ON THE

EMERGENT ASPECTS OF QUANTUM MECHANICS

IN RELATION TO

THE THERMODYNAMICS OF IRREVERSIBLE PROCESSES

AND EMERGENT GRAVITY

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Título: Aspectos emergentes de la mecánica cuántica: relación con la termodinámica de los procesos irreversibles y con la gravedad emergente.

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Abstract

This PhD thesis elaborates on a proposal made by the Dutch theoretical physicist G. 't Hooft (1999 Nobel prize in physics), to the effect that quantum mechanics is the emergent theory of some underlying, deterministic theory. According to this proposal, information–loss effects in the underlying deterministic theory lead to the arrangement of states of the latter into equivalence classes, that one identifies as quantum states of the emergent quantum mechanics. In brief, *quantisation is dissipation*, according to 't Hooft. In our thesis we present two mechanisms whereby quantum mechanics is explicitly seen to emerge, thus explicitly realising 't Hooft's proposal.

The first mechanism makes use of Verlinde's approach to classical mechanics and general relativity via holographic screens. This technique, first presented in 2010 in order to understand the emergent nature of spacetime and gravity, is applied in our thesis to the case of quantum mechanics.

The second mechanism presented to support 't Hooft's statement is based on a dictionary, also developed by the authors, between semiclassical quantum mechanics, on the one hand, and the classical theory of irreversible thermodynamics, on the other. This thermodynamical formalism, established by Nobel prize winners Onsager and Prigogine, can be easily mapped into that of semiclassical quantum mechanics.

Resumen

Esta tesis doctoral profundiza en una propuesta hecha por el físico teórico holandés G. 't Hooft (premio Nobel de física, 1999), en el sentido de que la mecánica cuántica es la teoría emergente de una teoría subyacente, determinista. Según esta propuesta, los efectos de pérdida de información en la teoría determinista subyacente conducen a la combinación de estados en clases de equivalencia, que se identifican como estados cuánticos de la mecánica cuántica emergente.

En resumen, la cuantización es disipación, de acuerdo con 't Hooft. En nuestra tesis se presentan dos mecanismos por los que la mecánica cuántica se ve emerger explícitamente, verificándose expresamente la propuesta de 't Hooft.

El primer mecanismo hace uso del enfoque de Verlinde a la mecánica clásica y la relatividad general a través de pantallas holográficas. Esta técnica, presentada por primera vez en 2010 con el fin de comprender la naturaleza emergente del espacio-tiempo y de la gravedad, se aplica en nuestra tesis al caso de la mecánica cuántica.

El segundo mecanismo presentado para apoyar la afirmación de 't Hooft se basa en un diccionario, también desarrollado por nosotros, entre la mecánica cuántica semiclásica, por un lado, y la teoría clásica de la termodinámica de los procesos irreversibles, por el otro. Este formalismo termodinámico, establecido por los premios Nobel Onsager y Prigogine, se puede trasladar fácilmente al caso de la mecánica cuántica semiclásica.

Resum

Aquesta tesi doctoral aprofundeix en una proposta feta pel físic teòric holandès G. 't Hooft (premi Nobel en física, 1999), en el sentit que la mecànica quàntica és la teoria emergent d'una teoría subjacent, determinista. Segons aquesta proposta, els efectes de pèrdua d'informació en la teoria determinista subjacent condueixen a la combinació d'estats en classes d'equivalència, que s'identifiquen com a estats quàntics de la mecànica quàntica emergent.

En resum, la quantització és dissipació, d'acord amb 't Hooft. En la nostra tesi es presenten dos mecanismes pels quals la mecànica quàntica es veu emergir explícitament, verificant expressament la proposta de 't Hooft.

El primer mecanisme fa ús de l'enfocament de Verlinde a la mecánica clássica i la relativitat general a través de pantalles holográfiques. Aquesta tècnica, presentada per primera vegada en el 2010 per tal de comprendre la naturalesa emergent de l'espaitemps i de la gravetat, es aplicada en la nostra tesi al cas de la mecànica quàntica.

El segon mecanisme presentat per recolzar la declaració de 't Hooft es basa en un diccionari, també desenvolupat pels autors, entre la mecànica quàntica semiclàssica, d'una banda, i la teoria clàssica de la termodinàmica dels processos irreversibles, de l'altra. Aquest formalisme termodinàmic, que va ser establert pels premis Nobel Onsager i Prigogine, es pot traslladar fàcilment al cas de la mecànica quàntica semiclàssica.

Contents

Chapter 1

Overview

1.1 The quantisation paradigm

It has become customary to classify the interactions known in Nature into four classes:
i) gravitational forces;
ii) electromagnetic forces;
iii) weak nuclear forces;
iv) strong nuclear forces.
According to the chronology of their discovery, the gravitational interaction was the first one to be discovered, followed by the electromagnetic force, and then the nuclear forces (weak and strong). The same order of presentation also applies to their relative strength, gravitational forces being the weakest and strong nuclear forces the strongest of all known interactions.

The two basic pillars on which modern physics stands are quantum theory and general relativity. These two theories were developed in the first quarter of the 20th century. Roughly speaking, general relativity accounts for the gravitational force, while quantum theory underlies the existing models (so–called Yang–Mills theories) of the electromagnetic force and the nuclear forces (both weak and strong).

In this way Einstein's theory of gravitation applies to the very large macroworld (*i.e.*, to astronomical scales, up to 10^{28} cm, the radius of the Universe). On the other hand quantum models (quantum Yang–Mills theories) describe the microworld (by which we mean length scales the size of an atom, typically 10^{-8} cm, and below).

Quantum theory and the theory of relativity have been experimentally tested innumerable times, always extremely successfully. As long as one remains within the limits of applicability of the corresponding theory, one can safely claim that both theories are right.

Now Einstein's theory of relativity leads to two remarkable predictions about the Universe. First, that the final state of massive stars is to undergo gravitational collapse behind an event horizon and form a black hole which will contain a singularity. Secondly, that there is a singularity in our past which constitutes, in some sense, a beginning to the Universe. One expects the physics of such singularities to be more

correctly described by some quantum version of gravity than it is by the Einstein theory alone. There are good reasons to believe that quantum effects should play a major role in explaining the true physics of black holes and the so-called Big Bang—the initial singularity at the beginning of time, out of which everything else in the Universe evolved. Refined over the course of the 20th century, these models start from an initial state of high density, high temperature and length scales of the order of the Planck length (10^{-33} cm). Such extreme conditions require the notions of quantum mechanics for their correct description.

However, severe technical difficulties arose as soon as one tried to apply the principles of quantum mechanics to the theory of relativity. This (failed) programme has come to be known as *the quantisation of gravity*; it is an outstanding problem in 20th–century theoretical physics that penetrates deeply into the 21st century. More than 80 years of hard work on a would–be quantum theory of gravity have produced no tangible result yet. Which is not to say that time and resources have been completely wasted: many interesting things have been learnt along the way. However, despite arduous efforts along many different lines of approach, a consistent theory of quantum gravity still eludes us. Those few approaches that have survived the test of time (most notably string theory and loop quantum gravity) still face enormous challenges, falling short of a providing completely satisfactory solution to the problem.

An obvious source of potential difficulties comes from the fact that in the quantisation of gravity one is trying to apply, at an astronomical scale, concepts drawn from the microworld: these two worlds differ by many orders of magnitude. There is no guarantee that inconsistencies will not arise when one tries to push a theory beyond its natural limits, beyond the scope of phenomena for whose explanation it was devised.

Difficult problems can sometimes be solved by application of existing techniques, often by painstakingly patient application, if necessary by brute–force application. However, conceptual difficulties often require a change of mind, a change in the paradigms that underlie one's whole intellectual framework. This viewpoint is based on the conviction that any theory necessarily has its limits, and that therefore a change in approach may be convenient, if not altogether necessary.

The quantisation of gravity is possibly one such case. The logic of the problem can be summarised as follows. One is given Einstein's *classical* theory of gravity. By *classical* one means that Planck's constant $\hbar$ is missing: it appears nowhere in the equations. Next one applies a heuristic set of rules known as *quantisation*. These rules have been successfully applied to explain electrons, atoms, molecules, nuclei, and the whole microworld known to us, so we have some degree of confidence in them. However, they remain *a set of heuristic rules* that one applies more or less automatically, without troubling oneself much about their range of applicability.

Quantisation, by its own definition, is applied on a *classical* theory to yield the corresponding *quantum* theory. The paradigm thus reads: start with a classical theory, then quantise. The quantum theory is the better theory, the classical one being just an approximation, sometimes a very coarse one. This paradigm has dominated much of 20th–century physics so successfully, that many physicists find it difficult to accept that there might be anything wrong with it. Perhaps the best example of the application of this paradigm is that of quantum electrodynamics, the theory of electrons and photons. In terms of explanatory power, predictive power and theoretical–numerical accuracy

checked against experiments, quantum electrodynamics stands out supreme. This success story bears the names of Dirac, Feynman and Schwinger, among others. After quantum electrodynamics (an abelian Yang–Mills theory) comes the standard model of particle physics (a nonabelian Yang–Mills theory), which unifies the electromagnetic force, the weak nuclear force, and the strong nuclear force under a single principle. This second success story, also based on quantum mechanics, is associated with the names of Yang, Mills, Weinberg, Glashow and Salam, among others.

So there must be something right in the quantisation paradigm—yet, the quantisation of gravity turned out to be a colossal failure. Could it be the quantisation paradigm that fails?

Needless to say, in this PhD thesis we will *not* tackle the quantisation of gravity. Instead we will be more modest and analyse the alternative possibility that the current paradigm (*start with a classical theory, then quantise*) fails for the problem at hand. It has been suggested by 't Hooft that a possible reason for this failure lies in the misconception that quantum mechanics is a *fundamental* theory. This point is elaborated throughout this PhD thesis in detail, starting in the next section.

1.2 Emergent quantum mechanics

The conceptual foundations of quantum mechanics have been the subject of heated dispute ever since the early 1930's, as the so–called Copenhagen interpretation was challenged by a few but very vocal (and very remarkable) physicists, Einstein being one of them. At stake was not the predictive power of the new theory (something everybody acknowledged without reservations) but, rather, its philosophical underpinning: *God doesn't play dice!* As time went by, the Copenhagen interpretation (as developed mainly by Bohr, Heisenberg and Born) won the day, and Einstein seemed to lose the game. In a nutshell, Einstein accepted Copenhagen quantum mechanics as a statistical theory, but refused to accept the loss of a fundamentally deterministic, ontological character that the Copenhagen interpretation brought about. For him, Copenhagen quantum mechanics was merely an effective probabilistic description of some deeper, deterministic theory, where physical entities play an ontological role independently of observation.

As quantum mechanics became established, more domains of classical physics came to be quantised. This, in turn, made the quantum theory even more successful. However, as already seen, general relativity adamantly resisted quantisation.

Classical Yang–Mills theory can be consistently quantised because, among other things, there is a systematic procedure for getting rid of some nasty infinities that arise in the corresponding quantum theory. This procedure is called *renormalisation*—much of the credit for this goes to 't Hooft, who proved the renormalisability of (nonabelian) quantum Yang–Mills theories.

General relativity, however, cannot be consistently quantised because it is not renormalisable. Yet one would like to have a quantum theory of gravity: a theory that describes the structure of spacetime at length scales as tiny as the Planck length. At the same time, this would–be quantum theory of gravity should be able to reproduce general relativity when considered at astronomical length scales.

So either *general relativity is not the ultimate theory of spacetime*, or *quantum mechanics is not the ultimate theory of the microscopic world*—or both.

't Hooft takes the viewpoint that quantum mechanics as we know it, is not the ultimate theory of the microscopic world. This does not invalidate quantum mechanics as we know it. Rather, according to 't Hooft, quantum mechanics *emerges* as a probabilistic description of an *underlying* deterministic theory. In this sense, 't Hooft's view agrees with Einstein's—but one still has to actually construct such a deterministic theory in the first place, and then describe how quantum mechanics emerges from it. Einstein fell short of achieving these two goals in his critique of the Copenhagen interpretation, while 't Hooft has taken a number of ground–breaking steps towards achieving them [56, 57, 58, 59].

This *emergence property* of quantum mechanics is analogous to that of classical thermodynamics as derived from, say, the classical kinetic theory of gases. In the passage from the kinetic theory to the thermodynamical description there is a great deal of information loss. At a microscopic level we can, at least in principle, follow the path of each and every single molecule of gas. This entails a huge amount of information. At a macroscopic level we renounce almost all this knowledge and satisfy ourselves with just a handful of variables such as pressure, volume and temperature. Macroscopic properties can be understood and explained microscopically, but we renounce this vast amount of information. The emergent theory has entirely different properties from those of its underlying microscopic theory. Of course, classical thermodynamics is also a deterministic theory, and in this sense the analogy with quantum mechanics breaks down. However, this example serves well to illustrate the process of information loss that, according to 't Hooft, characterises the passage from an underlying deterministic theory to the probabilistic quantum mechanics that we observe and verify in our labs.

The theory underlying Copenhagen quantum mechanics, called *deterministic quantum mechanics* by 't Hooft, must first and foremost be deterministic (as opposed to probabilistic), and it must also exhibit information loss. It is a quantum theory because it describes the microscopic world, but it is deterministic because it obeys some classical equation of motion. States in the deterministic theory are arranged, by a dissipative process of information loss, into equivalence classes that the Copenhagen interpretation calls quantum states. To revert to our thermodynamical analogy, pressure can be understood as arising from the collisions of gas molecules against the container walls. Many different motions of the molecules will give rise to the same overall momentum transfer to the wall and, therefore, to the same value of the pressure. Yet, a knowledge of the pressure is far less detailed than a knowledge of the precise molecules, and the precise paths they follow as they hit the wall. So one quantum state in the Copenhagen interpretation (pressure) is the result of arranging very many different deterministic states (configurations of molecules) into one equivalence class.

Specifically, 't Hooft proves the following existence theorem: *For any quantum system there exists at least one deterministic model that reproduces all its dynamics*.

As in the previous thermodynamical analogy, the underlying deterministic theory may have little in common (at least at first sight) with the emergent statistical theory. In this way not only quantum mechanics is emergent, but possibly also its symmetries. 't Hooft further argues that symmetries we are used to such as local gauge symmetry in Yang–Mills theory, or diffeomorphism invariance in general relativity, may be emer-

gent symmetries that need not be present, at least in their usual form, in the underlying deterministic models.

There exists a large body of literature on emergent quantum mechanics, some basic references being [4, 57, 80]; see also [5, 24, 28, 34, 51, 58, 74, 82, 104, 98, 105] for more recent work. The hypothesis of emergence and the holographic principle [55, 106] have been hailed as landmarks in the endeavour to arrive at a consistent theory of quantum gravity.

To summarise: if quantum mechanics truly is an emergent theory, then the quantisation paradigm alluded to in section 1.1 certainly breaks down, because quantum mechanics is not as fundamental as so far believed.

1.3 Emergent gravity

It turns out that the quantisation paradigm for gravity, mentioned in section 1.1, breaks down for two reasons. One has already been explained: quantum mechanics is not a fundamental theory, but rather an effective phenomenon. The other reason is that *spacetime and gravity are both emergent phenomena, too*. Although this thesis deals primarily with the notion of quantum mechanics as an emergent theory, here we would like to say a few words about the emergent nature of gravity and spacetime.

