The logistic map of matrices

Viviana Costinela Preduna

Submitted to obtain the degree of

"Master en investigación matemática"

Universitat Politècnica de València

Supervised by

Félix Martínez Jiménez and Alfred Peris Manguillot

Valencia - September 13, 2012

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

DEPARTAMENTO DE MATEMÁTICA APLICADA

Contents

Summary

The main goal of this work is to show what happens when instead of the logistic map with a scalar we use a matrix. This work is based on a paper of 2011 from a investigation team of Lithuania[1]. The logistic map is a polynomial mapping (equivalently, recurrence relation) of degree 2, often cited as an archetypal example of how complex, chaotic behaviour can arise from very simple non-linear dynamical equations. The map was popularized in a seminal 1976 paper by the biologist Robert May, in part as a discrete-time demographic model analogous to the logistic equation first created by Pierre François Verhulst.

The logistic map is one of the most simple forms of a chaotic process. Basically, this map, like any one-dimensional map, is a rule for getting a number from a number. The parameter a is fixed, but if one studies the dynamics of the the map for different values of a, it is found that this parameter is the catalyst to show a variety of possibilities (including chaos).

Dynamical properties of the iterative map are explored in detail when the order of matrices is 2. We will see that the evolution of the logistic map depends not only on the control parameter but also on the eigenvalues of the matrix of initial conditions. The second part is reserved for computational examples that are used to demonstrate the convergence to periodic attractors and the sensitivity of chaotic processes to initials conditions.

In the first chapter we are going to do a short introduction about the well-known logistic map with a scalar variable and a several properties of square matrices of order 2. We will see the algebraic representation of matrices and some properties of the matrix with a few corollaries.
The second chapter will include the dynamic of the logistic map with matrices and parametric expressions of idempotent and nilpotent matrix. The second part of this chapter will contain theorems and results about the dynamics of the logistic map of matrices, such as the theorem of *bounded four iterated sequences* in the case of idempotents and nilpotent matrices.

In the third chapter we illustrate the behavior of iterated matrices of order 2 with some computational experiments. We will to see how the nilpotent and idempotent matrix react by changing the initial conditions and the parameter a and the attractors in every case are the key point for beginning the study of these experiments. The last part of the third chapter will be the concluding remarks like advantages, disadvantages and a short comparator with the scalar form of the iterative logistic map.

CHAPTER 1

Preliminaries

1. The logistic map with a scalar variable

Definition 1.1. The dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space. The concept unifies very different types of such "rules" in mathematics: the different choices made for how time is measured and the special properties of the ambient space may give an idea of the vastness of the class of objects described by this concept. Time can be measured by integers, by real or complex numbers or can be a more general algebraic object, losing the memory of its physical origin, and the ambient space may be simply a set, without the need of a smooth space-time structure defined on it [2]

.

We are going to considerate a dynamical system composed by:

$$\begin{cases} f : I \subset \mathbb{R} \to \mathbb{R} \\ f(I) \subset I \end{cases}$$

where I is an interval of $\mathbb{R}$. Usually the function f depends of a parameter k, in this situation we note f_k. Then we choose an initial fixed point $x \in I$ and our objective is to study the behaviour of the orbit of x given by the function f. The orbit of the the point x is:

$$\text{Orb}(x, f) := \{x, f(x); f(f(x)); f(f(f(x))), \dots\}$$

In practice we can denote the Orb like this: $f^n = \underbrace{f \circ \dots \circ f}_{(n)}$. Now we have:

$$\text{Orb}(x) := \{x;, f(x), f^2(x), \dots\} = \{x_0, x_0 x_0, \dots\}$$

Given the dynamic system, we say that $x \in I$ is a fixed-point of the dynamic system if $f(x) = x$. It's obvious that the orbit of the fixed point x is the constant sequence$\{x, x \dots\}$. If $f(x) = x^2$ we are going to calculate two fixed points by solving the equation $x^2 - x = 0$. This points are$x = 0$ and $x = 1$. Their orbits are $\{0, 0 \dots\}$ and $\{1, 1 \dots\}$. We are asking about the behaviour of another point orbits and we

observe that $\text{Orb}(-1) = \{-1, 1, 1 \dots\}$, so we could say that the orbit of -1 is eventually fixed, even the point -1 isn't a fixed point. Now, for any other initial point x_0 we have two possibilities, if $0 < |x_0| < 1$, its orbit tends to the fixed point 0. If $|x_0| > 1$ its orbit tends to infinite.
Studing the dynamical systems we will find other cathegories of orbits. For example if $f(x) = x^2 - 1$ the the orbit of x_0 is $\{0, -1, 0, -1 \dots\}$. In this case the orbit of the fixed point is a 2-cycle.

1.1. Evolution of the logistic map. The evolution of the logistic map means, in fact, the orbit of fixed parameters x depending of the initial value and the a parameter between [0,4]. The mathematic expression of the logistic map is:

$$x^{(n+1)} = ax^{(n)}(1 - x^{(n)})$$

For following the evolution of the logistic map we need to introduce two definitions [**3**].

DEFINITION 1.2. Given the function $f(h)$ a fixed point is named "attractor" if x_0 is contained into a interval such that for every x in this interval its orbit continue inside the interval and the orbit tends to the fixed point x_0.

DEFINITION 1.3. Given the function $f(h)$ and a fixed point named "repeller"(repulsive) if x_0 is contained into a interval such that for every x in this interval(except x_0) its orbit gets out of the interval.

A point who's not attractor or repeller is called neutral. The experiments could give an idea about the spontaneity of attractor or repeller fixed points but we cannot iterate all the points close to the fixed point; we need another criterion for sorting if the fixed point is an attractor or a repeller. Returning at the lineal case where $f_k(x) = kx$ and we have $f'_k(x) = k$ and the family's dynamic could be expressed in function of the derivative f'_k on the fixed point. In the non-linear case this criterion is true and it's based on the absolute value of the function derivative on the fixed point. We have:

(1) The fixed point is an attractor if $|f'_{(x_0)}| < 1$
(2) The fixed point is a repulsive if $|f'_{(x_0)}| > 1$

Now we are going to make some observations. We cannot be sure about the nature of the fixed point x_0 if $|f'_{(x_0)}| = 1$. On the other hand the criterion gives the information for the behaviour of orbits concerning only "close" points to the fixed point and the behaviour of the "far away" points could be totaly different. The periodic points are divided into attractors,repellers or neutrals. Because the periodic

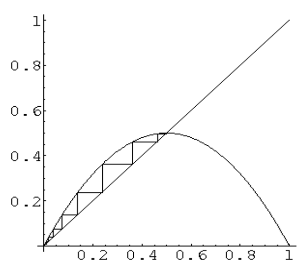
points create cycles, we can say that we have attractor cycles, repeller cycles or neutral cycles.
Now we have almost all the definitions for explain the behavoir of the logistic map. We need give another two definitions: the definition of a dense orbit and a chaotic function.

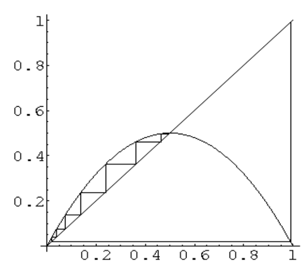
Definition 1.4. A dense orbit of a dynamical system on a set X is an orbit whose points form a dense subset of X.

Definition 1.5. (R.L. Devaney) A dynamical system F is chaotic if:

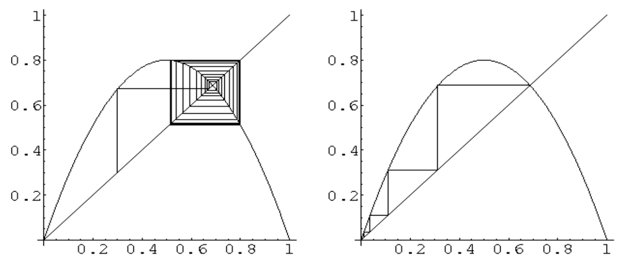
(1) Periodic points for F are dense.
(2) F is transitive.
(3) F depends sensitively on initial conditions.

The logistic map has a chaotic dynamic which is complex and rich because gives a lot of possibilities. We are going to show a few experiments for showing the chaotic behaviour or this family.
First, we choose an example without chaos. For example, the function $f_2(x) = 2x(1-x)$. This function have fixed points between 0 and $\frac{1}{2}$. Using the derivative, we have 0 as a repeller point and $\frac{1}{2}$ a attractor point. Moreover for any $x_0 \in (0,1)$, his orbit is $\frac{1}{2}$ or the orbit tends to $\frac{1}{2}$. We could make an analysis of this fact, but if we construct the function' s graphic we'll see the orbit because the diagonal $x = y$ cuts the function in the starting point and in the point $(\frac{1}{2}, \frac{1}{2})$ which is the maximum of the function. Finally the orbit of the point 1 stays fixed on 0.





Following the original idea, now let's try with another function, now the a parameter will be taken between [2,3,5]: $f_{3,2}(x) = 3,2x(1-x)$. Will see that our dependence is sensible on a few points but it isn't in all the points. This function has two fixed points 0 and 0,6875, both repellers. In the fist graph we see that if we take a point haphazardly and we iterate him,surely we'll obtain a 2-period cycle(the points of this orbit don't have sensitive dependence). An elaborate analysis gives us the opportunity to obtain orbits which fall over in a fixed point, not 0. This points show the sensitive dependence, because if we are changing the entries of a system, the orbit will have a different behaviour.



