THE ŁOJASIEWICZ EXPONENT OF A SET OF WEIGHTED HOMOGENEOUS IDEALS

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Abstract. We give an expression for the Łojasiewicz exponent of a set of ideals which are pieces of a weighted homogeneous filtration. We also study the application of this formula to the computation of the Łojasiewicz exponent of the gradient of a semi-weighted homogeneous function \((\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) with an isolated singularity at the origin.

1. Introduction

Let \(R\) be a Noetherian ring and let \(I\) be an ideal of \(R\). Let \(\nu_I\) be the order function of \(R\) with respect to \(I\), that is, \(\nu_I(h) = \sup \{ r : h \in I^r \}\), for all \(h \in R, h \neq 0\), and \(\nu(0) = \infty\). Let us consider the function \(\nu_I : R \to \mathbb{R}_{\geq 0} \cup \{\infty\}\) defined by \(\nu_I(h) = \lim_{s \to \infty} \frac{\nu_I(h^s)}{s}\), for all \(h \in R\). It was proven by Samuel [17] and Rees [14] that this limit exists and Nagata proved in [12] that, when finite, the number \(\nu_I(h)\) is a rational number. The function \(\nu_I\) is called the asymptotic Samuel function of \(I\). If \(J\) is another ideal of \(R\), then the number \(\nu_I(J)\) is defined analogously and if \(h_1, \ldots, h_r\) is a generating system of \(J\) then \(\nu_I(J) = \min\{\nu_I(h_1), \ldots, \nu_I(h_r)\}\). Let us denote by \(\overline{I}\) the integral closure of \(I\). As a consequence of the theorem of existence of the Rees valuations of an ideal (see for instance [8, p. 192]), it is known that, if \(J\) is another ideal and \(p, q \in \mathbb{Z}_{\geq 1}\), then \(J^q \subseteq \overline{I}^p\) if and only if \(\nu_I(J) \geq \frac{p}{q}\).

Let \(O_n\) denote the ring of analytic function germs \(f : (\mathbb{C}^n, 0) \to \mathbb{C}\) and let \(m_n\) denote its maximal ideal, that will be also denoted by \(m\) if no confusion arises. Let \(I\) be an ideal of \(O_n\) of finite colength. Lejeune and Teissier proved in [10, p. 832] that \(1/\nu_I(m)\) is equal to the Łojasiewicz exponent of \(I\) (in fact, this result was proven in a more general context, that is, for ideals in a structural ring \(O_X\), where \(X\) is a reduced complex analytic space). If \(g_1, \ldots, g_r\) is a generating system of \(I\), then the Łojasiewicz exponent of \(I\) is defined as the infimum of those \(\alpha > 0\) for which there exist a constant \(C > 0\) and an open neighbourhood \(U\) of \(0 \in \mathbb{C}^n\) with

\[\|x\|^\alpha \leq C \sup_i |g_i(x)|\]

for all \(x \in U\). Let us denote this number by \(L_0(I)\) and let \(e(I)\) denote the Samuel multiplicity of \(I\). Therefore we have that \(L_0(I) = \inf\{\frac{p}{q} : m^p \subseteq \overline{I}^q, p, q \in \mathbb{Z}_{\geq 0}\}\) and hence, by the Rees’
multiplicity theorem (see [8, p. 222]) it follows that $\mathcal{L}_0(I) = \inf\{\frac{r}{q} : e(I^q) = e(I^p + m^p), \ p, q \in \mathbb{Z}_{>0}\}$. This expression of $\mathcal{L}_0(I)$ is one of the motivations that lead the first author to introduce the notion of Łojasiewicz exponent of a set of ideals in [4]. This notion is based on the Rees’ mixed multiplicity of a set of ideals (Definition 2.1).

Łojasiewicz exponents have important applications in singularity theory. Here we recall one of them. If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is an analytic map germ such that $g^{-1}(0) = \{0\}$ then we denote by $\mathcal{L}_0(g)$ the Łojasiewicz exponent of the ideal generated by the component functions of $g$. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated singularity at the origin. Then $\nabla f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ denotes the gradient map of $f$, that is, $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$. The Jacobian ideal of $f$, that we will denote by $J(f)$, is the ideal generated by the components of $\nabla f$. The degree of $C^0$-determinacy of $f$, denoted by $s_0(f)$, is defined as the smallest integer $r$ such that $f$ is topologically equivalent to $f + g$, for all $g \in \mathcal{O}_n$ with $\nu_m(g) \geq r + 1$. Teissier proved in [19, p. 280] that $s_0(f) = [\mathcal{L}_0(\nabla f)] + 1$, where $[a]$ stands for the integer part of a given $a \in \mathbb{R}$. Despite the fact that this equality connects $\mathcal{L}_0(\nabla f)$ with a fundamental topological aspect of $f$, the problem of determining whether the Łojasiewicz exponent $\mathcal{L}_0(\nabla f)$ is a topological invariant of $f$ is still an open problem.

The effective computation of $\mathcal{L}_0(I)$ has proven to be a challenging problem in algebraic geometry that, by virtue of the results of Lejeune and Teissier is directly related with the computation of the integral closure of an ideal. In [5] the authors relate the problem of computing $\mathcal{L}_0(I)$ with the algorithms of resolution of singularities. The approach that we give in this paper is based on techniques of commutative algebra.

We recall that, if $w = (w_1, \ldots, w_n) \in \mathbb{Z}_{\geq 1}^n$, then a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is called weighted homogeneous of degree $d$ with respect to $w$ when $f$ is written as a sum of monomials $x_1^{k_1} \cdots x_n^{k_n}$ such that $w_1 x_1 + \cdots + w_n x_n = d$. This paper is motivated by the main result of Krasiński, Oleksik and Płoski in [9], which says that if $f : \mathbb{C}^3 \to \mathbb{C}$ is a weighted homogeneous polynomial of degree $d$ with respect to $(w_1, w_2, w_3)$ with an isolated singularity at the origin, then $\mathcal{L}_0(\nabla f)$ is given by the expression

$$\mathcal{L}_0(\nabla f) = \frac{d - \min\{w_1, w_2, w_3\}}{\min\{w_1, w_2, w_3\}}$$

provided that $d \geq 2w_i$, for all $i = 1, 2, 3$. That is, $\mathcal{L}_0(\nabla f)$ depends only on the weights $w_i$ and the degree $d$ in this case. Therefore it is concluded that $\mathcal{L}_0(\nabla f)$ is a topological invariant of $f$, by virtue of the results of Saeki [16] and Yau [21]. In view of the above equality it is reasonable to conjecture that the analogous result holds in general, that is, if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a weighted homogeneous polynomial, or even a semi-weighted homogeneous function (see Definition 4.1), with respect to $(w_1, \ldots, w_n)$ of degree $d$ with an isolated singularity at the origin, and if $d \geq 2w_i$, for all $i = 1, \ldots, n$, then

$$\mathcal{L}_0(\nabla f) = \frac{d - \min\{w_1, \ldots, w_n\}}{\min\{w_1, \ldots, w_n\}}. \tag{1}$$
We point out that inequality \((\leq)\) always holds in (1) for semi-weighted homogeneous functions (see Corollary 4.11).

In this paper we obtain the equality (1) for semi-weighted homogeneous germs \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) under a restriction expressed in terms of the supports of the component functions of \(\nabla f\) (see Corollary 4.11). This result arises as a consequence of a more general result involving the Lojasiewicz exponent of a set of ideals coming from a weighted homogeneous filtration (see Theorem 4.7). Our approach to Lojasiewicz exponents is purely algebraic and comes from the techniques developed in [3] and [4]. This new point of view of the subject has led us to detect a broad class of semi-weighted homogeneous functions where relation (1) holds.

For the sake of completeness we recall in Section 2 the definition of Rees’ mixed multiplicity and basic facts about this notion. In Section 3 we show some results about the notion of Lojasiewicz exponent of a set of ideals that will be applied in Section 4. The main results appear in Section 4.