Groundbreaking advances in our understanding of gravity have led to profound new insights into its nature (see [87, 88, 89, 90, 91, 113] and refs. therein). Perhaps the most relevant insight is the recognition that gravity cannot be a fundamental force, but rather must be an effective description of some underlying degrees of freedom. As such, gravity is amenable to a thermodynamical description. Although this fact had already been suspected for some time [11, 12, 53, 111, 65, 54], it is only more recently that it has been given due attention. The derivation of Newton's laws of motion and of Einstein's gravity, presented in ref. [113] from an entropic perspective, has triggered off an avalanche of research into the subject, ensueing papers being too numerous to quote here in detail; see however [77, 18, 108, 30, 76, 41, 43]. A feature of these developments is that, while offering insights into the quantum structure of spacetime, the treatment is largely classical, in that no specific microscopic model of spacetime is assumed. In other words, these developments refer not to the (microscopic) statistical mechanics of gravity and spacetime, but to its (macroscopic) thermodynamics instead. In this sense, notions usually considered to be *a priori*, such as inertia, force and spacetime, appear as phenomena arising from some underlying theory whose minutiæ are largely unknown—but fortunately also irrelevant for a thermodynamical description. Such *emergent* phenomena are no longer *a priori*, but derived. We refer readers to the comprehensive overview of emergent physics presented in the nice book [21]. Spacetime itself appears as an emergent phenomenon, with the holographic principle playing a key role [55, 106]. Developments in string theory also point in this direction [15, 99].

To summarise: Boltzmann's dictum, *If something heats up, it has microstructure*, applies to the spacetime continuum of general relativity, because many known spacetimes can be assigned thermodynamical properties like temperature, entropy, heat capacity, etc. We do not know yet what the atoms of spacetime look like—in fact we are probably centuries away from developing the necessary technology that would allow

one to probe spacetime at the Planck scale. However, the granularity of spacetime is a commonly accepted feature today, a feature that becomes invisible at the energies currently available, thus causing the impression of a continuum. In other words, spacetime is an emergent phenomenon, too, as much as quantum mechanics.

1.4   Structure of this PhD thesis

This PhD thesis is based on the three papers [1, 2, 3]. The publications [1] and [3] are original research articles, while the paper [2] contains a written version of the invited talk presented by J.M.I. at the *5th International Heinz von Foerster Congress: Emergent Quantum Mechanics*, Vienna, Austria, Nov. 11-13, 2011. This talk, being largely based on the previous publication [1], is not collected here to avoid repetitions, except for a brief section that did not appear in the initial paper [1].

Thus chapter 2 of this thesis contains an exact copy of paper [1] plus that section of [2] that did not appear in [1] (modulo some rearrangements of the material), while chapter 3 contains an exact copy of the article [3] (again up to minor rearrangements). Finally chapter 4 summarises our overall conclusions.

Chapter 2

An entropic picture

2.1 Introduction

It has been conjectured that quantum mechanics must be an emergent theory [81, 4, 105, 56, 57, 33, 34, 35, 36, 67]; see also [73, 38, 19, 20, 22, 23, 24, 69] for its close link with gravity theories, and [47, 48, 52, 49] for an interpretation in thermodynamical terms. The guiding principle at work in many of these approaches is the notion that quantum mechanics provides some coarse–grained description of an underlying deterministic theory. In some of these models [56], quantum states arise as equivalence classes of classical, deterministic states, the latter being grouped together into equivalence classes, or quantum states, due to our ignorance of the full microscopic description. Quantisation thus appears to be some kind of dissipation mechanism for information. In the presence of dissipation, entropy immediately comes to mind [25, 26, 27].

Thus the two research lines mentioned above, gravity and quantum mechanics, share the common feature of being effective, thermodynamical descriptions of their respective underlying theories. It is the purpose of this chapter to develop an approach to emergent quantum mechanics from the *entropic* point of view pioneered in ref. [113], with a quantum–mechanical particle replacing the classical particle considered in ref. [113]. Additionally, this will contribute towards clarifying the role played by Planck's constant $\hbar$ in the entropic derivation of classical gravity (Newton's and Einstein's) presented in [113]. Indeed, our results can be regarded as an entropic derivation of Planck's constant $\hbar$ from Boltzmann's constant k_B—at least conceptually if not numerically. Altogether, our approach will provide us with *a holographic, entropic picture of emergent quantum mechanics*.

Finally let us say a word on notation. Awkward though the presence of $\hbar, c, G, k_B$ in our equations may seem, our purpose of exhibiting how $\hbar$ emerges from k_B renders natural units inconvenient. Quantum operators will be denoted as $\hat{f}$, with f being the corresponding classical function.

2.2   Holographic screens as entropy reservoirs

2.2.1   A quantum of entropy

The starting point in ref. [113] is a classical point particle of mass M approaching a holographic screen $\mathcal{S}$, from that side of the latter on which spacetime has already emerged. At a distance from $\mathcal{S}$ equal to 1 Compton length, the particle causes the entropy S of the screen to increase by the amount

$$\Delta S = 2\pi k_B, \tag{2.1}$$

where k_B is Boltzmann's constant. The above can also be understood as meaning that $2\pi k_B$ is the *quantum* by which the entropy of the screen increases, whenever a particle crosses $\mathcal{S}$. The factor 2π on the right–hand side is conventional. Relevant is only the fact that the entropy increase of the screen appears quantised in units of k_B.

We call *bright* that side of the holographic screen on which spacetime has already emerged, whereas the other side might well be termed *dark*. One can also think of the holographic screen as being the horizon of some suitably picked observer $\mathcal{O}$ in space-time. For example, in the relativistic case, one can think of this observer as being a Rindler observer. The dark side might well be identified with the screen itself, as there is literally no spacetime beyond the bright side—this assertion is to be understood as relative to the corresponding observer, since different observers might perceive different horizons. In this way, for each fixed value of the time variable, a collection of observers $\mathcal{O}_j$, with the index j running over some (continuous) set $\mathcal{J}$, gives rise to a foliation of 3–space by 2–dimensional holographic screens $\mathcal{S}_j$: $\mathbb{R}^3 = \cup_{j \in \mathcal{J}} \mathcal{S}_j$. For reasons to be explained presently we will mostly restrict our attention to potentials such that the $\mathcal{S}_j$ are all closed surfaces; we denote the finite volume they enclose by $\mathcal{V}_j$, so $\partial \mathcal{V}_j = \mathcal{S}_j$.

2.2.2   Two thermodynamical representations

We will take (2.1) to hold for a quantum particle as well. A quantum particle hitting the holographic screen[1] exchanges entropy with the latter, *i.e.*, the wavefunction ψ exchanges information with $\mathcal{S}$. Just as information is quantised in terms of bits, so is entropy quantised, as per eqn. (2.1). The only requirement on this exchange is that the holographic screen act as an entropy reservoir. (See refs. [71, 78] for related proposals, with the mechanical action integral replacing the entropy).

Describing the quantum particle on the bright side of the screen we have the standard wavefunction ψ_+, depending on the spacetime coordinates and obeying the usual laws of quantum mechanics. On the other hand, the *entropic* wavefunction ψ_- describes the same quantum particle, as seen by an observer on the dark side of the holographic screen. If imagining an observer on the dark side of $\mathcal{S}$, where spacetime has not yet emerged, raises some concern, one can also think of ψ_- as being related, in a

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[1]Due to quantum delocalisation, statements such as *a quantum particle hitting the holographic screen* must be understood as meaning *a quantum–mechanical wavepacket, a substantial part of which has nonzero overlap with the screen.*

way to be made precise below, to the flow of entropy across the horizon $\mathcal{S}$, as measured by an observer on the bright side of the same horizon.

Our goal is to describe the laws of *entropic quantum mechanics*, that is, the laws satisfied by the entropic wavefunction ψ_-, and to place them in correspondence with those satisfied by the standard wavefunction ψ_+ on spacetime. The relevant thermodynamical formalism needed here can be found, *e.g.*, in the classic textbook [17]. However, for later use, let us briefly summarise a few basics. Any given thermodynamical system can be completely described if one knows its *fundamental equation*. The latter contains all the thermodynamical information one can obtain about the system. The fundamental equation can be expressed in either of two equivalent ways, respectively called the *energy representation* and the *entropy representation*. In the energy representation one has a fundamental equation $E = E(S, \ldots)$, where the energy E is a function of the entropy S, plus of whatever additional variables may be required. In the entropy representation one solves for the entropy in terms of the energy to obtain a fundamental equation $S = S(E, \ldots)$.

As an example, let there be just one extensive parameter, the volume V. Then the fundamental equation in the entropy representation will be an expression of the form $S = S(E, V)$, hence $dS = (\partial S/\partial E)\, dE + (\partial S/\partial V)\, dV$. We know that $\delta Q = T dS$, while the first law of thermodynamics reads, in this case, $\delta Q = dE + p dV$, with p the pressure. It follows that $T^{-1} = \partial S/\partial E$ and $p = T\, (\partial S/\partial V)$. This latter equation is the equation of state. For example, in the case of an ideal gas we have $S(E, V) = k_B \ln (V/V_0) + f(E)$, with $f(E)$ a certain function of the energy and V_0 a reference volume (that can be regarded as a constant contribution to S and thus neglected). It follows from $\partial S/\partial V = k_B V^{-1}$ that pV is proportional to T, as expected of an ideal gas.

In a sense to be made more precise presently, the bright side of the holographic screen corresponds to the energy representation, while the dark side corresponds to the entropy representation. Thus the energy representation will give us quantum mechanics on spacetime as we know it. One must bear in mind, however, that standard thermodynamical systems admit both representations (energy and entropy) simultaneously, which representation one uses being just a matter of choice. In our case this choice is dictated, for each fixed observer, by that side of the screen on which the observer wants to study quantum mechanics. For example there is no energy variable on the dark side, as there is no time variable, but an observer can assign the screen an entropy, measuring the observer's ignorance of what happens beyond the screen. By the same token, on the bright side we have an energy but there is no entropy[2]. In this case these two representations cannot be simultaneous.

The situation just described changes somewhat as soon as one considers two or more observers, each one of them perceiving a different horizon or holographic screen. Consider, for simplicity, two observers $\mathcal{O}_1, \mathcal{O}_2$ with their respective screens $\mathcal{S}_1, \mathcal{S}_2$, and assume the latter to be such that $\mathcal{S}_2$ gets beyond $\mathcal{S}_1$, in the sense that $\mathcal{S}_2$ encloses more emerged volume than $\mathcal{S}_1$. That is, the portion of emerged spacetime perceived by $\mathcal{O}_2$ includes all that perceived by $\mathcal{O}_1$, plus some volume that remains on the dark side of

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[2]We are considering the simplified case of a pure quantum state. Were our quantum state to be described by a density matrix, there would of course be an entropy associated.

$\mathcal{S}_1$. Call $\mathcal{V}_{12}$ this portion of spacetime that appears dark to $\mathcal{O}_1$ but bright to $\mathcal{O}_2$. Clearly, quantum mechanics on $\mathcal{V}_{12}$ will be described in the energy representation by $\mathcal{O}_2$ and in the entropy representation by $\mathcal{O}_1$. In this case the two representations can coexist simultaneously—not as corresponding to one observer, as in standard thermodynamics, but each one of them as *pertaining to a different observer*.

The differences just mentioned, as well as some more that will arise along the way, set us somewhat apart from the standard thermodynamical formalism. Nevertheless, the thermodynamical analogy can be quite useful if one bears these differences in mind.

2.2.3   A holographic dictionary

Let us recall that one can formulate a *holographic dictionary* between gravitation, on the one hand, and thermodynamics, on the other [87, 88, 89, 90, 91]. Let V_G denote the gravitational potential created by a total mass $M = \int_{\mathcal{V}} \mathrm{d}^3V \, \rho_M$ within the volume $\mathcal{V}$ enclosed by the holographic screen $\mathcal{S} = \partial\mathcal{V}$. Then the following two statements are equivalent [113, 60]:

i) there exists a gravitational potential V_G satisfying Poisson's equation $\nabla^2 V_G = 4\pi G \rho_M$, such that a test mass m in the background field created by the mass distribution ρ_M experiences a force $\mathbf{F} = -m\nabla V_G$;

ii) given a foliation of 3–space by holographic screens, $\mathbb{R}^3 = \cup_{j \in \mathcal{J}} \mathcal{S}_j$, there are two scalar quantities, called entropy S and temperature T, such that the force acting on a test mass m is given by $F\delta x = \int_{\mathcal{S}} T\delta \mathrm{d}S$. The latter integral is taken over a screen that does not enclose m.

Moreover, the thermodynamical equivalent of the gravitational theory includes the following *dictionary entries* [113]:

$$\frac{1}{k_B}S(x) = \frac{-1}{4\hbar c L_P^2}V_G(x)A(V_G(x)), \tag{2.2}$$

$$2\pi k_B T(x) = \frac{\mathrm{d}V_G}{\mathrm{d}n}, \tag{2.3}$$

$$\frac{k_B}{2}\int_{\mathcal{S}} \mathrm{d}^2a \, T = L_P^2 M c^2. \tag{2.4}$$

In (2.2), (2.3) and (2.4) we have placed all thermodynamical quantities on the left, while their mechanical analogues are on the right. As in ref. [113], the area element d^2a on $\mathcal{S}$ is related to the infinitesimal number of bits $\mathrm{d}N$ on it through $\mathrm{d}^2a = L_P^2 \mathrm{d}N$. We denote the area of the equipotential surface passing through the point x by $A(V_G(x))$, while $\mathrm{d}V_G/\mathrm{d}n$ denotes the derivative of V_G along the normal direction to the same equipotential. The above expressions tell us how, given a gravitational potential $V_G(x)$ and its normal derivative $\mathrm{d}V_G/\mathrm{d}n$, the entropy S and the temperature T can be defined *as functions of space*.

Specifically, eqn. (2.2) expresses the proportionality between the area A of the screen $\mathcal{S}$ and the entropy S it contains. This porportionality implies that gravitational equipotential surfaces get translated, by the holographic dictionary, as *isoentropic surfaces*, above called holographic screens $\mathcal{S}$.

Equation (2.3) expresses the Unruh effect: an accelerated observer experiences the vacuum of an inertial observer as a thermal bath at a temperature T that is proportional to the observer's acceleration $\mathrm{d}V_G/\mathrm{d}n$.

Finally, eqn. (2.4) expresses the first law of thermodynamics and the equipartition theorem. The right–hand side of (2.4) equals the total rest energy of the mass enclosed by the volume $\mathcal{V}$, while the left–hand side expresses the same energy content as spread over the bits of the screen $\mathcal{S} = \partial\mathcal{V}$, each one of them carrying an energy $k_B T/2$. It is worthwhile noting that equipartition need not be postulated. Starting from (2.3) one can in fact prove the following form of the equipartition theorem:

$$\frac{k_B}{2}\int_{\mathcal{S}}\mathrm{d}^2a\,T = \frac{A(\mathcal{S})}{4\pi}U(\mathcal{S}), \qquad A(\mathcal{S}) = \int_{\mathcal{S}}\mathrm{d}^2a. \tag{2.5}$$

The details leading up to (2.5) from (2.3) will be given in section 2.4.5. Above, U can be an arbitrary potential energy[3]. We will henceforth mean eqn. (2.5) when referring to the first law and the equipartition theorem. In all the above we are treating the area as a continuous variable, but in fact it is quantised [113]. If $N(\mathcal{S})$ denotes the number of bits of the screen $\mathcal{S}$, then

$$A(\mathcal{S}) = N(\mathcal{S})L_P^2. \tag{2.6}$$

However, in the limit $N \to \infty$, when $\Delta N/N << 1$, this approximation of the area by a continuous variable is accurate enough. We will see later on that letting $N \to \infty$ is equivalent to the semiclassical limit in quantum mechanics.

