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The combinatorial derivation

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ABSTRACT

Let G be a group, $\mathcal{P}_G$ be the family of all subsets of G. For a subset $A \subseteq G$, we put $\Delta(A) = \{g \in G : |gA \cap A| = \infty\}$. The mapping $\Delta : \mathcal{P}_G \to \mathcal{P}_G$, $A \mapsto \Delta(A)$, is called a combinatorial derivation and can be considered as an analogue of the topological derivation $d : \mathcal{P}_X \to \mathcal{P}_X$, $A \mapsto A^d$, where X is a topological space and A^d is the set of all limit points of A. Content: elementary properties, thin and almost thin subsets, partitions, inverse construction and Δ-trajectories, Δ and d.

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1. INTRODUCTION

Let G be a group with the identity e, $\mathcal{P}_G$ be the family of all subsets of G. For a subset A of G, we denote

$$\Delta(A) = \{g \in G : |gA \cap A| = \infty\},$$

observe that $\Delta(A) \subseteq AA^{-1}$, and say that the mapping

$$\Delta : \mathcal{P}_G \to \mathcal{P}_G, A \mapsto \Delta(A)$$

is the *combinatorial derivation*.

In this paper, on one hand, we analyze from the Δ-point of view a series of results from Subset Combinatorics of Groups (see the survey [9]), and point out some directions for further progress. On the other hand, the Δ-operation is interesting and intriguing by its own sake. In contrast to the trajectory $A \to$

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$AA^{-1} \to (AA^{-1})(AA^{-1})^{-1} \to \dots$, the Δ-trajectory $A \to \Delta(A) \to \Delta^2(A) \to \dots$ of a subset A of G could be surprisingly complicated: stabilizing, increasing, decreasing, periodic or chaotic. For a symmetric subset A of G with $e \in A$, there exists a subset $X \subseteq G$ such that $\Delta(X) = A$. We conclude the paper by demonstrating how Δ and a topological derivation d arise from some unified ultrafilter construction.

We note also that $\Delta(A)$ may be considered as some infinite version of the symmetry sets well-known in Additive Combinatorics [11, p. 84]. Given a finite subset A of an Abelian group G and $\alpha \geqslant 0$, the symmetry set $Sym_\alpha(A)$ is defined by

$$Sym_\alpha(A) = \{g \in G : |A \cap (A+g)| \geqslant \alpha|A|\}.$$

2. Elementary properties

Claim 2.1. $(\Delta(A))^{-1} = \Delta(A)$, $\Delta(A) \subseteq AA^{-1}$.

Claim 2.2. $\Delta(A) = \varnothing \Leftrightarrow e \notin \Delta(A) \Leftrightarrow A$ *is finite.*

Claim 2.3. *For subsets A, B of G, we let*

$$\Delta(A, B) = \{g \in G : |gA \cap B| = \infty\}$$

and note that

$$\Delta(A \cup B) = \Delta(A) \cup \Delta(B) \cup \Delta(A, B) \cup \Delta(B, A),$$
$$\Delta(A \cap B) \subseteq \Delta(A) \cap \Delta(B)$$

.

Claim 2.4. *If F is a finite subset of G then*

$$\Delta(FA) = F\Delta(A)F^{-1}.$$

Claim 2.5. *If A is an infinite subgroup then $A = \Delta(A)$ but the converse statement does not hold, see Theorem 6.2.*

3. Thin and almost thin subsets

A subset A of a group G is said to be [8]:

- *thin* if either A is finite or $\Delta(A) = \{e\}$;
- *almost thin* if $\Delta(A)$ is finite;
- *k-thin* ($k \in \mathbb{N}$) if $|gA \cap A| \leqslant k$ for each $g \in G \setminus \{e\}$;
- *sparse* if, for every infinite subset $X \subseteq G$, there exists a non-empty finite subset $F \subset X$ such that $\bigcap_{g \in F} gA$ is infinite;
- *k-sparse* ($k \in \mathbb{N}$) if, for every infinite subset $X \subseteq G$, there exists a subset $F \subset X$ such that $|F| \leqslant k$ and $\bigcap_{g \in F} gA$ is finite.

The following statements are from [8].