2. The Rees’ mixed multiplicity of a set of ideals

Let \((R, m)\) be a Noetherian local ring and let \(I\) be an ideal of \(R\). We denote by \(e(I)\) the Samuel multiplicity of \(I\). Let \(\dim R = n\) and let us fix a set of \(n\) ideals \(I_1, \ldots, I_n\) of \(R\) of finite colength. Then we denote by \(e(I_1, \ldots, I_n)\) the mixed multiplicity of \(I_1, \ldots, I_n\), as defined by Teissier and Risler in [20] (we refer to [8, §17] and [18] for fundamental results about mixed multiplicities of ideals). We recall that, if the ideals \(I_1, \ldots, I_n\) are equal to a given ideal, say \(I\), then \(e(I_1, \ldots, I_n) = e(I)\).

Let us suppose that the residue field \(k = R/m\) is infinite. Let \(a_{i1}, \ldots, a_{is_i}\) be a generating system of \(I_i\), where \(s_i \geq 1\), for \(i = 1, \ldots, n\). Let \(s = s_1 + \cdots + s_n\). We say that a property holds for sufficiently general elements of \(I_1 \oplus \cdots \oplus I_n\) if there exists a non-empty Zariski-open set \(U\) in \(k^s\) verifying that the said property holds for all elements \((g_1, \ldots, g_n) \in I_1 \oplus \cdots \oplus I_n\) such that \(g_i = \sum_j u_{ij}a_{ij}\), \(i = 1, \ldots, n\) and the image of \((u_{11}, \ldots, u_{1s_1}, \ldots, u_{ns_n})\) in \(k^s\) lies in \(U\).

By virtue of a result of Rees (see [15] or [8, p. 335]), if the ideals \(I_1, \ldots, I_n\) have finite colength and \(R/m\) is infinite, then the mixed multiplicity of \(I_1, \ldots, I_n\) is obtained as \(e(I_1, \ldots, I_n) = e(g_1, \ldots, g_n)\), for a sufficiently general element \((g_1, \ldots, g_n) \in I_1 \oplus \cdots \oplus I_n\).

Let us denote by \(\mathcal{O}_n\) the ring of analytic function germs \((\mathbb{C}^n, 0) \to \mathbb{C}\). Let \(g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)\) be a complex analytic map germ such that \(g^{-1}(0) = \{0\}\) and let \(g_1, \ldots, g_n\) denote the component functions of \(g\). We recall that \(e(I) = \dim_\mathbb{C} \mathcal{O}_n/I\), where \(I\) is the ideal of \(\mathcal{O}_n\) generated by \(g_1, \ldots, g_n\). It turns out that this number is equal to the geometric multiplicity of \(g\) (see [11, p. 258] or [13]).

Now we show the definition of a number associated to a family of ideals that generalizes the notion of mixed multiplicity. This number is fundamental in the results of this paper.

We denote by \(\mathbb{Z}_+\) the set of non-negative integers. Let \(a \in \mathbb{Z}\), we denote by \(\mathbb{Z}_{\geq a}\) the set of integers \(z \geq a\).
Proposition 2.2. [3, p. 393] Let \((R, m)\) be a Noetherian local ring of dimension \(n\). Let \(I_1, \ldots, I_n\) be ideals of \(R\). Then we define the Rees’ mixed multiplicity of \(I_1, \ldots, I_n\) as

\[
\sigma(I_1, \ldots, I_n) = \max_{r \in \mathbb{Z}_+} e(I_1 + m^r, \ldots, I_n + m^r),
\]

when the number on the right hand side is finite. If the set of integers \(\{e(I_1+m^r, \ldots, I_n+m^r) : r \in \mathbb{Z}_+\}\) is non-bounded then we set \(\sigma(I_1, \ldots, I_n) = \infty\).

We remark that if \(I_i\) is an ideal of finite colength, for all \(i = 1, \ldots, n\), then \(\sigma(I_1, \ldots, I_n) = e(I_1, \ldots, I_n)\). The next proposition characterizes the finiteness of \(\sigma(I_1, \ldots, I_n)\).

Proposition 2.2. [3, p. 393] Let \(I_1, \ldots, I_n\) be ideals of a Noetherian local ring \((R, m)\) such that the residue field \(k = R/m\) is infinite. Then \(\sigma(I_1, \ldots, I_n) < \infty\) if and only if there exist elements \(g_i \in I_i\), for \(i = 1, \ldots, n\), such that \((g_1, \ldots, g_n)\) has finite colength. In this case, we have that \(\sigma(I_1, \ldots, I_n) = e(g_1, \ldots, g_n)\) for sufficiently general elements \((g_1, \ldots, g_n) \in I_1 \oplus \cdots \oplus I_n\).

Remark 2.3. It is worth pointing out that, if \(I_1, \ldots, I_n\) is a set of ideals of \(R\) such that \(\sigma(I_1, \ldots, I_n) < \infty\), then \(I_1 + \cdots + I_n\) is an ideal of finite colength. Obviously the converse is not true.

The following result will be useful in subsequent sections.

Lemma 2.4. [4, p. 392] Let \((R, m)\) be a Noetherian local ring of dimension \(n \geq 1\). Let \(J_1, \ldots, J_n\) be ideals of \(R\) such that \(\sigma(J_1, \ldots, J_n) < \infty\). Let \(I_1, \ldots, I_n\) be ideals of \(R\) such that \(J_i \subseteq I_i\), for all \(i = 1, \ldots, n\). Then \(\sigma(I_1, \ldots, I_n) < \infty\) and

\[
\sigma(J_1, \ldots, J_n) \geq \sigma(I_1, \ldots, I_n).
\]

Now we recall some basic definitions. Let us fix a coordinate system \(x_1, \ldots, x_n\) in \(\mathbb{C}^n\). If \(k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n\), we will denote the monomial \(x_1^{k_1} \cdots x_n^{k_n}\) by \(x^k\). If \(h \in \mathcal{O}_n\) and \(h = \sum_k a_k x^k\) denotes the Taylor expansion of \(h\) around the origin, then the support of \(h\) is the set \(\text{supp}(h) = \{ k \in \mathbb{Z}_+^n : a_k \neq 0 \}\). If \(h \neq 0\), the Newton polyhedron of \(h\), denoted by \(\Gamma_+(h)\), is the convex hull of the set \(\{ k + v : k \in \text{supp}(h), v \in \mathbb{R}_+^n \}\). If \(h = 0\), then we set \(\Gamma_+(h) = \emptyset\). If \(I\) is an ideal of \(\mathcal{O}_n\) and \(g_1, \ldots, g_s\) is a generating system of \(I\), then we define the Newton polyhedron of \(I\) as the convex hull of \(\Gamma_+(g_1) \cup \cdots \cup \Gamma_+(g_r)\). It is easy to check that the definition of \(\Gamma_+(I)\) does not depend on the chosen generating system of \(I\). We say that \(I\) is a monomial ideal of \(\mathcal{O}_n\) when \(I\) admits a generating system formed by monomials.

Definition 2.5. Let \(I_1, \ldots, I_n\) be monomial ideals of \(\mathcal{O}_n\) such that \(\sigma(I_1, \ldots, I_n) < \infty\). Then we denote by \(\mathcal{S}(I_1, \ldots, I_n)\) the family of those maps \(g = (g_1, \ldots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)\) for which \(g^{-1}(0) = \{0\}\), \(g_i \in I_i\), for all \(i = 1, \ldots, n\), and \(\sigma(I_1, \ldots, I_n) = e(g_1, \ldots, g_n)\), where \(e(g_1, \ldots, g_n)\) stands for the multiplicity of the ideal of \(\mathcal{O}_n\) generated by \(g_1, \ldots, g_n\). The elements of \(\mathcal{S}(I_1, \ldots, I_n)\) are characterized in [3, Theorem 3.10].

We denote by \(\mathcal{S}_0(I_1, \ldots, I_n)\) the set formed by the maps \(g = (g_1, \ldots, g_n) \in \mathcal{S}(I_1, \ldots, I_n)\) such that \(\Gamma_+(g_i) = \Gamma_+(I_i)\), for all \(i = 1, \ldots, n\).
3. The Lojasiewicz exponent of a set of ideals

In this section we introduce some results concerning the notion of Lojasiewicz exponent of a set of ideals in a Noetherian ring. These results will be applied in the next section.