We intend to write a holographic dictionary between quantum mechanics, on the one hand, and thermodynamics, on the other. This implies that we will need to generalise eqns. (2.2), (2.3) and (2.5) so as to adapt them to our quantum–mechanical setup. Thus we will replace the classical particle of [113] with a quantum particle, subject to some potential energy U of nongravitational origin.

2.3 The energy representation

Let $H = K + U$ be the classical Hamiltonian function on $\mathbb{R}^3$ whose quantisation leads to the quantum Hamiltonian operator $\hat{H} = \hat{K} + \hat{U}$ that governs our quantum particle. The Hamiltonian $\hat{H}$ will be assumed to possess normalisable states. This condition on the potential was already reflected in the gravitational case of eqn. (2.2), where the negative sign of the gravitational potential led to a positive definite entropy.

On the bright side of the screen, spacetime has already emerged. This gives us the energy representation of quantum mechanics—the one we are used to: a time variable with a conserved Noether charge, the energy, and wavefunctions depending on the spacetime coordinates. We have the uncertainty relation

$$\Delta\hat{Q}\,\Delta\hat{P} \geq \frac{\hbar}{2}. \tag{2.7}$$

In the semiclassical limit we have a wavefunction

$$\psi_+ = \exp\left(\frac{\mathrm{i}}{\hbar}I\right), \tag{2.8}$$

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[3]The gravitational potential V_G appearing above is the gravitational energy U_G per unit test mass m.

where $I = \int \mathrm{d}t L$ is the action integral satisfying the Hamilton–Jacobi equation.

Let $\mathcal{V}$ denote the finite portion of 3–space bounded by the closed holographic screen $\mathcal{S} = \partial \mathcal{V}$. We can now posit the quantum–mechanical analogues of eqns. (2.2), (2.3) and (2.5). In the energy representation these analogues read, respectively,

$$\frac{1}{k_B}\hat{S}(x) = \frac{1}{4\hbar c L_P} A(U(x))|\hat{U}(x)|, \tag{2.9}$$

$$2\pi k_B \hat{T}(x) = L_P \frac{\mathrm{d}\hat{U}}{\mathrm{d}n}, \tag{2.10}$$

$$\frac{k_B}{2} \int_{\mathcal{S}} \mathrm{d}^2 a\, \hat{T} = \frac{A(\mathcal{S})}{4\pi}\hat{U}(\mathcal{S}). \tag{2.11}$$

Some comments are in order. We are considering the nonrelativistic limit, in which the rest energy of the particle can be ignored. We also neglect all gravitational effects, relativistic or not; we will limit ourselves to the external potential $\hat{U}$. Quantum operators such as $\hat{U}$, initially defined to act on wavefunctions in $L^2(\mathbb{R}^3)$, must now be restricted to act on wavefunctions in $L^2(\mathcal{V})$. Denote this restriction by $\hat{U}_{\mathcal{V}}$. By definition, its matrix elements $\langle f_+ | \hat{U}_{\mathcal{V}} | g_+ \rangle$ are

$$\langle f_+ | \hat{U}_{\mathcal{V}} | g_+ \rangle := \int_{\mathcal{V}} \mathrm{d}^3 V f_+^* \hat{U} g_+, \tag{2.12}$$

the integral extending over the finite volume $\mathcal{V}$ instead of all $\mathbb{R}^3$. For simplicity we have suppressed the subindex $_{\mathcal{V}}$ in (2.9), (2.10) and (2.11), but it must be understood that all operators are to be restricted as specified.

The right–hand side of (2.9) deserves more attention. $|\hat{U}|$ denotes the operator whose matrix elements are the absolute values of those of $\hat{U}$. Taking the absolute value ensures that the entropy is positive definite, given that the potential U need not have a constant sign, contrary to the gravitational case of (2.2).

It will also be observed that no carets stand above $A(U(x))$, $A(\mathcal{S})$, because they are c–numbers. They denote the area of the equipotential surface passing through the point x and the are of the screen $\mathcal{S}$, respectively. Also, the integral on the left–hand side of (2.11) is a standard surface integral, even if the integrand is the operator $\hat{T}$, because the latter depends on the c–number–valued coordinate functions x.

As a final remark, let us point out that the above equations (2.9), (2.10) and (2.11), as well as their classical counterparts (2.2), (2.3) and (2.5), are correctly understood as being expressed in the energy representation of thermodynamics. This is so despite the fact that one writes the entropy as an explicit function of the potential energy—would this not be the defining property of the entropy representation? The answer is negative for two reasons. First, one would need to express the entropy as a function of the total energy H, rather than as a function of just the potential energy U. Second, all the above expressions are functions defined on the emerged portion of space, where there exists a conserved Noether charge, the energy H, and its conjugate variable, the time t. The entropy representation will be introduced later on, when the absence of spacetime will make it necessary to eliminate the space dependence of quantities such as entropy and temperature. Such will be the case beyond the holographic screen.

2.4 The entropy representation

The entropy representation can also be thought of as quantum mechanics in the absence of spacetime, as we will come to recognise presently.

2.4.1 Action *vs.* entropy

It is well known, in the theory of thermodynamical fluctuations [17], that the probability density function d required to compute expectation values of thermodynamical quantities is given by the exponential of the entropy:

$$d = \exp\left(\frac{S}{k_B}\right). \tag{2.13}$$

Its square root, that one may call the amplitude for the probability density d, can therefore be identified with an entropic wavefunction $\psi_-^{(d)}$:

$$\psi_-^{(d)} = \exp\left(\frac{S}{2k_B}\right). \tag{2.14}$$

This identification is made up to a (possibly point–dependent) phase $e^{i\alpha}$, plus a normalisation. Comparing (2.14) with (2.8) we arrive at the correspondence

$$\frac{iI}{\hbar} \leftrightarrow \frac{S}{2k_B} \tag{2.15}$$

between the energy representation and the entropy representation, *both of them taken in the semiclassical limit*. This amounts to the statement that quantum–mechanical fluctuations can be understood thermodynamically, at least in the semiclassical limit.

We should note that the correspondence (2.15) is holographic in nature, because the action integral I is defined on space, while the entropy S is defined on the screen bounding it. Moreover, the above correspondence also implies that, in the entropic representation, the semiclassical limit (the one considered in (2.8)) corresponds to letting $k_B \to 0$.

The wavefunction (2.14) describes an *incoming* wave, from the point of view of the screen. An *outgoing* wave, from the point of view of the screen, would be described by $\exp\left(-S/2k_B\right)$.

It is reassuring to observe that the same correspondence (2.15) has been found in the context of gravity and black–hole thermodynamics [10, 9].

2.4.2 Quantum states *vs.* holographic screens

The equation $U(x^1, x^2, x^3) = U_0$, where U_0 is a constant, defines an equipotential surface in $\mathbb{R}^3$. As U_0 runs over all its possible values, we obtain a foliation of $\mathbb{R}^3$ by equipotential surfaces. Following [113], we will identify equipotential surfaces with holographic screens. Hence forces will arise as entropy gradients.

Assume that ψ_+ is nonvanishing at a certain point in space. Consider an infinitesimal cylinder around this point, with height L_P and base area equal to the area element

$\mathrm{d}^2 a$. Motivated by the proportionality between area and entropy, already mentioned, we postulate that there is an infinitesimal entropy flow $\mathrm{d}S$ *from the particle to the area element* $\mathrm{d}^2 a$:

$$\mathrm{d}S = C\, 2\pi k_B L_P |\psi_+|^2 \mathrm{d}^2 a. \tag{2.16}$$

Here C is a dimensionless numerical constant, to be determined presently. A closed surface Σ receives an entropy flux $S(\Sigma)$:

$$S(\Sigma) = C(\Sigma) 2\pi k_B L_P \int_\Sigma \mathrm{d}^2 a\, |\psi_+|^2. \tag{2.17}$$

The constant $C(\Sigma)$ will in general depend on the particular surface chosen; the latter may, but need not, be a holographic screen. The key notion here is that the integral of the scalar field $|\psi_+|^2$ over any surface carries an entropy flow associated. When the surface Σ actually coincides with a holographic screen $\mathcal{S}$, and when the latter is not a nodal surface of ψ_+, the constant $C(\mathcal{S})$ may be determined by the requirement that the entropy flux from the particle to the screen equal the quantum of entropy (2.1). Thus

$$\frac{1}{C(\mathcal{S})} = L_P \int_\mathcal{S} \mathrm{d}^2 a\, |\psi_+|^2. \tag{2.18}$$

Let us now read eqn. (2.18) in reverse, under the assumption that one knows the proportionality constants $C(\mathcal{S}_j)$ for a given foliation $\mathbb{R}^3 = \cup_{j \in \mathcal{J}} \mathcal{S}_j$. This amounts to a knowledge of the integrands, *i.e.*, of the probability density $|\psi_+|^2$ within the surface integral (2.18) on each and every $\mathcal{S}_j$. From these tomographic sections of all probability densities *there emerges the complete wavefunction ψ_+ on all of $\mathbb{R}^3$*, at least up to a (possibly point–dependent) phase $\mathrm{e}^{\mathrm{i}\alpha}$.

Thus the integrand of (2.18) gives the surface density of entropy flow into the holographic screen $\mathcal{S}_j$, and the wavefunction ψ_+ becomes (proportional to) the square root of this flow. The collection of all these tomographic sections of ψ_+ along all possible screens amounts to a knowledge of the complete wavefunction. Hence *a knowledge of the different surface densities of entropy flux across all possible screens is equivalent to a knowledge of the quantum–mechanical wavefunction ψ_+*. This is how the quantum–mechanical wavefunction ψ_+ emerges from the holographic screens. Close ideas concerning the wavefunction in relation to foliations of space have been put forward in ref. [19].

2.4.3 The entropic uncertainty principle

Let us define the dimensionless variable

$$s := \frac{S}{2\pi k_B}, \tag{2.19}$$

that we will call the *reduced entropy*. It is nonnegative: $s \geq 0$. For example, the semiclassical entropic wavefunction (2.14) can be expressed in terms of s as $\psi_-^{(d)}(s) = \mathrm{e}^{\pi s}$. We can consider arbitrary functions $f(s)$ on which we let the following operators $\hat{Q}_S, \hat{P}_S$ act:

$$\hat{Q}_S f(s) := s f(s), \qquad \hat{P}_S f(s) := 2\pi k_B \frac{\mathrm{d}f(s)}{\mathrm{d}s}. \tag{2.20}$$

For reasons that will become clear presently, $\hat{Q}_S$ will also be called the *normal, or entropic, position operator*, while $\hat{P}_S$ will be called the *normal, or entropic, momentum*[4]. One finds that $i\hat{P}_S$ and $\hat{Q}_S$ are Hermitian on $L^2[0, \infty)$. Unlike the usual case on $L^2(\mathbb{R})$, the Hermitian property of position and momentum on the semiaxis involves some nontrivial mathematical subtleties that will not be touched upon here; see [107]. Now the above operators satisfy the Heisenberg algebra

$$[\hat{Q}_S, \hat{P}_S] = 2\pi k_B \mathbf{1}. \tag{2.21}$$

Therefore the following *entropic uncertainty principle* holds:

$$\Delta\hat{Q}_S \, \Delta\hat{P}_S \geq \pi k_B. \tag{2.22}$$

The above uncertainty principle has been derived rather than postulated; this is in the spirit of refs. [44, 45].

2.4.4 The entropic Schroedinger equation

Since the screens $\mathcal{S}_j$ are isoentropic surfaces, the reduced entropy s can be regarded as a dimensionless coordinate orthogonal to all the $\mathcal{S}_j$. Multiplication by L_P gives a dimensionful coordinate ρ:

$$\rho := L_P s. \tag{2.23}$$

Modulo multiplication by a dimensionless numerical factor, and the possible addition of a constant, the above is an equivalent reexpression of the equation [113]

$$\Delta S = 2\pi k_B \frac{Mc}{\hbar} \Delta x, \tag{2.24}$$

where x is the distance measured normally to the screen—in turn, (2.24) is the same as (2.1). We can exploit this fact if we assume that the time–independent Schroedinger equation

$$-\frac{\hbar^2}{2M}\nabla^2\psi_+ + U\psi_+ = E\psi_+ \tag{2.25}$$

is separable in a coordinate system that includes ρ as one of its coordinate functions. So let us supplement ρ with two additional coordinates ξ, χ such that the triple ρ, ξ, χ provides an orthogonal set of curvilinear coordinates[5] in which (2.25) separates as per (2.27) below. Then the Euclidean line element on $\mathbb{R}^3$ will be given by

$$ds^2 = h_\rho^2 d\rho^2 + h_\xi^2 d\xi^2 + h_\chi^2 d\chi^2, \tag{2.26}$$

where the metric coefficients h_ρ, h_ξ, h_χ are functions of all three coordinates ρ, ξ, χ. We will call ρ the *normal coordinate* to the foliation, while ξ, χ will be called *tangential*

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[4]The missing factor of i in the definition of $\hat{P}_S$ is due to the correspondence (2.15).

[5]In general, ρ, ξ, χ are only local coordinates, and need not cover all of $\mathbb{R}^3$. In particular, ξ, χ need not cover a complete screen $\mathcal{S}_j$, nor need they be simultaneously defined on different screens $\mathcal{S}_j, \mathcal{S}_k$. However, to simplify our notation, we omit all the indices that would be necessary in order to take all these possibilities into account.

coordinates to the foliation. A more physical terminology, based on (2.23) and (2.10), could be *entropic coordinate* for ρ and *isothermal coordinates* for ξ, χ.

We recall that U depends only on the normal coordinate ρ, so equipotential surfaces are defined by $U(\rho) = U_0$, for any constant U_0. The tangential dimensions ξ, χ are purely spatial constructs: they encode the geometry of the equipotential surfaces. For example, in the particular case of a Coulomb potential, or also of an isotropic harmonic oscillator, the $\mathcal{S}_j$ are a family of concentric spheres of increasing radii. Then ρ can be identified with the usual radial coordinate r on $\mathbb{R}^3$, while ξ, χ can be taken as the usual polar angles θ, φ. In the general case ρ, ξ, χ need not coincide with any of the standard coordinate functions on $\mathbb{R}^3$. However, each screen $\mathcal{S}_j$ can be univocally identified by the equation $\rho = \rho_j$. The uncertainty principle (2.22) holds on the phase space corresponding to ρ, and the operator $\hat{Q}_S$ defined in (2.20) is nothing but *the position operator along the normal, or entropic, coordinate*.