Let's see what is happening when we are close enough to the chaos. For seeing the chaotic behaviour we have the next function: $f_4(x) = 4x(1-x)$. This functions has the fixed points 0 and $\frac{3}{4}$, both repulsive points. We can do iterations toward them and obtain orbits of "eventually" fixed points. For example:

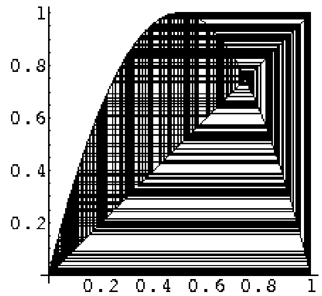
$$\text{Orb}(0,009607359798384785) =$$

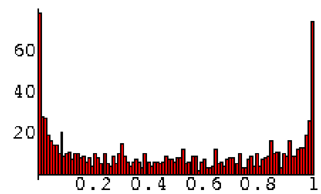
$$0,00960736; 0,0380602; 0,146447; 0; 5; 1; 0$$

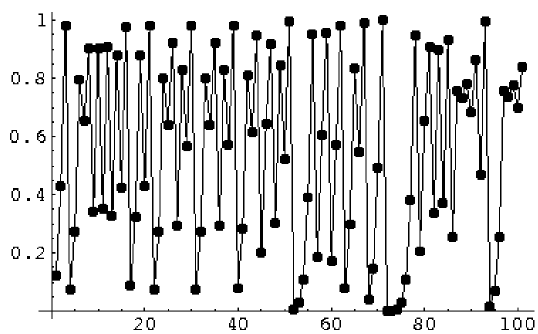
$$\text{Orb}(0,0010705383806982505) =$$

$$0,00107054; 0,00427757; 0,0170371; 0,0669873; 0,25; 0,75$$

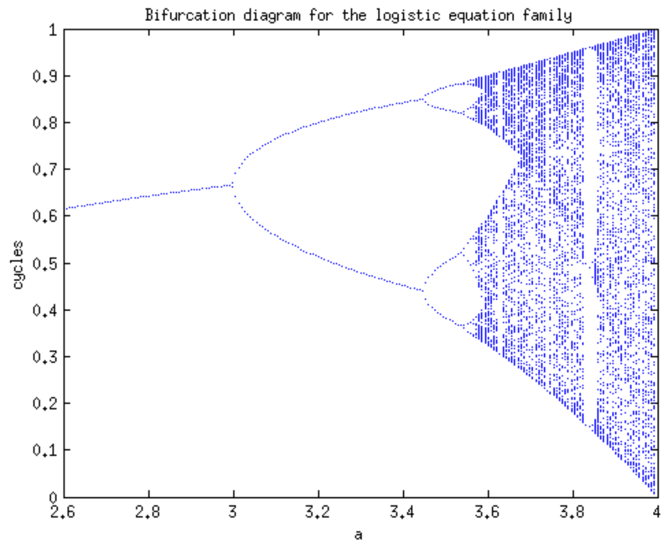
If we take another point haphazardly, we'll surely note that the destiny of the orbit is practically unexpected. Now we'll show the graphic iteration (200 iterations), the histogram (3000 iterations) and the time series (100 iterations) of the point 0,123.Analyzing the graphic iteration and the time series it's complicated to find a patron, the histogram confirms it by observing that the orbit seems to "visit" every small subinterval of [0,1]. It can be proved that $f_4(x)$ is chaotic in [0,1].







Finally, we will show a bifurcation example of the logistic family. In mathematics, particularly in dynamical systems, a bifurcation diagram shows the possible long-term values (equilibria/fixed points or periodic orbits) of a system as a function of a bifurcation parameter in the system. It is usual to represent stable solutions with a solid line and unstable solutions with a dotted line[4].The bifurcation parameter a is shown on the horizontal axis of the plot and the vertical axis shows the possible long-term population values of the logistic function. Only the stable solutions are shown here, there are many other unstable solutions which are not shown in this diagram.
The bifurcation diagram nicely shows the forking of the possible periods of stable orbits from 1 to 2 to 4 to 8 etc. Each of these bifurcation points is a period-doubling bifurcation. The ratio of the lengths of successive intervals between values of a for which bifurcation occurs converges to the first Feigenbaum constant (the Feigenbaum constants are two mathematical constants which both express ratios in a bifurcation diagram for a non-linear map. They are named after the mathematician Mitchell Feigenbaum).



2. Auxiliary results

In this section we focus on our model which is the logistic map with matrices. Now we know the behaviour of the classical logistic map. In the next part will see that the logistic map with matrices is even more complex. We start by studying different properties matrices that will be apply later.

The logistic map is a paradigmatic model usually used to demonstrate the onset of chaos and to illustrate the behavior can arise from very simple non-linear dynamical equation: $x^{(n+1)} = ax^{(n)}(1 - x^{(n)})$ where n is the iteration number; $n = 0, 1, 2, \cdot; a \in \mathbb{R}$ is the parameter of the logistic map and $x^{(0)}$ is the initial condition (the initial population at year 0). The logistic map is thoroughly explored and is used to model, encrypt, predict different physical systems and processes. The logistic map, for the last years, has been used to: generate fractals, study the effects of spatial heterogeneity on population dynamics,model a car following model,etc. In the first part will see some theoretical results very useful for understanding the last part, computational experiments. The object of this work is to investigate the extension of the logistic map when the discrete scalar variable $x^{(n)}$ is replaced by a square matrix of order 2; the n-th iterate of that matrix is denoted as $X^{(n)}$.We are going now to introduce the expression of the logistic map in two dimensions. Let the matrix of initial conditions read as:

$$X^{(0)} = \begin{pmatrix} x_{11}^{(0)} & x_{12}^{(0)} \\ x_{21}^{(0)} & x_{22}^{(0)} \end{pmatrix}; x_{kl}^{0} \in \mathbb{R};\ k,l = 1,2. \tag{1.1}$$

Then the iterate map

$$X^{(n+1)} = aX^{(n)}(I - X^{(n)}) := \begin{pmatrix} x_{11}^{(n+1)} & x_{12}^{(n+1)} \\ x_{21}^{(n+1)} & x_{22}^{(n+1)} \end{pmatrix} \tag{1.2}$$

represents a logistic map of square matrices of order 2. This equation produces four scalar time series $\{x_{(kl)}^{(j)}\}$;k,l=1,2, Explicitly,

$$\begin{cases} x_{11}^{(n+1)} = ax_{11}^{(n)}(1 - x_{11}^{(n)}) - ax_{12}^{(n)}x_{21}^{(n)} \\ x_{12}^{(n+1)} = ax_{12}^{(n)}(1 - x_{11}^{(n)} - x_{22}^{(n)}) \\ x_{21}^{(n+1)} = ax_{21}^{(n)}(1 - x_{11}^{(n)} - x_{22}^{(n)}) \\ x_{22}^{(n+1)} = ax_{22}^{(n)}(1 - x_{22}^{(n)}) - ax_{12}^{(n)}x_{21}^{(n)} \end{cases}$$

$n = 0,1,2...$; and $x_{11}^{(0)}, x_{12}^{(0)}, x_{21}^{(0)}, x_{22}^{(0)}$ are four scalar initial conditions. Though such an extension of the classical logistic map seems to be trivial, the apparent simplicity of the dynamical properties of such an iterative map is misguiding.

2.1. Properties Of Square Matrices. Several properties of square matrices of order 2 will be treated in this section. These properties are very important before continuing with the logistic map of matrices.

$$X := \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \tag{1.3}$$

where $x_{11}, x_{12}, x_{21}, x_{22} \in \mathbb{R}$ and its eigenvalues $\lambda_i \in \mathbb{C}$

$$\lambda_i = \frac{1}{2}(\operatorname{Tr} X + \varepsilon_i\sqrt{\operatorname{dsk} X}),$$

where $\varepsilon_i = (-1)^{i+1}$, $i = 1,2$ and Tr represents the trace of the matrix X, this is,

$$\operatorname{Tr} X := x_{11} + x_{22};$$

The discriminant of a polynomial is a function of its coefficients which gives information about the nature of its roots. In our case, we have the discriminant of the matrix

$$\operatorname{dsk} X := (x_{11} + x_{22}) + 4x_{12}x_{21}.$$

Lemma 1.6. *Suppose that the eigenvalues λ_1, λ_2 of the matrix X are not equal. Then it is possible to construct two idempotent matrices*

$$D_1 := \frac{1}{\lambda_1 - \lambda_2}(X - \lambda_2 I), \tag{1.4}$$

$$D_2 := \frac{1}{\lambda_2 - \lambda_1}(X - \lambda_1 I) \tag{1.5}$$

where I is the identity matrix, satisfying:

(i) $\det D_k = 0$, $k = 1, 2$,
(ii) $D_1 + D_2 = I$,
(iii) $D_1 \cdot D_2 = D_2 \cdot D_1 = \Theta$ *(where* $\Theta := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$*).*

Proof. We start showing that D_1 is an idempotent matrix. By the Theorem of Cayley-Hamilton and we have

$$\begin{aligned} D_1^2 =& \frac{1}{(\lambda_1 - \lambda_2)^2}(X - \lambda_2 I)^2 = \frac{1}{(\lambda_1 - \lambda_2)^2}(X^2 - 2\lambda_2 X + \lambda_2^2 I) = \\ & \frac{1}{(\lambda_1 - \lambda_2)^2}((X^2 - (\lambda_1 + \lambda_2)X + \lambda_1\lambda_2 I) - \lambda_2 X + \lambda_1 X + \lambda_2^2 I - \lambda_1\lambda_2 I = \\ & \frac{1}{(\lambda_1 - \lambda_2)^2}((\lambda_1 - \lambda_2)X - (\lambda_1 - \lambda_2)\lambda_2 I) = \\ & \frac{1}{(\lambda_1 - \lambda_2)}(X - \lambda_2 I) = D_1. \end{aligned}$$

The proof for D_2 is the same.

The equality (i) follows from the the definitions of D_1 and D_2 and the fact that $\det X - \lambda_i I = 0$.

For equality (ii), a direct computation shows that

$$\begin{aligned} D_1 + D_2 =& \frac{1}{\lambda_1 - \lambda_2}(X - \lambda_2 I) + \frac{1}{\lambda_2 - \lambda_1}(X - \lambda_1 I) = \\ & \frac{X}{\lambda_1 - \lambda_2} - \frac{X}{\lambda_1 - \lambda_2} - \frac{\lambda_2 I}{\lambda_1 - \lambda_2} - \frac{\lambda_1 I}{\lambda_1 - \lambda_2} = \\ & \frac{-(\lambda_2) I + \lambda_1}{\lambda_1 - \lambda_2} = I \end{aligned}$$

Finally, to se (iii)

$$\begin{aligned} D_1 \cdot D_2 =& \frac{1}{(\lambda_1 - \lambda_2)}(X - \lambda_2 I) \cdot \frac{1}{(\lambda_2 - \lambda_1)}(X - \lambda_1 I) = \\ & - \frac{1}{(\lambda_1 - \lambda_2)^2}(X^2 - (\lambda_1 + \lambda_2)X + \lambda_1 \cdot \lambda_2 I = \Theta \end{aligned}$$

, and analogously for $D_2 \cdot D_1 = \Theta$.

