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# A non parametric estimation of service level in a discrete context.

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**Abstract:** An exact method for the estimation of the cycle service level has been proposed for periodic review stock policies in a discrete demand context for any known i.i.d. demand distribution. However, the implementation of this method in real environments has previously to manage some important and eventually cumbersome issues such as: (i) the identification of the appropriate demand distribution and its validation; (ii) the estimation of the parameters of the demand distribution; and (iii) the calculation of temporal aggregates of the demand distribution in order to estimate the expected service level. This paper shows some difficulties linked to these issues and proposes an alternative approach based on the observed demand frequencies, so that these issues are avoided and the accuracy of the service level estimation seems to be improved.

**Key words:** periodic review, service level, demand distribution.

## 1. Introduction

Probably the most important and useful problem studied by inventory control is the selection of the stock policy and the estimation of its parameters. For example, in the periodic review (*R, S*) policy, the review period *R* is usually predetermined by factors like the transportation schedule, so in practice managers see this problem as the determination of the optimal base stock level *S* such that total costs are minimized or some target customer service level is fulfilled. Estimating the parameters of the stock policy subject to a target service constraint is by far the most frequent approach in practice.

This paper focuses on the cycle service level (*CSL*) and periodic review policy but the approach proposed in this paper also applies even if an alternative service metric or continuous review policy is selected. Finally, the demand distribution is assumed to be i.i.d. and discrete and sample demand data is available.

(Cardós *et al.* 2009) propose a comprehensive set of procedures for the exact calculation of *CSL* with backlog and lost sales, not only for the periodic review policy but also for the continuous review one and also approximate expressions in every case. These expressions apply for any i.i.d. demand distribution being discrete and known. For the sake of simplicity, this paper focuses on the case in which backlog is allowed and whose exact estimation of the *CSL*, according to (Cardós, Babiloni, Palmer, & Albarracín J.M. 2009) can be obtained by the expression

$$CSL = \frac{F_L(S) - F_R(0)}{1 - F_R(0)} \qquad (1)$$

being $F(\cdot)$ the cumulative distribution function of demand, *S* the base stock, *R* the cycle and *L* the lead time. The application of that formula, or the appropriate one in different circumstances such as for example in a demand lost context or when a base stock policy is used, is just the last step of the procedure to compute the service level using sample data, as shown in Figure 1. Usually it is assumed in the literature that the demand distribution is known, but in practice it is not the case so that we have to cope with the first steps of the estimation procedure.





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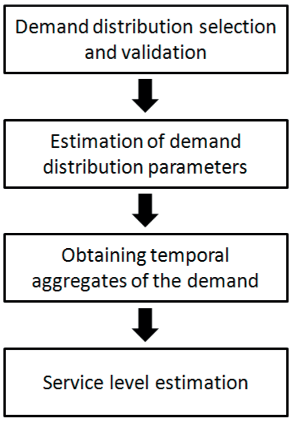



**Figure 1.** Procedure steps to estimate the service level from sample data.

The purpose of this paper is twofold: (i) to gain insight into the practical and technical difficulties of estimating the service level from demand sample data and its effect on the accuracy of the service level; and (ii) to propose an alternative non parametric approach so that these issues are avoided.

The rest of this paper is organized as follows. Section 2 presents the most important practical issues related with the estimation of the service level considering: (i) demand distribution selection and validation using sample data even when they are scarce, (ii) alternatives for the estimation of the demand distribution parameters, and (iii) calculation of the temporal aggregates of the demand. Section 3 is devoted to introduce our proposed non parametric approach and provide some illustrative examples. Finally, conclusions and further research are presented in Section 4.

## 2. Steps to Estimate the Service Level

### 2.1. Demand Distribution Selection and Validation

The demand distribution pattern has to be modelled from the available demand data. Continuous distributions are very used for modelling the demand pattern (Dunsmuir and Snyder 1989), (Schultz 1987), (Yeh *et al.* 1997) and normal distribution is especially frequent even for discrete demand. Although normal distributions may provide acceptable results even in the discrete demand case depending on its characteristics (average, variance, etc) it is more accurate modelling the discrete demand with a discrete distribution (Janssen *et al.* 1996), (Strijbosch *et al.* 2000), (Vereecke and Verstraeten 1994).