Thus separating variables as per

$$\psi_+(\rho, \xi, \chi) = R(\rho)Y(\xi, \chi), \tag{2.27}$$

and substituting into (2.25) leads to

$$\frac{1}{h_\rho h_\xi h_\chi} \left[\frac{1}{R} \frac{\partial}{\partial \rho} \left(\frac{h_\xi h_\chi}{h_\rho} \frac{\partial R}{\partial \rho} \right) + \frac{1}{Y} \frac{\partial}{\partial \xi} \left(\frac{h_\rho h_\chi}{h_\xi} \frac{\partial Y}{\partial \xi} \right) + \frac{1}{Y} \frac{\partial}{\partial \chi} \left(\frac{h_\rho h_\xi}{h_\chi} \frac{\partial Y}{\partial \chi} \right) \right]$$

$$+ \frac{2M}{\hbar^2}(E - U) = 0. \tag{2.28}$$

The precise way in which (2.28) separates into a ρ–dependent piece and a ξ, χ–dependent piece cannot be written down in all generality, as it varies according to the particular choice made for ρ, ξ, χ. This is due to our ignorance of the specific way in which the metric coefficients h_ρ, h_ξ, h_χ depend on all three variables ρ, ξ, χ. One can, however, outline some general features of the final outcome. Terms involving the Laplacian ∇^2 will decompose as a sum $\nabla_\rho^2 + \nabla_{\xi,\chi}^2$, where subindices indicate the variables being differentiated in the corresponding operators. Calling the separation constant λ, there will be two separate equations. The first equation will involve the normal Laplacian ∇_ρ^2, the potential energy $U(\rho)$, the energy eigenvalue E, the mass M and the separation constant λ. All these elements (with the exception of ∇_ρ^2) appear as a certain function F of ρ:

$$\nabla_\rho^2 R(\rho) + F(\rho, U(\rho), E, M, \lambda)R(\rho) = 0. \tag{2.29}$$

The unknown function F is explicitly computable once a specific choice has been made for the coordinates ξ, χ. The second equation involves only the tangential Laplacian $\nabla_{\xi,\chi}^2$ and the separation constant λ:

$$\nabla_{\xi,\chi}^2 Y(\xi, \chi) + \lambda Y(\xi, \chi) = 0. \tag{2.30}$$

It is important to note that (2.30) can be solved independently of (2.29)[6]. The eigenfunctions $Y(\xi, \chi)$ constitute a complete orthonormal system of eigenfunctions of the

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[6]Needless to say, in the case of a Coulomb field, (2.29) becomes the standard radial wave equation, while (2.30) becomes that satisfied by the usual spherical harmonics, with $\lambda = l(l+1)$.

tangential Laplacian within the *tangential Hilbert space* $L^2(\mathcal{S}_j)$. Moreover, since we have assumed the screens to be closed surfaces, the eigenvalues λ will be quantised. Once these eigenvalues have been determined, substitution into (2.29) allows the latter to be completely solved.

We are finally in a position to define the entropic wavefunction ψ_- in terms of its partner ψ_+. We take the entropic wavefunction to be the ρ–dependent piece in the factorisation (2.27),

$$\psi_-(\rho) := R(\rho). \tag{2.31}$$

Clearly the *entropic, or normal, Hilbert space* corresponding to the screen $\mathcal{S}_j$ will be $L^2[0, \rho_j)$. The latter is considered with respect to an integration measure that includes a certain Jacobian factor $J(\rho)$. In order to compute this Jacobian we proceed as follows. Apply the factorisation (2.27) to the normalisation condition for ψ_+ on $\mathcal{V}_j$:

$$\int_{\mathcal{V}_j} \mathrm{d}^3 V \, |\psi_+|^2 = \int_0^{\rho_j} \mathrm{d}\rho \int_{\mathcal{S}_j} \mathrm{d}\xi \mathrm{d}\chi \, h_\rho h_\xi h_\chi |R(\rho)|^2 |Y(\xi, \chi)|^2. \tag{2.32}$$

In general, the product $h_\rho h_\xi h_\chi$ depends on all three coordinates ρ, ξ, χ. The sought–for Jacobian $J(\rho)$ equals the ρ–dependent factor in the integration measure after the integral over ξ, χ has been carried out. As ρ_j becomes larger and larger, we obtain the entropic Hilbert space $L^2[0, \infty)$. The latter would correspond to an observer who perceives no horizon at all, thus extending his normalisation integral (2.32) over all of $\mathbb{R}^3$. We will come back to the issue of the different realisations of the entropic Hilbert space ($L^2[0, \rho_j)$ *vs.* $L^2[0, \infty)$) in section 2.5.2.

In the passage form the energy representation to the entropy representation we appear to have lost the information corresponding to the holographic screens one integrates over. However the screens carry no dynamics, because the force at point x is orthogonal to the screen passing through x. Thus a knowledge of the entropic wavefunction ψ_-, *plus of the foliation itself*, is equivalent to a knowledge of the wavefunction ψ_+ in the energy representation. That the foliation is a piece of information belonging to the entropy representation, was stated in assertion *ii)* of our section 2.2.3 following [113, 60].

It remains to identify the wave equation satisfied by the entropic wavefunction ψ_-. Obviously this equation is (2.29), which may thus be regarded as *the entropy–representation analogue of the time–independent Schroedinger equation $\hat{H}\psi_+ = E\psi_+$* on space. Recalling (2.9) and (2.23), this entropic Schroedinger equation reads

$$\nabla_s^2 \psi_-(s) + G(s, A(s), E, M, \lambda)\psi_-(s) = 0. \tag{2.33}$$

We have called $G(s, A(s), E, M, \lambda)$ the function that results from expressing the potential U as a function of the entropy S and the area A, and writing everything in terms of the reduced entropy s. As was the case with F in (2.29), the unknown function G is explicitly computable once a specific choice has been made for the coordinates ξ, χ.

2.4.5 The fundamental equation, the equation of state, and equipartition

In this section we will rewrite the dictionary entries (2.9), (2.10) and (2.11), found to hold in the energy representation, in the entropy representation. For this purpose we first need to solve the eigenvalue equation $\hat{S}\phi_- = S\phi_-$ on the screen, so the latter will be kept fixed. That is, we will not consider a variable surface $\mathcal{S}_j$ of the foliation, but rather a specific surface corresponding to a fixed value of the index j. Observe also a difference in notation: ϕ instead of ψ. This is to stress the fact that, by (2.9), entropy eigenstates ϕ cannot be eigenstates of the complete Hamiltonian $\hat{H}$, but only of the potential energy $\hat{U}$. Once $\hat{U}$ is diagonalised by a set of ϕ_+ defined on the bright side, *i.e.*, once we have solved the eigenvalue equation[7]

$$\hat{U}\phi_+ = U\phi_+, \tag{2.34}$$

then the corresponding ϕ_- on the screen are defined per continuity: $\phi_-(\mathcal{S}) = \phi_+(\mathcal{S})$. By (2.9), the same ϕ_- then diagonalise $\hat{S}$:

$$\hat{S}\phi_- = S\phi_-, \qquad S = \frac{k_B}{4\hbar c L_P} A(\mathcal{S})|U(\mathcal{S})|. \tag{2.35}$$

Thermodynamical quantities will now arise as expectation values of operators in the entropic eigenstates $\phi_-(\mathcal{S})$.

We first deal with (2.9). Clearly its reexpression in the entropy representation will be the thermodynamical fundamental equation $S = S(A)$ in the sense of ref. [17], since the extensive parameter corresponding to the holographic screen is the area A. Then we have

$$\langle \hat{S} \rangle = \frac{k_B}{4\hbar c L_P} A(\mathcal{S})|U(\mathcal{S})|. \tag{2.36}$$

Availing ourselves of the freedom to pick the origin of potentials at will, let us set $|U(\mathcal{S})| = \hbar c / L_P$. Thus

$$\langle \hat{S} \rangle = \frac{k_B}{4 L_P^2} A, \tag{2.37}$$

which is the celebrated Bekenstein–Hawking law. It arises as a thermodynamical fundamental equation in the entropy representation.

Our holographic screen is treated thermodynamically as a stretched membrane, so the generalised force conjugate to the extensive parameter A is the surface tension σ. Then the equation of state corresponding to (2.37) is

$$\sigma = \frac{k_B \langle \hat{T} \rangle}{4 L_P^2}. \tag{2.38}$$

Rewrite the above as $2\pi k_B \langle \hat{T} \rangle = 8\pi L_P^2 \sigma$ and recall that σ is the normal component of force per unit length on the screen. Since force is proportional to acceleration, the above equation of state turns out to be equivalent to the Unruh law.

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[7]Obviously the ϕ_+ are the well–known eigenfunctions of the position operator on the bright side, but this property is immaterial for our purposes.

Finally we turn to the first law of thermodynamics and the equipartition theorem. As already mentioned in section 2.2.3, it turns out that the equipartition theorem can be derived from the Unruh law. Since this fact is valid both in the classical case (2.5) and in its quantum counterpart (2.11), the derivation being exactly the same whatever the case, we will provide the details pertaining to the derivation of (2.11) from (2.10). Integrate the latter over a thin 3–dimensional slice of width $\mathrm{d}n$ bounded by two equipotentials $\mathcal{S}_1$ and $\mathcal{S}_2$. Now the Planck length L_P is extremely small, so we can safely set $\mathrm{d}n = L_P$, while the two screens $\mathcal{S}_1$ and $\mathcal{S}_2$ will not differ appreciably in their surface area. Then the volume integral of the left–hand side of (2.10) very approximately equals $2\pi k_B L_P \int_{\mathcal{S}} \mathrm{d}^2 a\, \hat{T}$. On the right–hand side, let us first integrate $\mathrm{d}\hat{U}/\mathrm{d}n$ along the normal direction, to obtain $L_P \hat{U}(\mathcal{S}_2) - L_P \hat{U}(\mathcal{S}_1)$. We can take the origin for the potential function such that it will vanish on $\mathcal{S}_1$. The remaining term is the surface integral $L_P \int_{\mathcal{S}} \mathrm{d}^2 a\, \hat{U}(\mathcal{S})$. The integrand can be pulled past the integration sign because $\mathcal{S}$ is an equipotential surface, thus yielding $L_P \hat{U}(\mathcal{S}) \int_{\mathcal{S}} \mathrm{d}^2 a$. This latter integral equals the surface area $A(\mathcal{S})$ of the screen, and (2.11) follows as claimed.

Taking the expectation value, in the entropic eigenstates ϕ_-, of the operator equation (2.11), produces the thermodynamical expression for the equipartition theorem:

$$\frac{k_B}{2} \int_{\mathcal{S}} \mathrm{d}^2 a\, \langle \hat{T} \rangle = \frac{A(\mathcal{S})}{4\pi} \langle \hat{U}(\mathcal{S}) \rangle. \tag{2.39}$$

2.4.6 Planck *vs.* Boltzmann, or $\hbar$ *vs.* k_B

Planck's quantum of action $\hbar$ gets replaced, in the entropic picture, with Boltzmann's constant k_B. This explains the presence of $\hbar$ in the entropic derivation of classical gravity (Newton's and Einstein's) given in ref. [113]: by the correspondence (2.15), *the presence of $\hbar$ is an unavoidable consequence of the presence of k_B, and viceversa*. We find this dichotomy between the energy and the entropy representations very suggestive—it appears to be a sort of complementarity principle, in Bohr's sense of the word. For example, this dichotomy allows one to write a quantum of energy in the form $E = \hbar\omega$, or else in the alternative form $E = Ck_BT$ (C being a dimensionless number). It also allows one to express a quantum of entropy in the form $S = \hbar\omega/T$, or else as $S = 2\pi k_B$. This dichotomy exchanges frequency ω with temperature T, thus time t maps to inverse temperature T^{-1}, which is reminiscent of the Tolman–Ehrenfest relation [110] and also of thermal time [96].

2.4.7 The second law of thermodynamics, revisited

As a minor technical point, we have restricted our analysis to closed holographic screens enclosing a finite 3–dimensional volume. Quantum–mechanically this corresponds to normalisable states in the energy representation. Nonnormalisable states correspond to open holographic screens without a boundary (thus having an infinite surface area and enclosing an infinite volume). Our analysis can be extended to the latter by replacing absolute quantities with densities (per unit surface or unit volume as the case may be). The connection with the second law of thermodynamics comes about as follows. The second law of thermodynamics, $\Delta S \geq 0$, lies hidden within

the quantum theory. Of course, one can derive it from statistical mechanics, but our purpose here is the opposite. We have seen that the domain of the reduced entropy s is the half axis $s \geq 0$, and that this fact led to the entropic Hilbert space $L^2[0, \infty)$ (instead of $L^2(\mathbb{R})$) for the wavefunctions $\psi_-(s)$. All this is a quantum–mechanical rewriting of the second law. One could ask, under what conditions will the entropic coordinate ρ be nonnegative? This is certainly the case when the holographic screens are all closed, but what happens in case they are open? The geometry of the screens is dictated by the potential U. If the latter has flat directions, then its equipotentials will no longer be closed surfaces—instead they will have an infinite surface area and will enclose an infinite volume. As mentioned above, one appropriately replaces quantities like entropy and energy with the corresponding densities. However, the corresponding screens must be such that the normal coordinate to their bright side, ρ, runs over the half axis $\rho \geq 0$. This latter condition will be satisfied whenever the potential is such that it possesses a centre of force, or an axis, or a plane, or possibly a more general surface of symmetry, with respect to which one can define a nonnegative normal coordinate. This appears to be the case in all physically interesting situations, thus staying in agreement with the second law of thermodynamics. Only the free particle lacks a *canonical* definition of a normal coordinate—but then again the second principle holds in the form $\Delta S = 0$, due to the absence of forces.

The second law of thermodynamics,

$$\Delta S \geq 0, \tag{2.40}$$

has been related to the Heisenberg uncertainty principle in ref. [84]. In ref. [32] it has been argued that the second law of thermodynamics has a quantum–mechanical reexpression in the Bell inequalities. Above we have established a link between (2.40) and the Hilbert space of entropic quantum mechanics. Next we would like to propose yet another quantum–mechanical interpretation of the second law, one that combines the uncertainty principle with the notion of emergence.

From eqn. (2.1) one derives the obvious inequality

$$\Delta S \geq \pi k_B \tag{2.41}$$

which looks like some refinement of the second law (2.40)—the latter would be recovered in the semiclassical limit $k_B \rightarrow 0$. Therefore let us, for the sake of the argument, consider eqn. (2.41) as a more precise statement of the second law than (2.40). As such (2.41) is reminiscent the uncertainty principle (2.7) of quantum mechanics. However the left–hand side of (2.41) contains just one uncertainty, instead of a product of two uncertainties as usual. This reflects the fact that the variable on the left, S, is *selfconjugate*—its dimension equals that of the quantum k_B on the right–hand side[8]. We can include a dimensionless formal parameter τ in the left–hand side that will make (2.41) resemble the uncertainty principle in its standard form. This can be done as follows.

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[8]Compare this situation with (q, p) and (H, t), which are conjugate pairs: the product of the two components of each pair has the dimension of $\hbar$. Angular momentum L is selfconjugate, in the sense that it carries the dimension of $\hbar$, but one writes the corresponding uncertainty principle as $\Delta L \Delta \varphi \geq \hbar/2$, where the dimensionless variable φ is an angle.