LEMMA 1.7. *Suppose that the eigenvalues of the matrix X are equal to λ_0. Then the matrix N defined as*

$$N := X - \lambda_0 I \tag{1.6}$$

satisfies the properties:

(i) $N^2 = \Theta$
(ii) $\det N = 0$

Proof. The equality (i) is easy since by hypothesis the characteristic polynomial of the matrix X may be factorized as $(x-\lambda_0)^2$ and since X is a root of this polynomial by the Theorem of Cayley-Hamilton. The property (ii) is immediate.

LEMMA 1.8. *Let λ_1, and λ_2 be the eigenvalues of the matrix X.*

(1) *If $\lambda_1 \neq \lambda_2$ then*

$$X = \lambda_1 D_1 + \lambda_2 D_2 \tag{1.7}$$

.

(2) *If $\lambda_0 := \lambda_1 = \lambda_2$ then*

$$X = \lambda_0 I + N \tag{1.8}$$

Proof. (ii) is clear and (i) is just a direct computation:

$$\lambda_1 D_1 + \lambda_2 D_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2}(X - \lambda_2 I) + \frac{\lambda_2}{\lambda_2 - \lambda_1}(X - \lambda_1 I) =$$
$$(\frac{\lambda_1}{\lambda_1 - \lambda_2} + \frac{\lambda_2}{\lambda_2 - \lambda_1})X = X$$

REMARK 1.9. Observe that D_1 and D_2 are similar idempotent matrices and that N and cN, $c \in \mathbb{C}$ are also similar nilpotent matrices.

Next definition describes the two possibilities when the eigenvalues of the matrix are distinct or equal.

DEFINITION 1.10.

(1) The matrix X is said to be of **type 1** if it can be expressed as

$$\lambda_1 D_1 + \lambda_2 D_2,$$

where λ_1 and λ_2 are the two **distinct** eigenvalues of X, and $D_1 + D_2 = I$. Matrices D_1 and D_2 are called the **conjugate idempotents** of X.

(2) The matrix X is said to be of **type 2** if its eigenvalues are equal. In this situation X can be expressed as

$$X = \lambda_0 I + N,$$

where $\lambda_0 := \lambda_1 = \lambda_2$ and $N := X - \lambda_0 I$. The matrix N is called the **nilpotent** of X.

REMARK 1.11. Clearly matrices D_1 and D_2 are of type 1 since $D_1 = 1 \cdot D_1 + 0 \cdot D_2$ and $D_2 = 0 \cdot D_1 + 1 \cdot D_2$. On the other hand, matrix N is clearly of type 2 since $X = 0 \cdot I + N$.

REMARK 1.12. Let us notice that a scalar matrix $X = \lambda_0 I$ can be expressed in the form $\lambda_0 I = \lambda_0 D_1 + \lambda_0 D_2$ where D_1, D_2 is a pair of conjugate idempotent matrices. Thus,$\lambda_0 I$ can be seen as a type 1 matrix or as a type 2 matrix.

EXAMPLE 1.13. The next example will illustrate that a scalar matrix is of type 1 and type 2. Let us assume that $\lambda_0 = 0.25$. Then,

$$\begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{4}\left(\begin{bmatrix} \frac{3}{2} & \frac{4}{2} \\ -(\frac{3}{8}) & -(\frac{1}{2}) \end{bmatrix} + \begin{bmatrix} -(\frac{1}{2}) & -(\frac{4}{2}) \\ \frac{3}{8} & \frac{3}{2} \end{bmatrix}\right) \tag{1.9}$$

It can be noted that matrices $\begin{bmatrix} \frac{3}{2} & \frac{4}{2} \\ -(\frac{3}{8}) & -(\frac{1}{2}) \end{bmatrix}$ and $\begin{bmatrix} -(\frac{1}{2}) & -(\frac{4}{2}) \\ \frac{3}{8} & \frac{3}{2} \end{bmatrix}$ are conjugate idempotents.

LEMMA 1.14. *If the conjugate idempotent matrices of type 1 matrices X_1' and X_1'' are the same, then the conjugate idempotent matrices of $X_1' \cdot X_1''$ and $X_1' + X_1''$ are also the same. Analogously, if the nilpotent matrix of type 2 matrices X_2' and X_2'' are similar, then the nilpotent matrices of $X_2' \cdot X_2''$ and $X_2' + X_2''$ are similar to the nilpotent matrix of X_2' and the nilpotent matrix of X_2''.*

Proof. Let

$$\begin{aligned} X_1' + X_1'' &= (\lambda_1' D_1 + \lambda_2' D_2) + (\lambda_1'' D_1 + \lambda_2'' D_2) \\ &= (\lambda_1' + \lambda_1'')D_1 + (\lambda_2' + \lambda_2'')D_2 \end{aligned}$$

$$\begin{aligned} X_1' \cdot X_1'' &= (\lambda_1' D_1 + \lambda_2' D_2) \cdot (\lambda_1'' D_1 + \lambda_2'' D_2) \\ &= \lambda_1' \cdot \lambda_1'' D_1^2 + \lambda_2' \cdot \lambda_1'' D_1 D_2 + \lambda_1' \cdot \lambda_2'' D_1 D_2 + \lambda_2' \cdot \lambda_2'' D_2^2 \\ &= \lambda_1' \cdot \lambda_1'' D_1^2 + \lambda_2' \cdot \lambda_2'' D_2^2 \end{aligned}$$

Analogously let $X_2' = \lambda_0' I + c_1 N$ and $X_2'' = \lambda_0'' I + c_2 N$

$$\begin{aligned} X_2' + X_2'' &= (\lambda_0' I + c_1 N + X_2'') + (\lambda_0'' I + c_2 N) \\ &= (\lambda_0' + \lambda_0'')I + (c_1 + c_2)N \end{aligned}$$

$$\begin{aligned} X_2' \cdot X_2'' &= (\lambda_0' I + c_1 N)(\lambda_0'' I + c_2 N) \\ &= \lambda_0'\lambda_0'' I \cdot I + c_1\lambda_0'' N + \lambda_0' c_2 N + c_1 c_2 N^2 \\ &= \lambda_0'\lambda_0'' I + (c_1\lambda_0'' + c_2\lambda_0')N \end{aligned}$$

PROPOSITION 1.15. *If X_1 is a type 1 matrix and X_2 is a type 2 matrix, then their powers X_1^n and X_2^n; $n = 0, 1, 2, ..$ read as*

$$X_1^n = \lambda_1^n D_1 + \lambda_2^n D_2; \tag{1.10}$$

$$X_2^n = \lambda_0^n I + n\lambda_0^{n-1} N; \tag{1.11}$$

where D_1 and D_2 are the idempotent matrices of X_1 and N is the nilpotent matrix of X_2.

Proof.

$$\begin{aligned} X_1^2 =&(\lambda_1 D_1 + \lambda_2 D_2)^2 = \\ &\lambda_1^2 D_1^2 + \lambda_1\lambda_2 D_1 \cdot D_2 + \lambda_2\lambda_2 D_2 \cdot D_1 + \lambda_2^2 D_2^2 = \\ &\lambda_1^2 D_1^2 + \lambda_2^2 D_2^2; \\ X_2^2 =&(\lambda_0 I + N) = \\ &\lambda_0^2 I + 2\lambda_0 N + N^2 = \\ &\lambda_0^2 I + 2\lambda_0 N; \end{aligned}$$

and for powers greater than 2 the proof is the same.

LEMMA 1.16. *Suppose that D_1 and D_2 are two conjugate idempotent matrices ($D_1 + D_2 = I$); and that λ_1 and λ_2 are two complex numbers. Then λ_1 and λ_2 are the eigenvalues and D_1 and D_2 are the conjugate idempotent matrices of the matrix $X := \lambda_1 D_1 + \lambda_2 D_2$.*

Proof. For the case $\lambda_1 \neq \lambda_2$ we have that the matrix characteristic equation

$$\det[(\lambda_1 D_1 + \lambda_2 D_2) + \lambda I] = \det[(\lambda_1 - \lambda) D_1 + (\lambda_2 - \lambda) D_2] = 0$$

yields two solutions $\lambda_1 = \lambda$ and $\lambda_2 = \lambda$. Therefore λ_1 and λ_2 are the eigenvalues of X. Also,

$$\frac{1}{\lambda_1 - \lambda_2}(\lambda_1 D_1 + \lambda_2 D_2) + \lambda_2 I) = \frac{1}{\lambda_1 - \lambda_2}(\lambda_1 - \lambda_2) D_1 = D_1.$$

Hence D_1 and D_2 are the idempoten matrices of X.

Finally, if $\lambda_1 = \lambda_2 = \lambda_0$, then X is a scalar matrix and it has the eigenvalue λ_0.

Lemma 1.17. *Let N be a nilpotent matrix and $\lambda_0 \in \mathbb{C}$. Then the matrix $X := \lambda_0 I + N$ has the single recurrent eigenvalue λ_0 and its nilpotent matrix is N.*

Proof. Since N is a nilpotent matrix $N^2 = (X - \lambda_0 I)^2 = \Theta$, so

$$X^2 = \lambda_0^2 I + 2\lambda_0 I N = \lambda_0 I(\lambda_0 + 2N)$$

and it's obvious that X has a single recurrent eigenvalue which is λ_0.

CHAPTER 2

The dynamics of the logistic map of matrices.

In this chapter we will see basic definitions and some results of the dynamics of the logistic map of matrices.

