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Operation of the Discrete Wavelet Transform: basic overview with examples

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1 Summary

The aim of this paper is to present a didactic overview of the operation of the Discrete Wavelet Transform (DWT). Unlike the Fast Fourier Transform (FFT), extensively exploited to analyze stationary quantities, the DWT is a signal processing tool which is especially suited for the analysis of non-stationary signals. These signals are present in a countless number of everyday applications and processes. Hence, the use of suitable tools for their processing, such as the DWT, is a topic of increasing relevance.

In this particular work, the bases of the DWT are reviewed. The operation of the tool is briefly described under an engineering perspective, without deepening in complex mathematical details, which are easily available in well-known references. Some illustrative examples of the practical operation of the tool are included in the final part of the document.

2 Introduction

Stationary signals are signals whose spectral characteristics do not change with time. Nonetheless, most signals in the nature do not have this characteristic. Instead, these signals have a time-varying spectral content. A very obvious one is the human speech, in which frequencies change as we speak. In fact, the transmitted message relies upon the frequency change and on the time sequencing of the frequencies [5]. These signals are known as '*non-stationary*' signals, due to the fact that their basic spectral features do not remain constant but change with time.

In order to analyze non-stationary signals, the FFT is not longer suitable. Note that in the case, since the frequencies change with time, the analysis tool must be capable of extracting the time evolution of the frequency components present in the analyzed signal. FFT analysis does not enable this, since it implies a loss of time information. In order words, FFT only extracts the frequency content of the analyzed signal, but it does not inform on when each frequency occurs.

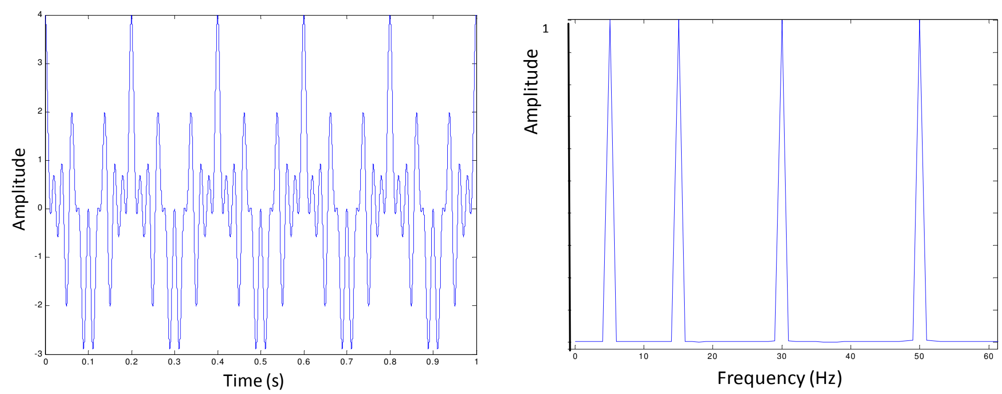
As an example, consider the following function, which has been built as an addition of four sinusoidal signals with similar amplitudes and with frequencies 5, 15, 30 and 50 Hz:

$$f(t) = \cos(2\cdot\pi\cdot 5\cdot t) + \cos(2\cdot\pi\cdot 15\cdot t) + \cos(2\cdot\pi\cdot 30\cdot t) + \cos(2\cdot\pi\cdot 50\cdot t)$$

Equation 1. Function based on the addition of four sinusoidal signals.

