Research Article

On Exact Series Solution of Strongly Coupled Mixed Parabolic Problems

Vicente Soler,1 Emilio Defez,2 M. V. Ferrer,3 and J. Camacho2

1 Departamento de Matemática Aplicada, Universitat Politècnica de València, Camino de Vera S/N, 46022 Valencia, Spain
2 Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera S/N, 46022 Valencia, Spain
3 Departamento de Matemática e Informática, Universidad Jaume I de Castellón, Avenida de Vicent Sos Baynat S/N, 12071 Castellón de la Plana, Spain

Correspondence should be addressed to Emilio Defez; edefez@imm.upv.es

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This paper studies the construction of the exact solution for parabolic coupled systems of the type

\[ u_t(x, t) - Au_{xx}(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \]  

\[ A_1 u(0, t) + B_1 u_x(0, t) = 0, \quad t > 0, \]  

\[ A_2 u(1, t) + B_2 u_x(1, t) = 0, \quad t > 0, \]  

\[ u(x, 0) = f(x), \quad 0 \leq x \leq 1, \]  

where the unknown \( u = (u_1, u_2, \ldots, u_m)^T \) and the initial condition \( f = (f_1, f_2, \ldots, f_m)^T \) are \( m \)-dimensional vectors, \( A_1, B_1, i = 1, 2, \) are \( m \times m \) complex matrices, elements of \( \mathbb{C}^{m \times m} \), and \( A \) is a matrix which satisfies the condition

\[ \operatorname{Re}(z) > 0 \quad \text{for all eigenvalues } z \text{ of } A, \]  

and we say that \( A \) is a positive stable matrix (where \( \operatorname{Re}(z) \) denotes the real part of \( z \in \mathbb{C} \)). We assume that the block matrix

\[ \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \]  

is regular

\[ (6) \]

and also that

the matrix pencil \( A_1 + \rho B_1 \) is regular.

\[ (7) \]

Condition (7) is well known from the literature of singular systems of differential equations, and it involves the existence of some \( \rho_0 \in \mathbb{C} \) such that matrix \( A_1 + \rho B_1 \) is invertible [11].

Problem (1)–(4) with the less restrictive condition that (7) was solved in [12], but not with all of its blocks \( A_1, A_2, B_1, B_2 \), is singular (in particular \( A_1 = I \)). Mixed problems of the previously mentioned type, but with the Dirichlet conditions \( u(0, t) = 0, u(1, t) = 0 \) instead of (2) and (3), have been treated in [13, 14].

Throughout this paper, and as usual, matrix \( I \) denotes the identity matrix. The set of all the eigenvalues of a matrix \( C \) in \( \mathbb{C}^{m \times m} \) is denoted by \( \sigma(C) \), and its 2-norm \( \|C\| \) is defined by [15, page 56]

\[ \|C\| = \sup_{x \neq 0} \frac{\|Cx\|}{\|x\|}, \]  

\[ (8) \]
where for vector \( y \in \mathbb{C}^m \), the Euclidean norm of \( y \) is \( \|y\| \). By [15, page 556], it follows that
\[
\|e^{At}\| \leq e^{\alpha(A)t} \sum_{k=0}^{m-1} \| \sqrt{\nu} A \|^k t^k / k!, \quad t \geq 0,
\]
where \( \alpha(A) = \max \{ \Re(\omega); \omega \in \sigma(A) \} \). We say that a subspace \( E \) of \( \mathbb{C}^m \) is invariant by the matrix \( A \in \mathbb{C}^{m \times m} \), if \( A(E) \subset E \). If \( B \) is a matrix in \( \mathbb{C}^{m \times m} \), we denote by \( B^* \) its Moore-Penrose pseudoinverse. A collection of examples, properties, and applications of this concept may be found in [11, 16], and \( B^* \) can be efficiently computed with the MATLAB and Mathematica computer algebra systems.

2. Preliminaries and Notation

In [17], eigenfunctions of problem (1)–(3) were constructed assuming other additional conditions besides (6) and (7). We recall in this section the notation and results needed. Let \( \tilde{A}_1 \) and \( \tilde{B}_1 \) be matrices defined by
\[
\tilde{A}_1 = (A_1 + \rho_0 B_1)^{-1} A_1, \quad \tilde{B}_1 = (A_1 + \rho_0 B_1)^{-1} B_1,
\]
fulfilling the relation: \( \tilde{A}_1 + \rho_0 \tilde{B}_1 = I \). Under hypothesis (6), matrix \( B_2 = (A_2 + \rho_0 B_2) \tilde{B}_1 \) is regular; see [17, page 431], and let \( \tilde{A}_2 \) and \( \tilde{B}_2 \) be the matrices defined by
\[
\tilde{A}_2 = [B_2 - (A_2 + \rho_0 B_2) \tilde{B}_1]^{-1} A_2,
\]
\[
\tilde{B}_2 = [B_2 - (A_2 + \rho_0 B_2) \tilde{B}_1]^{-1} B_2,
\]
so that they satisfy the relationships
\[
\tilde{B}_2 - (\tilde{A}_2 + \rho_0 \tilde{B}_2) \tilde{B}_2 = I, \quad \tilde{B}_2 \tilde{A}_1 - \tilde{A}_2 \tilde{B}_1 = I.
\]