We start by rewriting the expressions of the idempotent and nilpotent matrices of type 1 and 2 matrices. Recall that eigenvalues λ_1 and λ_2 of a type 1 matrix X satisfy the following relationships:

$$\lambda_1 = \frac{1}{2}(\operatorname{Tr} X + \sqrt{\operatorname{dsk} X})$$
$$\lambda_2 = \frac{1}{2}(\operatorname{Tr} X - \sqrt{\operatorname{dsk} X})$$

Subtracting this two expression we obtain

$$\lambda_1 - \lambda_2 = \sqrt{\operatorname{dsk} X}. \tag{2.1}$$

Substituting also the trace we get

$$x_{11} - \lambda_2 = x_{11} - \frac{1}{2}(x_{11} + x_{22} - \sqrt{\operatorname{dsk} X}) = \frac{1}{2}(\sqrt{\operatorname{dsk} X} + (x_{11} - x_{22})); \tag{2.2}$$

and analogously,

$$x_{22} - \lambda_1 = -\frac{1}{2}(\sqrt{\operatorname{dsk} X} - (x_{22} - x_{11})). \tag{2.3}$$

Using the definition of the matrices D_1 and D_2 we have

$$D_1 = \frac{1}{2\sqrt{\operatorname{dsk} X}} \begin{bmatrix} \sqrt{\operatorname{dsk} X} + (x_{11} - x_{22}) & 2x_{12} \\ 2x_{21} & \sqrt{\operatorname{dsk} X} + (x_{22} - x_{11}) \end{bmatrix} \tag{2.4}$$

and

$$D_2 = \frac{1}{2\sqrt{\operatorname{dsk} X}} \begin{bmatrix} \sqrt{\operatorname{dsk} X} + (x_{22} - x_{11}) & -2x_{12} \\ -2x_{21} & \sqrt{\operatorname{dsk} X} + (x_{11} - x_{22}) \end{bmatrix} \tag{2.5}$$

when $\operatorname{dsk} X \neq 0$. The introduction of new parameters $\alpha := \frac{x_{11} - x_{22}}{\sqrt{\operatorname{dsk} X}}$ and $\beta := \frac{2x_{12}}{\sqrt{\operatorname{dsk} X}}$ yields the parametric expressions of the idempotent matrices of X as

$$D_1 = \frac{1}{2}\begin{bmatrix} 1+\alpha & \beta \\ \frac{1+\alpha^2}{\beta} & 1-\alpha \end{bmatrix}; D_2 = \frac{1}{2}\begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix} \tag{2.6}$$

where we have used the equalities

$$(\frac{(x_{11}+x_{22})^2}{\sqrt{\operatorname{dsk} X}})^2+\frac{2x_{12}}{\sqrt{\operatorname{dsk} X}}\cdot\frac{2x_{21}}{\sqrt{\operatorname{dsk} X}}=\frac{(x_{11}+x_{22})^2+4x_{12}x_{21}}{\sqrt{\operatorname{dsk} X}}=\frac{\sqrt{\operatorname{dsk} X}}{\sqrt{\operatorname{dsk} X}}=1$$

and

$$\frac{2x_{21}}{\sqrt{\operatorname{dsk} X}}=\frac{1-\alpha^2}{\beta}$$

The expression of the nilpotent matrix when X of type 2 is as

$$N=\frac{1}{2}\begin{bmatrix} x_{11}-x_{22} & 2x_{12} \\ 2x_{21} & x_{22}-x_{11} \end{bmatrix} \tag{2.7}$$

when $\operatorname{dsk} X=0$ and $\lambda_0=\frac{x_{11}+x_{22}}{2}$. In this case introducing parameters $\widehat{\alpha}=x_{11}-x_{22}$ and $\widehat{\beta}=2x_{12}$, since $\operatorname{dsk} X=0$ we have

$$N=\frac{1}{2}\begin{bmatrix} \widehat{\alpha} & \widehat{\beta} \\ -\frac{\widehat{\alpha}^2}{\widehat{\beta}} & -\widehat{\alpha} \end{bmatrix} \tag{2.8}$$

We observe that following conditional parametric limits exist

$$\lim_{\alpha\to 1\beta\to 0\frac{1-\alpha^2}{\beta}\to 0} D_1=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\lim_{\alpha\to 1\beta\to 0\frac{1-\alpha^2}{\beta}\to 0} D_2=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};$$

$$\lim_{\widehat{\alpha}\to 1\widehat{\beta}\to 0\frac{1-\widehat{\alpha}^2}{\widehat{\beta}}\to b} N=\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix},$$

$$\lim_{\widehat{\alpha}\to 1\frac{\widehat{\beta}}{2}\to b} N=\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix};$$

where $b\in\mathbb{C}$.

1. Dynamic dependence

In this section we will highlight the main features of the dynamics of the logistic map of square matrices of order 2.

THEOREM 2.1. *The four iterated sequences* $\left\{x_{kl}^{(n)}\right\}_{n=0}^{+\infty}; k,l=1,2$ *generated by the logistic map of matrices defined by Eq. 1.2 will stay bounded for all* $n=0,1,2...$ *if the following statements hold:*

(i) $0\le a\le 4$,
(ii) *the matrix of initial conditions* $X^{(0)}$ *is a type 1 idempotent matrix,*
(iii) *the eigenvalues* $\lambda_1^{(0)}$ *and* $\lambda_2^{(0)}$ *of* $X^{(0)}$ *belong to the interval* $[0;1]$.

Proof. Since $X^{(0)}$ is a type 1 matrix we may write

$$X^{(0)} = \begin{bmatrix} x_{11}^{(0)} & x_{12}^{(0)} \\ x_{21}^{(0)} & x_{22}^{(0)} \end{bmatrix}$$
$$= \lambda_1^{(0)} \frac{1}{2} \begin{bmatrix} 1+\alpha & \beta \\ \frac{1+\alpha^2}{\beta} & 1-\alpha \end{bmatrix} + \lambda_2^{(0)} \frac{1}{2} \begin{bmatrix} 1+\alpha & -\beta \\ -\frac{1+\alpha^2}{\beta} & 1+\alpha \end{bmatrix}$$

Straightforward computations give $X^{(1)} = \lambda_1^1 D_1 + \lambda_2^1 D_2$ and suppose that

$$X^{(n)} = \begin{bmatrix} x_{11}^{(n)} & x_{12}^{(n)} \\ x_{21}^{(n)} & x_{22}^{(n)} \end{bmatrix} \tag{2.9}$$
$$= \lambda_1^{(n)} \frac{1}{2} \begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix} + \lambda_2^{(n)} \frac{1}{2} \begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix} \tag{2.10}$$

Computing $X^{(n+1)}$ we get

$$\begin{aligned} X^{(n+1)} =& a(\lambda_1^{(n)} D_1 + \lambda_2^{(n)} D_2) + (I - \lambda_1^{(n)} D_1 - \lambda_2^{(n)} D_2) = \\ & a(\lambda_1^{(n)} D_1 + \lambda_2^{(n)} D_2 + (\lambda_1^{(n)})^2 (D_1)^2 - \lambda_1^{(n)} \lambda_2^{(n)} D_1 D_2 - \lambda_1^{(n)} \lambda_2^{(n)} D_1 D_2 - (\lambda_2^{(n)})^2 (D_2)^2) = \\ & a((\lambda_1^{(n)} D_1 + \lambda_2^{(n)} D_2 - (\lambda_1^{(n)})^2 (D_1)^2 - (\lambda_2^{(n)})^2 (D_2)^2) = \\ & a(\lambda_1^{(n)} - (\lambda_1^{(n)})^2) D_1 + a(\lambda_2^{(n)} - (\lambda_2^{(n)})^2) D_2 = \\ & \lambda_1^{(n+1)} D_1 + \lambda_2^{(n+1)} D_2 \end{aligned}$$

where we used equalities:

$$\lambda_1^{(n+1)} \frac{1}{2} \begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix} + \lambda_2^{(n+1)} \frac{1}{2} \begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix} = \tag{2.11}$$
$$a\left(\left(\lambda_1^{(n)} - (\lambda_1^{(n)})^2\right) \frac{1}{2} \begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix} + \left(\lambda_2^n - (\lambda_2^n)^2\right) \frac{1}{2} \begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix} \right)$$

that yield

$$\begin{cases} \lambda_1^{(n+1)} = a\lambda_1^{(n)}(1-\lambda_1^{(n)}); \\ \lambda_2^{(n+1)} = a\lambda_2^{(n)}(1-\lambda_2^{(n)}); \end{cases} n = 0, 1, 2... \tag{2.12}$$

where $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ are eigenvalues of $X^{(n)}$.

and

$$(2.13)\qquad \begin{cases} x_{11}^{(n)} = \frac{1}{2}(\lambda_1^{(n)} + \lambda_2^{(n)}) + \frac{\alpha}{2}(\lambda_1^{(n)} - \lambda_2^{(n)}); \\ x_{12}^{(n)} = \frac{1}{2}\beta(\lambda_1^{(n)} - \lambda_2^{(n)}); \\ x_{21}^{(n)} = \frac{1}{2} \cdot \frac{1-\alpha^2}{\beta}(\lambda_1^{(n)} - \lambda_2^{(n)}); \\ x_{22}^{(n)} = \frac{1}{2}(\lambda_1^{(n)} + \lambda_2^{(n)}) - \frac{\alpha}{2}(\lambda_1^{(n)} - \lambda_2^{(n)}); \end{cases} \qquad n = 0, 1, 2...$$

It is clear that if $\lambda_1^{(0)}, \lambda_2^{(0)} \in [0,1]$ and $0 \le a \le 4$, then $\lambda_1^{(n)}, \lambda_2^{(n)} \in [0,1]$ for all $n = 1, 2, ...$ This is sufficient to have $x_{kl}^{(n)}; k, l = 1, 2$ bounded for all $n = 0, 1, 2....$

To summarize, when we iterate by the logistic map matrices of type 1, the formulas that describe the iterated matrices and eigenvalues are

$$(2.14)\qquad X^{(n)} = \lambda_1^{(n)} \frac{1}{2} \begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix} + \lambda_2^{(n)} \frac{1}{2} \begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix},\ n = 0, 1, \ldots$$

$$(2.15)\qquad \lambda_1^{(n)} = a\lambda_1^{(n-1)}(1 - \lambda_1^{(n-1)}),\ n = 0, 1, \ldots$$

$$(2.16)\qquad \lambda_2^{(n)} = a\lambda_2^{(n-1)}(1 - \lambda_2^{(n-1)}),\ n = 0, 1, \ldots$$

