), 132-158.

[15] Engelking, R. *General Topology,* Polish. Sci. Publ. Warsaw, 1977.

[16] Escardo, M.H. *PCF extended with real numbers,* Theoretical Computer Science **162** (1996), 79-115.

[17] Escardo, M.H. Introduction to Real PCF, Notes for an invited speach at the 3rd Real Numbers and Computers Conference (RNC3), l'Universite Pierre et Marrie Curie, Paris, April 1998. Available at: http://www.cs.bham.ac.uk/ mhe/papers.html.

[18] Ferrer, J., Gregori, V., Alegre, C. *Quasi-uniform structures in linear lattices.* Rocky Mountain J. Math. **23** (1993), 877-884.

[19] Fletcher, P., Lindgren, W.F. *Quasi-Uniform Spaces*. Marcel Dekker. New York. 1982.

[20] Gregori, V., Romaguera, S. *Fixed point theorems for fuzzy mappings in quasi-metric spaces,* Fuzzy Sets and Systems **115** (2000), 477-483.

[21] García Raffi, L.M., Romaguera, S., Sánchez Pérez, E.A. *The bicompletion of an asymmetric normed linear space*. Acta Math. Hungar. **97** (2002), 183-191.

[22] García Raffi, L.M., Romaguera, S., Sánchez Pérez, E.A. *On Hausdorff asymmetric normed linear spaces.* Houston J. Math., to appear.

[23] García Raffi, L.M., Romaguera, S., Sánchez Pérez, E.A. *Extensions of asymmetric norms to Linear Spaces.* Rend. Ist. Mat. Univ. Trieste **33** (2001), 113-125.

[24] García Raffi, L.M., Romaguera, S., Sánchez-Pérez, E.A. *Sequence spaces and asymmetric norms in the theory of computational complexity,* Math. Comput. Model. **36** (2002), 1-11.

[25] Grothendieck, A. *Critères de compacité dans les espaces fonctionnels généraux,* Amer. Math. J. **74** (1952), 168-186.

[26] Holá, L., Levi, S. *Decomposition properties of hyperspace topologies,* Set-Valued Anal. **5** (1997), 309-321.

[27] Keimel K., Roth, W. *Ordered Cones and Approximation*, Springer-Verlag, Berlin, Heidelberg, 1992.

[28] Künzi, H.P.A. *Nonsymmetric topology,* In: Proc. Colloquium on Topology, 1993, Szekszárd, Hungary, Colloq. Math. Soc. János Bolyai Math. Studies **4** (1995), 303-338.

[29] Künzi, H.P.A. *Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology,* in: Handbook of the History of General Topology, ed. by C.E. Aull and R. Lowen, vol. 3, Kluwer, Dordrecht, 2001, 853-968.

[30] Künzi, H.P.A., Marín, J., Romaguera, S. *Quasi-uniformities on topological semigroups and bicompletion,* Semigroup Forum **62** (2001), 403-422.

[31] Künzi H.P.A., Ryser, C. *The Bourbaki quasi-uniformity,* Topology Proc. **20** (1995), 161-183.

[32] Künzi, H.P.A., Schellekens, M. *The Yoneda completion of a quasi-metric space,* Theoretical Comput. Sci. **278** (2002), 159-194.

[33] Lindenstrauss, J., Tzafriri, L. *Classical Banach Spaces I and II.* Springer. Berlin. 1996.

[34] Marín, J., Romaguera, S. *A bitopological view of quasi-topological groups,* Indian J. Pure Appl. Math. **27** (1996), 393-405.

[35] Marín, J., Romaguera, S. *Bicompleting the left quasi-uniformity of a paratopological group,* Archiv. Math. (Basel) **70** (1998), 104-110.

[36] Matthews, S.G. *Partial metric topology,* In: Proc. 8th Summer Conference on General Topology and Appl., Ann. New York Acad. Sci. **728** (1994), 183-197.