Poisson distribution is recommended by (Silver *et al.* 1998) for slow moving class A items. Compound Poisson distribution is also used when the probability of zero demand is significant but (Strijbosch, Heuts, & van der Schoot 2000) explain that this compound distribution is the result of modelling the number of received orders as Poisson and also the size of the orders, but this is often an unrealistic starting point

**Table 1.** Data of three class A items from an spare parts system.

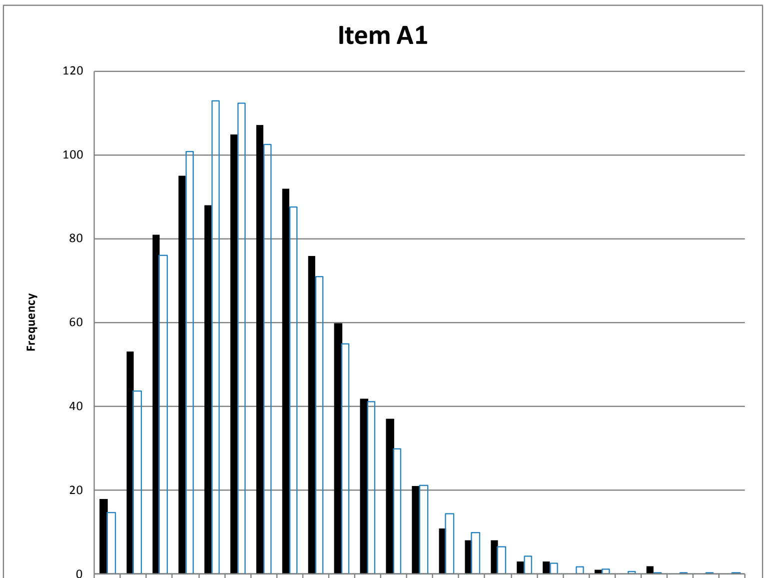
| Item | Ranked | Average | Variance | *r* | *p* | Pearson's test |
|---|---|---|---|---|---|---|
| A1 | 1 | 5,9550 | 11,9221 | 5,9429 | 0,4995 | Pass |
| A2 | 13 | 0,2415 | 4,6251 | 0,0522 | 0,0133 | Fail |
| A3 | 47 | 0,0549 | 0,3069 | 0,1789 | 0,0120 | Fail |

because data demand is usually aggregated on a daily basis becoming into a compound Bernouilli distribution.

Negative binomial distribution can be used as an alternative to the Poisson distribution especially when the sample variance exceeds the sample mean. Not surprisingly (Syntetos and Boylan 2006) point out that the negative binomial distribution is able to model demand patterns belonging to every demand category (smooth, erratic, intermittent and lumpy).

First of all, from a practical point of view, the most suitable distributions are Poisson and negative binomial. Usually the demand pattern is selected as Poisson when the variance differs from the average in no more than 10 per cent; if not so a negative binomial distribution is preferred. Bernouilli compound distributions are rarely used because of the kind of difficulties explained below.

In order to illustrate the main difficulties related with the distribution function selection, we consider the daily demands of three items of class A from the spare parts of an airline company during 911 consecutive days (see Table 1). There are 941 items and selected ones are ranked 1, 13 and 47 respectively considering the number of units demanded during the period. Obviously items A1, A2 and A3 belong to class A. In these three cases variance is much higher than the average, so following the usual rule a negative binomial distribution is selected and its parameters $r$ and $p$ are estimated. Last column shows the results of the Pearson's chi-squared test.