THEOREM 2.2. *The four iterated sequences* $\left\{x_{kl}^{(n)}\right\}_{n=0}^{+\infty}; k, l = 1, 2$ *generated by the logistic map of matrices defined by Eq. 1.2 will stay bounded for all* $n = 0, 1, 2...$ *if the following statements hold:*

(i) $0 \le a \le 4$,
(ii) *the matrix of initial conditions* $X^{(0)}$ *is a type 2 matrix,*
(iii) *the eigenvalue* $\lambda_0^{(0)}$ *of* $X^{(0)}$ *belongs to the interval* $[0; 1]$,
(iv) *elements of the sequence* $\left\{a^{(n+1)} \prod_{i=1}^{n}(1 - 2\lambda_0^{(k)})\right\}_{n=0}^{+\infty}$ *remain in the interval* $[-M; M]$, *with* $0 \le M < +\infty$, *where* $\lambda_0^{(k)}$ *is the recurrent eigenvalue of* $X^{(k)}$, $k = 0, 1, 2....$

Proof. Since $X^{(0)}$ is a type 2 matrix

$$(2.17)\qquad X^{(0)} = \begin{bmatrix} x_{11}^{(0)} & x_{12}^{(0)} \\ x_{21}^{(0)} & x_{22}^{(0)} \end{bmatrix} = \lambda_0^{(0)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \widehat{\alpha} & \widehat{\beta} \\ -\frac{\widehat{\alpha}^2}{\widehat{\beta}} & -\widehat{\alpha} \end{bmatrix}$$

In a similar way as we did in the previous theorem we may get an iterative sequence of matrices where balancing appropriate components yield

$$(2.18)\qquad \begin{cases} \lambda_0^{(n+1)} = a\lambda_0^{(n)}\left(1 - \lambda_0^{(n)}\right); \\ \mu_0^{(n+1)} = a\mu_0^{(n)}\left(1 - \lambda_0^{(n)}\right); \end{cases} \quad n = 0, 1, 2...$$

where $\mu_0^{(0)} = 1$ and $\left\{\mu_0^{(n)}\right\}_{n=0}^{+\infty}$ are coefficients of the nilpotent matrix of type 2 matrix

$$\frac{1}{2}\begin{bmatrix} \widehat{\alpha} & \widehat{\beta} \\ -\frac{\widehat{\alpha}^2}{\widehat{\beta}} & -\widehat{\alpha} \end{bmatrix}$$

in the decomposition of the type 2 matrices $\left\{X^{(n)}\right\}_{n=0}^{+\infty}$. Now, four iterative sequences take the following form

$$(2.19)\qquad \begin{cases} x_{11}^{(n)} = \lambda_0^n + \frac{\widehat{\alpha}}{2}\mu_0^{(n)}; \\ x_{12}^{(n)} = \frac{\widehat{\beta}}{2}\mu_0^{(n)}; \\ x_{21}^{(n)} = -\frac{(\widehat{\alpha})^2}{2\widehat{\beta}}\mu_0^{(n)}; \\ x_{22}^{(n)} = \lambda_0^n - \frac{\widehat{\alpha}}{2}\mu_0^{(n)}; \end{cases} \qquad n = 0, 1, 2...$$

Since $0 \leq \lambda_0^{(0)} \leq 1$, $0 \leq a \leq 4$, and $|\mu_0^{(n)}| < M$ we have that $x_{kl}^{(n)}; k, l = 1, 2$ are bounded for all $n = 1, 2...$

The following corollary describes the evolution of the logistic map for square matrices of order 2 from initial conditions.

Corollary 2.3.

(i) *If the matrix of initial conditions $X^{(0)}$ is a scalar matrix then iterated matrices $X^{(n)}$ will stay scalar matrices for all $n = 1, 2, ...$*

(ii) *If the matrix of initial conditions $X^{(0)}$ is a type 1 matrix then iterated matrices $X^{(n)}$ stay type 1 matrices for all $n = 1, 2, ...$ or may become scalar matrices from $n = m, m+1, ...m \geq 1$.*

(iii) *If the matrix of initial conditions $X^{(0)}$ is a type 2 matrix then iterated matrices $X^{(n)}$ stay type 2 matrices for all $n = 1, 2, ...$ or may become scalar matrices from $n = m, m+1, ...m \geq 1$.*

Proof. (i) If the matrix of initial conditions is a scalar matrix $X^{(0)} = \lambda_0^{(0)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then iterated matrices stay scalar matrices $X^{(n)} = \lambda_0^{(n)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for all $n = 1, 2, ..$ because $\lambda_0^{(n+1)} = a\lambda_0^{(n)}(1 - \lambda_0^{(n)})$.

(ii) If the matrix of initial conditions $X^{(0)}$ is an type 1 matrix then iterated matrices $X^{(n)}$ stay type 1 matrices because

$$(2.20)\qquad X^{(n)} = \lambda_1^{(n)}\frac{1}{2}\begin{bmatrix} 1+\alpha & \beta \\ \frac{1+\alpha^2}{\beta} & 1-\alpha \end{bmatrix} + \lambda_2^{(n)}\frac{1}{2}\begin{bmatrix} 1+\alpha & -\beta \\ -\frac{1+\alpha^2}{\beta} & 1+\alpha \end{bmatrix}$$

if $\lambda_1 \neq \lambda_2$ for $n = 1, 2, ..$ (Theorem 2.1).

On the other hand, if eigenvalues of the iterated matrix $X^{(m)}$ become equal for some $m > 0$, then $\lambda_1^{(m)} = \lambda_2^{(m)} = \lambda_0^{(m)}$ and hence the iterated matrices $X^{(k)}$ become scalar matrices of the form

$$X^{(k)} = \lambda_0^{(k)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ k = m, m+1, m+2, ...$$

(iii) If the matrix of initial conditions $X^{(0)}$ is a type 2 matrix, then iterated matrices $X^{(n)}$ stay nilpotent matrices (Theorem 2.2)

$$X^{(n)} = \lambda_0^{(n)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mu_0^{(n)} \frac{1}{2} \begin{bmatrix} \widehat{\alpha} & \widehat{\beta} \\ -\frac{\widehat{\alpha}^2}{\widehat{\beta}} & -\widehat{\alpha} \end{bmatrix} \tag{2.21}$$

if $\mu_0^{(n)} \neq 0$ for $n = 1, 2, ...$

On the other hand, if $\mu_0^{(n)}$ becomes equal to 0 for some $m > 0$, then iterated matrices $X^{(k)}$ become scalar matrices of the form

$$X^{(k)} = \lambda_0^{(k)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ k = m, m+1, m+2, ...$$

CHAPTER 3

Computational experiments

The main purpose of this chapter is to illustrate with examples behavior of iterated matrices of order two. As we have seen, this behavior is governed by initial conditions and eigenvalues. For all the computational experiments we are going to use Matlab.

First of all it can be noted that the qualitative behavior of iterated matrices of order 2 is governed by Eq. (2.14) or Eq. (2.15) depending of the type of the matrix of initial conditions. It is important to remark that the evolution of the logistic map differs substantially if $X^{(0)}$ is of type 1 or type 2.

If $X^{(0)}$ has two distinct eigenvalues in [0,1], it is enough to have $0 \leq a \leq 4$ to get that the elements of iterated matrices remain bounded. If $X^{(0)}$ is a type 2 matrix and has one recurrent eigenvalue in [0,1], one can be sure that the elements of iterated matrices would be bounded if $0 \leq a \leq 1$. A separate investigation must be done for higher values of the parameter a.

1. Asymptotic versus nonasymptotic convergence when $X^{(0)}$ is a type 1 matrix: $2 < a < 3$.

In this section we will focus on the situation when the matrix of initial conditions is a type 1 matrix and the parameter of the logistic map a is bounded in the interval $1 < a < 3$ (this means that the scalar logistic map converges to a stable fixed point).

According to the system of equations (2.15), both eigenvalues $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ will converge to $1 - a^{-1}$ at increasing of the n if, of course,$\lambda_1^{(0)}$ and $\lambda_1^{(0)}$ are bounded in the interval [0,1]). In other words, the iterated matrix when the initial condition matrix is of type 1 will eventually be transformed into a scalar matrix at sufficiently large n. First of all it can be noted that the convergence of a scalar logistic map to a stable fixed point $1 - a^{-1}$ can be asymptotic or nonasymptotic.

As we did in the case of logistic family with a scalar we'll do the same in this situation. First, we are going to start the iterations of a type 1 matrix, our initial condition taking the parameter $a \in [1, 2]$. We consider two examples.

The first example will have the following initial condition

$$X^{(0)} = \begin{bmatrix} 0.25 & 0.38 \\ 0.35 & 0.8 \end{bmatrix}$$

Let's see what is happening whit the matrix coefficients and with its eigenvalues after 40 iterations:

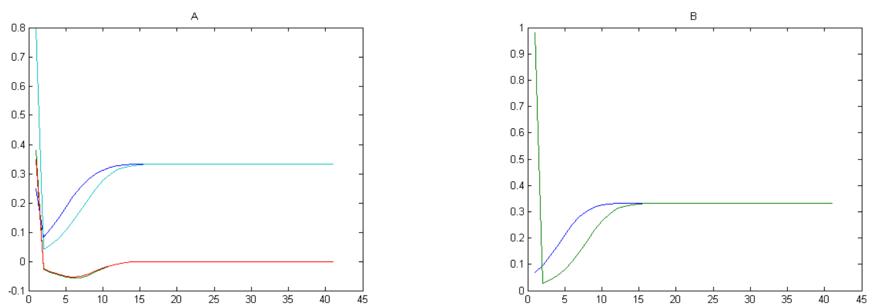


FIGURE 1. The matrix $X^{(0)} = \left(\begin{smallmatrix} 0.25 & 0.38 \\ 0.35 & 0.8 \end{smallmatrix}\right)$ coefficient results into asymptotic convergence (A showing the evolution of $x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}$ and B showing the evolution of eigenvalues $\lambda_1^{(n)}, \lambda_2^{(n)}$)

Fig. 1 is used to illustrate asymptotic and convergence of eigenvalues to a fixed point when $a = 1.5$. The type 1 matrix of initial conditions $\begin{bmatrix} 0.25 & 0.38 \\ 0.35 & 0.8 \end{bmatrix}$ is gradually transformed into a scalar matrix:

$$\lim_{n\to\infty} X^{(n)} = \begin{bmatrix} 0.33 & 0 \\ 0 & 0.33 \end{bmatrix}$$

and the coefficients x_{11},x_{22} converge to $0,33$ and the other two coefficients x_{12},x_{21} converge to 0 (Fig.1 A), while its eigenvalues $\lambda_1^{(0)} = 0.0682$ and $\lambda_2^{(0)} = 0.9818$ converge asymptotically to the fixed point $1 - a^{-1} = 0.33$ (Fig 1.B).