[37] Murdeshwar, M. G., Naimpally, S.A. *Quasi-uniform Topological Spaces*, Nordhoff, Groningen. 1966.

[38] Nagel, R., ed., in Arendt, W., et al., *One-parameter Semigroups of Positive Operators,* Springer Lecture Notes Math. 1184, Berlin, 1986.

[39] Reilly, I.L., Subhramanyam, P.V., Vamanamurthy, M.K. *Cauchy sequences in quasi-pseudo-metric spaces,* Monatsh. Math. **93** (1982), 127-140.

[40] Render, H. *Generalized uniform spaces and applications to function spaces,* Studia Sci. Math. Hungar. **34** (1998), 429-443.

[41] Romaguera, S., Sánchez-Pérez, E.A., Valero, O. *Quasi-normed monoids and quasi-metrics,* Publ. Math. Debrecen, to appear.

[42] Romaguera, S., Sanchis, M. *Semi-Lipschitz functions and best approximation in quasi-metric spaces.* J. Approx. Theory **103** (2000), 292-301.

[43] Romaguera, S., Schellekens, M. *On the structure of the dual complexity space: The general case,* Extracta Math. **13** (1998), 249-253.

[44] Romaguera, S., Schellekens, M. *Quasi-metric properties of complexity spaces.* Topology Appl. **98** (1999), 311-322.

[45] Romaguera, S., Schellekens, M. *Weightable quasi-metric semigroups and semilattices,* Electronic Notes in Theoretical Computer Science **40** (2001), 12 pages.

[46] Romaguera, S., Schellekens, M. *Duality and quasi-normability for complexity spaces,* Appl. Gen. Topology **3** (2002), 91-112.

[47] Romaguera, S., Schellekens, M. *The quasi-metric of complexity convergence,* Quaestiones Math. **23** (2000), 359-374.

[48] Romaguera, S., Schellekens, M. *Cauchy filters and strong completeness of quasi-uniform spaces,* Rostock. Math. Kolloq. **54** (2000), 69-79.

[49] Rudin, W. *Functional Analysis.* McGraw-Hill. New York. 1973.

[50] Salbany, S. *Bitopological Spaces, Compactifications and Completions.* Math. Monographs, No.1, Dept. Math. Univ. Cape Town. 1974

[51] Schellekens, M. *The Smyth completion: a common foundation for denonational semantics and complexity analysis,* In: Proc. MFPS 11, Electronic Notes in Theoretical Computer Science **1** (1995), 211-232.

[52] Schellekens, M. *Complexity spaces revisited. Extended abstract,* In: Proc. 8th Prague Topological Symposium, Topology Atlas, 1996, 337-348.

[53] Smyth, M.B. *Quasi-uniformities: Reconciling domains with metric spaces,* In: Proc. MFPS 3, Lecture Notes in Computer Science **298** (M. Main et al., eds.), Springer, Berlin 1988, 236-253.

[54] Smyth, M.B. *Totally bounded spaces and compact ordered spaces as domains of computation,* In: Topology and Category Theory in Computer Science, (G.M. Reed, A.W. Roscoe and R.F. Wachter, eds.), Clarendon Press, Oxford, 1991, 207-229.

[55] Sünderhauf, Ph. *Constructing a quasi-uniform function space,* Topology Appl. **67** (1995), 1-27.

[56] Tix, R. *Continuous D-Cones: Convexity and Powerdomain Constructions,* Thesis, Darmstadt University, 1999.

[57] Tix, R. *Some results on Hahn-Banach type theorems for continuous d-cones,* Theoretical Comput. Sci. **264** (2001), 205-218.

[58] Vitolo, P. *A representation theorem for quasi-metric spaces,* Topology Appl. **65** (1995), 101-104.

[59] Wojtaszczyk, P. *Banach Spaces for Analysts.* Cambridge University Press. Cambridge. 1991.