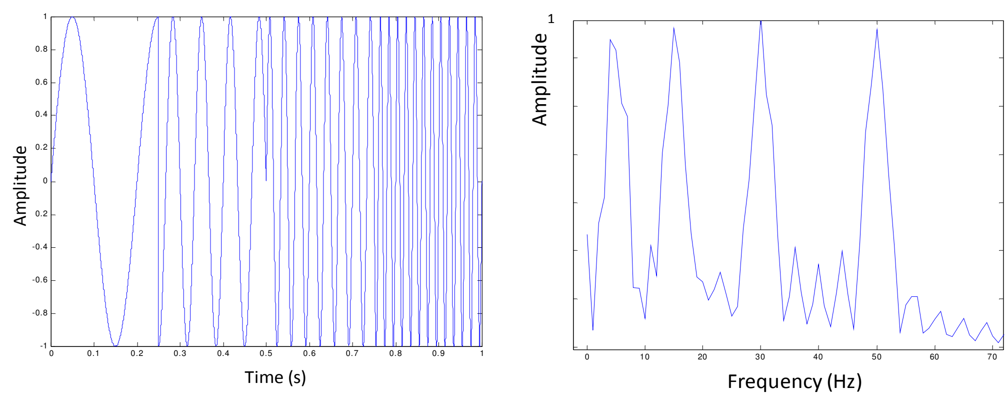
Figure 1(a) shows the representation of the function $f(t)$, given by Equation 1. It can be observed that all frequencies are present at every time. Hence, the signal has a stationary nature, considered as the invariability of its basic spectral features with time. Figure 1(b) shows its FFT analysis: it reveals four frequency 'peaks' at the aforementioned frequencies.

On the other hand, consider now the function depicted in Figure 2(a); in that function, the same frequency components appear but, in this case, they occur at different time instants. The signal in this case is no longer stationary. Figure 2(b) shows the FFT analysis of this signal: it also reveals the presence of four 'peaks' at the corresponding frequencies.



(a) *(b)*

Figure 1. (a) Representation of $f(t)$, (b)FFT analysis of $f(t)$



(a) *(b)*

Figure 2. (a) Different frequencies at different times, (b)FFT analysis

These examples illustrate one of the drawbacks of the FFT analysis: since the transform implies a loss of time information, simply extracting the frequency components, two rather different signals (such as those plotted in Figure 1(a) and Figure 2(a)) can have similar representations in terms of their FFT spectra. In other words, the FFT only extracts the frequency content of a signal, which may be enough for stationary signals, but not for non-stationary, in which the knowledge of the time at which each frequency occurs is fundamental for the comprehension of the signal structure.

In this context is where novel time-frequency decomposition (TFD) tools, suited for the analysis of non-stationary signals rise. These tools enable to extract, not only the frequency content present in a certain signal, but also the time information (i.e. when the frequencies occur). The TFD tools lead to a time-frequency representation of the analyzed signal.

3 Objetives

The present work has three main goals:

- To explain the operation of a particular TFD tool, the Discrete Wavelet Transform, under an engineering perspective, without getting into complex mathematical details that are available in well-known references.
- To provide several illustrative examples facilitating the comprehension of the DWT operation.
- To review the advantages and drawbacks of the transform, in comparison with classical FFT analysis and with other TFD tools.

The present work can be especially useful for students or researchers involved in the study of applications implying the analysis of non-stationary signals. In this context, the DWT has recently revealed itself as a very powerful tool, providing important advantages versus other techniques.

4 Development

4.1 Foundations

When the Discrete Wavelet Transform (DWT) is applied to a certain sampled function $s(t)$, this function is decomposed as the addition of a set of signals, named *wavelet signals*: an *approximation signal* at a certain decomposition level *n* (a_n) plus *n detail signals* (d_j with j varying from 1 to *n*). The mathematical expression characterizing this process is given by Equation 2, where α_i^n, β_i^j are the scaling and wavelet coefficients, $\varphi^n(\mathrm{t})$, $\psi^j(\mathrm{t})$ are the scaling function at level *n* and wavelet function at level *j*, respectively, and *n* is the decomposition level [1-3].

$$s(t) = \sum_i \alpha_i^n \cdot \boldsymbol{\varphi_i^n(t)} + \sum_{j=1}^{n} \sum_i \beta_i^j \cdot \boldsymbol{\psi_i^j(t)} = a_n + d_n + \ldots + d_1$$

Equation 2. Decomposition of the signal $s(t)$ in terms of wavelet signals.