Let N denote the total number of bits on $\mathcal{S}$. Whenever a quantum particle hits the screen we have $\Delta N = 1$, and the ratio $\Delta N/N$ will be small if N is large enough. In this limit we can treat N as a continuous variable, that we redenote by τ in order to interpret it as a continuous, dimensionless parameter:

$$\tau := N, \qquad \text{when} \qquad \frac{\Delta N}{N} << 1. \tag{2.42}$$

This is the limit $N \to \infty$ referred to in (2.6). Compatibility with all the above requires this limit to correspond to $k_B \to 0$ or, equivalently, to $\hbar \to 0$. In other words, the large area limit for a holographic screen corresponds to the semiclassical approximation in quantum mechanics.

We have $\Delta \tau \geq 1$, the inequality allowing for the possibility of more than just one particle hitting $\mathcal{S}$. Thus multiplying the two inequalities $\Delta \tau \geq 1$ and $\Delta S \geq \pi k_B$ together we arrive at the following uncertainty principle *on the holographic screen*:

$$\Delta S \Delta \tau \geq \pi k_B. \tag{2.43}$$

The fact that k_B, though small, is nonvanishing, leads to the impossibility of having strictly reversible processes; reversibility is possible only in the limiting case of a vanishing value for the quantum k_B. We conclude that quantisation appears as dissipative mechanism. The notion that information loss leads to a quantum behaviour lies at the heart of the notion of emergence [13, 14, 33, 34, 55, 56, 97].

We have derived the uncertainty principle (2.43) starting from the second law of thermodynamics (2.40). Let us now prove that the reverse path is also possible: from the uncertainty principle to the second law of thermodynamics. We start from (2.7) in the bulk rewritten as $\Delta I/\hbar \geq 1$, where $I = \int p \mathrm{d}q$ is the action. On the boundary, the correspondence (2.15) allows to reexpress the above inequality as in (2.41). Along the way we have dropped irrelevant numerical factors.

Altogether, we have an equivalence between the uncertainty principle of quantum mechanics (either in the bulk (2.7) or on the boundary (2.43)), and a refined version of the second law of thermodynamics, one that includes a small but nonvanishing value of the corresponding quantum ($\hbar$ or k_B) on the right–hand side. This is in agreement with the results of [84]—now with the added bonus that our equivalence has the properties of emergence and holography.

2.5 Conclusions to chapter 2

2.5.1 Quantum mechanics as a holographic, emergent phenomenon

Classical thermodynamics can be conveniently expressed in either of two equivalent languages, respectively called the energy representation and the entropy representation [17]. Here we have argued that quantum mechanics as we know it (*i.e.*, on spacetime) corresponds to the energy representation, while quantum mechanics beyond a holographic screen (*i.e.*, in the absence of spacetime) corresponds to the entropy representation. In this paper we have developed the formalism of entropic quantum mechanics and placed it in correspondence with that of standard quantum mechanics on spacetime.

In particular, we have formulated the entropic uncertainty principle (2.22) for the (reduced) entropy variable s that the entropic wavefunction $\psi_-(s)$ (sometimes also denoted $R(\rho)$) depends on; see (2.23). The latter arises as the result of factoring out the part of the wavefunction that depends on the tangential coordinates to the screen, the normal coordinate being proportional to the entropy itself. We have also written down a differential equation satisfied by the entropic wavefunction, that one may well call the entropic Schroedinger equation; see (2.33).

Moreover, we have identified the explicit expression (2.14) as corresponding to the entropic wavefunction in the semiclassical limit $k_B \to 0$. There is a nice map, given by (2.15), between the semiclassical wavefunction in the energy representation and the corresponding semiclassical wavefunction in the entropy representation. This map exchanges the classical action integral with the entropy of the screen, while at the same time introducing a relative factor of i. It also exchanges Planck's constant $\hbar$ with Boltzmann's constant k_B. In so doing, this map succeeds in explaining why Planck's constant $\hbar$ had to appear in the derivation of classical gravity (Newton's and Einstein's) given in ref. [113]. Namely, the presence of $\hbar$ is an inescapable consequence of the presence of k_B, and viceversa, since $\hbar$ is required by the energy representation, while k_B is required by the entropy representation.

If spacetime is an emergent phenomenon, then everything built on it necessarily becomes emergent [40]. This applies to quantum mechanics in particular. However, in the entropy representation developed here, the emergence property of quantum mechanics becomes a much sharper feature. Indeed, one usually associates entropy with lack of information, while energy (*e.g.*, a sharp energy eigenvalue) is thought of as providing definite information. Now the correspondence (2.15) implies that, if the entropy representation is emergent, then so is the energy representation, and viceversa. In this sense, the information content carried by entropy is no more diffuse than that carried by energy, nor is the information encoded by energy more sharply defined than that encoded in entropy. In other words, the correspondence (2.15) confirms what we already knew from other sources—namely, that quantum mechanics is definitely an emergent phenomenon.

We have also succeeded in writing a holographic dictionary between quantum mechanics, on the one hand, and thermodynamics, on the other. An analogous holographic dictionary was presented, in the gravitational case, in ref. [113]. Some key entries in this gravitational/thermodynamical dictionary are summarised in eqns. (2.2), (2.3) and (2.5), preceded by the equivalence between statements *i)* and *ii)* of section 2.2.3. As a novelty, here we have presented the corresponding entries in our quantum–mechanical/thermodynamical dictionary. These entries include the equivalence between the analogues of statements *i)* and *ii)* of section 2.2.3. In our setup, this is expressed in the assertion that the energy representation of quantum mechanics (statement *i)*) is equivalent to the entropy representation of quantum mechanics (statement *ii)*). Further entries in this dictionary of equivalences are the analogues of eqns. (2.2), (2.3) and (2.5), respectively given by our eqns. (2.9), (2.10) and (2.11) when working in the energy representation. Our eqns. (2.9), (2.10) allow one to define an entropy field and a temperature field as (operator–valued) functions on $\mathbb{R}^3$, whereas (2.11) is a reexpression of the first law of thermodynamics and of the equipartition theorem. Their respective vacuum expectation values give rise to the corresponding equations in the

entropy representation, (2.37), (2.38) and (2.39), where the space dependence disappears. Their respective interpretations are the proportionality between the area and the entropy of the screen (the Bekenstein–Hawking law), the thermodynamical equation of state of the screen (the Unruh law), and the equipartition theorem.

2.5.2   Quantum mechanics in the absence of spacetime

Entropic quantum mechanics can be thought of as describing quantum mechanics *in the absence of spacetime*. This latter statement must be understood as meaning that the tangential coordinates to the holographic screens, as well as functions thereof, have been factored out, while the normal coordinate and functions thereof remain—though no longer as a *spatial coordinate*, but rather as a *measure of entropy*. This viewpoint is motivated in eqn. (2.24), that we have borrowed directly from [113]. Now in the absence of time there is no Hamiltonian. In the absence of space there are also no paths to sum over *à la* Feynman. One might thus conclude that there can be no quantum mechanics in the absence of spacetime. This is however not true, as shown here and as shown also by independent analyses. For example, quantum mechanics without spacetime has been proposed as a case for noncommutative geometry [100, 101, 64]. Without resorting to noncommutative geometry, one can also argue as follows.

   We have seen that the Hilbert space of entropic quantum states is $L^2[0, \rho_j)$ for an observer who perceives space terminating at the screen $\mathcal{S}_j$, and $L^2[0, \infty)$ for an observer who perceives no screen at all, or horizon. Given the two screens $\mathcal{S}_j$ and $\mathcal{S}_k$, respectively located at $\rho = \rho_j$ and $\rho = \rho_k$ with $\rho_j < \rho_k$, it holds that the two spaces $L^2[0, \rho_j)$ and $L^2[0, \rho_k)$ are unitarily isomorphic because both are infinite–dimensional and separable [107]. Now let $\rho_k \rightarrow \infty$. The isomorphism between $L^2[0, \rho_j)$ and $L^2[0, \infty)$, plus the identification (2.23) between entropy and normal coordinate, allows the observer who perceives the screen $\mathcal{S}_j$ to extend his wavefunctions $R(\rho)$ beyond his boundary at ρ_j. His wavefunctions are now understood as $\psi_-(s)$, *i.e.*, as functions of the reduced entropy s—indeed the latter is not bounded from above. It is in this sense that this second observer can be said to be doing *quantum mechanics in the absence of spacetime*.

   It is right to observe that the unitary isomorphism between the two different realisations of the entropic Hilbert space, $L^2[0, \infty)$ and $L^2[0, \rho_j)$, need not map the semiclassical regime of the one into the semiclassical regime of the other, nor the strong–quantum regime of the one into the corresponding regime of the other. An analogous statement applies to the spaces $L^2[0, \rho_j)$ and $L^2[0, \rho_k)$ corresponding to the screens $\mathcal{S}_j$, $\mathcal{S}_k$. The observation just made will become relevant in section 2.5.3.

2.5.3   Open questions

We can summarise our conclusions so far with the assertion that entropic quantum mechanics is a holographic phenomenon, as emergent as spacetime itself. To round up our discussion we would like to present some thoughts of a more speculative nature.

   As a first thought we would like to state that *entropic quantum mechanics is an observer–dependent phenomenon*. That measurement disturbs any quantum system

is, of course, a basic tenet of quantum mechanics. The statement just made, however, refers to something different. The concept that quantum mechanics is observer–dependent has also appeared, in different guises, in [112, 102, 103] under the name of *duality*. Under duality one understands that *the notion of classical vs. quantum is relative to which theory one measures from* (see section 6 of ref. [112]). This is also the interpretation advocated in refs. [63] by one of the present authors.

An idea that lies close to the above notions is the statement that *the entropy of a horizon is an observer–dependent quantity* (see section 3 of ref. [89]). In view of our correspondence (2.15), this latter assertion turns out to be equivalent to the one above defining duality.

Thus the statement that quantum mechanics is observer–dependent, is an equivalent reexpression of duality, *i.e.*, of the relativity of the notion of a quantum. In the entropic picture developed here, this relativity presents itself as the different realisations of the entropic Hilbert space, explained in section 2.5.2. Equivalently, this relativity of the notion of a quantum arises here as the relativity of the entropy.

The previous statements may at first sound surprising. Classic treatises such as, *e.g.*, ref. [110], teach that the Lorentz transformation laws for the heat energy and the temperature are such that their ratio (the entropy) is a scalar. Moreover, in principle one expects physical constants such as k_B and $\hbar$ to be observer–independent. However, let us note that a totally analogous phenomenon has been reported in refs. [87, 88, 89, 90, 91], where the entropy of the screen has been argued to be an observer–dependent quantity. That the entropy of a thermodynamical system becomes an observer–dependent quantity has also been concluded in an information–theoretical context [95]. Upon transforming back to the energy representation, the dependence just described can be recast as the dependence of Planck's constant $\hbar$ upon the observer. Exactly this latter conclusion concerning $\hbar$ has been reported in [114].

Given that the equations of motion for Einstein's gravity can be recast as thermodynamical equations of state, it has been claimed that the canonical quantisation of gravity makes as little sense as *quantising sound waves in air* [65]. This remark makes it clear that quantising Einstein's gravity may be attempting to quantise the wrong classical theory, but it casts no doubt yet on the validity of quantum theory. However, doubts concerning the microscopic fundamentality of the latter arise once one realises that *quantum theory, too, is a thermodynamics in disguise*...

Chapter 3

A picture of irreversibility

3.1 Introduction

The purpose of this chapter is twofold:

i) to establish an explicit correspondence between quantum mechanics on the one hand, and the classical thermodynamics of irreversible processes on the other. We claim validity for this correspondence at least in the Gaussian approximation (which corresponds to the linear response regime in thermodynamics, and to the semiclassical approximation in quantum mechanics);

ii) to use the correspondence just mentioned in order to provide an independent proof of the statement that *quantum mechanics is an emergent phenomenon, at least in the semiclassical limit*.

    With hindsight, once one has realised that quantum mechanics in the Gaussian approximation is a classical thermodynamics in disguise, the emergent nature of quantum theory becomes selfevident—after all, thermodynamics is a paradigm of emergent theories.

3.2 The Chapman–Kolmogorov equation in quantum mechanics

To begin with we present a collection of *purely* quantum–mechanical expressions, for which there will be *purely* thermodynamical reexpressions using the correspondence we are about to develop. Although the material of this section is standard, a good general reference is [115]. For simplicity we will restrict to a 1–dimensional configuration space X coordinatised by x.

    The quantum–mechanical propagator $K\left(x_2, t_2 | x_1, t_1\right)$ is defined as the amplitude for the conditional probability that a particle starting at (x_1, t_1) end at (x_2, t_2):

$$K\left(x_2, t_2 | x_1, t_1\right) = \langle x_2 | U(t_2 - t_1) | x_1 \rangle, \quad U(t) = \exp\left(-\frac{\mathrm{i}}{\hbar} t H\right). \qquad (3.1)$$

Above, $U(t)$ is the unitary time–evolution operator, and H is the quantum Hamiltonian operator. The time–evolution operators satisfy *the group property*,

$$U(t_1)U(t_2) = U(t_1 + t_2),\tag{3.2}$$

an equation known in statistics already since the 1930's as *the Chapman–Kolmogorov equation* [31]. Its solutions satisfy the differential equation

$$i\hbar\frac{\mathrm{d}U}{\mathrm{d}t} = HU(t), \qquad H = i\hbar\frac{\mathrm{d}U}{\mathrm{d}t}\Big|_{t=0}.\tag{3.3}$$

Using (3.1) we obtain an alternative reexpression of the Chapman–Kolmogorov equation:

$$K\left(x_3, t_3 | x_1, t_1\right) = \int \mathrm{d}x_2\, K\left(x_3, t_3 | x_2, t_2\right) K\left(x_2, t_2 | x_1, t_1\right).\tag{3.4}$$

Since wavefunctions ψ are *unconditional* probability amplitudes, they are related to propagators K (which are *conditional* probability amplitudes) as follows:

$$\psi(x_2, t_2) = \int \mathrm{d}x_1\, K\left(x_2, t_2 | x_1, t_1\right) \psi(x_1, t_1).\tag{3.5}$$

Propagators can be computed via path integrals over configuration space X,

$$K\left(x_2, t_2 | x_1, t_1\right) = \int_{x(t_1)=x_1}^{x(t_2)=x_2} \mathrm{D}x(t)\, \exp\left\{\frac{i}{\hbar}\int_{t_1}^{t_2}\mathrm{d}t\, L\left[x(t), \dot{x}(t)\right]\right\},\tag{3.6}$$

where L is the classical Lagrangian function. Two simple examples in which the path integral (3.6) can be evaluated exactly are the free particle and the harmonic potential. For a free particle we have

$$K^{(\mathrm{free})}\left(x_2, t_2 | x_1, t_1\right) = \sqrt{\frac{m}{2\pi i\hbar\left(t_2 - t_1\right)}}\, \exp\left[\frac{im}{2\hbar}\frac{\left(x_2 - x_1\right)^2}{t_2 - t_1}\right],\tag{3.7}$$

while for a harmonic potential we have, ignoring the caustics,

$$K^{(\mathrm{harmonic})}\left(x_2, t_2 | x_1, t_1\right) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\left(\omega(t_2 - t_1)\right)}}\tag{3.8}$$

$$\times \exp\left\{\frac{im}{2\hbar}\frac{\omega}{\sin\left(\omega(t_2 - t_1)\right)}\left[(x_2^2 + x_1^2)\cos\left(\omega(t_2 - t_1)\right) - 2x_1 x_2\right]\right\}.$$

When the path integral (3.6) cannot be computed exactly, an approximate evaluation can still be helpful. For $\hbar \to 0$ we have the semiclassical approximation to the propagator, denoted by K_{cl}:

$$K_{\mathrm{cl}}\left(x_2, t_2 | x_1, t_1\right) = Z^{-1}\, \exp\left\{\frac{i}{\hbar}\int_{t_1}^{t_2}\mathrm{d}t\, L\left[x_{\mathrm{cl}}(t), \dot{x}_{\mathrm{cl}}(t)\right]\right\},\tag{3.9}$$

where $x_{\mathrm{cl}}(t)$ stands for the classical trajectory between (x_1, t_1) and (x_2, t_2), and Z^{-1} is some normalisation factor.[1]

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[1]We will henceforth use the collective notation Z^{-1} to denote all the different normalisation factors that we will not keep track of.