Assuming that the following condition
\[
\exists b_1 \in \sigma(\tilde{B}_1) \cap \mathbb{R}, \quad b_2 \in \sigma(\tilde{B}_2) \cap \mathbb{R}, \quad \nu \in \mathbb{C}^m \cap \mathbb{R},
\]
exists \( \nu \in \mathbb{C}^m \cap \mathbb{R} \), such that \( (\tilde{B}_1 - b_1 I) \nu = (\tilde{B}_2 - b_2 I) \nu = 0 \),

and that values \( b_1, b_2 \) of condition (13) satisfy
\[
b_1 b_2 \in \mathbb{R}, \quad \text{where } b_1 \in \mathbb{R} \text{ or } 2b_1 b_2 (\Re (b_1^{-1}) - \rho_0) = 1,
\]

we can define the function
\[
\alpha(\rho_0, b_1, b_2, \lambda) = \frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1} - b_1 b_2 \lambda^2,
\]

where \( \lambda > 0 \).

Note that under hypothesis (14) we have guaranteed the existence of the solutions for
\[
\lambda \cot(\lambda) = \frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1} - b_1 b_2 \lambda^2. \tag{16}
\]

Equation (16) has a unique solution \( \lambda_k \) in each interval \( (k\pi, (k + 1)\pi) \) for \( k \geq 1 \), as seen in Figure 1. Also, it is straightforward to prove the following lemma.

**Lemma 1.** Under hypothesis (14), the roots \( \lambda_k \) of \( (16) \) satisfy
\[
\lim_{n \to \infty} \sin(\lambda_n) = 0, \quad \lim_{n \to \infty} |\cos(\lambda_n)| = 1. \tag{17}
\]

Otherwise, if \( b_1 b_2 = 0 \), then
\[
\lim_{n \to \infty} \sin(\lambda_n) = 1, \quad \lim_{n \to \infty} |\cos(\lambda_n)| = 0. \tag{18}
\]

However, in all cases it is
\[
\lim_{n \to \infty} (\lambda_{n+1} - \lambda_n) = \pi. \tag{19}
\]

**Proof.** Function \( f(\lambda) = \lambda \cot(\lambda) \) has vertical asymptotes at the points \( \lambda = k\pi, k \in \mathbb{N} \), and \( f(\lambda) \) has zeros at the points \( \lambda = (\pi/2) + k\pi \), \( k \in \mathbb{N} \). Thus, as we have stated, the real coefficient function \((1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)\) intersects the graph of the function \( f(\lambda) \) in each interval \((k\pi, (k + 1)\pi)\), where \( \lambda_k \in (k\pi, (k + 1)\pi) \) is the point of intersection. Thus, the sequence \( \{\lambda_k\}_{k \geq 1} \) is monotonically increasing with \( \lim_{k \to \infty} \lambda_k = \infty \). We have to consider two possibilities.

(i) \( b_1 b_2 > 0 \). Function \((1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)\) is therefore decreasing, and as seen in Figure 1, for large enough \( k \), then \( \lambda_k \in (\pi/2 + k\pi, (k + 1)\pi) \).

(ii) \( b_1 b_2 < 0 \). Function \((1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)\) is therefore increasing, and as seen in Figure 1, for large enough \( k \), then \( \lambda_k \in (k\pi, (\pi/2 + k\pi)) \).

Thus, observe that if \( b_1 b_2 \neq 0 \), then \( \pi/2 < \lambda_{k+1} - \lambda_k < (3\pi/2) \) for large sufficiently \( k \). For \( \lambda_k \), reemplying in (16), one gets
\[
\lambda_k \cot(\lambda_k) = \frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1} - b_1 b_2 \lambda_k^2. \tag{20}
\]
dividing by $\lambda_k^2$ and taking limits where $k \to \infty$:

$$
\lim_{k \to \infty} \cot \left( \frac{\lambda_k}{\lambda_k^2} \right) = -b_1 b_2 \neq 0.
$$

(21)

This demonstrates that sequences $\{\lambda_k\}_{k \geq 1}$ and $\{\cot(\lambda_k)\}_{k \geq 1}$ are infinite equivalents and

$$
\lim_{k \to \infty} \cot \left( \lambda_k \right) = \infty,
$$

(22)

where $\lim_{k \to \infty} \tan(\lambda_k) = 0$. Moreover, as $\{\cos(\lambda_k)\}_{k \geq 1}$ is bounded, one gets that $\lim_{k \to \infty} \sin(\lambda_k) = 0$ and $\lim_{k \to \infty} |\cos(\lambda_k)| = 1$. Taking into account that

$$
\tan(\lambda_{k+1} - \lambda_k) = \frac{\tan(\lambda_{k+1}) - \tan(\lambda_k)}{1 + \tan(\lambda_{k+1})\tan(\lambda_k)},
$$

(23)

considering limits where $k \to \infty$, one gets

$$
\lim_{k \to \infty} \tan(\lambda_{k+1} - \lambda_k) = 0, \quad \text{and with } (\pi/2) < \lambda_{k+1} - \lambda_k < (3\pi/2), \quad \text{then } \lim_{k \to \infty} \tan(\lambda_{k+1} - \lambda_k) = \pi.
$$

If $b_1 b_2 = 0$, then one obtains two possibilities.