Theorem 3.1. *Every almost thin subset A of a group G can be partitioned in $3^{|\Delta(A)|-1}$ thin subsets. If G has no elements of odd order, then A can be partitioned in $2^{|\Delta A|-1}$ thin subsets.*

Theorem 3.2. *A subset A of a group G is 2-sparse if and only if $X^{-1}X \nsubseteq \Delta(A)$ for every infinite subset X of G.*

Theorem 3.3. *For every countable thin subset A of a group G, there is a thin subset B such that $A \cup B$ is 2-sparse but not almost thin.*

Theorem 3.4. *Suppose that a group G is either torsion-free or, for every $n \in \mathbb{N}$, there exists a finite subgroup H_n of G such that $|H_n| > n$. Then there exists a 2-sparse subset of G which cannot be partitioned in finitely many thin subsets.*

By Theorem 3.2, every almost thin subset is 2-sparse. By Theorems 3.3, 3.4, the class of 2-sparse subsets is wider than the class of almost thin subsets. By Theorem 3.3, a union of two thin subsets needs not to be almost thin. By Theorem 2.3, a union $A_1 \cup \ldots \cup A_n$ of almost thin subset is almost thin if and only if $\Delta(A_i, A_j)$ is finite for all $i, j \in \{1, \ldots, n\}$, By Claim 2.4, if A is almost thin and K is finite then KA is almost thin.

The following statements are from [7].

Theorem 3.5. *For every infinite group G, there exists a 2-thin subset such that $G = XX^{-1} \cup X^{-1}X$.*

Theorem 3.6. *For every infinite group G, there exists a 4-thin subset such that $G = XX^{-1}$.*

Since $\Delta(X) = \{e\}$ for each infinite thin subset of G, Theorem 3.6 gives us X with $\Delta(X) = \{e\}$ and $XX^{-1} = G$.

4. Large and small subsets

A subset A of a group G is called [8]:

- *large* if there exists a finite subset F of G such that $G = FA$;
- Δ-*large* if $\Delta(A)$ is large;
- *small* if $(G \setminus A) \cap L$ is large for each large subset L of G;
- *P-small* if there exists an injective sequence $(g_n)_{n\in\omega}$ in G such that the subsets $\{g_nA : n \in \omega\}$ are pairwise disjoint;
- *almost P-small* if there exists an injective sequence $(g_n)_{n\in\omega}$ in G such that the family $\{g_nA : n \in \omega\}$ is almost disjoint, i.e. $g_nA \cap g_mA$ is finite for all distinct $n, m \in \omega$.
- *weakly P-small* if, for every $n \in \omega$, one can find distinct elements $g_1, \ldots, g_n$ of G such that the subsets $g_1A, \ldots, g_nA$ are pairwise disjoint.

Let G be a group, A is a large subset of G. We take a finite subset F of G, $F = \{g_1, \ldots, g_n\}$ such that $G = FA$. Take an arbitrary $g \in G$. Then $g_iA \cap gA$ is infinite for some $i \in \{1, \ldots, n\}$, so $g_i^{-1}g \in \Delta(A)$. Hence, $G = F\Delta(A)$ and A is Δ-large. By Theorem 3.6, the converse statement is very far from being true.

If A is not small then FA is thick (see Definition 5.2) for some finite subset F. It follows that $\Delta(FA) = G$. By Claim 2.4, $\Delta(FA) = F\Delta(A)F^{-1}$, so if G is Abelian then A is Δ-large.

J. Erde proved that every non-small subset of an arbitrary infinite non-Abelian group G Δ-large.

It is easy to see that A is P-small (almost P-small) if and only if there exists an infinite subset X of G such that $X^{-1}X \cap PP^{-1} = \{e\}$ ($X^{-1}X \cap \Delta(X) = \{e\}$). A is weakly P-small if and only if, for every $n \in \omega$, there exists $F \subset G$ such that $|F| = n$ and $F^{-1}F \cap PP^{-1} = \{e\}$.

By [8, Lemma 4.2], if AA^{-1} is not large then A is small and P-small. Using the inverse construction from Section 6, we can find A such that A is not Δ-large and A is not P-small.

Every infinite group G has a weakly P-small not P-small subsets [1]. Moreover, G has almost P-small not P-small subset and , if G is countable, weakly P-small not almost P-small subset. By [8], every almost P-small subset can be partitioned in two P-small subsets. If A is either almost or weakly P-small then $G \setminus \Delta(A)$ is infinite, but a subset A with infinite $G \setminus \Delta(A)$ could be large: $G = \mathbb{Z}$, $A = 2\mathbb{Z}$.