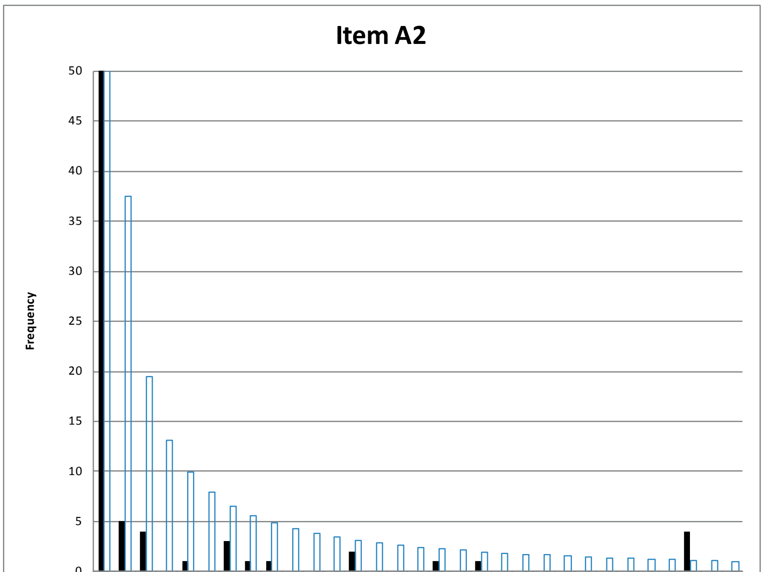



| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| data | 18 | 53 | 81 | 95 | 88 | 105 | 107 | 92 | 76 | 60 | 42 | 37 | 21 | 11 | 8 | 8 | 3 | 3 | 0 | 1 | 0 | 2 | 0 | 0 | 0 |
| NB fitted | 15 | 44 | 76 | 101 | 113 | 112 | 102 | 88 | 71 | 55 | 41 | 30 | 21 | 15 | 10 | 7 | 4 | 3 | 2 | 1 | 1 | 0 | 0 | 0 | 0 |

**Figure 2.** Demand Histogram for item A1 and negative binomial expected frequencies.

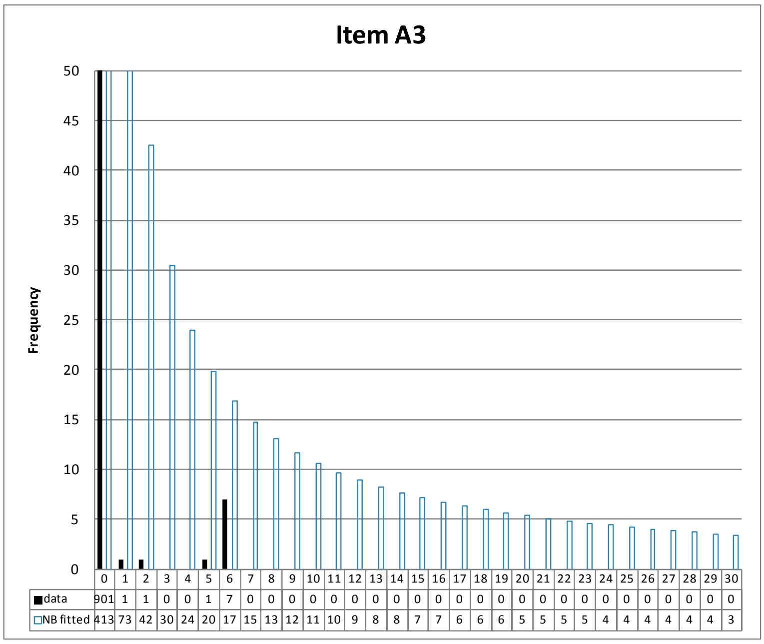
Demand histogram of item A1 seems to fit to a negative binomial distribution (see Figure 1) confirmed by Pearson's chi-squared test.