For our second example we again the parameter $a = 1.5$ and the initial condition

$$X^{(0)} = \begin{bmatrix} 2.2 & -0.7 \\ 3.5 & -1 \end{bmatrix}$$

Fig. 2 shows the results after 40 iterations. As we can observe the situation is similar to the previous example. For this case we make the histograms of the iterated matrix coefficients.

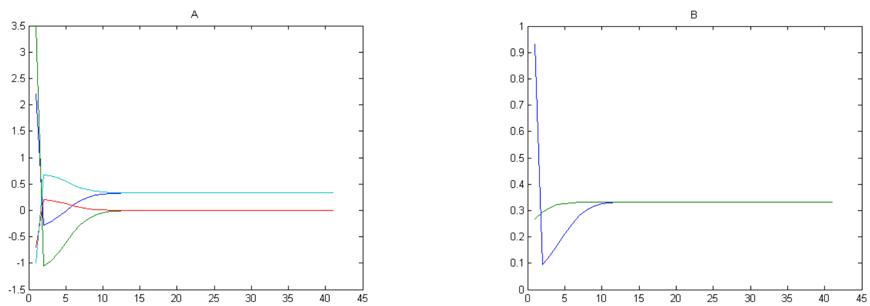
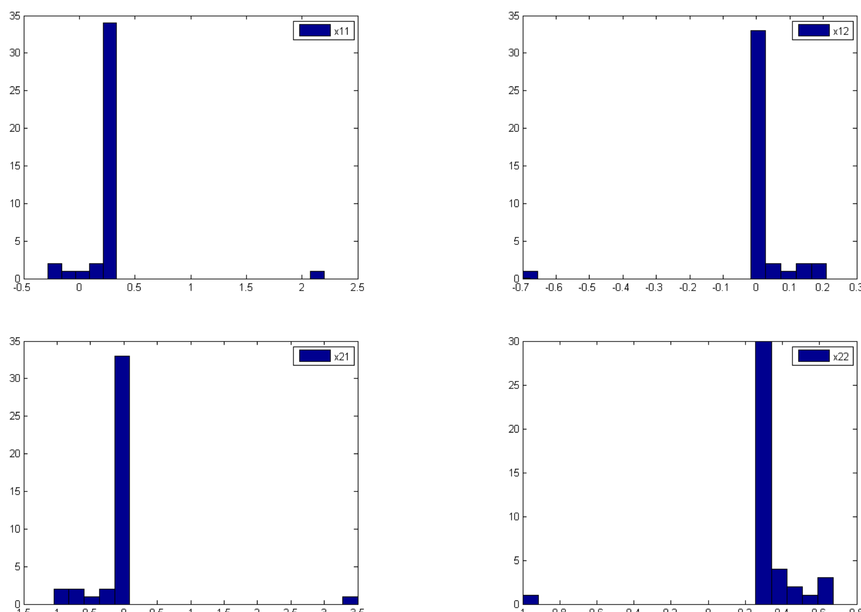


FIGURE 2. The matrix coefficient results into a close orbit at the biggining, very similar with a=1.5 (A showing the evolution of $x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}$ and B showing the evolution of eigenvalues $\lambda_1^{(n)}, \lambda_2^{(n)}$)



Therefore there exist such points which would yield the exact value of the stable fixed point $1-a^{-1}$ in a finite number of forward iterations (asymptotic convergence). For all other initial conditions (where the coefficients of the initial matrix lie in the interval [0,1]) converge to a fixed point asymptotically [[6]].

Now, we will consider the same examples with same initial condition matrix but with a different value of the parameter a. We will illustrate asymptotic and nonasymptotic convergence of eigenvalues for the case $a = 2.5$.

For the next example we take the type 1 matrix of initial condition

$$X^{(0)} = \begin{bmatrix} 0.25 & 0.38 \\ 0.35 & 0.8 \end{bmatrix}$$

The iterations are gradually transformed into a scalar matrix

$$\lim_{n\to\infty} X^{(n)} = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix},$$

while its eigenvalues $\lambda_1^{(0)} = 0.0682$ and $\lambda_2^{(0)} = 0.9818$ converge asymptotically to the fixed point $1 - a^{-1} = 0.6$. This behavior can be seen in Fig. 3.

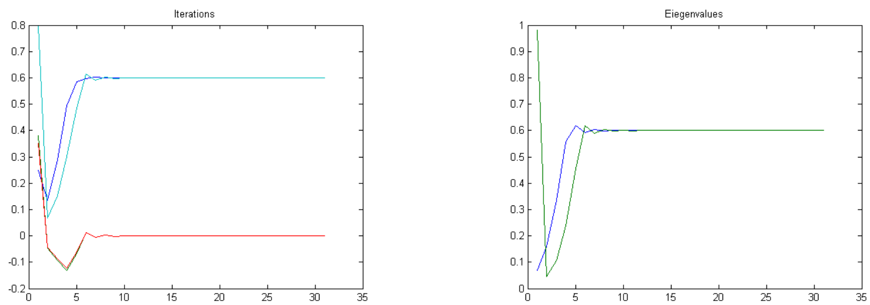


FIGURE 3. Asymptotic versus nonasymptotic convergence to a period-1 attractor: The initial matrix $X^{(0)} = \left(\begin{smallmatrix} 0.25 & 0.38 \\ 0.35 & 0.8 \end{smallmatrix}\right)$ results into asymptotic convergence.'Iterations' is showing the evolution of $x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}$ and 'Eigenvalues' is showing the evolution of eigenvalues

Alternatively, if we take the type 1 matrix of initial condition

$$X^{(0)} = \begin{bmatrix} 2.2 & -0.7 \\ 3.8 & -1.2 \end{bmatrix},$$

their iterations are transformed into a scalar matrix in few steps, namely

$$X^{(8)} = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix},$$

while its eigenvalues $\lambda_1^{(0)} = 0.02$ and $\lambda_2^{(0)} = 0.97$ converge nonasymptotically to 0.6 in the same number of iterations that the matrix did. See Fig. 4B for more details.

More complex examples of nonasymptotic convergence could be used to illustrate the transition from a type 1 matrix to a scalar matrix.

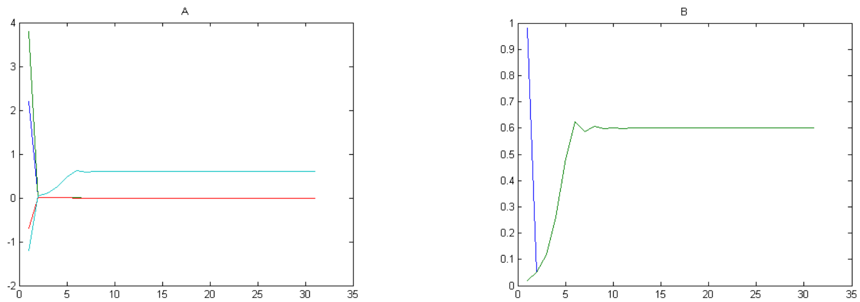


FIGURE 4. The initial matrix $X^{(0)} = \begin{pmatrix} 2.2 & -0.7 \\ 3.8 & -1.2 \end{pmatrix}$ results into nonasymptotic convergence (A showing the evolution of elements of the matrix and B showing the evolution of its eigenvalues)

2. Periodic attractors at $a = 3.5$

As we are getting close to the chaotic case we will illustrate the behavior of the iterates when the value of the parameter is $a = 3,5$. In this situation a 4-period attractor appears in the iterated eigenvalues; the convergence may be asymptoticor nonasymptotic. The following question arises! Will any type 1 matrix of initial condition evolve into a scalar matrix when eigenvalues will be gradually (or in a finite number of steps) attracted to an eventual 4-period attractor (eigenvalues of $X^{(0)}$ are bounded in [0,1] of course)? The answer is negative. Eigenvalues of $X^{(n)}$ will be attracted to an eventual 4-period attractor in any case, but a phase difference between iterated eigenvalues can be not necessarily equal to zero.

For example, for the type 1 initial matrix

$$X^{(0)} = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{bmatrix},$$

is gradually transformed into a sequence of scalar matrices (4 different scalar matrices in a 4-period) while its eigenvalues asymptotically converge to the period-4 attractor without a phase difference (Fig.5A and 5B).

For the type 1 initial matrix

$$X^{(0)} = \begin{bmatrix} 2.5 & 3.8 \\ 3.5 & 8 \end{bmatrix} \cdot \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix} \cdot \begin{bmatrix} 2.5 & 3.8 \\ 3.5 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -0.29 & 0.28 \\ -0.83 & 0.69 \end{bmatrix},$$

we get a sequence of type 1 matrices and the eigenvalues converge to the 4-period attractor with a constant phase difference not equal to 0 (Fig. 5C and 5C).