(i) If $((1 - b_2 + \rho_0 b_1 b_2 - \rho_0 b_1) b_2) > 0$, as one can see in Figure 1, for large enough $k, \lambda_k \in (\pi k, (\pi/2) + k\pi)$.

(ii) If $((1 - b_2 + \rho_0 b_1 b_2 - \rho_0 b_1) b_2) < 0$, as one can see in Figure 1, for large enough $k, \lambda_k \in (\pi/2 + k\pi, (k+1)\pi)$.

Thus, observe that if $b_1 b_2 = 0$, then also $\lambda_{k+1} - \lambda_k < (3\pi/2)$ for $k$ sufficiently large. For $\lambda_k$, reemerging in

(16), one gets

$$
\lambda_k \cot(\lambda_k) = \frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1},
$$

(24)

dividing by $\lambda_k$ and taking limits where $k \to \infty$, one gets

$$
\lim_{k \to \infty} \cot(\lambda_k) = 0, \quad \text{and as the sequence } \{\sin(\lambda_k)\}_{k \geq 1}
$$

is bounded, one gets that $\lim_{k \to \infty} \sin(\lambda_k) = 0$ and $\lim_{k \to \infty} |\sin(\lambda_k)| = 1$. Moreover, one gets that

$$
\cot(\lambda_{k+1} - \lambda_k) = \frac{\cot(\lambda_{k+1}) \cot(\lambda_k) + 1}{\cot(\lambda_k) - \cot(\lambda_{k+1})},
$$

(25)

considering limits where $k \to \infty$, one gets

$$
\lim_{k \to \infty} \cot(\lambda_{k+1} - \lambda_k) = \infty,
$$

(26)

as the sequence $\{\cos(\lambda_{k+1} - \lambda_k)\}_{k \geq 1}$ is bounded, we have that $\lim_{k \to \infty} \sin(\lambda_{k+1} - \lambda_k) = 0$, and with $\lambda_{k+1} - \lambda_k < (\pi/2)$, one gets that $\lim_{k \to \infty} \cos(\lambda_k) = 0$.

If $b_1 b_2 = 0$ and $((1 - b_2 + \rho_0 b_1 b_2 - \rho_0 b_1) b_2) = 0$, (16) reduces to $\lambda \cot(\lambda) = 0$, whose roots are $\lambda_k = (\pi/2) + k\pi$, $k \in \mathbb{N}$, and trivially $\lambda_{k+1} - \lambda_k = \pi$. Then

$$
\lim_{k \to \infty} (\lambda_{k+1} - \lambda_k) = \pi.
$$

(27)

The eigenfunctions associated to the problem (1) are then given by

$$
\mathcal{F} = \{ \lambda_k \in (k\pi, (k+1)\pi) \}
$$

and under condition (32), the solution of (30) is given by

$$
C(\lambda_k) = (I - G(\rho_0, b_1, b_2, \lambda_k))^{-1} G(\rho_0, b_1, b_2, \lambda_k) J \in \mathbb{C}^m.
$$

(33)
one gets that function
\[ u(x, 0) = \left( x\overline{A}_1 - \overline{B}_1 \right) C(0) \] (37)
is an eigenfunction of problem (1) associated to eigenvalue \( \lambda = 0 \).

All these results are summarized in Theorem 2.1 of [17, page 434]. Our goal is to find the exact solution of the problem (1)–(4). We provide conditions for the function \( f(x) \) and the matrix coefficients in order to ensure the existence of a series solution of the problem. The paper is organized as follows. In Section 3 a series solution for the problem is presented. In Section 4 we proceed with an algorithm and give an illustrative example.

3. A Series Solution

By the superposition principle, a possible candidate to the series solution of problem (1)–(4) is given by
\[ u(x, t) = \sum_{\lambda_n \in \mathcal{F}} u(x, t, \lambda_k), \quad 0 \notin \mathcal{F}, \] (38)
where \( u(x, t, \lambda_k) \) and \( u(x, 0) \) are defined by (34) and (37), respectively, for suitable vectors \( C(\lambda_n) \) and \( C(0) \).

Assuming that series (38) and the corresponding derivatives \( u_t(x, t), u_{xx}(x, t), \) and \( u_{tx}(x, t) \) are convergent (we will demonstrate this later), (38) will be a solution of (1)–(3). Now, we need to determine vectors \( C(\lambda) \) and \( C(0) \) so that (38) satisfies (4).