5. Partitions

Let G be a group and let $G = A_1 \cup \dots A_n$ be a finite partition of G. In section 7, we show that at least one cell A_i is Δ-large, in particular, $A_iA_i^{-1}$ is large. If G is infinite amenable group and μ is a left invariant Banach measure on G, we can strengthened this statement: there exist a cell A_i and a finite subset F such that $|F| \leqslant n$ and $G = F\Delta(A_i)$. To verify this statement, we take A_i such that $\mu(A_i) \geqslant \frac{1}{n}$ and choose distinct $g_1, \dots, g_m$ such that $\mu(g_kA_i \cap g_lA_i) = 0$ for all distinct $k, l \in \{1, \dots, m\}$, and the family $\{g_1A_i, \dots, g_mA_i\}$ is maximal with respect to this property. Clearly, $m \leqslant n$. For each $g \in G$, we have $\mu(gA_i \cap g_kA_i) > 0$ for some $k \in \{1, \dots, m\}$ so $g_k^{-1}g \in \Delta(A_i)$ and $G = \{g_1, \dots, g_m\}\Delta(A_i)$.

By [10, Theorem 12.7], for every partition $A_1 \cup \dots \cup A_n$ of an arbitrary group G, there exist a cell A_i and a finite subset F of G such that $G = FA_iA_i^{-1}$ and $|F| \leqslant 2^{2^{n-1}-1}$. S. Slobodianiuk strengthened this statement: there are F and A_i such that $|F| \leqslant 2^{2^{n-1}-1}$ and $G = F\Delta(A_i)$.

It is an old unsolved problem [5, Problem 13.44] whether i and F can be chosen so that $G = FA_iA_i^{-1}$ and $|F| \leqslant n$, see also [10, Question 12.1].

Question 5.1. *Given any partition $G = A_1 \cup \dots \cup A_n$, do there exist F and A_i such that $G = F\Delta(A_i)$ and $|F| \leqslant 2^n$?*

Definition 5.2. A subset A of a group G is called [11]:

- *thick* if $G \setminus A$ is not large;
- *k-prethick* ($k \in \mathbb{N}$) if there exists a subset F of G such that $|F| \leqslant k$ and FA is thick;
- *prethick* if A is k-prethick for some $k \in \mathbb{N}$.

By [3, Theorem 5.3.2], for a group G, the following two conditions (i) and (ii) are equivalent:

(i) for every partition $G = A \cup B$, either $G = AA^{-1}$ or $G = BB^{-1}$;
(ii) each element of G has odd order.

If G is infinite, we can show that these conditions are equivalent to

(iii) for every partition $G = A \cup B$, either $G = \Delta(A)$ or $G = \Delta(G)$.

6. Inverse construction and Δ-trajectories

Theorem 6.1. *Let G be an infinite group, $A \subseteq G$, $A = A^{-1}$, $e \in A$. Then there exists a subset X of G such that $\Delta(X) = A$.*

Proof. First, assume that G is countable and write the elements of A in the list $\{a_n : n < \omega\}$, if A is finite then all but finitely many a_n are equal to e. We represent $G \setminus A$ as a union $G \setminus A = \bigcup_{n\in\omega} B_n$ of finite subsets such that $B_n \subseteq B_{n+1}$, $B_n^{-1} = B_n$. Then we choose inductively a sequence $(X_n)_{n\in\omega}$ of finite subsets of G,

$$X_n = \{x_{n0}, x_{n1}, \ldots, x_{nn}, a_0 x_{n0}, \ldots, a_n x_{nn}\}$$

such that $X_m X_n^{-1} \cap B_n = \{e\}$ for all $m \leqslant n < \infty$.

After ω steps, we put $X = \bigcup_{n\in\omega} X_n$. By the construction, $\Delta(X) = A$.

If $|A| \leqslant \aleph_0$ but G is not countable, we take a countable subgroup H of G such that $A \subseteq H$, forget about G and find a subset $X \subseteq H$ such that $\Delta(X)$ is equal to A in H. Since $gA \cap A = \varnothing$ for each $g \in G \setminus H$, we have $\Delta(X) = A$.

At last, let $|A| > \aleph_0$. By above paragraph, we may suppose that $|A| = |G|$. We enumerate $A = \{a_\alpha : \alpha < |G|\}$ and construct inductively a sequence $(X_\alpha)_{\alpha<|G|}$ of finite subsets of G and an increasing sequence $(H_\alpha)_{\alpha<|G|}$ of subgroup of G such that if $\alpha = 0$ or α is a limit ordinal, $n \in \omega$,

$$X_{\alpha+n} = \{x_{\alpha+n,0}, x_{\alpha+n,1}, \ldots, x_{\alpha+n,n}, a_\alpha x_{\alpha+n,0}, \ldots, a_{\alpha+n} x_{\alpha+n,n}\},$$

$$X_{\alpha+n} \subseteq H_{\alpha+n+1} \setminus H_{\alpha+n},\ X_{\alpha+n} X_{\alpha+n}^{-1} \subseteq A \cup (H_{\alpha+n+1} \setminus H_{\alpha+n}).$$