Let \( I_1, \ldots, I_n \) be ideals of a local ring \((R,m)\) such that \( \sigma(I_1, \ldots, I_n) < \infty \). Then we define

\[
 r(I_1, \ldots, I_n) = \min \{ r \in \mathbb{Z}_+ : \sigma(I_1, \ldots, I_n) = e(I_1 + m^r, \ldots, I_n + m^r) \}.
\]

**Theorem 3.1.** [4, p. 398] Let \( I_1, \ldots, I_n \) be monomial ideals of \( \mathcal{O}_n \) such that \( \sigma(I_1, \ldots, I_n) \) is finite. If \( g \in \mathcal{S}_0(I_1, \ldots, I_n) \), then \( L_0(g) \) depends only on \( I_1, \ldots, I_n \) and it is given by

\[
 L_0(g) = \min_{s \geq 1} \frac{r(I^s_1, \ldots, I^s_n)}{s}.
\]

By the proof of the above theorem it is concluded that the infimum of the sequence \( \{ \frac{r(I^s_1, \ldots, I^s_n)}{s} \}_{s \geq 1} \) is actually a minimum. Theorem 3.1 motivates the following definition.

**Definition 3.2.** Let \((R,m)\) be a Noetherian local ring of dimension \( n \). Let \( I_1, \ldots, I_n \) be ideals of \( R \). Let us suppose that \( \sigma(I_1, \ldots, I_n) < \infty \). We define the *Lojasiewicz exponent of \( I_1, \ldots, I_n \)* as

\[
 L_0(I_1, \ldots, I_n) = \inf_{s \geq 1} \frac{r(I^s_1, \ldots, I^s_n)}{s}.
\]

As we will see in Lemma 3.3, we have that \( r(I^s_1, \ldots, I^s_n) \leq sr(I_1, \ldots, I_n) \), for all \( s \in \mathbb{Z}_{\geq 1} \). Therefore \( L_0(I_1, \ldots, I_n) \leq r(I_1, \ldots, I_n) \).

We can extend Definition 2.1 by replacing the maximal ideal \( m \) by an arbitrary ideal of finite colength, but the resulting number is the same. That is, under the hypothesis of Definition 2.1, let us denote by \( J \) an ideal of \( R \) of finite colength and let us suppose that \( \sigma(I_1, \ldots, I_n) < \infty \). Then we define

\[
 \sigma_J(I_1, \ldots, I_n) = \max_{r \in \mathbb{Z}_+} e(I_1 + J^r, \ldots, I_n + J^r).
\]

An easy computation reveals that \( \sigma_J(I_1, \ldots, I_n) = \sigma(I_1, \ldots, I_n) \). We also define

\[
 r_J(I_1, \ldots, I_n) = \min \{ r \in \mathbb{Z}_+ : \sigma(I_1, \ldots, I_n) = e(I_1 + J^r, \ldots, I_n + J^r) \}.
\]

Let \( I \) be an ideal of \( R \) of finite colength. Then we denote by \( r_J(I) \) the number \( r_J(I, \ldots, I) \), where \( I \) is repeated \( n \) times. We deduce from the Rees’ multiplicity theorem that, if \( R \) is quasi-unmixed, then \( r_J(I) = \min \{ r \geq 1 : J^r \subseteq T \} \).

**Lemma 3.3.** Let \((R,m)\) be a Noetherian local ring of dimension \( n \). Let \( I_1, \ldots, I_n \) be ideals of \( R \) such that \( \sigma(I_1, \ldots, I_n) < \infty \) and let \( J \) be an \( m \)-primary ideal. Then

\[
 r_J(I^s_1, \ldots, I^s_n) \leq sr_J(I_1, \ldots, I_n)
\]

\[
 r_J(I_1, \ldots, I_n) \geq \frac{1}{s} r_J(I_1, \ldots, I_n)
\]

for all integer \( s \geq 1 \).
Remark 3.5. Let $\epsilon > 0$ and an integer $n$ is repeated $m$ times.

Proof. For the first inequality, set $r = r_J(I_1, \ldots, I_n)$. Thus $\sigma(I_1, \ldots, I_n) = e(I_1 + J^r, \ldots, I_n + J^r)$. It is enough to prove that $\sigma(I_1^n, \ldots, I_n^n) = e(I_1^n + J^{rs}, \ldots, I_n^n + J^{rs})$:

\[
e(I_1^n + J^{rs}, \ldots, I_n^n + J^{rs}) = e(I_1^n + J^{rs}, \ldots, I_n^n + J^{rs}) = e((I_1 + J^r)^s, \ldots, (I_n + J^r)^s) = e(I_1 + J^r, \ldots, I_n + J^r) = s^n \sigma(I_1, \ldots, I_n) = \sigma(I_1^n, \ldots, I_n^n),\]

where last equality comes from [4, Lemma 2.6].

The second inequality comes directly from the definition of $r_J(I_1, \ldots, I_n)$. \qed

It is easy to find examples of ideals $I$ and $J$ such that $r_J(I_1, \ldots, I_n) \neq r(I_1, \ldots, I_n)$ in general. This fact motivates the following definition.

Definition 3.4. Let $(R, m)$ be a Noetherian local ring of dimension $n$. Let $I_1, \ldots, I_n$ be ideals of $R$ such that $\sigma(I_1, \ldots, I_n) < \infty$. Let $J$ be an $m$-primary ideal of $R$. We define the Lojasiewicz exponent of $I_1, \ldots, I_n$ with respect to $J$, denoted by $L_J(I_1, \ldots, I_n)$, as

\[
L_J(I_1, \ldots, I_n) = \inf_{s \geq 1} \frac{r_J(I_1^n, \ldots, I_n^n)}{s}.
\]

If $I$ is an $m$-primary ideal of $R$, then we denote by $L_J(I)$ the number $L_J(I_1, \ldots, I_n)$, where $I$ is repeated $n$ times.

Remark 3.5. Under the conditions of the previous definition, we observe that $L_J(I_1, \ldots, I_n)$ can be seen as a limit inferior:

\[
L_J(I_1, \ldots, I_n) = \liminf_{s \to \infty} \frac{r_J(I_1^n, \ldots, I_n^n)}{s}.
\]

Set $\ell = L_J(I_1, \ldots, I_n)$. In order to prove the equality above, it is enough to see that for all $\epsilon > 0$ and all $p \in \mathbb{Z}_+$, there exists an integer $m \geq p$ such that

\[
\frac{r_J(I_1^m, \ldots, I_n^m)}{m} \leq \ell + \epsilon.
\]

Let us fix an $\epsilon > 0$ and an integer $p \in \mathbb{Z}_+$. By definition, there exists $q \in \mathbb{Z}_+$ such that

\[
\frac{r_J(I_1^q, \ldots, I_n^q)}{q} \leq \ell + \epsilon.
\]

Let $s \in \mathbb{Z}_+$ such that $sq \geq p$. Then, from Lemma 3.3 we obtain that

\[
\frac{r_J(I_1^{sq}, \ldots, I_n^{sq})}{sq} \leq \frac{r_J(I_1^n, \ldots, I_n^n)}{q} \leq \ell + \epsilon.
\]

If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ denotes an analytic map germ such that $g^{-1}(0) = \{0\}$ and $J$ is an ideal of $\mathcal{O}_n$ of finite colength, then we denote the number $L_J(I)$, where $I$ is the ideal generated by the component functions of $g$, by $L_J(g)$. A straightforward reproduction of the argument in the proof of Theorem 3.1 consisting of replacing the powers of the maximal ideal by the powers of a given ideal of finite colength leads to the following result, which is analogous to Theorem 3.1.
Theorem 3.6. Let $I_1, \ldots, I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \ldots, I_n)$ is finite and let $J$ be a monomial ideal of $\mathcal{O}_n$ of finite colength. Then the sequence $\{r_{J^s(I_1, \ldots, I_n)}\}_{s \geq 1}$ attains a minimum and if $g \in S_0(I_1, \ldots, I_n)$ then

(7) \[ \mathcal{L}_J(g) = \mathcal{L}_J(I_1, \ldots, I_n) = \min_{s \geq 1} \frac{r_{J^s(I_1, \ldots, I_n)}^s}{s}. \]

Lemma 3.7. Under the hypothesis of Lemma 3.3 we have

\[ \mathcal{L}_J(I_1^s, \ldots, I_n^s) = s\mathcal{L}_J(I_1, \ldots, I_n) \]

for all $s \in \mathbb{Z}_{\geq 1}$.