Note that, taking \( v \) to satisfy (13), from (12) one gets
\[ \overline{A}_2 v = \left( \frac{b_1 - \rho_0 b_1 b_2}{b_1} \right) v, \quad \overline{A}_1 v = (1 - \rho_0 b_1) v. \] (39)
Under condition (39), we will consider the scalar Sturm-Liouville problem:
\[ X''(x) + \lambda^2 X(x) = 0, \]
\[ (1 - \rho_0 b_1) X(0) + b_1 X'(0) = 0, \]
\[ - \left( \frac{1 - b_2 + \rho_0 b_1 b_2}{b_1} \right) X(1) + b_2 X'(1) = 0, \] (40)
which provides a family of eigenvalues \( \mathcal{F} \) given in (27). Then, the associated eigenfunctions are
\[ X_{\lambda_n}(x) = (1 - \rho_0 b_1) \sin(\lambda_n x) - b_1 \lambda_n \cos(\lambda_n x), \quad \lambda_n > 0, \]
\[ X_0(x) = (1 - \rho_0 b_1) x - b_1, \quad \text{if } \lambda_0 = 0. \] (41)

By the theorem of convergence of the Sturm-Liouville for functional series [18, chapter 11], with the initial condition
\[ f(x) = (f_1(x), \ldots, f_n(x))^T \] given in (4) satisfying the following properties:
\[ f \in C^2([0,1]), \]
\[ (1 - \rho_0 b_1) f(0) + b_1 f'(0) = 0, \]
\[ - \left( \frac{1 - b_2 + \rho_0 b_1 b_2}{b_1} \right) f(1) + b_2 f'(1) = 0, \] (42)
each component \( f_i \) of \( f \), for \( 1 \leq i \leq m \), has a series expansion which converges absolutely and uniformly on the interval \([0,1]\); namely,
\[ f_i(x) = \alpha ((1 - \rho_0 b_1) x - b_1) e_{\lambda_0} \]
\[ + \sum_{\lambda_n \in \mathcal{F}} ((1 - \rho_0 b_1) \sin(\lambda_n x) - b_1 \lambda_n \cos(\lambda_n x)) e_{\lambda_n}, \] (43)
where
\[ \alpha = \begin{cases} 
1 & \text{if } \frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1} = 1 \\
0 & \text{if } \frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1} \neq 1,
\end{cases} \]
\[ e_{\lambda_0} = b_1 \int_0^1 \left( (1 - \rho_0 b_1) x - b_1 \right) f_i(x) \, dx \\
= b_1 \frac{\int_0^1 \left( (1 - \rho_0 b_1) \sin(\lambda_n x) - b_1 \lambda_n \cos(\lambda_n x) \right) f_i(x) \, dx}{\int_0^1 \left( (1 - \rho_0 b_1) \sin(\lambda_n x) - b_1 \lambda_n \cos(\lambda_n x) \right)^2 \, dx} \] if \( \lambda_0 = 0, \]
\[ e_{\lambda_n} = b_1 \lambda_n \int_0^1 \left( (1 - \rho_0 b_1) \sin(\lambda_n x) - b_1 \lambda_n \cos(\lambda_n x) \right) f_i(x) \, dx \] if \( \lambda_n > 0. \] (44)
Thus,
\[ f(x) = \alpha ((1 - \rho_0 b_1) x - b_1) E(0) + \sum_{\lambda_n \in \mathcal{F}} ((1 - \rho_0 b_1) \sin(\lambda_n x) - b_1 \lambda_n \cos(\lambda_n x)) e_{\lambda_n} \]
\[ - b_1 \lambda_n \cos(\lambda_n x) E(\lambda_n), \] (45)
where \( E(0) = \begin{pmatrix} e_{\lambda_1} \\ \vdots \\ e_{\lambda_m} \end{pmatrix} \) and \( E(\lambda_n) = \begin{pmatrix} e_{\lambda_1} \\ \vdots \\ e_{\lambda_n} \end{pmatrix} \). On the other hand, from (38) and taking into account (34) and (37), one gets
\[ f(x) = u(x, 0) = \alpha \left( x\overline{A}_1 - \overline{B}_1 \right) C(0) + \sum_{\lambda_n \in \mathcal{F}} \left( \sin(\lambda_n x) \overline{A}_1 - \lambda_n \cos(\lambda_n x) \overline{B}_1 \right) C(\lambda_n). \] (46)
We can equate the two expressions; if \( C(0) \) and \( C(\lambda_n) \), apart from conditions \( (33) \) and \( (36) \), satisfy \( \{ C(0), C(\lambda) \} \subset \text{Ker}(\tilde{B}_1 - b_1 I) \). Then, we have

\[
C(\lambda_n) = E(\lambda_n)
\]

\[
= \int_0^1 \left( (1 - \rho b_1) \sin(\lambda_n x) - b_1 \lambda_n \cos(\lambda_n x) \right) f(x) \, dx
\]

\[
\int_0^1 \left( (1 - \rho b_1) \sin(\lambda_n x) - b_1 \lambda_n \cos(\lambda_n x) \right)^2 \, dx,
\]

if \( \lambda_n > 0 \),

\[
C(0) = E(0)
\]

\[
= \int_0^1 \left( (1 - \rho b_1) x - b_1 \right) f(x) \, dx
\]

\[
\int_0^1 \left( (1 - \rho b_1) x - b_1 \right)^2 \, dx
\]

if \( \lambda_0 = 0 \). (47)