Each one of the wavelet signals (approximation and detail) has an associated frequency band, the limits of which are well-established, once the sampling rate (f_s) of the original analyzed signal is known, in accordance with an algorithm enunciated by S. Mallat (*Subband Coding Algorithm*) [2]. The expressions used to calculate the limits of the frequency bands associated with each wavelet signal, according to the Mallat algorithm, are specified in Figure 3 [4]. It is observed how the limits of the frequency band for each wavelet signal depend on the sampling rate (f_s) as well as on the level of the corresponding wavelet signal (j). As an example, if the sampling rate used for capturing $s(t)$ is f_s=10000 samples/second, and we perform the DWT decomposition in *n*=8 levels, the frequency bands associated with each wavelet signal are those specified in Table I.

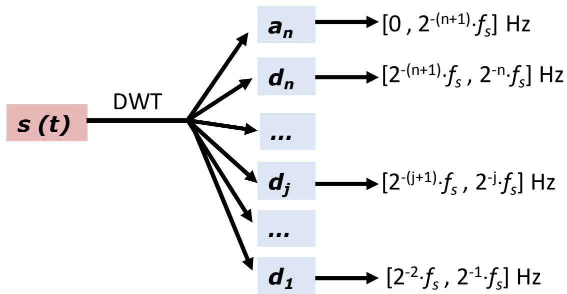


Figure 3. DWT decomposition in wavelet signals and associated frequency bands

Wavelet signal	Frequency band
a8	[0-19'5] Hz
d8	[19'5-39] Hz
d7	[39-78'1] Hz
d6	[78'1-156'2] Hz
d5	[156'2-312'5] Hz
d4	[312'5-625] Hz
d3	[625-1250] Hz
d2	[1250-2500] Hz
d1	[2500-5000] Hz

Table 1. Frequency bands associated with wavelet signals for f_s=10 kHz and n=8

The intuitive idea underlying the application of the DWT relies on the following fact: each one of the wavelet signal acts as a filter, extracting the temporal evolution of the components of the original signal contained within the frequency band associated with that wavelet signal. For instance, in the previous example, the wavelet signal d7 (*detail signal 7*) will reflect the time evolution of every harmonic component of the original signal when its frequency falls in the band [39-78'1] Hz. For instance, if the signal is a pure 50 Hz sinusoidal waveform, the whole signal evolution would be reflected in that signal d7.

In conclusion, the DWT performs a dyadic band-pass filtering process in frequency bands whose limits depend on f_s and on n. This filtering is illustrated in Figure 4.

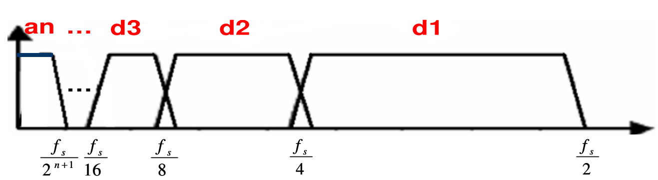


Figure 4. Dyadic filtering process carried out by the DWT

4.2 Examples

In this section, several didactic examples of the operation of the DWT are explained. They are useful to understand how the transform works in a very simple way. In all three examples, the DWT decomposition is carried out in *n*=9 levels and DB-44 is used as mother wavelet for the analyses. The corresponding frequency bands associated with each wavelet signal are specified beside each figure.

4.2.1 Example 1: DWT analysis of a pure sinusoidal signal

Figure 5 shows the DWT decomposition for the case of a 50 Hz pure sinusoidal signal (signal s, plotted at the top of the figure). It is observed how, in accordance with the filtering process carried out by the transform, the whole signal is filtered into the detail signal d7. This is due to the fact that this signal reflects the evolution of every component evolving within the range [39-78,1]Hz. Since there is a single 50 Hz component in the original signal, d7 exactly reflects the evolution of the whole component and, hence, of the signal. The rest of wavelet signals are approximately zero, since no other frequency component exist in the original signal.