Proof. For the first equality

\[ \mathcal{L}_J(I_1^s, \ldots, I_n^s) = \inf_{p \geq 1} \frac{r_{J^p(I_1, \ldots, I_n)}}{p} = s \inf_{p \geq 1} \frac{r_{J^p(I_1, \ldots, I_n)}}{sp} \geq s\mathcal{L}_J(I_1, \ldots, I_n). \]

On the other hand, by Lemma 3.3 we obtain

\[ \inf_{p \geq 1} \frac{r_{J^p(I_1, \ldots, I_n)}}{p} \leq s \inf_{p \geq 1} \frac{r_{J^p(I_1, \ldots, I_n)}}{p} = s\mathcal{L}_J(I_1, \ldots, I_n). \]

Let us see the second equality. Applying Lemma 3.3 we have

\[ \mathcal{L}_{J^s}(I_1, \ldots, I_n) = \inf_{p \geq 1} \frac{r_{J^s(I_1, \ldots, I_n)}}{p} \geq \frac{1}{s} \inf_{p \geq 1} \frac{r_{J^s(I_1, \ldots, I_n)}}{p} = \frac{1}{s} \mathcal{L}_J(I_1, \ldots, I_n). \]

Let us denote the number $r_{J^s}(I_1^p, \ldots, I_n^p)$ by $r_p$, for all $p \geq 1$. Then

\[ \sigma(I_1^p, \ldots, I_n^p) > e(I_1^p + J^{s(r_p-1)}, \ldots, I_n^p + J^{s(r_p-1)}). \]

In particular

\[ r_{J^s}(I_1^p, \ldots, I_n^p) > s(r_p - 1) \]

for all $p \geq 1$. Dividing the previous inequality by $p$ and taking $\liminf_{p \to \infty}$ we obtain by Remark 3.5, that

\[ \mathcal{L}_J(I_1, \ldots, I_n) = \liminf_{p \to \infty} \frac{r_{J^s(I_1, \ldots, I_n)}}{p} \geq \liminf_{p \to \infty} \left( \frac{r_p - 1}{p} \right) = \mathcal{L}_{J^s}(I_1, \ldots, I_n). \]

\[ \square \]

Lemma 3.8. Let $(R, m)$ be a quasi-unmixed Noetherian local ring of dimension $n$. Let $I_1, \ldots, I_n$ be ideals of $R$ such that $\sigma(I_1, \ldots, I_n) < \infty$. If $J_1, J_2$ are $m$-primary ideals of $R$ then

\[ \mathcal{L}_{J_1}(I_1, \ldots, I_n) \leq \mathcal{L}_{J_1}(J_2)\mathcal{L}_{J_2}(I_1, \ldots, I_n). \]
Proof. By (5) we have that
\[ r_{J_1}(J_2) = \min \{ r \geq 1 : e(J_2) = e(J_2 + J_1^r) \} . \]

Given an integer \( r \geq 1 \), the condition \( e(J_2) = e(J_2 + J_1^r) \) is equivalent to saying that \( J_1^r \subseteq J_2 \), by the Rees’ multiplicity theorem (see [8, p. 222]). Therefore, an elementary computation shows that
\[ r_{J_1}(I_1, \ldots, I_n) \leq r_{J_1}(J_2)r_{J_2}(I_1, \ldots, I_n) . \]

By the generality of the previous inequality, we have
\[ r_{J_1}(I_1^p, \ldots, I_n^p) \leq r_{J_1}(J_2^p)r_{J_2}(I_1^p, \ldots, I_n^p) \]
for all integers \( p, s \geq 1 \). The inequality (9) shows that
\[ \mathcal{L}_{J_1}(I_1, \ldots, I_n) = \inf_{s \geq 1} \frac{r_{J_1}(I_1^s, \ldots, I_n^s)}{s} \leq \inf_{s \geq 1} \frac{r_{J_1}(J_2^s)r_{J_2}(I_1^s, \ldots, I_n^s)}{s} = \]
\[ = r_{J_1}(J_2^p)\mathcal{L}_{J_2}(I_1, \ldots, I_n) = r_{J_1}(J_2^p)\frac{1}{p}\mathcal{L}_{J_2}(I_1, \ldots, I_n) \]
for all integer \( p \geq 1 \), where the last equality comes from Lemma 3.7. Then
\[ \mathcal{L}_{J_1}(I_1, \ldots, I_n) \leq \left( \inf_{p \geq 1} \frac{r_{J_1}(J_2^p)}{p} \right) \mathcal{L}_{J_2}(I_1, \ldots, I_n) = \mathcal{L}_{J_1}(J_2)\mathcal{L}_{J_2}(I_1, \ldots, I_n) . \]

We recall the following two results, which will be applied in the next section.

**Proposition 3.9.** [4] Let \((R, m)\) be a Noetherian local ring of dimension \( n \). For each \( i = 1, \ldots, n \) let us consider ideals \( I_i \) and \( J_i \) such that \( I_i \subseteq J_i \). Let suppose that \( \sigma(I_1, \ldots, I_n) < \infty \) and that \( \sigma(I_1, \ldots, I_n) = \sigma(J_1, \ldots, J_n) \). Then
\[ \mathcal{L}_0(I_1, \ldots, I_n) \leq \mathcal{L}_0(J_1, \ldots, J_n) . \]

Let us denote the canonical basis in \( \mathbb{R}^n \) by \( e_1, \ldots, e_n \).

**Proposition 3.10.** [2] Let \( J \) be an ideal of finite colength of \( \mathcal{O}_n \) and set \( r_i = \min \{ r : re_i \in \Gamma_+(J) \} \), for all \( i = 1, \ldots, n \). Then
\[ \max\{r_1, \ldots, r_n\} \leq \mathcal{L}_0(J) \]
and equality holds if \( J \) is a monomial ideal.

### 4. Weighted homogeneous filtrations

Let us fix a vector \( w = (w_1, \ldots, w_n) \in \mathbb{Z}_{\geq 1}^n \). We will usually refer to \( w \) as the vector of weights. Let \( h \in \mathcal{O}_n, h \neq 0 \), the degree of \( h \) with respect to \( w \), or \( w \)-degree of \( h \), is defined as
\[ d_w(h) = \min \{ \langle k, w \rangle : k \in \text{supp}(h) \} , \]
where \( \langle \cdot, \cdot \rangle \) stands for the usual scalar product. In particular, if \( x_1, \ldots, x_n \) denotes a system of coordinates in \( \mathbb{C}^n \) and \( x_1^{k_1} \cdots x_n^{k_n} \) is a monomial in \( \mathcal{O}_n \), then \( d_w(x_1^{k_1} \cdots x_n^{k_n}) = w_1k_1 + \)
\[ \cdots + w_n k_n. \] By convention, we set \( d_w(0) = +\infty. \) If \( h \in \mathcal{O}_n \) and \( h = \sum_k a_k x^k \) is the Taylor expansion of \( h \) around the origin, then we define the principal part of \( h \) with respect to \( w \) as the polynomial given by the sum of those terms \( a_k x^k \) such that \( \langle k, w \rangle = d_w(h) \). We denote this polynomial by \( p_w(h) \).

**Definition 4.1.** We say that a function \( h \in \mathcal{O}_n \) is weighted homogeneous of degree \( d \) with respect to \( w \) if \( \langle k, w \rangle = d, \) for all \( k \in \text{supp}(h) \). The function \( h \) is said to be semi-weighted homogeneous of degree \( d \) with respect to \( w \) when \( p_w(h) \) has an isolated singularity at the origin. Note that \( p_w(h) \) is weighted homogeneous with respect to \( w \).

It is well-known that, if \( h \) is a semi-weighted homogeneous function, then \( h \) has an isolated singularity at the origin and that \( h \) and \( p_w(h) \) have the same Milnor number (see for instance [1, §12]). Let \( g = (g_1, \ldots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) be an analytic map germ, let us denote the map \( (p_w(g_1), \ldots, p_w(g_n)) \) by \( p_w(g) \). The map \( g \) is said to be semi-weighted homogeneous with respect to \( w \) when \( (p_w(g))^{-1}(0) = \{0\} \).