Note that \( C(0) \) and \( C(\lambda) \) \( \in \text{Ker}(\tilde{B}_1 - b_1 I) \), if

\[
f(x) \in \text{Ker} \left( \tilde{B}_1 - b_1 I \right).
\]

(48)

Then \( u(x, t) \) defined by

\[
u(x, t) = \alpha \left( (1 - \rho b_1) x - b_1 \right) C(0)
\]

\[
+ \sum_{\lambda_n \in \mathcal{F}} e^{-\lambda_n At} \left( (1 - \rho b_1) \sin(\lambda_n x) - b_1 \lambda_n \cos(\lambda_n x) \right)(\lambda_n) \cos(\lambda_n x),
\]

(49)

where \( \alpha \) and \( C(\lambda_n) \) are defined by \( (44) \) and \( (47) \), satisfies the initial condition \( (4) \). Note that conditions \( (30)-(32) \) hold if

\[
G(\rho_0, b_1, b_2, \lambda_k) f(x) = 0,
\]

(50)

and then

\[
(\tilde{B}_1 - b_1 I) A^j f(x) = 0, \quad 0 \leq j < p,
\]

\[
\left( \tilde{A}_2 \tilde{A}_1 + \lambda_n^2 \tilde{B}_2 \tilde{B}_1 + \alpha (\rho_0, b_1, b_2, \lambda) I \right) A^j f(x) = 0,
\]

(51)

0 \leq j < p.

It is easy to check that conditions \( (48), (51) \) are equivalent to the condition

\[
A^j f(x) \in \text{Ker}(\tilde{B}_1 - b_1 I) \cap \text{Ker}(\tilde{B}_2 - b_2 I), \quad \text{if} \quad 0 \leq j < p.
\]

(52)

Condition \( (52) \) holds if

\[
f(x) \in \text{Ker}(\tilde{B}_1 - b_1 I) \cap \text{Ker}(\tilde{B}_2 - b_2 I), \quad 0 \leq x \leq 1,
\]

\[
\text{Ker}(\tilde{B}_1 - b_1 I) \cap \text{Ker}(\tilde{B}_2 - b_2 I),
\]

(53)

is an invariant subspace with respect to matrix \( A \).

Now we study the convergence of the solution given by \( (49) \) with \( \alpha \) defined by \( (44) \) and \( C(\lambda_n) \) by \( (47) \). Using Parseval’s identity for scalar Sturm–Liouville problems \( [19] \), there exists a positive constant \( M_1 > 0 \) so that \( \| C(\lambda_n) \| \leq M_1 \). Taking formal derivatives in \( (49) \), one gets

\[
u_t(x, t) = \sum_{\lambda_n \in \mathcal{F}} \left( -\lambda_n^2 \right) e^{-\lambda_n At} \left( \sin(\lambda_n x) - \lambda_n \cos(\lambda_n x) \right)(\lambda_n) \cos(\lambda_n x),
\]

\[
u_x(x, t) = \sum_{\lambda_n \in \mathcal{F}} \lambda_n e^{-\lambda_n At} \left( \cos(\lambda_n x) - \lambda_n \sin(\lambda_n x) \right)(\lambda_n) \cos(\lambda_n x) + \lambda_n \cos(\lambda_n x)(\lambda_n) C(\lambda_n)
\]

(54)

These series are all bounded in their respective norms:

\[
\| u(x, t) \|
\]

\[
\leq \sum_{\lambda_n \in \mathcal{F}} \left[ \left\| e^{-\lambda_n At} \right\| \right] \left| 1 - \rho b_1 \right| M_1 + \left\| \lambda_n e^{-\lambda_n At} \right\| \left| b_1 \right| M_1
\]

\[
+ \alpha \left| 1 - \rho b_1 \right| \| C(0) \|,
\]

\[
\| u_t(x, t) \|
\]

\[
\leq \sum_{\lambda_n \in \mathcal{F}} \left[ \left\| e^{-\lambda_n At} \right\| \right] \left| 1 - \rho b_1 \right| M_1 + \left\| \lambda_n e^{-\lambda_n At} \right\| \left| b_1 \right| M_1
\]

\[
+ \alpha \left| 1 - \rho b_1 \right| \| C(0) \|,
\]

\[
\| u_x(x, t) \|
\]

\[
\leq \sum_{\lambda_n \in \mathcal{F}} \left[ \left\| e^{-\lambda_n At} \right\| \right] \left| 1 - \rho b_1 \right| M_1 + \left\| \lambda_n e^{-\lambda_n At} \right\| \left| b_1 \right| M_1
\]

(55)

To check that the series is uniformly convergent in each domain \([0, 1] \times [c, d]\), it is sufficient to verify that the series

\[
\sum_{\lambda_n \in \mathcal{F}} \lambda_n^3 e^{-\lambda_n At}
\]