Demand histograms of items A2 and A3 do not seem to fit a negative binomial distribution (see Figures 2 and 3) and obviously Pearson's chi-squared test fails in both cases. Unfortunately usually the validity of the demand distribution is not checked because: (i) the application of this test is quite cumbersome and impractical for large inventories; (ii) Poisson and negative binomial are usually the only available options; and (iii) there is not a manageably distribution function able to fit demand patterns with so many peaks. It could be argued that these peaks could be anomalies in the demand but this is not the case for items A2 and A3.




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| data | 888 | 5 | 4 | 0 | 1 | 0 | 3 | 1 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 |
| NB fitted | 727 | 37 | 19 | 13 | 10 | 8 | 7 | 6 | 5 | 4 | 4 | 3 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Figure 3.** Demand Histogram for item A2 and negative binomial expected frequencies.




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| data | 901 | 1 | 1 | 0 | 0 | 1 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| NB fitted | 413 | 73 | 42 | 30 | 24 | 20 | 17 | 15 | 13 | 12 | 11 | 10 | 9 | 8 | 8 | 7 | 7 | 6 | 6 | 6 | 5 | 5 | 5 | 5 | 4 | 4 | 4 | 4 | 4 | 4 | 3 |

**Figure 4.** Demand Histogram for item A3 and negative binomial expected frequencies.

Additionally, the validation of a demand distribution using a statistical test requires a number of non zero demand periods but it is not always possible as it happens with A3 item even being a class A item.

## 2.2. Estimation of Parameters

Second, the estimation of the parameters of a demand distribution may be obtained using maximum likelihood estimators which estimate the parameters in order to make the observed data the most probable. These estimators have a number of desirable properties, but in some cases the estimators are unsuitable or do not exist. For example, the estimation of the parameters of a Bernouilli compound distribution requires the use of a computer to solve simultaneously two equations

$$p = \frac{n-m}{1-e^{-\lambda}}\frac{1}{n}$$
$$\lambda = (1-e^{-\lambda})\frac{\sum_{i=1}^{n} x_i}{n-m} \qquad (2)$$

being $p$ the probability of zero demand, $\lambda$ the Poisson rate, $n$ the size of the sample, $x_i$ the demand data and $m$ the number of non zero demands. This situation also applies to the negative binomial distribution.

Another estimation approach is the method of moments which uses as many moments as parameters have to be estimated, replaces the moments by the sample moments and derives the expressions of the parameters. For example, for the negative binomial distribution with parameters $r$ and $p$

$$p = 1 - \frac{\hat{x}}{\hat{s}^2}$$
$$r = \frac{1-p}{p}\hat{x} \qquad (3)$$

being $\hat{x}$ and $\hat{s}^2$ the sample moments of first and second order. Maximum likelihood estimators tend to offer better estimations than the moments method, but the estimators based on moments can be quickly and easily calculated.

## 2.3. Obtaining Temporal Aggregates of the Demand

First of all, it should be noted that the probability of no demand during R consecutive periods is needed in expression (1) and can be calculated directly whatever the demand distribution would be as

$$F_R(0) = F(0)^R \qquad (4)$$

Once the demand distribution has been selected and its parameters have been estimated, the last step before applying expression (1) is to develop temporal aggregates of the demand. For example, the cumulative distribution in $L$ consecutive periods $F_L(S)$ is the kind of temporal aggregate needed to compute

CSL and it can be obtained using the properties of the sum of i.i.d. distribution functions. We only need to develop an expression for $F_L(S)$ in a convenient form to be used. This is quite straightforward for Poisson and negative binomial distributions since both maintain their own distribution and

$$\sum_{i=1}^{L} PS(\lambda) = PS(L\lambda)$$
$$\sum_{i=1}^{L} NB(r,p) = NB(Lr,p) \tag{5}$$

However it is not so easy for many other distributions such as Bernouilli compound that becomes into a binomial compound distribution and its aggregates involve quite complex and long calculations. Anyway, $F_L(S)$ can always be obtained based on *F(.)* using the convolution of two discrete distributions

$$h = f + g$$
$$h(i) = \sum_{j=1}^{i} f(j)g(i-j) \qquad i \in \mathbb{N} \tag{6}$$