If \( I \) is an ideal of \( \mathcal{O}_n \), then we define the degree of \( I \) with respect to \( w \), or \( w \)-degree of \( I \), as

\[
d_w(I) = \min \{d_w(h) : h \in I \}.
\]

If \( g_1, \ldots, g_r \) constitutes a generating system of \( I \), then it is straightforward to see that \( d_w(I) = \min \{d_w(g_1), \ldots, d_w(g_r) \} \).

Let \( r \in \mathbb{Z}_+ \), then we denote by \( \mathcal{B}_r \) the set of all \( h \in \mathcal{O}_n \) such that \( d_w(h) \geq r \) (therefore \( 0 \in \mathcal{B}_r \)). We observe that

(a) \( \mathcal{B}_r \) is an integrally closed monomial ideal of finite colength, for all \( r \geq 1 \);

(b) \( \mathcal{B}_r \mathcal{B}_s \subseteq \mathcal{B}_{r+s} \), \( r, s \geq 1 \);

(c) \( \mathcal{B}_0 = \mathcal{O}_n \).

The family of ideals \( \{\mathcal{B}_r\}_{r \geq 1} \) is called the weighted homogeneous filtration induced by \( w \). We denote by \( A_r \) the ideal of \( \mathcal{O}_n \) generated by the monomials \( x^k \) such that \( d_w(x^k) = r \). If there is not any monomial \( x^k \) such that \( d_w(x^k) = r \) then we set \( A_r = 0 \). Given an integer \( r \geq 1 \), we observe that \( A_{r-s} \subset \mathcal{B}_r \) and that \( A_r \neq \mathcal{B}_r \) in general. Moreover it follows easily that \( A_r = \mathcal{B}_r \) if and only if \( A_r \) is an ideal of finite colength of \( \mathcal{O}_n \).

If \( r_1, \ldots, r_n \in \mathbb{Z}_{\geq 1} \), then it is not true in general that \( \sigma(A_{r_1}, \ldots, A_{r_n}) < \infty \), even if \( A_{r_i} \neq 0 \), for all \( i = 1, \ldots, n \). However \( \sigma(\mathcal{B}_{r_1}, \ldots, \mathcal{B}_{r_n}) < \infty \), since \( \mathcal{B}_{r_i} \) has finite colength, for all \( i = 1, \ldots, n \). For instance, let us consider the vector \( w = (3, 1) \). Then we have

\[
A_4 = \langle xy, y^4 \rangle, \quad A_5 = \langle xy^2, y^5 \rangle.
\]

We observe that the ideal \( A_4 + A_5 \) does not have finite colength, therefore \( \sigma(A_4, A_5) \) is not finite (see Remark 2.3).

**Proposition 4.2.** Let \( r_1, \ldots, r_n \in \mathbb{Z}_{\geq 1} \). If \( \sigma(A_{r_1}, \ldots, A_{r_n}) < \infty \) then \( \sigma(\mathcal{B}_{r_1}, \ldots, \mathcal{B}_{r_n}) < \infty \) and

\[
\sigma(A_{r_1}, \ldots, A_{r_n}) = \sigma(\mathcal{B}_{r_1}, \ldots, \mathcal{B}_{r_n}) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n}.
\]
Then the result follows. □

Proof. By Proposition 2.2, there exists a sufficiently general element \( (h_1, \ldots, h_n) \in \mathcal{B}_{r_1} \oplus \cdots \oplus \mathcal{B}_{r_n} \) such that

\[
\sigma(\mathcal{B}_{r_1}, \ldots, \mathcal{B}_{r_n}) = e(h_1, \ldots, h_n).
\]

The condition \( \sigma(\mathcal{A}_{r_1}, \ldots, \mathcal{A}_{r_n}) < \infty \) implies that \( \mathcal{A}_{r_i} \neq 0 \), for all \( i = 1, \ldots, n \). The ideal \( \mathcal{A}_{r_i} \) is generated by the monomials of \( w \)-degree \( r_i \), for all \( i = 1, \ldots, n \), then \( h_i \) can be written as \( h_i = g_i + g'_i \), for all \( i = 1, \ldots, n \), where \( (g_1, \ldots, g_n) \) is a sufficiently general element of \( \mathcal{A}_{r_1} \oplus \cdots \oplus \mathcal{A}_{r_n} \) and \( g'_i \in \mathcal{O}_n \) verifies that \( d_w(g'_i) > r_i \), for all \( i = 1, \ldots, n \). Therefore \( p_w(h_i) = g_i \), for all \( i = 1, \ldots, n \).

Let \( g \) denote the map \( (g_1, \ldots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \). The condition \( \sigma(\mathcal{A}_{r_1}, \ldots, \mathcal{A}_{r_n}) < \infty \) and the genericity of \( g \) imply that \( g \) is finite, that is, \( g^{-1}(0) = \{0\} \) and \( \sigma(\mathcal{A}_{r_1}, \ldots, \mathcal{A}_{r_n}) = e(g_1, \ldots, g_n) \). Consequently the map \( h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) is semi-weighted homogeneous with respect to \( w \). By [1, §12] (see also [7] for a more general phenomenon), this implies that

\[
e(h_1, \ldots, h_n) = e(g_1, \ldots, g_n) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n}.
\]

Then the result follows.

Definition 4.3. Let \( J_1, \ldots, J_n \) be a family of ideals of \( \mathcal{O}_n \) and let \( r_i = d_w(J_1) \), for all \( i = 1, \ldots, n \). We say that \( J_1, \ldots, J_n \) admits a \( w \)-matching if there exists a permutation \( \tau \) of \( \{1, \ldots, n\} \) and an index \( i_0 \in \{1, \ldots, n\} \) such that

(a) \( w_{i_0} = \min\{w_1, \ldots, w_n\} \),
(b) \( r_{\tau(i_0)} = \max\{r_1, \ldots, r_n\} \) and
(c) the pure monomial \( x_i^{r_{\tau(i)}/w_i} \) belongs to \( J_{\tau(i)} \), for all \( i \neq i_0 \).

Remark 4.4. If \( r \in \mathbb{Z}_{\geq 1} \) then we observe that \( \mathcal{A}_r \) has finite colength if and only if \( w_i \) divides \( r \), for all \( i = 1, \ldots, n \). Let \( r_1, \ldots, r_n \in \mathbb{Z}_{\geq 1} \) such that \( \mathcal{A}_{r_i} \) has finite colength, for all \( i = 1, \ldots, n \). Then condition (c) of the above definition is not a restriction in this case and therefore \( \mathcal{A}_{r_1}, \ldots, \mathcal{A}_{r_n} \) admits a \( w \)-matching.

Let us consider the case \( n = 2 \) of the previous definition. Therefore, let \( r_1, r_2 \in \mathbb{Z}_{\geq 1} \) with \( r_1 \succ r_2 \) and let us suppose that \( w_1 < w_2 \). Let \( J_1, J_2 \) be ideals of \( \mathcal{O}_2 \) such that \( d_w(J_i) = r_i \), \( i = 1, 2 \). Then \( J_1, J_2 \) admits a \( w \)-matching if and only if \( y^{r_2/w_2} \in J_2 \).

Example 4.5. Set \( w = (1, 2, 3, 4) \) and \( r_1 = 10, r_2 = 9, r_3 = 8, r_4 = 6 \). The family of ideals given by

\[
J_1 = \langle x_1 x_3^3 \rangle, \quad J_2 = \langle x_3^3, x_1 x_4^2 \rangle, \quad J_3 = \langle x_4^2, x_1^2 x_3^2 \rangle, \quad J_4 = \langle x_2^3, x_2 x_4 \rangle,
\]

admits a \( w \)-matching. Observe that here \( i_0 = 1 \) and the permutation \( \tau \) is defined by \( \tau(1) = 1, \tau(2) = 4, \tau(3) = 2, \tau(4) = 3 \).