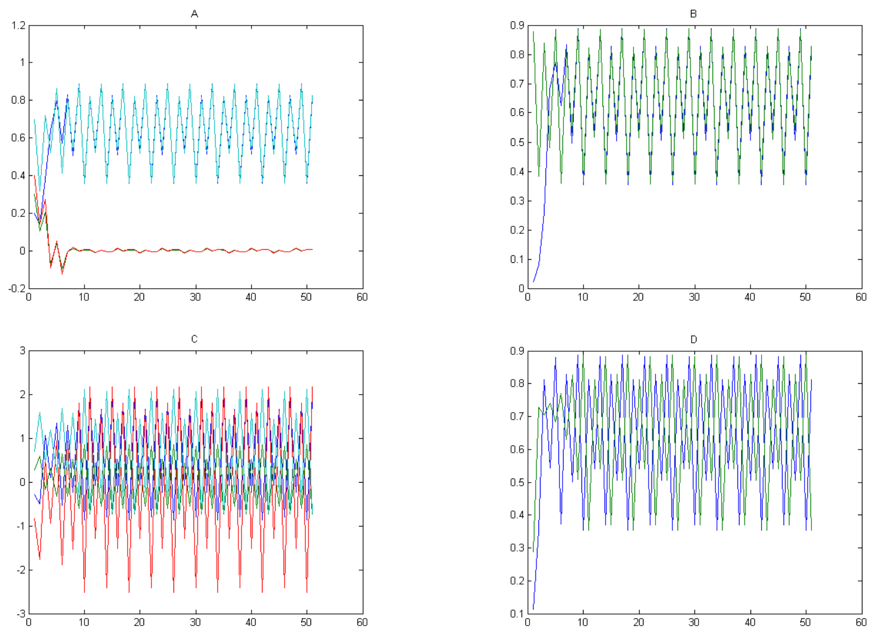
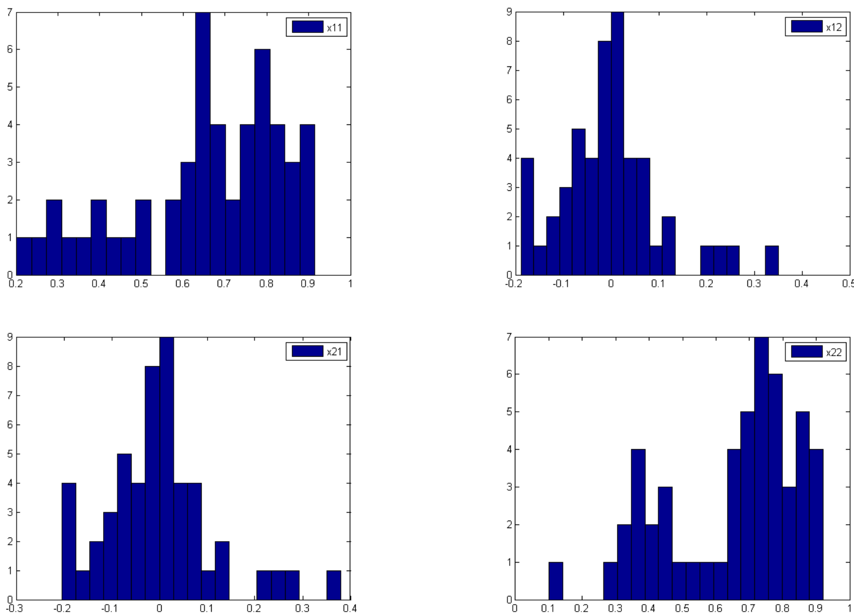


FIGURE 5. The type 1 matrix $X^{(0)} = \left(\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right)$ can yields a sequence of scalar matrices or a sequence of type 1 matrices: the initial matrix converges to a sequence of scalar matrices (A showing the evolution of elements of the matrix and B showing the evolution of its eigenvalues) the phase difference between eigenvalues in the 4-period regime is equal to 0; The matrix $X^{(0)} = \left(\begin{smallmatrix} 2.2 & 3.8 \\ -0.7 & -1.2 \end{smallmatrix}\right)$ yields a sequence of type 1 matrices (C showing the evolution of elements of the matrix $x_{11}, x_{12}, x_{21}, x_{22}$ and D showing the evolution of its eigenvalues λ_1, λ_2) because eigenvalues converge to the 4-period regime with a phase difference. The parameter $a = 3,55$ in the first case and $a = 3.5$ in the second.

Moreover, the cycles of the type 1 matrix : $\begin{bmatrix} 2.2 & 3.8 \\ -0.7 & -1.2 \end{bmatrix}$ could be observed better by showing the parameters evolution using the histograms. In our case is important to see the cycles for every coefficient of the matrix, even that the evolutions of x_{12}, x_{21} are very similar as we can see in the figure 5. The histogram (Fig. 5) represents the evolution of the elements $x_{11}, x_{12}, x_{21}, x_{22}$ of a type 1 matrix.



3. The evolution of the logistic map of matrices when $X^{(0)}$ is a type 2 matrix.

In this section we will focus on the situation when the matrix of initial condition is a type 2 matrix. Values of parameters $\lambda_{(0)} = 0.4, \widehat{\alpha} = 2, \widehat{\beta} = 10$ yield the next matrix:

$$X^{(0)} = \begin{bmatrix} 1.4 & 5 \\ -0.2 & -0.6 \end{bmatrix}$$

Fig. 6A and 6B show respectively the strong fluctuations of four scalar time coefficient and eigenvalues sequences for the first 50 iterations, but the fluctuations calm down as n increase. We observe that iterated matrices become scalar matrices and that the eigenvalues $\lambda_0^{(n)}$ oscillate in the interval [0,1]. It is interesting to note that the sequence of parameters $\mu_0^{(n)}$ and the sequence of coefficients of the type 2 matrix tend to zero after less then 100 iterations thus, ensuring the boundedness of $\left\{x_{kl}^{(n)}\right\}_{n=0}^{+\infty}; k, l = 1, 2..$ In Fig.6C and D we give a closer view of the iterations when $180 \leq n \leq 200$. We observe that some sequences remain close to 0 and others have a cyclic behavior.

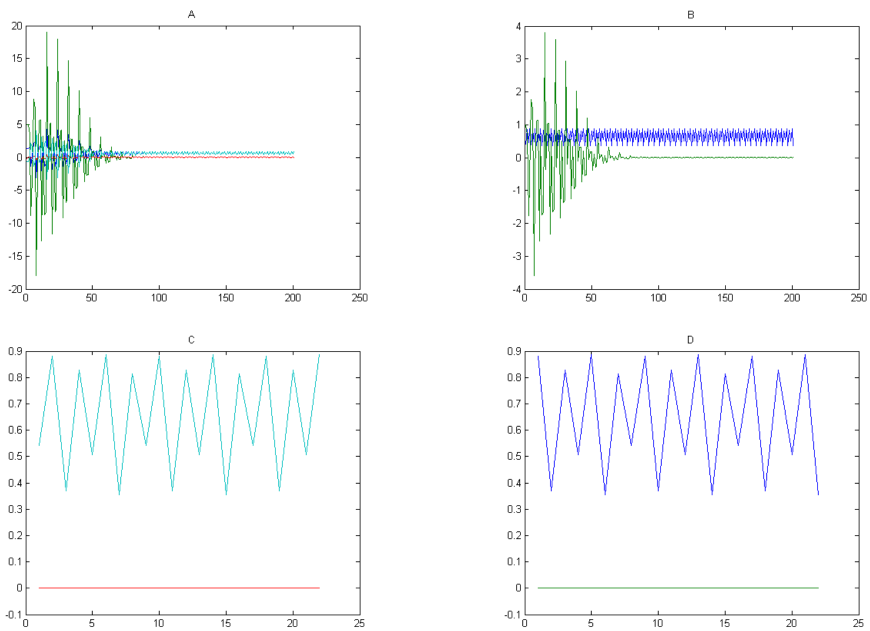


FIGURE 6. The evolution of the logistic map of the type 2 matrix $X^{(0)} = \left(\begin{smallmatrix} 1.4 & 5 \\ -0.2 & -0.6 \end{smallmatrix}\right)$ at $a = 3,55$.A and C showing the evolution of the matrix coefficients;B and D showing the evolution of the eigenvalue (a solid line) and the parameter $\mu_0^{(n)}$ deffined by Eq. (3.17) (a dashed line). Evolutions in C and D are displayed in the interval $180 \leq n \leq 200$ where n is the iteration number.

Now, we consider the value of the parameter $a = 3.7$. This value yields a violent divergence of the iterative process.

If we take a look of the histograms of the coefficients we see that there is not a clear cycle revealing that the orbits of the coefficient are not regular (see Fig. 8).

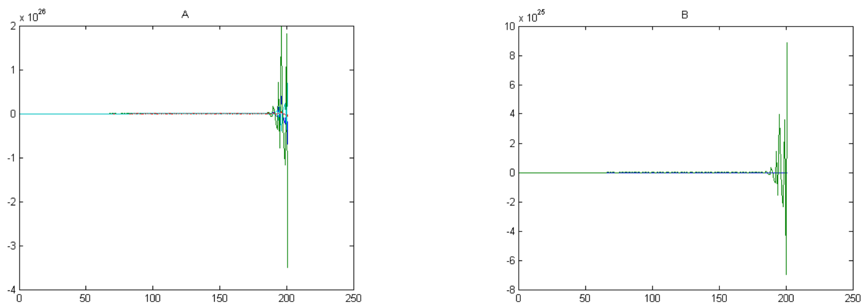


FIGURE 7. The evolution of the logistic map of matrices from $X^{(0)} = \left(\begin{smallmatrix} 1.4 & 5 \\ -0.2 & -0.6 \end{smallmatrix}\right)$ with $a = 3,7$. (A shows the evolution of $x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}$); B shows the evolution of the eigenvalue (a solid line) and the parameter $\mu_0^{(n)}$ defined by the equation 2.15 (a dashed line).