## 3. Proposed non Parametric Approach

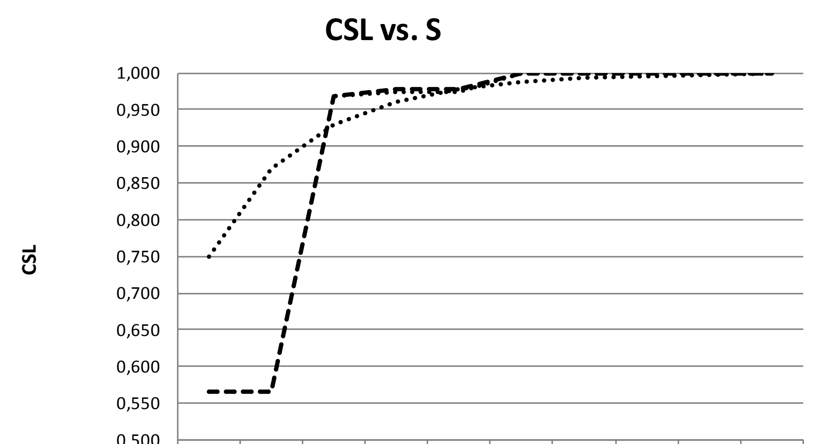
The difficulties explained below appear when demand data do not fit a convenient distribution function such as Poisson or negative binomial. These problems are difficult to manage when they occur, but there are also interactions among them making it harder. For example, if you improve the fitting of the distribution function using a complex compound distribution, then it leads to an impractical analytical expression for the demand during the lead time.

We propose a different approach by defining the demand distribution as

$$f(i) = fr_i \qquad i \in \mathbb{N} \tag{7}$$

being $fr_i$ the sample relative frequency. This formulation avoids the need of identifying and validating the demand distribution and has no parameters to be estimated.

The performance of this approach can be illustrated with an example where we know the demand distribution and we compare the performance of parametric and non parametric approaches. First, the demand is the sum of a Poisson distribution with *λ=0.01* and three times a Bernouilli distribution with *p=0.1* being *p* the probability of non zero demand. Second, we compute the exact CSL using expressions (1), (4) and (6). Third, demand is simulated 30 times for 1,000 consecutive days and parametric and non parametric estimations of CSL are obtained each time using the generated sample data. Finally the average CSL estimation for each base stock and procedure is obtained (see Figure 5) resulting in better estimates from the non parametric one.




| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| exact | 0,566 | 0,567 | 0,969 | 0,977 | 0,977 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 |
| non parametric | 0,565 | 0,565 | 0,969 | 0,975 | 0,975 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 |
| negative binomial | 0,749 | 0,870 | 0,929 | 0,961 | 0,978 | 0,987 | 0,993 | 0,996 | 0,998 | 0,999 |

**Figure 5.** Illustrative example comparing the parametric and non parametric procedures for different base stocks (S=1..10)

## 4. Conclusions and Practical Implications

Although exact estimation procedures are available for computing CSL and other service level metrics in discrete contexts, its estimation remains being a challenge due to the practical difficulties involved in identifying and validating the demand distribution, estimating the parameters of the distribution and developing temporal aggregates.

When demand patterns can be modelled using Poisson or negative binomial distributions, then the estimation of the service level is quite simple because of their distribution properties. Other discrete distributions like Bernouilli compound may be an interesting alternative when the probability of no demand is quite high, but the calculation of temporal aggregates become much more complex. Anyway, the validation of the demand distribution is always very time-consuming so that usually it is not performed extensively.

As a consequence, we propose a non parametric approach that outperforms the conventional parametric estimation: (i) it is not necessary to estimate the parameters of the distribution or to validate the distribution pattern; and (ii) the estimation of the service level seems to outperform the parametric one.

Given the impact and practical implications of these results in operational management, further research will focus on developing an extensive experiment in order to check the influence of sample size and other factors including the demand characteristics.

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