Let us observe that, if \( J_1, \ldots, J_n \) admits a \( w \)-matching, then it is always possible to reorder the ideals \( J_i \) in such a way that \( \tau(i_0) = i_0 \), and therefore one could restrict to the case \( \tau = \text{id} \) after a permutation of the ideals \( J_i \). But the permutation \( \tau \) is specially relevant when considering ideals coming from the gradient of a function \( f \) (see Example 4.12).
Lemma 4.6. Let \( r_1, \ldots, r_n \in \mathbb{Z}_{\geq 1} \) and let \( I_1, \ldots, I_n \) be monomial ideals of \( \mathcal{O}_n \) such that \( d_w(I_i) = r_i \), for all \( i = 1, \ldots, n \), and \( \sigma(I_1, \ldots, I_n) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n} \). Let \( J \) be an ideal of \( \mathcal{O}_n \) such that \( J = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \), for some \( r \geq 1 \), where \( \alpha_i = \frac{r - 1}{w_i} \) and \( w = w_1 \cdots w_n \). Then

\[
e(I_1 + J, \ldots, I_n + J) = \min\{r_1, \overline{wr}\} \cdots \min\{r_n, \overline{wr}\}
\]

where the first equality comes from [1, §12] (see also [6, Theorem 3.3]). Let \( e(I_1 + J, \ldots, I_n + J) = e(I_1 + J), \ldots, I_n + J) \).

By Proposition 2.2, there exist an element \((g_1, \ldots, g_n) \in I_1 + \cdots + I_n \) such that \( d_w(g_i) = r_i \), for all \( i = 1, \ldots, n \), and

\[
e(g_1, \ldots, g_n) = \sigma(I_1, \ldots, I_n) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n}.
\]

Let us denote by \( R \) the quotient ring \( \mathcal{O}_n/(p_w(g_1), \ldots, p_w(g_n)) \) and let \( H \) denote the ideal of \( \mathcal{O}_n \) generated by \( x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} \).

Relation (13) implies, by [6, Theorem 3.3], that the ideal generated by \( p_w(g_1), \ldots, p_w(g_n) \) has finite colength. In particular, these elements form a regular sequence and then \( \dim(R) = n - s \). Hence there exists a sufficiently general element \((h_1, \ldots, h_{n-s}) \in H + \cdots + H \) such that the images of the \( h_i \) in \( R \) generate a reduction of the image of \( J \) in \( R \), by the theorem of existence of reductions (see [8, p. 166]). In particular, the ideal \( K = \langle p_w(g_1), \ldots, p_w(g_n), h_1, \ldots, h_{n-s} \rangle \) has finite colength.

Since \( h_i \) is a generic \( \mathcal{O} \)-linear combination of \( x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} \), for all \( i = 1, \ldots, n \), we have that \( p_w(h_i) = h_i \), for all \( i = 1, \ldots, n \). Then \( K = \langle p_w(g_1), \ldots, p_w(g_n), p_w(h_1), \ldots, p_w(h_{n-s}) \rangle \).

Therefore

\[
e(K) = \frac{r_1 \cdots r_n (\overline{wr})^{n-s}}{w_1 \cdots w_n} = \frac{\min\{r_1, \overline{wr}\} \cdots \min\{r_n, \overline{wr}\}}{w},
\]

where the first equality comes from [1, §12] (see also [6, Theorem 3.3]).

Since \( I_i \) is a monomial ideal, for all \( i = 1, \ldots, n \), we have that \( p_w(g_i) \in I_i \), for all \( i = 1, \ldots, n \). In particular we have \( e(K) \geq e(I_1 + J, \ldots, I_n + J) \), by Lemma 2.4. Then

\[
e(K) \geq e(I_1 + H, \ldots, I_n + H) \geq \frac{\min\{r_1, \overline{wr}\} \cdots \min\{r_n, \overline{wr}\}}{w},
\]

where the second inequality follows from [6, Theorem 3.3].

The hypothesis \( J = \overline{J} \) implies that

\[
e(I_1 + J, \ldots, I_n + J) = e(I_1 + H, \ldots, I_n + H).
\]

Then the result follows by joining (14), (15) and (16).

\[\square\]

Theorem 4.7. Let \( r_1, \ldots, r_n \in \mathbb{Z}_{\geq 1} \) such that \( \sigma(A_{r_1}, \ldots, A_{r_n}) < \infty \). Let \( J_1, \ldots, J_n \) be a set of ideals of \( \mathcal{O}_n \) with \( d_w(J_i) = r_i \), for all \( i = 1, \ldots, n \), and \( \sigma(J_1, \ldots, J_n) = \sigma(A_{r_1}, \ldots, A_{r_n}) \).

Then

\[
\mathcal{L}_0(J_1, \ldots, J_n) \leq \mathcal{L}_0(B_{r_1}, \ldots, B_{r_n}) \leq \frac{\max\{r_1, \ldots, r_n\}}{\min\{w_1, \ldots, w_n\}}
\]
and the above inequalities turn into equalities if \( J_1, \ldots, J_n \) admit a \( w \)-matching.

**Proof.** The condition \( \sigma(A_{r_1}, \ldots, A_{r_n}) < \infty \) and the equality \( \sigma(J_1, \ldots, J_n) = \sigma(A_{r_1}, \ldots, A_{r_n}) \) imply that

\[
\sigma(J_1, \ldots, J_n) = \sigma(B_{r_1}, \ldots, B_{r_n}) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n},
\]

by Proposition 4.2. Then we can apply Proposition 3.9 to deduce that

\[
\mathcal{L}_0(J_1, \ldots, J_n) \leq \mathcal{L}_0(B_{r_1}, \ldots, B_{r_n}).
\]

Let us denote max\( \{r_1, \ldots, r_n\} \) and min\( \{w_1, \ldots, w_n\} \) by \( p \) and \( q \), respectively. Let us see that \( \mathcal{L}_0(B_{r_1}, \ldots, B_{r_n}) \leq \frac{p}{q} \).

Let us denote by \( \overline{w} \) the product \( w_1 \cdots w_n \) and let us consider the ideal \( J = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \), where \( \alpha_i = \frac{p}{w_i} \), for all \( i = 1, \ldots, n \). Since \( \sigma(B_{r_1}, \ldots, B_{r_n}) < \infty \), it makes sense to compute the number \( r_J(B_{r_1}^s, \ldots, B_{r_n}^s) \), for all \( s \geq 1 \):

\[
r_J(B_{r_1}^s, \ldots, B_{r_n}^s) = \min \left\{ r \geq 1 : r \sigma(B_{r_1}^s, \ldots, B_{r_n}^s) = e(B_{r_1}^s + J', \ldots, B_{r_n}^s + J') \right\}
\]

\[
= \min \left\{ r \geq 1 : \frac{sr_1 \cdots sr_n}{\overline{w}} = \frac{\min\{sr_1, \overline{w}r\} \cdots \min\{sr_n, \overline{w}r\}}{\overline{w}} \right\}
\]

\[
= \min \left\{ r \geq 1 : r \geq \frac{\max\{sr_1, \ldots, sr_n\}}{\overline{w}} \right\} = \left\lceil \frac{\max\{sr_1, \ldots, sr_n\}}{\overline{w}} \right\rceil,
\]

where \( \lceil a \rceil \) denotes the least integer greater than or equal to \( a \), for any \( a \in \mathbb{R} \), and the second equality is a direct application of Lemma 4.6. Therefore

\[
\mathcal{L}_J(B_{r_1}, \ldots, B_{r_n}) = \inf_{s \geq 1} \frac{r_J(B_{r_1}^s, \ldots, B_{r_n}^s)}{s} \leq \inf_{a \geq 1} \frac{r_J(B_{r_1}^{a\overline{w}}, \ldots, B_{r_n}^{a\overline{w}})}{a\overline{w}}
\]

\[
= \inf_{a \geq 1} \frac{1}{a\overline{w}} \left( \max\{a\overline{w}r_1, \ldots, a\overline{w}r_n\} \right) = \max\{r_1, \ldots, r_n\} / \overline{w}.
\]

Moreover, by Proposition 3.10 we have

\[
\mathcal{L}_0(J) = \max\{\alpha_1, \ldots, \alpha_n\} = \frac{\overline{w}}{\min\{w_1, \ldots, w_n\}},
\]

since \( J \) is a monomial ideal. Therefore, by Lemma 3.8 we obtain

\[
\mathcal{L}_0(B_{r_1}, \ldots, B_{r_n}) \leq \mathcal{L}_0(J) \mathcal{L}_J(B_{r_1}, \ldots, B_{r_n})
\]

\[
\leq \frac{\overline{w}}{\min\{w_1, \ldots, w_n\}} \max\{r_1, \ldots, r_n\} / \overline{w} = \max\{r_1, \ldots, r_n\} / \min\{w_1, \ldots, w_n\}.
\]