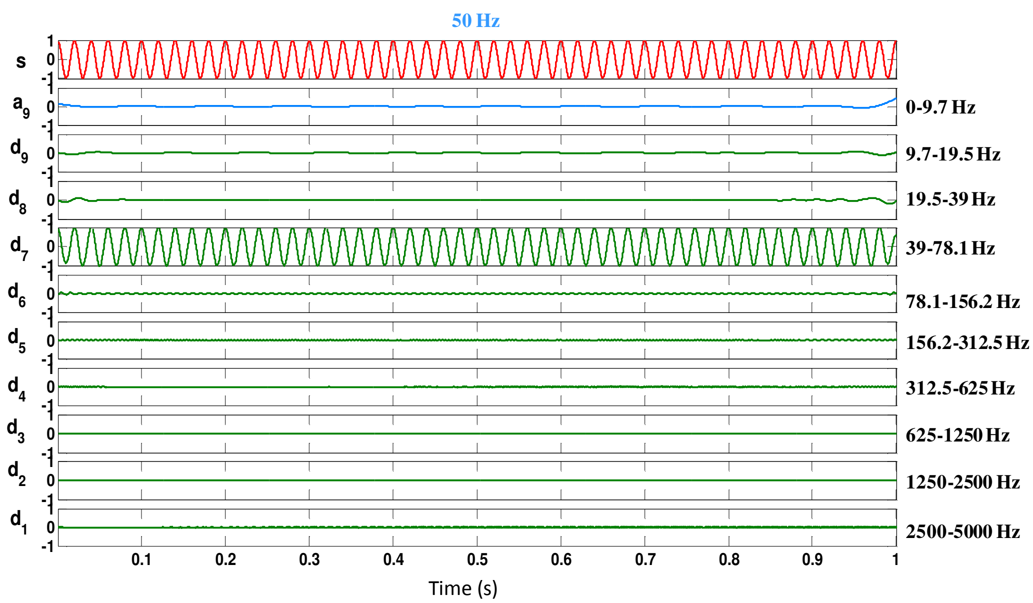


Figure 5. DWT analysis of a 50 Hz pure sinusoidal signal

4.2.2 Example 2: Superposition of sinusoidal signals

Figure 6 shows the DWT analysis of a signal s (plotted at the top of that figure) which has been built by adding four sinusoidal signals with respective frequencies 5 Hz, 15 Hz, 30 Hz and 50 Hz. The result is a stationary signal in which all four frequencies are present at every time. The filtering nature of the DWT enables to extract each frequency component in a separate wavelet signal, in agreement with the values of their respective band limits. As it is observed, the 5Hz component is filtered in a9, the 15 Hz component in d9, the 30 Hz component in d8 and the 50 Hz component in d7, remaining almost zero the rest of signals, since no other components exist within their bands. This example illustrates the filtering process carried out by the transform and its ability to separate the different components of the signals, provided that they fall in different frequency bands covered by the wavelet signals.