After $|G|$ steps, we put $X = \bigcup_{\alpha<|G|} X_\alpha$. By the construction, $\Delta(X) = A$. □

Let $A_1, \ldots, A_m$ be subsets of an infinite group G such that $G = A_1 \cup \ldots \cup A_m$. By the Hindman theorem [4, Theorem 5.8], there are exists $i \in \{1, \ldots, m\}$ and an injective sequence $(g_n)_{n\in\omega}$ in G such that $FP(g_n)_{n\in\omega} \subseteq A_i$, where $FP(g_n)_{n\in\omega}$ is a set of all element of the form $g_{i_1} g_{i_2} \ldots g_{i_l}$, $i_1 < \ldots, i_k < \omega$, $k \in \omega$.

We show that there exists $X \subseteq FP(g_n)_{n\in\omega}$ such that $\Delta(X) = \{e\} \cup FP(g_n)_{n\in\omega} \cup (FP(g_n)_{n\in\omega})^{-1}$. We note that if G is countable, at each step n of the inverse construction, the elements $x_{n0}, \ldots, x_{nn}$ can be chosen from any pregiven infinite subset Y of G. We enumerate $FP(g_n)_{n\in\omega}$ in a sequence $(a_n)_{n\in\omega}$ and put $Y = \{g_n : n \in \omega\}$. Using above observation, we get the desired X.

If G is countable, we can modify the inverse construction to get X such that $\Delta(X) = A$ and $|X \cap g_1 \cap g_2 X| < \infty$ for all distinct $g_1, g_2 \in G \setminus \{e\}$, in particular, X is 3-sparse and, in particular, small.

Another modification, we can choose X such that $X \cap gX \neq \varnothing$ for each $g \in G$. If we take A not large, then we get X which is not P-small and X is not Δ-large, see Section 4.

Theorem 6.2. *Let G be a countable group such that, for each $g \in G \setminus \{e\}$, the set $\sqrt{g} = \{x \in G : x^2 = g\}$ is finite. Then the following statements hold:*

- (Tr_1) *Given any subset $X_0 \subseteq G$, $X_0 = X_0^{-1}$, $e \in X_0$, there exists a sequence $(X_n)_{n\in\omega}$ of subsets of G such that $\Delta(X_{n+1}) = X_n$ and $X_m \cap X_n = \{e\}$, $0 < m < n < \omega$.*
- (Tr_2) *There exists a sequence $(X_n)_{n\in\mathbb{Z}}$ of subsets of G such that $\Delta(X_n) = X_{n+1}$, $X_m \cap X_n = \{e\}$, $m, n \in \mathbb{Z}, m \neq n$.*
- (Tr_3) *There exists a subset A of G such that $\Delta(A) = A$ but A is not a subgroup.*
- (Tr_4) *There exists a subset A such that $A \supset \Delta(A) \supset \Delta^2(A) \supset \ldots$.*
- (Tr_5) *There exists a subset A such that $A \subset \Delta(A) \subset \Delta^2(A) \subset \ldots$.*
- (Tr_6) *For each natural natural number n, there exists a periodic Δ-trajectory $X_0, \ldots, X_{n-1}$ of length n: $X_1 = \Delta(X_0), X_2 = \Delta(X_1), \ldots, X_n = \Delta(X_{n-1})$ such that $X_i \cap X_j = \{e\}$, $i < j < n$.*

Proof. We use the following simple observation

(*) if F is a finite subset of an infinite group G and $g \notin F$ then the set $\{x \in G : x^{-1}gx \notin F\}$ is infinite.

In constructions of corresponding trajectories, at each inductive step, we use a finiteness of $\sqrt{g}$ and (*) in the following form:

(**) if $a \in G$, F is a finite subset of G, $F \cap \{e, a^{\pm 1}\} = \varnothing$ then there exists $x \in G$ such that

$$\{x^{\pm 1}, (ax)^{\pm 1}\}\{x^{\pm 1}, (ax)^{\pm 1}\} \cap F = \varnothing.$$

We show how to get a 2-periodic trajectory: X, Y, $\Delta(X) = Y$, $\Delta(Y) = X$, $X \cap Y = \{e\}$. We write G as a union $G = \bigcup_{n\in\omega} F_n$ of increasing chain $\{F_n : n \in \omega\}$ of finite symmetric subsets $F_0 = \{e\}$. We put $X_0 = Y_0 = \{e\}$ and construct inductively with usage of (**) two chains $(X_n)_{n\in\omega}$, $(Y_n)_{n\in\omega}$ of finite subsets of G such that, for each $n \in \omega$,