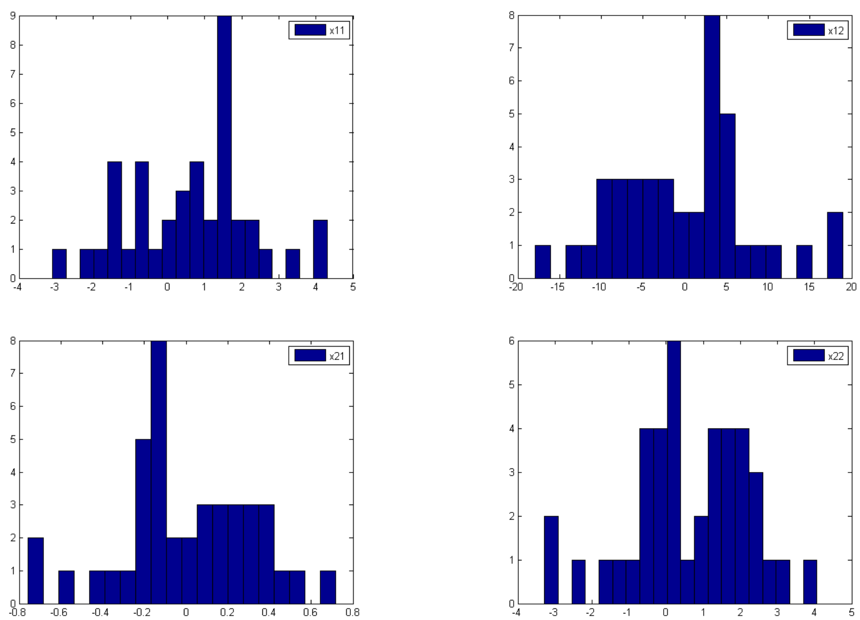


FIGURE 8. The evolution of the logistic map of the nilpotent matrix $X^{(0)} = \left(\begin{smallmatrix} 1.4 & 5 \\ -0.2 & -0.6 \end{smallmatrix}\right)$ with $a = 3,55$ whit histograms

4. The sensitivity to initial conditions when $a = 3.8$.

It is well known that a scalar logistic map evolves to chaos after a cascade of period doubling bifurcations. When $a = 3.8$, the dynamics of the scalar logistic map is almost chaotic. We will illustrate this feature for the logistic map of matrices. The matrix of initial condition is

$$X^{(0)} = \begin{bmatrix} 0.25 & 0.38 \\ 0.35 & 0.8 \end{bmatrix},$$

the sequences of eigenvalues yield seem to be chaotic (Fig. 10A and 10B).

To illustrate sensitive dependence on initial conditions we construct a perturbed initial matrix

$$X^{(0)} = \begin{bmatrix} 0.25+\varepsilon & 0.38+\varepsilon \\ 0.35+\varepsilon & 0.8+\varepsilon \end{bmatrix},$$

where $\varepsilon = 10^6$. The iterated process can be see in Fig. 10C and 10D). Differences between values of iterated elements and iterated eigenvalues of these matrices are shown in Fig. 10E and 10F. We observe that the difference between the two matrices became obvious after the thirtieth iteration. We may do more experiments but it is clear that for smaller ε we need more iterations to get the orbits to get apart. For this case we also construct the histograms in order to see the almost chaotic evolution (see Fig. 9).

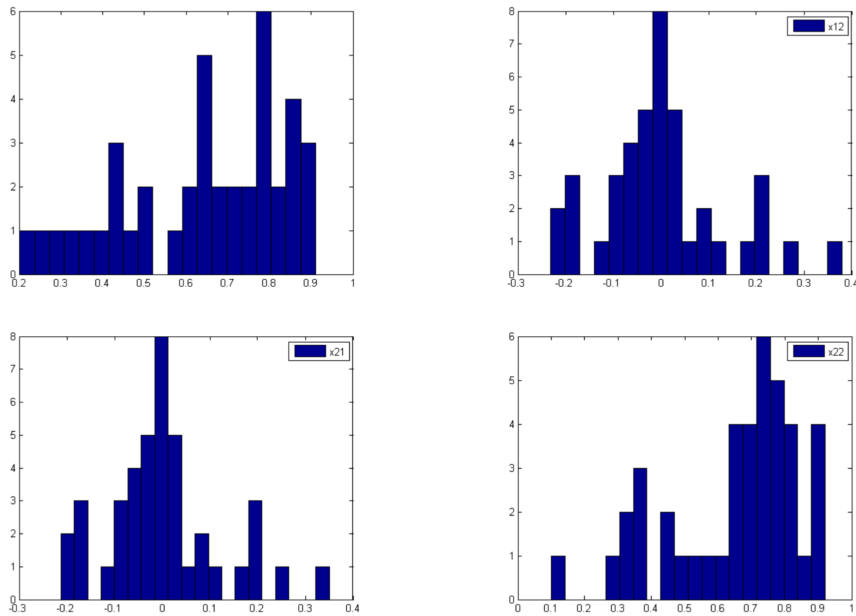


FIGURE 9. The illustration of the histogram of the matrix coefficients $X^{(0)} = \begin{pmatrix} 0.25 & 0.38 \\ 0.35 & 0.8 \end{pmatrix}$

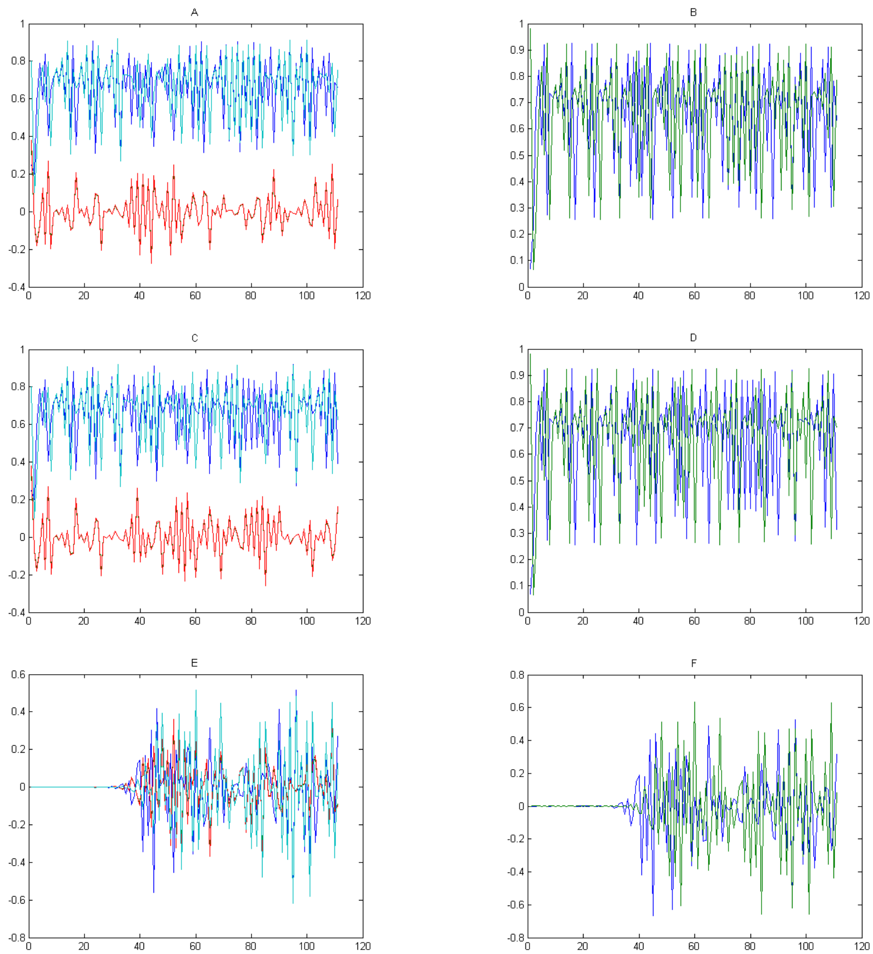


FIGURE 10. The illustration of the sensitivity to initial conditions at $a = 3.8$; $X^{(0)} = \begin{pmatrix} 0.25 & 0.38 \\ 0.35 & 0.8 \end{pmatrix}$ yields chaotic sequences (A-the evolution of elements; B-the evolution of eigenvalues);$X^{(0)} = \begin{pmatrix} 0.25+\varepsilon & 0.38+\varepsilon \\ 0.35+\varepsilon & 0.8+\varepsilon \end{pmatrix}$ also yields chaotic sequences (C-the evolution of elements; D-the evolution of eigenvalues); E shows differences between appropriate elements of matrices; F shows differences between appropriate eigenvalues.

5. Final conclusions and possible future work

The standard one dimensional logistic map is extended to square matrices by replacing the scalar iterative variable by a square matrix of variables. Main dynamical features of this iterative map are discussed in detail and illustrated by numerical examples.

It appears that the dynamics of the logistic map of matrices of order 2 can be interpreted exploiting the concept of independent one dimensional logistic maps of the eigenvalues of the matrix X_0. Two independent scalar logistic maps govern the dynamics of the logistic map of matrices if the matrix of initial conditions is a type 1 matrix. Explicit formulas for the iterations are provided in this work.

Alternatively, one scalar logistic map and another special (non-logistic) iterative relationship govern the evolution of the system if the matrix of initial conditions is a type 2 matrix. Also, in this case formulas to write the n-th iterated matrix are obtained.

It is important to remark here that the eigenvalues of the matrices involved in the process play an important role from the very beginning. In fact, the existence of two different eigenvalues or a single double one change completely the behavior of the iteration.

These results are not trivial at all and explain complex behavior of this relatively simple dynamical system.

To finish, let me highlight several future directions related with the work presented in this Master's Thesis. All these questions pose a definite interest in this topic and may be object of future work.

(1) The first question is about the size of the matrix, what are dynamical properties of the logistic map of square matrices of order n.
(2) The second question is regarding the convergence of the logistic map of matrices. Clocking convergence is a well explored topic in nonlinear dynamics of the standard logistic map; the Lyapunov exponent is used to characterize its stability. The computation of the spectrum of Lyapunov exponents and the measurement of the speed of convergence towards a stable attractor for the logistic map of matrices are rather subtle problems. It is not enough to perturb eigenvalues (consider the nilpotent matrix of initial conditions).
(3) The third (and probably the most important) question is about the potential of applicability of the logistic map of matrices for similar problems the standard logistic map has been used to modeling, encryption and prediction of different systems and processes.

Bibliography

[1] Z. Navickas,R. Smidtaite,A. Vainoras,Minvydas Ragulskis *The Logistic map of matrices* Discrete and Continuous Dynamical Systems series B,Volume 16,Nr. 3, October 2011

[2] Vladimir Igorevic Arnold *Ordinary differential equations,chapter 1 "Fundamental concepts"*.. various editions from MIT Press and from Springer Verlag.

[3] R.L. Devaney An introduction to chaotic dynamical systems. Addison Wesley, Reading, MA, 1989

[4] Paul Glendinning *Stability, Instability and Chaos*. Cambridge University Press,1994

[5] D.S.Bernstein *Matrix Mathematics:Theory, Facts and Formulas with Application to Linear Systems Theory* Princenton University Press,2005

[6] M. Ragulskis, Z. Navickas *The rank of a sequence as an indicator of chaos in discrete nonlinear dynamical systems* Commun. Nonlinear Sci., 16 (2011)