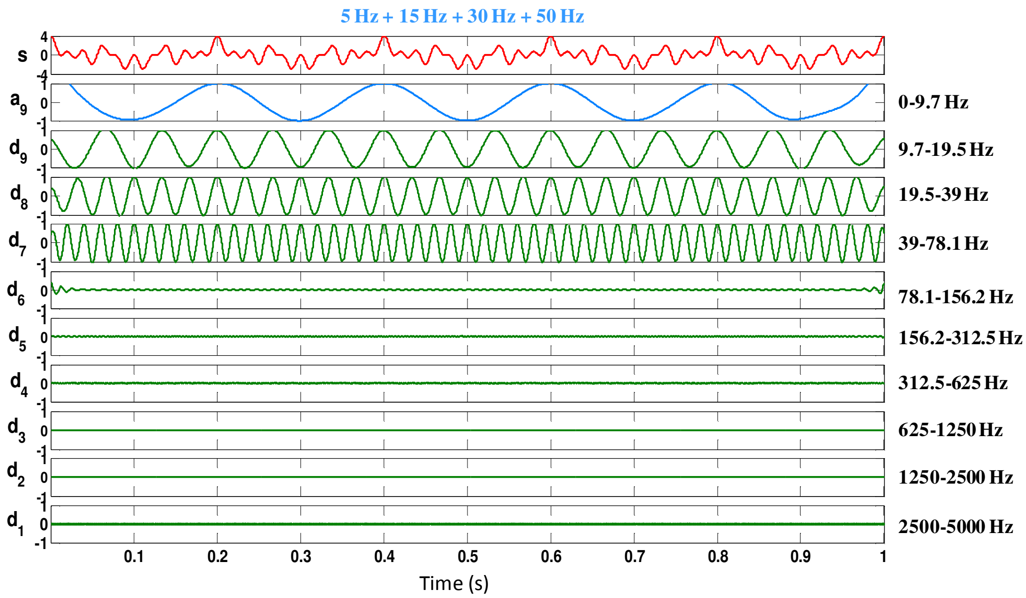


Figure 6. DWT analysis of a signal based on the superposition of four sinusoidal signals with frequencies 5 Hz, 15 Hz, 30 Hz and 50 Hz.

4.2.3 Example 3: Concatenation of sinusoidal signals

Figure 7 represents the DWT analysis of a signal s (plotted at the top of the figure) which has been built by concatenating four sinusoidal signals with respective frequencies 5 Hz, 15 Hz, 30 Hz and 50 Hz. The result is a non-stationary signal, in which each frequency component is present only during its corresponding time interval.

The application of the DWT leads to filter each component in the wavelet signal covering the frequency band in which it is included. Hence, the 5Hz component is filtered in a9, the 15 Hz component in d9, the 30 Hz component in d8 and the 50 Hz component in d7, remaining almost zero the rest of signals since no components exist within their bands. Moreover, the transform indicates when each component starts and ends in the analyzed signal; for instance, a9 shows how the 5 Hz component is present during the initial 0,25 seconds, d9 shows that the 15 Hz component is present between 0,25 and 0,5 s, d8 reveals that the 30 Hz component occurs between 0,5 and 0,75 seconds and, finally, d7 shows that the 50 Hz component is present between 0,75 and 1 second.

This example illustrates a clear advantage of the DWT versus the classical FFT approach. Whereas with the FFT, the time information was lost and two rather different signals (such as those analyzed in Examples 2 and 3) could be represented by similar FFT spectra (see Figures 1 and 2), the DWT preserves the time information, enabling to identify not only which frequencies are present but also when they occur. Therefore, DWT leads to a three-dimensional representation of the analyzed signal: *frequency* (because each wavelet signal covers a frequency band), *time* (since each wavelet signal is represented versus time) and *amplitude* (the amplitude of the wavelet signal informs on the corresponding amplitude of its filtered components in the analyzed signal). This is why DWT is known as a time-frequency decomposition tool.