(56)

is uniformly convergent in this domain. This is trivial because, using \( (9) \), one gets

\[
\left\| \lambda_n^3 e^{-\lambda_n At} \right\| \leq e^{-\lambda_n^2 M_1 t} \sum_{k=0}^{m-1} \left( \frac{\sqrt{m} \| A \| t^k}{k!} \right)^3 \lambda_n^{2k+3},
\]

(57)
and from the d’Alembert test series applied to each summand, taking into account (5) and the relation (19), \( \lim_{n \to \infty} (\lambda_{n+1} - \lambda_n) = \pi \), given in Lemma 1, one gets for \( 3 \leq r \leq 2 \ (m - 1) + 3 \) that

\[
\lim_{n \to \infty} e^{(\lambda_{n+1} - \lambda_n)\alpha(A) t n} \left( \lambda_{n+1} \right)^r \\
\leq \lim_{n \to \infty} e^{(\lambda_{n+1} - \lambda_n)\alpha(A) \left( n + 2 \right)} n^r \\
= e^{-\alpha(A) t \lim_{n \to \infty} (\lambda_{n+1} - \lambda_n) = 0 < 1}.
\]

(58)

Thus, the series (56) is convergent.

Independence of the series solution (49) with respect to the chosen \( \rho_0 \in \mathbb{R} \) can be demonstrated using the same technique as given in [20].

We can summarize the results in the following theorem.

**Theorem 2.** Consider the homogeneous problem with homogeneous conditions (1)–(4) under hypotheses (5), (6), and (7) verifying conditions (13) and (14). Let \( f(x) \) be a vectorial function satisfying (42). Let \( \mathcal{F} \) be the set defined by (27), and let \( G(\rho_0, b_1, b_2, \lambda) \) be the matrix defined by (31), taking as eigenvalues of problems \( \lambda \in \mathcal{F} \) satisfying

\[
\text{rank} \left( G(\rho_0, b_1, b_2, \lambda) \right) = m,
\]

including the eigenvalue \( \lambda = 0 \) if \( 1 \in \sigma(-\bar{A} \bar{A}_1) \), and taking as eigenfunctions \( u(x, t, \lambda) \) defined by (34). Let \( \alpha \) be given by (44) and vectors \( C(\lambda) \) defined by (47). Then, \( u(x, t, \lambda) \), as defined in (49), is a series solution of problem (1)–(4).

**4. Algorithm and Example**

We can summarize the process to calculate the solution of the homogeneous problem with homogeneous conditions (1)–(4) in Algorithm 1.

**Example 1.** We will consider the homogeneous parabolic problem with homogeneous conditions (1)–(4), where the matrix \( A \in \mathbb{C}^{4 \times 4} \) is chosen as

\[
A = \begin{pmatrix}
2 & 0 & 0 & -1 \\
1 & 2 & 1 & -2 \\
-1 & 0 & 2 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(60)

and the \( 4 \times 4 \) matrices \( A_i, B_i, i \in \{1, 2\} \), are

\[
A_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(61)

and

\[
B_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Also, the vectorial valued function \( f(x) \) will be defined as

\[
f(x) = \begin{pmatrix}
x^2 - 1 \\
0 \\
0
\end{pmatrix}.
\]

(62)

Observe that the method proposed in [12] cannot be applied to solve this problem.

We will follow Algorithm 1 step by step.

(1) Matrix \( A \) satisfies the condition (5), because \( \sigma(A) = \{1, 2\} \). That is, \( A \) is positive stable.

(2) Each of the matrices \( A_i, B_i, i \in \{1, 2\} \), is singular, and the block matrix

\[
\begin{pmatrix}
A_1 & B_1 \\
A_2 & B_2
\end{pmatrix}
\]

(63)

is regular.

(3) Note that although \( A_1 \) is singular, taking \( \rho_0 = 1 \in \mathbb{R} \), the matrix pencil

\[
A_1 + \rho_0 B_1 = I_{4 \times 4}
\]

(64)

is regular. Therefore, we take \( \rho_0 = 1 \).

(4) By (10) we have

\[
\bar{A}_1 = (A_1 + \rho_0 B_1)^{-1} A_1 = A_1,
\]

\[
\bar{B}_1 = (A_1 + \rho_0 B_1)^{-1} B_1 = B_1.
\]

(65)

(5) By (11) we have

\[
\bar{A}_2 = \left( B_2 - (A_2 + \rho_0 B_2) \bar{B}_1 \right)^{-1} A_2 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\bar{B}_2 = \left( B_2 - (A_2 + \rho_0 B_2) \bar{B}_1 \right)^{-1} B_2 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(66)

(6) We have \( \sigma(\bar{B}_1) = \{0, 1\} \) and \( \sigma(\bar{B}_2) = \{0, 1, -1\} \). Note that in this case the condition (13) holds because with \( b_1 = 1 \) and \( b_2 = 0 \in \sigma(\bar{B}_1) \) there exists a common eigenvector \( v \in \mathbb{C}^4 \), \( v = (0, 1, 0, 0)^t \), and thus \( \ker(\bar{B}_1 - I) \cap \ker(\bar{B}_2) \neq (0, 0, 0, 0)^t \). We are therefore in Case 1 of Algorithm 1.
Abstract and Applied Analysis