Let us prove that \( \mathcal{L}_0(J_1, \ldots, J_n) \geq \frac{p}{q} \) supposing that \( J_1, \ldots, J_n \) admit a \( w \)-matching. This inequality holds if and only if

\[
\frac{r(J_1^s, \ldots, J_n^s)}{s} \geq \frac{p}{q}
\]
for all \( s \geq 1 \). By Lemma 3.3 we have that \( qr(J_1^n, \ldots, J_n^n) \geq r(J_1^{sq}, \ldots, J_n^{sq}) \), for all \( s \geq 1 \). Therefore it suffices to show that
\[
(18) \quad r(J_1^{sq}, \ldots, J_n^{sq}) > sp - 1,
\]
for all \( s \geq 1 \). Let us fix an integer \( s \geq 1 \), then relation \( 18 \) is equivalent to saying that
\[
(19) \quad \sigma(J_1^{sq}, \ldots, J_n^{sq}) > e(J_1^{sq} + m^{sp-1}, \ldots, J_n^{sq} + m^{sp-1}).
\]

Since \( J_1, \ldots, J_n \) admits a \( w \)-matching, let us consider a permutation \( \tau \) of \( \{1, \ldots, n\} \) such that
(a) \( w_{i_0} = \min\{w_1, \ldots, w_n\} \),
(b) \( r_{\tau(i_0)} = \max\{r_1, \ldots, r_n\} \) and
(c) the pure monomial \( x_{r_{\tau(i_0)}}^{w_{i_0}} \) belongs to \( J_{\tau(i)} \) for all \( i \neq i_0 \).

Let us define the ideal
\[
H = \left\langle x_i^{r_{\tau(i)}}^{w_i} : i \neq i_0 \right\rangle + \langle x_{i_0}^{sp-1} \rangle.
\]
Then
\[
e(H) = e\left(x_1^{w_1}, \ldots, x_{i_0-1}^{w_{i_0-1}}, x_{i_0}^{sp-1}, x_{i_0+1}^{w_{i_0+1}}, \ldots, x_n^{w_n}\right)
= (sq)^{n-1}r_1\cdots r_n \frac{w_{i_0}}{w_1\cdots w_n}(sp - 1).
\]

Since \( x_i^{w_i} \in J_{\tau(i)} \) for all \( i \in \{1, \ldots, n\} \setminus \{i_0\} \), and \( x_{i_0}^{sp-1} \in m^{sp-1} \), we can apply Lemma 2.4 to conclude that
\[
(20) \quad e(H) \geq e(J_1^{sq} + m^{sp-1}, \ldots, J_n^{sq} + m^{sp-1}) = e(J_1^{sq} + m^{sp-1}, \ldots, J_n^{sq} + m^{sp-1}).
\]

Hence, if we prove that \( \sigma(J_1^{sq}, \ldots, J_n^{sq}) > e(H) \) then the result follows.

By [4, Lemma 2.6], we have that \( \sigma(J_1^{sq}, \ldots, J_n^{sq}) = (sq)^n \sigma(J_1, \ldots, J_n) \). Then, using the hypothesis \( \sigma(J_1, \ldots, J_n) = \sigma(\mathcal{A}_{r_1}, \ldots, \mathcal{A}_{r_n}) \) and Proposition 4.2, we obtain that
\[
(21) \quad \sigma(J_1^{sq}, \ldots, J_n^{sq}) = (sq)^n \frac{r_1\cdots r_n}{w_1\cdots w_n}.
\]

Thus, since we assume that \( r_{\tau(i_0)} = p \) and \( w_{i_0} = q \), we have that \( \sigma(J_1^{sq}, \ldots, J_n^{sq}) > e(H) \) if and only if
\[
sq > \frac{q}{p}(sp - 1),
\]
which is to say that \( sq > spq - q \). Therefore relation \( 19 \) holds for all integer \( s \geq 1 \) and consequently the inequality \( L_0(J_{r_1}, \ldots, J_{r_n}) \geq \frac{p}{q} \) follows. Thus relation \( 17 \) is proven. \( \square \)

**Remark 4.8.** We observe that the condition that \( J_1, \ldots, J_n \) admits a \( w \)-matching can not be removed from the hypothesis of the previous theorem. Let us consider now the weighted homogeneous filtration in \( \mathcal{O}_2 \) induced by the vector of weights \( w = (1, 4) \) and let \( J_1, J_2 \) be the ideals of \( \mathcal{O}_2 \) given by \( J_1 = \langle x^4 \rangle, J_2 = \langle y^2 \rangle \). We observe that \( d_w(x^4) = 4, d_w(y^2) = 8 \) and consequently the right hand side of \( 17 \) would lead to the conclusion that \( L_0(J_1, J_2) = 8, \)
which is not the case, since clearly $L_0(x^4, y^2) = 4$. We also observe that the system of ideals $J_1, J_2$ does not admit a $w$-matching.

In order to simplify the exposition, we need to introduce the following definition.

**Definition 4.9.** If $f \in \mathcal{O}_n$, $f(0) = 0$, then $f$ is termed convenient when $\Gamma_+(f)$ intersects each coordinate axis. Let $J_i$ denote the ideal of $\mathcal{O}_n$ generated by all monomials $x^k$ such that $k \in \Gamma_+(\partial f/\partial x_i)$, $i = 1, \ldots, n$. Let us fix a vector of weights $w = \mathbb{Z}_{\geq 1}^n$. Then we say that $f$ admits a $w$-matching when the family of ideals $J_1, \ldots, J_n$ admits a $w$-matching (see Definition 4.3).

If a function $f \in \mathcal{O}_n$ is convenient and quasi-homogeneous, then $f$ admits a $w$-matching. Observe that in this case the monomials $x^{d_i w_i}$ are in the support of $f$, for $i = 1, \ldots, n$. Then there is a pure monomial in $x_i$ belonging to the support of the partial derivative $\partial f/\partial x_i$ and one could take $\tau = \text{id}$ in the definition of $w$-matching (see Definition 4.3).

Let us fix a vector of weights $w = (w_1, \ldots, w_n) \in \mathbb{Z}_{\geq 1}^n$ and an integer $d \geq 1$. Then we denote by $\mathcal{O}(w; d)$ the set of all functions $f \in \mathcal{O}_n$ such that $f$ is semi-weighted homogeneous with respect to $w$ of degree $d$.

**Remark 4.10.** From Definition 4.3 we observe that a function $f \in \mathcal{O}(w; d)$ admits a $w$-matching if and only if $p_w(f)$ admits a $w$-matching, since the ideals $J_i$ introduced in Definition 4.9 have the same $w$-degree as the analogous ideals defined for $p_w(f)$.

**Corollary 4.11.** Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a semi-weighted homogeneous function of degree $d$ with respect to the weights $w_1, \ldots, w_n$. Then
\begin{equation}
L_0(\nabla f) \leq \frac{d - \min\{w_1, \ldots, w_n\}}{\min\{w_1, \ldots, w_n\}}
\end{equation}
and equality holds if $f$ admits a $w$-matching.

**Proof.** Let $J_i$ denote the ideal of $\mathcal{O}_n$ generated by all monomials $x^k$ such that $k \in \Gamma_+(\partial f/\partial x_i)$, $i = 1, \ldots, n$. Since $f$ has an isolated singularity at the origin (that is, the ideal $J(f)$ has finite colength) then $\sigma(J_1, \ldots, J_n) < \infty$, by Proposition 2.2. Then Theorem 3.1 shows that $L_0(\nabla f) = L_0(J_1, \ldots, J_n)$. We observe that $d_w(J_i) = d - w_i$, for all $i = 1, \ldots, n$. Then the result arises as a direct application of Theorem 4.7.

It has been proven recently by Płoski et al. [9] that equality holds in (22) for all weighted homogeneous functions $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that $f$ has an isolated singularity at the origin, under the hypothesis that $2w_i \leq d$ for all $i$.

The result of Corollary 4.11 holds for any number of variables.