$$X_{n+1} = \{(x(y))^{\pm 1}, (yx(y))^{\pm 1} : y \in Y_0 \cup \ldots \cup Y_n\},$$

$$Y_{n+1} = \{(y(x))^{\pm 1}, (xy(x))^{\pm 1} : x \in X_0 \cup \ldots \cup X_n\},$$

$$(X_0 \cup \ldots \cup X_n) \cap (Y_0 \cup \ldots \cup Y_n) = \{e\},$$

$$X_{n+1}X_{n+1} \cap (F_{n+1} \setminus (Y_0 \cup \ldots \cup Y_n)) = \varnothing,$$

$$Y_{n+1}Y_{n+1} \cap (F_{n+1} \setminus (X_0 \cup \ldots \cup X_n)) = \varnothing,$$

$$(X_0 \cup \ldots \cup X_n)X_{n+1} \cap (F_{n+1} \setminus (Y_0 \cup \ldots \cup Y_n)) = \varnothing,$$

$$(Y_0 \cup \ldots \cup Y_n)Y_{n+1} \cap (F_{n+1} \setminus (X_0 \cup \ldots \cup X_n)) = \varnothing.$$

After ω steps, we put $X = \bigcup_{n\in\omega} X_n$, $Y = \bigcup_{n\in\omega} Y_n$. □

7. Δ and d

For a subset A of a topological space X, the subset A^d of all limit points of A is called *a derived subset*, and the mapping $d : \mathcal{P}(X) \to \mathcal{P}(X)$, $A \to A^d$, defined on the family of $\mathcal{P}(X)$ of all subsets of X, is called *the topological derivation*, see [6, §9].

Let X be a discrete set, βX be the Stone-Čech compactification of X. We identify βX with the set of all ultrafilters on X, X with the set of all principal ultrafilters, and denote $X^* = \beta X \setminus X$ the set of all free ultrafilters. The topology of βX can be defined by the family $\{\overline{A} : A \subseteq X\}$ as a base for open sets, $\overline{A} = \{p \in \beta X : A \in p\}$, $A^* = \overline{A} \cap G^*$. For a filter φ on X, we put $\overline{\varphi} = \{p \in \beta X : \varphi \subseteq p\}$, $\varphi^* = \overline{\varphi} \cap G^*$.

Let G be a discrete group, $p \in \beta G$. Following [2, Chapter 3], we denote

$$cl(A,p) = \{g \in G : A \in gp\},\ gp = \{gP : P \in p\},$$

say that $cl(A,p)$ is *a closure of* A *in the direction of* p, and note that

$$\Delta(A) = \bigcap_{p \in A^*} cl(A,p).$$

A topology τ on a group G is called *left invariant* if the mapping $l_g : G \to G$, $l_g(x) = gx$ is continuous for each $g \in G$. A group G endowed with a left invariant topology τ is called *left topological*. We note that a left invariant topology τ on G is uniquely determined by the filter φ of neighbourhoods of the identity $e \in G$, $\overline{\varphi}$ and φ^* are the sets of all ultrafilters an all free ultrafilters of G converging to e. For a subset A of G, we have

$$A^d = \bigcap_{p \in (\tau^*)} cl(A,p),$$

and note that $A^d \subseteq \Delta(A)$ if A is a neighbourhood of e in (G,τ).

Now we endow G with the discrete topology and, following [4, Chapter 4], extend the multiplication on G to βG. For $p, q \in \beta G$, we take $P \in p$ and, for each $g \in P$, pick some $Q_g \in q$. Then $\bigcup_{g \in p} gQ_p \in pq$ and each member of pq contains a subset of this form. With this multiplication, βG is a compact right topological semigroup. The product pq can also be defined by the rule [2, Chapter 3]:

$$A \subseteq G,\ A \in pq \Leftrightarrow cl(A,q) \in p.$$

If (G,τ) is left topological semigroup then $\overline{\tau}$ is a subsemigroup of βG. If an ultrafilter $p \in \overline{\tau}$ is taken from the minimal ideal $K(\overline{\tau})$ of $\overline{\tau}$, by [2, Theorem 5.0.25]. there exists $P \in p$ and finite subset F of G such that $Fcl(P,p)$ is neighbourhood of e in τ. In particular, if τ indiscrete ($\tau = \{\varnothing, G\}$), $p \in K(\beta G)$) and $P \in p$ then $cl(P,p)$ is large. If G is infinite, $p \in K(\beta G)$ is free, so $cl(P,p) \subseteq \Delta(P)$ and P is Δ-large. If a group G is finitely partitioned $G = A_1 \cup \ldots \cup A_n$, then some cell A_i is a member of p, hence A_i is Δ-large.

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