**Input data:** $A, A_1, A_2, B_1, B_2 \in \mathbb{C}^{m \times m}$, $f(x) \in \mathbb{C}^m$.

**Result:** $u(x, t)$.

(1) Check that matrix $A$ satisfies (5).
(2) Check that matrices $A_i, B_i \in \mathbb{C}^{m \times m}, i \in \{1, 2\}$ are singular, and check that the block matrix $(A_i, B_i)$ is regular.
(3) Determine a number $\rho_0 \in \mathbb{R}$ so that the matrix pencil $A_1 + \rho_0 B_1$ is regular.
(4) Determine matrices $A_2$ and $B_2$ defined by (10).
(5) Determine matrices $A_i$ and $B_i$ defined by (11).
(6) Consider the following cases:
    (i) Case 1. Condition (13) holds, that is, matrices $\tilde{B}_1$ and $\tilde{B}_2$ have a common eigenvector $\nu \neq 0$ associated with eigenvalues $b_1 \in \sigma(\tilde{B}_1) - \{0\}$ and $b_2 \in \sigma(\tilde{B}_2)$. In this case continue with step (7).
    (ii) Case 2. Condition (13) does not hold. In this case the algorithm stops because it is not possible to find the solution of (1)–(4) for the given data.
(7) Determine $b_i \in \sigma(\tilde{B}_1), b_i \neq 0, b_2 \in \sigma(\tilde{B}_2)$ and vector $\nu \neq 0$ verifying $\nu \in \text{Ker}(\tilde{B}_1 - b_1 I) \cap \text{Ker}(\tilde{B}_2 - b_2 I)$ such that:
    (i) Conditions (53) hold, that is:
    (1.1) Ker($\tilde{B}_1 - b_1 I$) $\cap$ Ker($\tilde{B}_2 - b_2 I$) is an invariant subspace respect matrix $A$.
    (1.2) $f(x) \in \text{Ker}(\tilde{B}_1 - b_1 I)$ $\cap$ Ker($\tilde{B}_2 - b_2 I$), $\forall x \in [0, 1]$.
    (ii) Conditions (14) hold, that is:
    (1.3) The vectorial function $f(x)$ satisfies (42), that is:
    (1.4) $f \in \mathbb{C}^2([0, 1])$.
    (1.5) $(1 - \rho_0 b_1) f(0) + b_1 f'(0) = 0$.
    (1.6) $-\frac{(1 - b_1 + \rho_0 b_1 b_2)}{b_1} f(1) + b_1 f'(1) = 0$.

If these conditions are not satisfied, return to step (6) of Algorithm 1 discarding the values taken for $b_i$ and $b_2$.
(8) Determine the positive solutions of (16) and determine $\mathcal{F}$ defined by (27).
(9) Determine degree $p$ of minimal polynomial of matrix $A$.
(10) Building block matrix $G(\rho_0, b_1, b_2, \lambda_1)$ defined by (31).
(11) Determine $\lambda \in \mathcal{F}$ so that rank $G(\rho_0, b_1, b_2, \lambda_1) < m$.
(12) Include the eigenvalue $\lambda_1 = 0$ if $1 \in \sigma(-A_1 A_1)$.
(13) Determine $\sigma$ given by (44).
(14) Determine vectors $C(\lambda_1)$ defined by (47).
(15) Determine functions $u(x, t, \lambda_1)$ defined by (34).
(16) Determine the series solution $u(x, t)$ of problem (1)–(4) defined by (49).

**Algorithm 1:** Solution of the homogeneous problem with homogeneous conditions (1)–(4).

(7) We take the values $b_1 = 1$ and $b_2 = 0$ and will check the conditions given in step 7 of the algorithm.
(1.1) One gets that

$$\text{Ker}(\tilde{B}_1 - I) \cap \text{Ker}(\tilde{B}_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$  

(67)

Let $x \in \text{Ker}(\tilde{B}_1 - I) \cap \text{Ker}(\tilde{B}_2)$. Then $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{C}$. In this case one gets

$$Ax = \begin{pmatrix} 0 \\ 2\lambda \\ 0 \end{pmatrix} \in \text{Ker}(\tilde{B}_1 - I) \cap \text{Ker}(\tilde{B}_2),$$  

(68)

and then the subspace Ker($\tilde{B}_1 - I$) $\cap$ Ker($\tilde{B}_2$) is invariant by matrix $A$.
(1.2) It is trivial to check that

$$f(x) \in \text{Ker}(\tilde{B}_1 - I) \cap \text{Ker}(\tilde{B}_2), \quad \forall x \in [0, 1].$$  

(69)

(1.3) With these values $\rho_0, b_1$, and $b_2$, one gets that

$$\frac{(1 - b_2 + \rho_0 b_1 b_2) (1 - \rho_0 b_1)}{b_1} = 0 \in \mathbb{R}. $$  

(70)

With these values $b_1$ and $b_2$, one gets

$$b_1 b_2 = 0 \in \mathbb{R}. $$  

(71)

(1.4) It is trivial to check that $f(x) \in \mathbb{C}^2([0, 1])$. 