3.3 Fluctuations and irreversible processes

For the benefit of the reader, with an eye on later applications, we include below a summary of ref. [86].

3.3.1 Thermodynamic forces

Let a thermodynamical system be given. If we are interested in only a single instant, the probability P of a given state is given by Boltzmann's principle,

$$k_B \ln P = S + \text{const}, \tag{3.10}$$

where S is the entropy of that state. If we are interested in two instants widely separated in time, the probability of given states at each instant is equal to the product of the individual probabilities. A long time lapse makes the states statistically independent. Hence the joint probability of the succession is related to the sum of the two entropies. But if the time lapse is not long, the states will be statistically correlated. It is precisely the laws for irreversible behaviour which tell us the correlations.

Let the thermodynamical state of our system be defined by a set of extensive variables $y^1, \ldots, y^N$. The entropy $S = S(y^1, \ldots, y^N)$ will be a function of all the y^k. Its maximum (equilibrium) value will be denoted by S_0, and the y^k will be redefined to vanish for the equilibrium state: $S_0 = S(0, \ldots, 0)$. The tendency of the system to seek equilibrium is measured by the *thermodynamic forces* Y_k defined as

$$Y_k = \frac{\partial S}{\partial y^k}, \quad k = 1, \ldots, N. \tag{3.11}$$

The Y_k are *restoring forces* that vanish with the y^k.

Fluxes are measured by the time derivatives of the y^k. The essential physical assumption made here is that *irreversible processes are linear, i.e., they depend linearly on the forces that cause them.* Therefore we have[2]

$$\dot{y}^i = \frac{\mathrm{d}y^i}{\mathrm{d}\tau} = \sum_{j=1}^{N} L^{ij} Y_j, \qquad i = 1, \ldots, N. \tag{3.12}$$

Onsager's reciprocity theorem states that L is a symmetric matrix [85],

$$L^{ij} = L^{ji}. \tag{3.13}$$

Further assuming that L is nonsingular one can solve for the forces in terms of the fluxes:

$$Y_i = \sum_{j=1}^{N} R_{ij} \dot{y}^j, \qquad i = 1, \ldots, N. \tag{3.14}$$

---

[2]We use τ to denote time in the theory of irreversible thermodynamics, and t to denote time in the quantum theory. As will be seen in (3.44), τ and t are related by a Wick rotation.

Thus the rate of production of entropy,

$$\dot{S} = \sum_{j=1}^{N} \frac{\partial S}{\partial y^j} \dot{y}^j = \sum_{j=1}^{N} Y_j \dot{y}^j, \tag{3.15}$$

can be expressed in either of two equivalent ways:

$$\dot{S} = \sum_{i,j=1}^{N} R_{ij} \dot{y}^i \dot{y}^j = \sum_{i,j=1}^{N} L^{ij} Y_i Y_j. \tag{3.16}$$

One defines the *dissipation function* Φ as the following quadratic form in the fluxes:[3]

$$\Phi := \frac{1}{2} \sum_{i,j=1}^{N} R_{ij} \dot{y}^i \dot{y}^j. \tag{3.17}$$

This function is a potential for the Y_k, because $\partial \Phi / \partial \dot{y}^j = R_{jk} Y_k$. The corresponding quadratic form of the forces,

$$\Psi := \frac{1}{2} \sum_{i,j=1}^{N} L^{ij} Y_i Y_j, \tag{3.18}$$

has a similar property, but it should be noticed that it is a function of the *state* (since the Y_k depend only on the y^j), whereas the numerically equal Φ is a function of its *rate of change*.

If we expand the entropy in a Taylor series around equilibrium we have

$$S = S_0 - \frac{1}{2} \sum_{i,j=1}^{N} s_{ij} y^i y^j + \dots \tag{3.19}$$

The matrix s_{ij} is symmetric and positive definite. Neglect of the higher terms in y^k means the assumption that fluctuations are Gaussian: for Boltzmann's principle (3.10) states that the logarithm of the probability of a given fluctuation is proportional to its entropy, or

$$P(y^1, \dots, y^N) = Z^{-1} \exp \left(\frac{S}{k_B} \right) = Z^{-1} \exp \left(-\frac{1}{2k_B} \sum_{i,j=1}^{N} s_{ij} y^i y^j \right). \tag{3.20}$$

The assumption of Gaussianity (3.19) then implies that the Y_i are linear in the y^j:

$$Y_i = -\sum_{j=1}^{N} s_{ij} y^j. \tag{3.21}$$

---

[3]We assume R_{ij} to be positive definite. This ensures that $\dot{S} > 0$ as expected of a dissipative process. Indeed, the dissipation function Φ can be identified with a kinetic energy, $T = \sum_{i,j=1}^{N} g_{ij} \dot{x}^i \dot{x}^j / 2$, where g_{ij} is a certain Riemannian metric on the space spanned by the velocities $\dot{x}^j$. Identifying $\dot{x}^j$ with $\dot{y}^j$ we have $g_{ij} = R_{ij}$.

Thus the phenomenological laws (3.14) become

$$\sum_{j=1}^{N} \left(R_{ij}\dot{y}^j + s_{ij}y^j \right) = 0. \tag{3.22}$$

3.3.2 Fluctuations

Let us now modify the deterministic equations (3.14) to include fluctuations by the addition of a random force ξ_i,

$$\sum_{j=1}^{N} R_{ij}\dot{y}^j = Y_i + \xi_i, \tag{3.23}$$

which turns (3.14) into the set of stochastic equations (3.23). We require that the ξ_i have zero means, which implies that the right–hand side of (3.23) is a random force with means Y_i. For simplicity, as in the quantum–mechanical case, let us set $N = 1$, so we have a single variable y obeying the stochastic equation

$$R\dot{y} + sy = \xi. \tag{3.24}$$

We will be concerned with the path of y in time under the influence of these random forces. Our aim is to calculate the probability of any path. For n instants of time $\tau_1 < \tau_2 < \ldots < \tau_n$ we denote the *cumulative distribution function* by F_n:

$$F_n \begin{pmatrix} y_1 \cdots y_n \\ \tau_1 \cdots \tau_n \end{pmatrix} = P\left(y(\tau_k) \le y_k, \ k = 1, \ldots, n \right). \tag{3.25}$$

The function F_n tells the probability that the thermodynamical path $y(\tau)$ lie below the barriers $y_1, \ldots, y_n$ at the corresponding instants $\tau_1, \ldots, \tau_n$. A *stationary* process is defined as one whose cumulative distribution function F_n is invariant under arbitrary time shifts $\delta\tau$:

$$F_n \begin{pmatrix} y_1 \cdots y_n \\ \tau_1 \cdots \tau_n \end{pmatrix} = F_n \begin{pmatrix} y_1 & \cdots & y_n \\ \tau_1 + \delta\tau & \cdots & \tau_n + \delta\tau \end{pmatrix}, \qquad \forall\, \delta\tau \in \mathbb{R}. \tag{3.26}$$

Physically this describes an *aged* system, one that has been left alone long enough that any initial conditions have worn off, or been forgotten. Thus we consider entropy creation as a loss of information: a dissipative system forgets its past.

Alongside F_n, the *probability density function* f_n is defined such that the product

$$f_n \begin{pmatrix} y_1 \cdots y_n \\ \tau_1 \cdots \tau_n \end{pmatrix} \mathrm{d}y_1 \cdots \mathrm{d}y_n \tag{3.27}$$

gives the probability that a thermodynamical path pass through gates of width $\mathrm{d}y_k$.

We will also be interested in conditional probabilities. The *conditional probability function* for the $(n + 1)$th event given the previous n,

$$F_1 \begin{pmatrix} y_{n+1} \,\big|\, y_1 \cdots y_n \\ \tau_{n+1} \,\big|\, \tau_1 \cdots \tau_n \end{pmatrix} = P\left(y(\tau_{n+1}) = y_{n+1} \,\Big|\, y(\tau_k) = y_k, \ k = 1, \ldots, n \right), \tag{3.28}$$

is defined implicitly as follows:

$$F_{n+1} \begin{pmatrix} y_1 \cdots y_{n+1} \\ \tau_1 \cdots \tau_{n+1} \end{pmatrix} \tag{3.29}$$

$$= \int_{-\infty}^{y_1} \mathrm{d}\tilde{y}_1 \cdots \int_{-\infty}^{y_n} \mathrm{d}\tilde{y}_n \, F_1 \begin{pmatrix} y_{n+1} \Big| \tilde{y}_1 \cdots \tilde{y}_n \\ \tau_{n+1} \Big| \tau_1 \cdots \tau_n \end{pmatrix} \mathrm{d}F_n \begin{pmatrix} \tilde{y}_1 \cdots \tilde{y}_n \\ \tau_1 \cdots \tau_n \end{pmatrix}.$$

Correspondingly, the *conditional probability density function* f_1 is defined such that

$$f_1 \begin{pmatrix} y_k \Big| y_{k-1} \\ \tau_k \Big| \tau_{k-1} \end{pmatrix} \mathrm{d}y_k \, \mathrm{d}y_{k-1} \tag{3.30}$$

equals the probability that a thermodynamical path pass through a gate of width $\mathrm{d}y_k$ at time τ_k, *given* that it passed through a gate of width $\mathrm{d}y_{k-1}$ at time τ_{k-1}.

3.3.3 Markov processes

A Markov process is defined as one whose conditional probabilities are independent of all but the immediately preceding instant [31]:

$$F_1 \begin{pmatrix} y_{n+1} \Big| y_1 \cdots y_n \\ \tau_{n+1} \Big| \tau_1 \cdots \tau_n \end{pmatrix} = F_1 \begin{pmatrix} y_{n+1} \Big| y_n \\ \tau_{n+1} \Big| \tau_n \end{pmatrix}. \tag{3.31}$$

Intuitively: a Markov system has a short memory. For a Markov process (3.29) and (3.31) imply

$$f_n \begin{pmatrix} y_1 \cdots y_n \\ \tau_1 \cdots \tau_n \end{pmatrix} = f_1 \begin{pmatrix} y_n \Big| y_{n-1} \\ \tau_n \Big| \tau_{n-1} \end{pmatrix} \cdots f_1 \begin{pmatrix} y_2 \Big| y_1 \\ \tau_2 \Big| \tau_1 \end{pmatrix} f_1 \begin{pmatrix} y_1 \\ \tau_1 \end{pmatrix}. \tag{3.32}$$

Now $f_1 \begin{pmatrix} y_1 \\ \tau_1 \end{pmatrix}$ is known from Boltzmann's principle (3.10). Hence, by stationarity, all that is needed in order to obtain the distribution function for an arbitrary number of gates is to evaluate the conditional probability density function

$$f_1 \begin{pmatrix} y_2 \Big| y_1 \\ \tau + \delta\tau \Big| \tau \end{pmatrix}, \tag{3.33}$$

which depends only on $\delta\tau$, being independent of τ. Thus the n–gate problem reduces to the 2–gate problem.

3.3.4 Gaussian processes

A Gaussian stochastic process is one whose probability density function is a Gaussian distribution. Let us set, in (3.24),

$$\gamma := \frac{s}{R}. \tag{3.34}$$

Then the conditional probability function for a Gaussian process is given by [86]

$$f_1 \begin{pmatrix} y_2 \Big| y_1 \\ \tau + \delta\tau \Big| \tau \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \frac{s/k_B}{\sqrt{1 - \mathrm{e}^{-2\gamma\delta\tau}}} \exp\left[-\frac{s}{2k_B} \frac{\left(y_2 - \mathrm{e}^{-\gamma\delta\tau} y_1\right)^2}{1 - \mathrm{e}^{-2\gamma\delta\tau}} \right]. \tag{3.35}$$

Now eqn. (3.35), together with (3.32), constitutes the solution to the problem of finding the probability of any path in a Gaussian Markov process. We also remark that (3.35) correctly reduces to the one–gate distribution function (3.20) for $\delta\tau \to \infty$.

Next let us divide the interval $(\tau, \tau + \delta\tau)$ into n equal subintervals of length $\delta\tau/n$:

$$\tau_1 = \tau, \quad \tau_2 = \tau_1 + \frac{\delta\tau}{n}, \quad \ldots, \quad \tau_{n+1} = \tau + \delta\tau. \tag{3.36}$$

Then we have

$$f_1\left(\begin{matrix} y_{n+1} \\ \tau_{n+1} \end{matrix}\middle|\begin{matrix} y_1 \\ \tau_1 \end{matrix}\right) = \int \mathrm{d}y_n \cdots \int \mathrm{d}y_2\, f_1\left(\begin{matrix} y_{n+1} \\ \tau_{n+1} \end{matrix}\middle|\begin{matrix} y_n \\ \tau_n \end{matrix}\right) \cdots f_1\left(\begin{matrix} y_2 \\ \tau_2 \end{matrix}\middle|\begin{matrix} y_1 \\ \tau_1 \end{matrix}\right). \tag{3.37}$$

This is again the Chapman–Kolmogorov equation. The integral above extends over all the $n-1$ intermediate gates. Using (3.37) one can reexpress (3.35) in the following alternative form [86]:

$$f_1\left(\begin{matrix} y_{n+1} \\ \tau_{n+1} \end{matrix}\middle|\begin{matrix} y_1 \\ \tau_1 \end{matrix}\right) = Z^{-1} \exp\left\{-\frac{1}{4k_B}\int_{\tau_1}^{\tau_{n+1}} \mathrm{d}\tau\, R\left[\dot{y}(\tau) + \gamma y(\tau)\right]^2\right\}_{\mathrm{min}}, \tag{3.38}$$

subject to $y(\tau_1) = y_1$, $y(\tau_{n+1}) = y_{n+1}$. The subscript *min* refers to the fact that argument of the exponential is to be evaluated along the trajectory that minimises the integral.