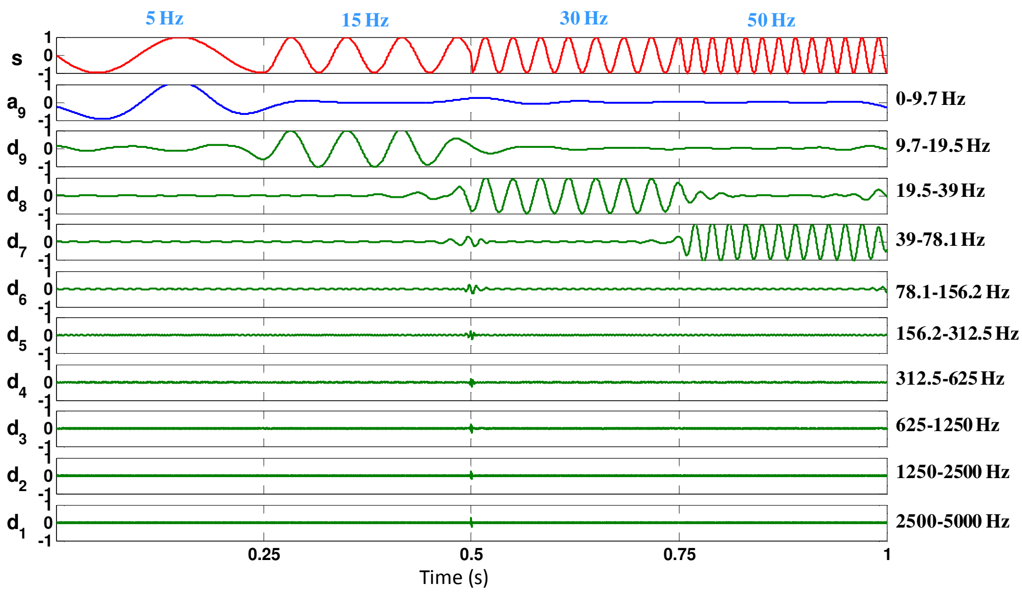


Figure 7. DWT analysis of a signal based on the concatenation of four sinusoidal signals with frequencies 5 Hz, 15 Hz, 30 Hz and 50 Hz.

5 Closing

This work has presented a basic overview of the operation of the Discrete Wavelet Transform (DWT). This is a time-frequency decomposition tool which has been used with success for the analysis of non-stationary signals, overcoming some drawbacks of the FFT when analyzing such signals, such as the loss of time information. The DWT enables a band pass filtering of the analyzed signal in well-established frequency bands. Moreover, it preserves the time information, since each wavelet signal in represented versus time.

The intention of the work has been to introduce the operation of the transform under a simple engineering perspective, without deepening in its mathematical background, which is easily accessible in well-known textbooks. In this regard, the work emphasizes the filtering process carried out by the transform, detailing the expressions to calculate the limits of the bands associated with the different wavelet signals. Moreover, some examples illustrating the operation of the transform are included: the analysis of a pure sinusoidal signal, the analysis of a signal based on the superposition of sinusoidal signals and the analysis of a signal based on the concatenation of sinusoidal signals. All three didactic examples are useful to show how the wavelet signals operate.

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As a conclusion of the ideas exposed in this work we can summarize some of the the *advantages* of the DWT in the following points:

- Simplicity
- General availability of the DWT algorithm in conventional software packages.
- Easy interpretation of the results
- Low computational burden

With regards to its *drawbacks* we can remark, among other, the following ones:

- Lower flexibility (limits of the bands are fixed, once the sampling rate is known)
- Reduced frequency resolution for the high frequencies
- Possible difficult discrimination of components when they fall within the same band.

6 References

6.1 Textbooks:

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[2] Mallat, S.: "A Wavelet Tour of Signal Processing, Third Edition". Academic Press, dec. 2008.

[3] Chui, C.K.: "Wavelets: A Mathematical Tool for Signal Analysis", SIAM, 1997.

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[4] Antonino-Daviu, J.; Riera-Guasp, M.; Roger-Folch, J.; Molina, M.P.: "Validation of a New Method for the Diagnosis of Rotor bar Failures via Wavelet Transformation in Industrial Induction Machines," IEEE Transactions on Industry Applications, Vol. 42, No. 4, July/August 2006, pp. 990-996.

6.3 Electronic source references:

[5] Sala Mayato, R.F.; Trujillo González, R. (2004). Análisis de señales Tiempo-Frecuencia. Curso organizado por el Departamento de Física Fundamental. Disponible en: http://www.iac.es/ensenanza/tercer-c/11-20.htm