(1.5) It is trivial to check that \((1 - b_2 + \rho b_1 b_2) f(1) + b_2 f'(1) = (0, 0, 0, 0)^t\).

(1.6) It is trivial to check that \(-((1 - b_2 + \rho b_1 b_2) f(1) + b_2 f'(1) = (0, 0, 0, 0)^t\).

(8) Equation (16) is of the form

\[ \lambda \cot(\lambda) = 0 \]  

(72)

We can solve (72) exactly, \( \lambda_k = (\pi/2) + k\pi \), with an additional solution \( \lambda_0 \in [0, \pi] \), because

\[ \frac{(1 - b_2 + \rho b_1 b_2)(1 - \rho b_1)}{b_1} < 0, \]  

(73)

and then \( \lambda_0 = (\pi/2) \). Thus, we have a numerable family of solutions of (72) which we denote by \( \mathcal{F} \), given by.

\[ \mathcal{F} = \left\{ \lambda_k = \frac{\pi}{2} + k\pi; \; \lambda_k \in (k\pi, (k + 1)\pi), \; k \geq 1 \right\} \cup \mathcal{F}_0, \]

(74)

We have a numerable family of solutions of (72) which we denote by \( \mathcal{F} \), given by.

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(74)

(9) The minimal polynomial of matrix \( A \) is given by

\[ \rho(x) = (x - 2)^2(x - 1). \]  

(10) If \( \lambda_k \) is a positive solution of (72), the matrix \( G(\rho_0, b_1, b_2, \lambda_k) \) given by (31) takes the form

\[ G(1, 1, 0, \lambda_k) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}, \]

\[ \begin{pmatrix} 0 & 0 & -7 \\ 0 & 0 & -13 \\ 12 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ \begin{pmatrix} 0 & 0 & 0 \\ -\lambda_k^2 & 0 & 0 \\ -\lambda_k^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ \begin{pmatrix} -2\lambda_k^2 & 0 & 0 & \lambda_k^2 \\ 2\lambda_k & 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ \begin{pmatrix} -4\lambda_k^2 & 0 & 0 & 3\lambda_k^2 \\ -4\lambda_k & 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ \begin{pmatrix} -8\lambda_k^2 & 0 & 0 & 7\lambda_k^2 \\ -8\lambda_k & 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]  

(75)

(11) Since the second column \( G(1, 1, 0, \lambda_k) \) is zero, we have that \( \text{rank}(G(1, 1, 0, \lambda_k)) < 4 \). Thus, each one of the positive solutions given by (74) is an eigenvalue.

(12) It is trivial to check that \( 1 \notin \sigma(-\tilde{A}_2 \tilde{A}_1) \), because

\[ -\tilde{A}_2 \tilde{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]  

\[ \sigma(-\tilde{A}_2 \tilde{A}_1) = \{0\}. \]  

(76)

Then we do not include 0 as an eigenvalue.

(13) Taking into account that \((1 - b_2 + \rho b_1 b_2)(1 - \rho b_1)/b_1 = 0 < 1, \) one gets \( \alpha = 0 \).

(14) Vectors \( C(\lambda_n) \) defined by (47) take the values

\[ C(\lambda_n) = \frac{64(-1)^n}{\pi^2(2n+1)^4} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \]  

(77)

(15) Using the minimal theorem [21, page 571], one gets that

\[ e^{A t} = \begin{pmatrix} e^{2u} & 0 & 0 & -e^{u}(e^{(1/2)(\pi+2n\pi) t}) \\ 0 & e^{2u} & 0 & 0 \\ 0 & 0 & e^{2u} & 0 \\ 0 & 0 & 0 & e^{2u} \end{pmatrix}. \]  

(78)

Next, by considering (78) with \( u = -(\pi/2 + n\pi)^2 t \) and simplifying, we obtain the value of \( e^{-(\pi/2 + n\pi)^2 A t} \).

Taking into account that all eigenvalues \( \lambda_n \) are positive, the associated eigenfunctions are

\[ u(x, t, \lambda_n) = e^{-\lambda_n^2 A t} ((1 - \rho_0 b_1) \sin(\lambda_n x) - b_1 \lambda_n \cos(\lambda_n x)) C(\lambda_n). \]

(79)

(16) We replace the values of \( C(\lambda_n) \) given by (77) in (79) and take into account the value of the matrix \( e^{-(\pi/2 + n\pi)^2 A t} \). After simplification, we finally obtain the solution of (1)–(4) given by

\[ u(x, t) = \sum_{n=0}^{\infty} \frac{32(-1)^n e^{-(1/2)(n\pi + 2n\pi)^2 t} \cos((1/2)(\pi + 2n\pi) x)}{\pi^2(2n+1)^3} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \]  

(80)

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References