The one–gate distribution is obtained from the conditional distribution $f_1\left(\begin{smallmatrix} y_2 \\ \tau_2 \end{smallmatrix}\middle|\begin{smallmatrix} y_1 \\ \tau_1 \end{smallmatrix}\right)$ by taking $\tau_1 = -\infty$ and $y_1 = 0$ (because the aged system certainly was at equilibrium long ago). Thus we set $n = 1$ in (3.38) and define the *thermodynamical Lagrangian function* $\mathcal{L}$ as

$$\mathcal{L}\left[\dot{y}(\tau), y(\tau)\right] := \frac{R}{2}\left[\dot{y}(\tau) + \gamma y(\tau)\right]^2. \tag{3.39}$$

The dimension of $\mathcal{L}$ is entropy per unit time, instead of energy. However, our map between mechanics and thermodynamics will justify the denomination "Lagrangian". The Euler–Lagrange equation for a minimum value of the integral in (3.38) is

$$\ddot{y} - \gamma^2 y = 0. \tag{3.40}$$

The solution to the above that satisfies the boundary conditions $y(\tau = -\infty) = 0$ and $y(\tau = \tau_2) = y_2$ is

$$y(\tau) = y_2 e^{\gamma(\tau - \tau_2)}. \tag{3.41}$$

Evaluating the integral in (3.38) along this extremal trajectory leads to

$$f_1\left(\begin{matrix} y_2 \\ \tau_2 \end{matrix}\middle|\begin{matrix} 0 \\ -\infty \end{matrix}\right) = f_1\left(\begin{matrix} y_2 \\ \tau_2 \end{matrix}\right) = Z^{-1} \exp\left[-\frac{s}{2k_B}(y_2)^2\right]. \tag{3.42}$$

This result is in agreement with what one expects from Boltzmann's principle (3.10) in the Gaussian approximation (3.19).

Finally substituting (3.42) into (3.37), we obtain the thermodynamical analogue of the quantum–mechanical relation (3.5):

$$f_1\left(\begin{matrix} y_2 \\ \tau_2 \end{matrix}\right) = \int \mathrm{d}y_1\, f_1\left(\begin{matrix} y_2 \\ \tau_2 \end{matrix}\middle|\begin{matrix} y_1 \\ \tau_1 \end{matrix}\right) f_1\left(\begin{matrix} y_1 \\ \tau_1 \end{matrix}\right). \tag{3.43}$$

This concludes our summary of ref. [86].

3.4 Quantum mechanics from irreversible thermodynamics

The Wick rotation

$$\tau = \mathrm{i}t \tag{3.44}$$

between the thermodynamical evolution parameter τ and the quantum–mechanical time variable t is the first entry in our dictionary between classical irreversible thermodynamics and quantum mechanics.

3.4.1 Path integrals in irreversible thermodynamics

The concept of a path integral can be traced back to the Chapman–Kolmogorov equation. Indeed letting $n \to \infty$ in (3.36) and using (3.37), the right–hand side of (3.38) becomes a path integral *over the thermodynamical configuration space Y*:

$$f_1 \begin{pmatrix} y_2 \,\big|\, y_1 \\ \tau_2 \,\big|\, \tau_1 \end{pmatrix} = \int_{y(\tau_1)=y_1}^{y(\tau_2)=y_2} \mathrm{D}y(\tau) \, \exp \left\{ -\frac{1}{4k_B} \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \, R \left[\dot{y}(\tau) + \gamma y(\tau) \right]^2 \right\}. \tag{3.45}$$

Thus it turns out that (3.38) actually equals the semiclassical approximation (as per (3.9)) to the path integral (3.45). This latter expression for the distribution function f_1 in terms of a path integral is implicit in ref. [86]—but actually never written down explicitly in that paper; see however [46].

Dropping in (3.39) the term proportional to $\dot{y}y$ (a total derivative), we redefine the thermodynamical Lagrangian function $\mathcal{L}$ to be

$$\mathcal{L} \left[\dot{y}(\tau), y(\tau) \right] = \frac{R}{2} \left[\dot{y}^2(\tau) + \gamma^2 y^2(\tau) \right]. \tag{3.46}$$

We observe that $\dot{y}^2(\tau)$ and $y^2(\tau)$ in $\mathcal{L}$ carry the same relative sign. Similarly dropping in (3.45) the term proportional to $\dot{y}y$, we can rewrite the path integral using (3.46) as

$$f_1 \begin{pmatrix} y_2 \,\big|\, y_1 \\ \tau_2 \,\big|\, \tau_1 \end{pmatrix} = \int_{y(\tau_1)=y_1}^{y(\tau_2)=y_2} \mathrm{D}y(\tau) \, \exp \left\{ -\frac{1}{2k_B} \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \, \mathcal{L} \left[\dot{y}(\tau), y(\tau) \right] \right\}. \tag{3.47}$$

The path integral (3.47) is the thermodynamical analogue of the path integral (3.6) that defines the quantum–mechanical propagator. Thus setting $n = 1$ in (3.38), dropping the total derivative $\dot{y}y$, and replacing the integrand with the thermodynamical Lagrangian (3.46) leads to the Gaussian approximation to (3.47):

$$f_1 \begin{pmatrix} y_2 \,\big|\, y_1 \\ \tau_2 \,\big|\, \tau_1 \end{pmatrix} = Z^{-1} \, \exp \left\{ -\frac{1}{2k_B} \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \, \mathcal{L} \left[\dot{y}_{\mathrm{cl}}(\tau), y_{\mathrm{cl}}(\tau) \right] \right\}. \tag{3.48}$$

Here $\mathcal{L} \left[\dot{y}_{\mathrm{cl}}(\tau), y_{\mathrm{cl}}(\tau) \right]$ stands for the evaluation of (3.46) along the classical trajectory $y_{\mathrm{cl}}(\tau)$ that satisfies the equations of motion (3.40). In this way (3.48) is seen to correspond to the semiclassical approximation for the quantum–mechanical propagator, given in (3.9). On the thermodynamical side, the quantum–mechanical semiclassical approximation translates as the assumption of Gaussianity for the stochastic forces ξ and for the entropy S, as well as the assumption of linearity between forces and fluxes (which leads up to the quadratic forms (3.17) and (3.18)).

3.4.2 Propagators from thermodynamical distributions

The next entry in our dictionary relates quantum–mechanical wavefunctions and propagators to thermodynamical distribution functions. Within the Gaussian approximation we use throughout, this entry will refer to the free particle and the harmonic oscillator. We first we need to identify certain mechanical variables with their thermodynamical partners. Specifically, we will make the following replacements:[4]

$$\omega \leftrightarrow \gamma, \qquad \frac{m\omega}{\hbar} \leftrightarrow \frac{s}{2k_B}, \qquad x \leftrightarrow y. \tag{3.49}$$

To begin with, one expects the squared modulus of the wavefunction $|\psi|^2$ to be related to the 1–gate distribution function $f_1 \left({y \atop \tau} \right)$, while the propagator K must correspond to a 2–gate distribution function $f_1 \left({y_2 \atop \tau_2} \Big| {y_1 \atop \tau_1} \right)$. Indeed the 1–gate distribution function (3.42) gives the squared modulus of the ground state $\psi_0(x) = \exp\left(-m\omega x^2/2\hbar \right)$ of the harmonic oscillator once the replacements (3.44), (3.49) are applied:

$$f_1 \left({x \atop \mathrm{i}t} \right) = Z^{-1} \exp\left(-\frac{m\omega}{\hbar} x^2 \right) = |\psi_0^{(\mathrm{harmonic})}(x)|^2. \tag{3.50}$$

With the appropriate choices for the constants m and ω, (3.50) can also represent a free wavepacket. Next we turn to propagators K. Elementary algebra brings the conditional probability function for a Gaussian process (3.35) into the form

$$f_1 \left({y_2 \atop \tau} \Big| {y_1 \atop 0} \right) = \frac{s}{2k_B} \frac{\mathrm{e}^{\gamma\tau/2}}{\sqrt{\pi \sinh(\gamma\tau)}} \exp\left[-\frac{s}{2k_B} \frac{\left(\mathrm{e}^{\gamma\tau/2} y_2 - \mathrm{e}^{-\gamma\tau/2} y_1 \right)^2}{2 \sinh(\gamma\tau)} \right]. \tag{3.51}$$

We will also be interested in the limit $\gamma \to 0$ of the above:

$$f_1 \left({y_2 \atop \tau} \Big| {y_1 \atop 0} \right)_{\gamma\to 0} \simeq \frac{s}{2k_B} \frac{1}{\sqrt{\pi \gamma\tau}} \exp\left[-\frac{s}{2k_B} \frac{(y_2 - y_1)^2}{2\gamma\tau} \right]. \tag{3.52}$$

Using (3.44) and (3.49), the free quantum–mechanical propagator (3.7) follows from (3.52):

$$K^{(\mathrm{free})}(x_2, t|x_1, 0) = \sqrt{\frac{k_B}{s}} f_1 \left({x_2 \atop \mathrm{i}t} \Big| {x_1 \atop 0} \right)_{\gamma\to 0}. \tag{3.53}$$

The case when γ is nonvanishing requires some more work. Again (3.44) and (3.49) allow one to relate the conditional probability (3.51) to the harmonic propagator (3.8) as follows:

$$f_1 \left({x_2 \atop \mathrm{i}t} \Big| {x_1 \atop 0} \right) = \exp\left(\frac{\mathrm{i}\omega t}{2} - \frac{\Delta V}{\hbar\omega} \right) \sqrt{\frac{2m\omega}{\hbar}} K^{(\mathrm{harmonic})}(x_2, t|x_1, 0), \tag{3.54}$$

where $V(x) = kx^2/2$ is the harmonic potential and $\Delta V = V(x_2) - V(x_1)$. As had to be the case, (3.54) correctly reduces to (3.53) when $\omega \to 0$. The square roots present in (3.53) and (3.54) ensure that these two equations are dimensionally correct.

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[4]A dimensionful conversion factor must be understood as implicitly contained in the replacement $x \leftrightarrow y$, whenever needed.

3.4.3   Integrability *vs.* square–integrability

Under our correspondence, the squared modulus of the wavefunction $|\psi|^2$ gets mapped into the *unconditional* probability density $f_1\left(\begin{smallmatrix} y_1 \\ \tau_1 \end{smallmatrix}\right)$, while the propagator K gets mapped into the *conditional* probability density $f_1\left(\begin{smallmatrix} y_2 \\ \tau_2 \end{smallmatrix} \middle| \begin{smallmatrix} y_1 \\ \tau_1 \end{smallmatrix}\right)$. One should bear in mind, however, that the quantum–mechanical objects ψ, K are probability *amplitudes*, while the thermodynamical objects f_1 are true probabilities. Therefore quantum mechanics is not just the Wick rotation of classical, irreversible thermodynamics—it is also the *square root* thereof, so to speak, because of the Born rule. In order to address this question in mode detail we need to recall some background mathematics; see ref. [107] for a physics–oriented approach, and also [8] for a recent discussion of some of the issues analysed later in this section.

Let M be a measure space, and denote by $L^p(M)$ the Banach space[5]

$$L^p(M) = \{f : M \to \mathbb{C}, \; ||f||_p < \infty\}, \quad ||f||_p := \left(\int_M |f|^p\right)^{1/p}, \quad 0 < p < \infty. \tag{3.55}$$

It turns out that $L^p(M)$ is a Hilbert space only when $p = 2$. Moreover, $L^p(M)$ and $L^q(M)$ are linear duals of each other whenever $1/p + 1/q = 1$. Two particular cases of this duality will interest us. The first one is $p = 2, q = 2$, the other one is $p = 1, q = \infty$.

When $p = 2$ we have that $L^2(M)$ is selfdual, the duality being given by the scalar product: $\langle\cdot|\cdot\rangle : L^2(M) \times L^2(M) \longrightarrow \mathbb{C}$. The corresponding algebra of bounded operators is $\mathcal{L}(L^2(M))$, a noncommutative C^*–algebra with respect to operator multiplication. Complex conjugation in $\mathcal{L}(L^2(M))$ consists in taking the adjoint operator, while the noncommutativity is that of matrix multiplication.

The operator algebra $\mathcal{L}(L^p(M))$ is also a Banach algebra for any $p > 0$, and not just for $p = 2$. However, only when $p = 2$ is a $\mathcal{L}(L^p(M))$ a C^*–algebra, because only when $p = 2$ does $\mathcal{L}(L^p(M))$ possess a complex conjugation.

Set now $p = 1$. The dual of $L^1(M)$ is $L^\infty(M)$. Elements of the latter are measurable, essentially bounded functions f with a finite norm $||f||_\infty$:

$$L^\infty(M) = \{f : M \to \mathbb{C}, \; ||f||_\infty < \infty\}, \qquad ||f||_\infty := \sup_{z \in M}\{|f(z)|\}. \tag{3.56}$$

The duality between $L^1(M)$ and $L^\infty(M)$ is

$$(\cdot|\cdot) : L^\infty(M) \times L^1(M) \longrightarrow \mathbb{C}, \qquad (f|\rho) := \int_M f\rho, \tag{3.57}$$

for any $f \in L^\infty(M)$ and any $\rho \in L^1(M)$. Now $L^\infty(M)$ also qualifies as a C^*–algebra, the multiplication law being pointwise multiplication of functions (hence commutative), and the complex conjugation being that of the functions f. An important difference with respect to the previous case is that $\mathcal{L}(L^2(M))$ is noncommutative, whereas $L^\infty(M)$ is commutative.

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[5]The space $L^p(M)$ is complex or real according to whether its elements f are taken to be complex–valued or real–valued functions on M. For quantum–mechanical applications we will consider the complex case, while thermodynamical applications require the real case. For generality, this summary assumes all spaces complex.

We will henceforth write X for the space M when dealing with the mechanical configuration space, and Y when referring to the thermodynamical configuration space.

Textbook quantum mechanics regards quantum states as unit rays within $L^2(X)$, while physical observables $\mathcal{O}$ are represented by selfadjoint operators $\mathcal{O} \in \mathcal{L}(L^2(X))$.[6] On the other hand, the natural framework for the theory of irreversible thermodynamics is the *real* Banach space $L^1(Y)$ and its dual, the *real* Banach algebra $L^\infty(Y)$. Thermodynamical states are probability distributions $\rho \in L^1(Y)$, that is, *real* functions, normalised as per $\int_Y \rho = 1$. Thermodynamical observables are *real* functions $f \in L^\infty(Y)$. Thus $\int_Y f\rho$ in (3.57) equals the average value of the physical quantity f in the state described by ρ.

Clearly the thermodynamical setup is not quite as sophisticated as its mechanical counterpart. As opposed to the *complex* Hilbert space $L^2(X)$, the *real* Banach space $L^1(Y)$ does not know about the existence of the imaginary unit i. In the absence of a complex conjugation to implement time reversal, the thermodynamical setup necessarily describes *irreversible* processes. Moreover, there exists no scalar product on $L^1(Y)$. Correspondingly there is no notion of a selfadjoint operator in $\mathcal{L}(L^1(Y))$—in fact, thermodynamical observables are elements of a very different space, $L^\infty(Y)$.[7]

The previous differences notwithstanding, we can establish a map between quantum–mechanical states/observables and their thermodynamical counterparts, as we do next. We treat observables first, and discuss states later.

It is reasonable to identify real thermodynamical averages $(f|\rho)$ with quantum mechanical expectation values $\langle\psi|\mathcal{O}|\psi\rangle$ of selfadjoint operators $\mathcal{O}$, something like

$$\int_Y f\rho = (f|\rho) \leftrightarrow \langle\psi|\mathcal{O}|\psi\rangle = \int_X \psi^*\mathcal{O}\psi, \tag{3.58}$$

where the correspondence denoted by $\leftrightarrow$ has yet to be given a precise meaning. For this we can assume diagonalising $\mathcal{O}$ by a (complete, orthonormal) set of eigenstates $\psi_i \in L^2(X)$, so we can replace the right–hand side of (3.58) with the corresponding eigenvalue λ_i. We want to define a functional f for the left–hand side of (3.58). A sensible definition actually involves a collection of constant functionals f_i, each one of them equal to the corresponding eigenvalue λ_i:

$$f_i : Y \longrightarrow \mathbb{R}, \qquad f_i(y) = \lambda_i, \qquad \forall y \in Y. \tag{3.59}$$

Since the eigenvalues λ_i are constants and the density ρ can be normalised to unity, the imprecise correspondence (3.58) can be replaced with the precise dictionary entry

$$\int_Y f_i\rho = (f_i|\rho) = \lambda_i = \langle\psi_i|\mathcal{O}|\psi_i\rangle = \int_X \psi^*\mathcal{O}\psi. \tag{3.60}$$

This generalises in the obvious way to the case of a set of commuting observables $\mathcal{O}_k$. Noncommuting observables, not being simultaneously diagonalisable, lead to the impossibility of simultaneously defining the corresponding thermodynamical functionals

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[6] We ignore the mathematical subtleties due to the fact that $\mathcal{O}$ is generally an unbounded operator, hence generally not an element of $\mathcal{L}(L^2(X))$, because this fact is immaterial to the discussion.

[7] In particular, the *real* space $L^\infty(Y)$ is a Banach algebra but not a C^*–algebra.

f on the left–hand side of (3.60). We will examine the thermodynamical analogue of
quantum commutators in a forthcoming publication.

So much for the observables; now we turn to the states. Since thermodynamical
probabilities are elements of $L^1(Y)$ while quantum–mechanical amplitudes belong to
$L^2(X)$, we would like to define some map of $L^2(X)$ into $L^1(Y)$, or viceversa. Given
$\psi \in L^2(X)$, one's first instinct is to set $\rho := |\psi|^2$ because then $\rho \in L^1(X)$; this is
of course the Born rule. The attentive reader will have noticed that we actually need
$\rho \in L^1(Y)$: it is generally meaningless to equate ρ to $|\psi|^2$—or to any other function
of ψ, for that matter. We will proceed ahead under the simplifying assumption that
$X = Y$.

The usual Born map b is defined as

$$b : L^2(X) \longrightarrow L^1(X), \quad b(\psi) := |\psi|^2. \tag{3.61}$$

This map is obviously not 1–to–1, so it fails to be an injection. As such it possesses no
inverse. We will however use the formal notation b^{-1} to denote the map

$$b^{-1} : L^1(X) \longrightarrow L^2(X), \quad b^{-1}(\rho) := \sqrt{\rho}\, e^{\frac{i}{\hbar}\varphi}, \tag{3.62}$$

where φ is taken as the solution to the continuity equation

$$\dot{\rho} + \nabla \cdot (\rho \nabla \varphi) = 0 \tag{3.63}$$

that is well known from the Madelung transformation. Moreover, if $b^{-1}(\rho)$ satisfies
the Schroedinger equation, then φ must of course equal the action integral $I = \int dt\, L$,
and thus satisfy the *quantum* Hamilton–Jacobi equation [38]. Although the map b^{-1}
also fails to be an injection, we use the notation b^{-1} because $bb^{-1}(\rho) = \rho$. Aside
from this difficulty about the lack of injectivity, b and b^{-1} provide us with the required
maps from quantum–mechanical states into thermodynamical distribution functions,
and viceversa.

The Chapman–Kolmogorov equation (3.37), written below for $n = 2$,

$$f_1 \begin{pmatrix} y_3 \\ \tau_3 \end{pmatrix} \begin{vmatrix} y_1 \\ \tau_1 \end{vmatrix} = \int dy_2\, f_1 \begin{pmatrix} y_3 \\ \tau_3 \end{pmatrix} \begin{vmatrix} y_2 \\ \tau_2 \end{vmatrix} f_1 \begin{pmatrix} y_2 \\ \tau_2 \end{pmatrix} \begin{vmatrix} y_1 \\ \tau_1 \end{vmatrix}, \tag{3.64}$$

is the thermodynamical analogue of the quantum–mechanical equation (3.4). This
leads us to the following point. Our correspondence maps $f_1 \left(\begin{smallmatrix} y_2 \\ \tau_2 \end{smallmatrix} \middle| \begin{smallmatrix} y_1 \\ \tau_1 \end{smallmatrix} \right)$, which is a
conditional probability, into $K(x_2, t_2 | x_1, t_1)$, which is an *amplitude* for a conditional
probability. In other words, under our correspondence, the Born rule does *not* apply to
the map between conditional probabilities, although it does apply to the map between
unconditional probabilities. There is nothing wrong with this. Indeed, f_1 and K satisfy
the respective Chapman–Kolmogorov equations (3.64) and (3.4). Regarding the latter
as matrix equations (which is what they are), they read formally $f_1 \times f_1 = f_1$ and
$K \times K = K$. That is, squaring f_1 and K as matrices (which is how they should be
squared, since f_1 and K are operators), they are idempotent. It therefore makes sense
not to impose the Born rule on the map between K and f_1.

3.4.4 Entropy *vs.* action

To complete our dictionary between quantum mechanics and irreversible thermodynamics we postulate the following correspondence between the action integral I and the entropy S:

$$\text{(mechanics)} \quad \frac{i}{\hbar} I \leftrightarrow \frac{1}{k_B} S \quad \text{(thermodynamics)}, \tag{3.65}$$

up to a numerical, dimensionless factor. Now the Wick rotation (3.44) replaces iI with the Euclidean action I_E, so we could just as well write

$$\text{(mechanics)} \quad \frac{1}{\hbar} I_E \leftrightarrow \frac{1}{k_B} S \quad \text{(thermodynamics)}, \tag{3.66}$$

again up to a numerical, dimensionless factor. We observe that both I and S independently satisfy an extremum principle. We also note that the respective fluctuation theories[8] in the Gaussian approximation are obtained upon taking the exponential. Thus exponentiating (3.65) we arrive at the wavefunction

$$\psi = \sqrt{\rho} \, \exp\left(\frac{i}{\hbar} I\right) \tag{3.67}$$

and at the Boltzmann distribution function (3.10):

$$\rho_B = Z^{-1} \exp\left(\frac{1}{k_B} S\right). \tag{3.68}$$

We should point out that the correspondence (3.65), (3.66) has also been found to hold in independent contexts, long ago by de Broglie [16] and more recently *e.g.* in [1, 9].

Applying the Born rule we set the Boltzmann probability density ρ_B equal to the quantum–mechanical probability density $|\psi|^2$:

$$\rho_B = |\psi|^2 = \rho. \tag{3.69}$$

(See ref. [7] for distributions other than the *squared* modulus). Hence

$$\rho = Z^{-1} \exp\left(\frac{1}{k_B} S\right). \tag{3.70}$$

Substitution of (3.70) into (3.67) yields an elegant expression for the wavefunction

$$\psi = Z^{-1/2} \exp\left(\frac{1}{2k_B} S\right) \exp\left(\frac{i}{\hbar} I\right), \tag{3.71}$$

combining thermodynamics and quantum mechanics into a single formula.

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[8]These fluctutations are of course measured with respect to the corresponding mean values of I and S as given by their extremals.

Implicitly assumed in (3.71) is the identification of mechanical variables x and thermodynamical variables y, as already done in (3.49). One can now define the *complex–valued action* $\mathcal{I}(x)$[9]

$$\mathcal{I}(x) := \frac{1}{2k_B} S + \frac{\mathrm{i}}{\hbar} I. \tag{3.72}$$

in order to write

$$\psi(x) = Z^{-1/2} \exp\left(\mathcal{I}(x)\right) \tag{3.73}$$

as the semiclassical wavefunction (3.71), where

$$Z = \int \mathrm{d}x \, |\exp\left(\mathcal{I}(x)\right)|^2. \tag{3.74}$$

We realise that the correspondence (3.65), (3.66) leads naturally to the existence of a complexified action such as (3.72), which expresses *a fundamental symmetry between entropy and mechanical action*.

Finally we would like to point out that complexified action functionals have also been considered recently in ref. [79].

3.5   Conclusions to chapter 3

We can summarise this chapter in the following statements:

i) we have succeeded in formulating a correspondence between standard quantum mechanics, on the one hand, and the classical thermodynamics of irreversible processes, on the other;

ii) this correspondence holds at least in the Gaussian approximation (the latter being defined in quantum mechanics as the semiclassical limit, and in thermodynamics as the regime of linearity between forces and fluxes);

iii) this possibility of encoding of quantum–mechanical information in thermodynamical terms provides an independent proof of the statement that quantum mechanics is an an emergent phenomenon.

Specifically, our correspondence between semiclassical quantum mechanics and Gaussian irreversible thermodynamics includes the following points of section 3.4:

i) we have shown that the path–integral representation for quantum–mechanical propagators is already present in the thermodynamical description of classical dissipative phenomena (section 3.4.1);

ii) we have mapped thermodynamical distribution functions into quantum–mechanical propagators (section 3.4.2);

iii) we have constructed an explicit correspondence between quantum–mechanical states and thermodynamical states, and also an analogous correspondence between quantum–mechanical observables and thermodynamical observables (section 3.4.3);

iv) we have grounded our correspondence in the existence of a fundamental symmetry

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[9]While the entropy S is a true function of x, the action integral I is actually a *functional* of $x(t)$. However, in (3.72) we need I within the exponential defining ψ. To this end, I is to be evaluated along *the* classical trajectory starting at a certain given point and ending at a variable endpoint x. This amounts to regarding I as a true function of x and no longer as a functional.

between mechanical action and entropy (section 3.4.4).

In order to make this chapter selfcontained we have also included, in section 3.3, a crash course in classical irreversible thermodynamics, the latter considered in the linear approximation. Presumably, the theory of irreversible thermodynamics beyond the linear regime should allow one to extend the present correspondence beyond the semi-classical approximation of quantum mechanics.

Having mapped *quantum* mechanics into *classical* irreversible thermodynamics raises another old question, *viz.*, the issue of how sharply, how univocally defined is the divide between *quantumness* and *classicality*. This issue has also been addressed, from the viewpoint of emergent theories, in ref. [37]; we defer our own contribution to the subject until a forthcoming publication. However we would like to briefly touch upon the emergence property of *spacetime*—not from a gravitational perspective, but from a purely quantum–mechanical viewpoint. If spacetime is an emergent phenomenon, as widely conjectured, then everything that makes use of spacetime concepts must necessarily be emergent, too. Quantum mechanics is no exception, unless one succeeds in constructing a quantum–mechanical formalism that is entirely free of spacetime notions. Progress towards this latter goal has been achieved along lines based on noncommutative geometry (see [42] and references therein). A more modest approach is to try and directly map quantum mechanics into thermodynamics, as done here and elsewhere. It turns out that spacetime arises as an emergent concept *also* in our quantum–mechanical approach, if only because our correspondence has required replacing space variables x with thermodynamical variables y. Thus, indirectly, we have also furnished (admittedly cirmcumstantial) evidence of the emergence property of spacetime.

Chapter 4

Overall summary

Emergent physics as a research topic has drawn a lot of attention recently [21, 62]. The very spacetime we live in, as well as the gravitational force that governs it, both appear to be emergent phenomena [61, 91, 113]. Quantum mechanics has also been conjectured to be the emergent theory of some underlying deterministic model, in part because of its long–standing conflict with general relativity. The guiding principle in all emergent theories is the fact that they provide a coarse–grained description of some underlying theory [21]. Due to our ignorance of a full microscopic description, emergent phenomena are in principle amenable to a thermodynamical description.

Without touching on the difficulties facing quantum gravity, a number of interpretational questions and foundational issues arise and remain within a purely quantum–mechanical setup (or, eventually, within a quantum field theory setup, see [59]). In this thesis, following our papers [1, 2, 3], we have focused on *the emergent aspects of quantum mechanics applying a thermodynamical approach*. In fact the classical thermodynamics of irreversible processes and fluctuation theory turns out to share many common features with quantum mechanics—surprisingly, with Feynman's path integral approach to quantum mechanics. Some basic references on the subject of fluctuations and irreversible thermodynamics are [70, 85, 86, 93, 109]; intriguing questions such as the emergence of macroscopic irreversibility from microscopic reversibility, the arrow of time, and other related puzzles are analysed in [75, 94]. A more complete list of references can be found in [84].

We take the view that, apart from other important reasons [6, 66, 67, 77, 78, 92], quantum theory must be an emergent phenomenon *also* because the spacetime it is defined on is an emergent concept. There exist in the literature a number of different approaches to account for the emergent nature of spacetime, too numerous to quote here in detail. Here we have followed the holographic [55, 106] proposal presented in ref. [113]. Thus gravity and quantum mechanics share the common feature of being effective, thermodynamical descriptions of their respective underlying theories.

The entropy representation of quantum mechanics, as presented here, is a holographic projection of the energy representation of the same theory, as defined on spacetime. Our central claim, summarised by eqn. (2.15), expresses this holographic property.

There is, however, one additional property of quantum mechanics that is deeply encoded in eqn. (2.15); as such it is not immediately recognised. Namely, quantum mechanics is an emergent phenomenon *also* because quantum mechanics is defined on spacetime, and spacetime itself is an emergent phenomenon. Let us analyse this latter point in more detail.

Any model of emergent gravity must ultimately account for the laws governing the motion of material bodies. Thus, *e.g.*, the proposal made in [113] allows for a (somewhat heuristic) derivation of Newton's law of motion, $F = ma$, and of the relativistic generalisations thereof, as emergent, thermodynamical laws. Moreover, the intriguing presence of Planck's constant $\hbar$ [29] in the purely classical setup of ref. [113] makes one suspect that quantum mechanics also has a role to play in that setup. On the other hand, it is well known that Newton's law $F = ma$ can be recovered in the semiclassical limit of quantum mechanics, as being satisfied by the expectation values of certain operators (Ehrenfest's theorem). Last but not least, thermodynamics is the paradigm of emergent phenomena.

All these different pieces of evidence point toward one and the same conclusion— *viz.*, that if classical mechanics follows from the emergence property of spacetime, then the same should be true of quantum mechanics. Here we have exploited this point of view. We would like to stress that this conclusion is ultimately independent of the precise mechanism whereby spacetime emerges. Thus, although the holographic dictionary presented in previous sections hinges crucially on the emergence mechanism being precisely that of ref. [113], the holographic correspondence (2.15) is independent of that mechanism. As such, the holographic correspondence (2.15) should hold just as well in any other specific model for the emergence of spacetime (say, string theory, loop quantum gravity or any alternative thereto such as [39, 72]).

It was Einstein's dream to see quantum mechanics formulated as an ensemble theory in which uncertainties would *not* have a fundamental ontological status. Instead, Einstein would have uncertainties and fluctuations arise as a consequence of *the statistical nature* of the description of an underlying *deterministic* system (see [68, 83] and refs. therein). Thermodynamical fluctuation theory thus appears to be the archetypal example that Einstein would presumably have liked for quantum mechanics to be modelled upon.

Actually it has been known since the early days of quantum mechanics that the (free) Schroedinger equation can be interpreted as the standard heat equation in imaginary time, so the thermodynamical connection has always existed. An unavoidable consequence of imaginary time is that real (decaying) exponentials replace imaginary (oscillatory) exponentials. This is the hallmark of dissipation. Thus quantum mechanics can be thought of as a dissipative phenomenon that becomes conservative only in stationary states [13, 14, 57]—that little i in the Schroedinger equation makes a big difference [66].

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