**Example 4.12.** Let us consider the vector of weights $w = (1, 2, 3, 5)$ and the polynomial $f : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ given by $f(x_1, x_2, x_3, x_4) = x_3^3 - x_2^{11} x_4 + x_2^6 x_3^3 + x_1^{27}$. Then $f$ is weighted homogeneous with $w$-degree 27 and $f$ has an isolated singularity at the origin. The ideals $J_i$ introduced in Definition 4.9 are given by
\begin{align*}
J_1 &= \langle x_1^{20} \rangle & J_2 &= \langle x_2^{10} x_4, x_4^5 \rangle & J_3 &= \langle x_3^8 \rangle & J_4 &= \langle x_2^{11}, x_2 x_4^4 \rangle.
\end{align*}
Then we observe that the polynomial \( f \) admits \( w \)-matching. Here the permutation \( \tau \) of Definition 4.3 is \( \tau(1) = 1, \tau(2) = 4, \tau(3) = 3, \tau(4) = 2 \). Then it follows from Corollary 4.11 that \( L_0(\nabla f) = 26 \).

Given a vector of weights \( w = (w_1, \ldots, w_n) \) and a degree \( d \), then it is not always possible to find a weighted homogeneous function \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) of degree \( d \) with respect to \( w \) such that \( f \) admits a \( w \)-matching, as the following example shows.

**Example 4.13.** Let \( w = (1, 2, 3) \) and \( d = 16 \). Let \( f \) be a weighted homogeneous function of degree \( d \) with respect to \( w \). Let \( J_i \) denote the ideal of \( \mathcal{O}_3 \) generated by all monomials \( x^k \) such that \( k \in \Gamma_+ (\partial f / \partial x_i) \), for all \( i = 1, 2, 3 \). As a direct consequence of Definition 4.3, if \( J_1, J_2, J_3 \) admits a \( w \)-matching, then \( J_3 \) contains a pure monomial of \( x_2 \) or a pure monomial of \( x_3 \), which is impossible since \( d_w(J_3) = 13 \) and neither 2 nor 3 are divisors of 13.

However we observe that \( \mathcal{O}(w; d) \neq \emptyset \), since the function \( f(x_1, x_2, x_3) = x_1^4 + x_2^8 + x_1 x_3^5 \) belongs to \( \mathcal{O}(w; d) \).

**Proposition 4.14.** Let \( d, w_1, \ldots, w_n \) be non-negative integers such that \( w_i \) divides \( d \) for all \( i = 1, \ldots, n \). Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a weighted homogeneous function of degree \( d \) with respect to the weights \( w_1, \ldots, w_n \). Let us assume that \( f \) has an isolated singularity at the origin. Then there exists a change of coordinates \( x \) in \( (\mathbb{C}^n, 0) \) of the form \( x_i = y_i + h_i(y_1, \ldots, y_n) \), where \( h_i \) is a polynomial in \( y_1, \ldots, y_n, i = 1, \ldots, n \), such that:

1. the function \( f \circ x \) is convenient;
2. if \( h_i \neq 0 \), then the polynomial \( h_i \) is weighted homogeneous of degree \( w_i \) with respect to \( w \) and therefore \( f \circ x \) is weighted homogeneous of degree \( d \) with respect to \( w \).

**Proof.** Since \( f \) has an isolated singularity at the origin, for any \( i = 1, \ldots, n \) we can fix an index \( k_i \in \{1, \ldots, n\} \) such that \( x_i^{m_i} \) appears in the support of \( \frac{\partial f}{\partial x_k} \), where \( m_i = \frac{d-w_k}{w_i} \), which is to say that the monomial \( x_{k_i} x_i^{m_i} \) appears in the support of \( f \). Then \( w_i \) divides \( d - w_k \) and consequently \( w_i \) divides \( w_k \), since \( w_i \) divides \( d \) by assumption.

For all \( j = 1, \ldots, n \), we set \( L_j = \{i : k_i = j, i \neq j\} \). Let us define

\[
(23) \quad h_j = \begin{cases} 
\sum_{i \in L_j} a_{j,i} y_i^{w_i/w_i} & \text{if } L_j \neq \emptyset \\
0 & \text{otherwise},
\end{cases}
\]

where we suppose that \( \{a_{j,i}\}_{j,i} \) is a generic choice of coefficients in \( \mathbb{C} \). It is straightforward to see that, given an index \( j \in \{1, \ldots, n\} \) such that \( h_j \neq 0 \), the polynomial \( h_j \) is weighted homogeneous of degree \( w_j \).

Let us consider the map \( x : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0), x(y_1, \ldots, y_n) = (x_1, \ldots, x_n) \), given by

\[
x_j = y_j + h_j(y) \quad \text{for all } j = 1, \ldots, n.
\]

We conclude that \( x \) is a local biholomorphism, the function \( f \circ x \) is weighted homogeneous with respect to \( w \) of degree \( d \) and, by the genericity of the coefficients \( a_{j,i} \) in (23), the pure monomial \( y_i^{d/w_i} \) appears in the support of \( f \circ x \), for all \( i = 1, \ldots, n \). Hence the function \( f \circ x \) is convenient. \( \square \)
Example 4.15. Set \( w = (1, 2, 3, 4, 6) \) and \( d = 12 \). The polynomial \( f = x_1^3 + x_2^3 x_4 + x_3^3 + x_3^2 x_5 + x_3^2 \) is weighted homogeneous of degree 12. Let \( J_i \) denote the ideal of \( \mathcal{O}_5 \) generated by all monomials \( x^k \) such that \( k \in \Gamma_+(\partial f/\partial x_i), i = 1, \ldots, 5 \). A straightforward computation shows that

\[
J_1 = \langle x_1^3 \rangle, \quad J_2 = \langle x_2^3 x_4 \rangle, \quad J_3 = \langle x_3 x_5 \rangle, \quad J_4 = \langle x_4^3, x_4^2 \rangle, \quad J_5 = \langle x_5^3, x_5 \rangle.
\]

Since the ideals \( J_2 \) and \( J_3 \) do not contain any pure monomial, the family of ideals \( \{ J_i : i = 1, \ldots, 5 \} \) does not admit a \( w \)-matching.

Following the proof of Proposition 4.14, we consider the coordinate change \( x : (\mathbb{C}^5, 0) \to (\mathbb{C}^5, 0) \), given by: \( x_1 = y_1, \ x_2 = y_2, \ x_3 = y_3, \ x_4 = y_4 + y_5^2, \ x_5 = y_5 + y_3^2 \). Let \( g = f \circ x \) and let \( J'_i \) denote the ideal of \( \mathcal{O}_5 \) generated by all monomials \( y^k \) such that \( k \in \Gamma_+ (\partial g/\partial y_i), i = 1, \ldots, 5 \). Then, as shown in that proof, the function \( g \) is convenient and therefore the family of ideals \( \{ J'_i : i = 1, \ldots, 5 \} \) admits a \( w \)-matching.

Corollary 4.16. Let \( d, w_1, \ldots, w_n \) be non-negative integers such that \( w_i \) divides \( d \) for all \( i = 1, \ldots, n \). Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a semi-weighted homogeneous function of degree \( d \) with respect to the weights \( w_1, \ldots, w_n \). Then

\[
\mathcal{L}_0 (\nabla f) = \frac{d - \min\{w_1, \ldots, w_n\}}{\min\{w_1, \ldots, w_n\}}
\]

Proof. Since \( f \) is semi-weighted homogeneous, the principal part \( p_w(f) \) has an isolated singularity at the origin. Let \( x : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) denote the analytic coordinate change obtained in Proposition 4.14 applied to \( p_w(f) \). The function \( p_w(f) \circ x \) is weighted homogeneous of degree \( d \) with respect to \( w \). Therefore

\[
p_w(f) \circ x = p_w(f \circ x),
\]

which implies that \( f \circ x \) is a semi-weighted homogeneous function. Then, by Proposition 4.14 and Remark 4.10, the function \( f \circ x \) admits a \( w \)-matching. Thus we obtain, by Corollary 4.11, that

\[
\mathcal{L}_0 (\nabla (f \circ x)) = \frac{d - \min\{w_1, \ldots, w_n\}}{\min\{w_1, \ldots, w_n\}}
\]

Then the result follows, since the local Lojasiewicz exponent is a bianalytic invariant.

We remark that in Corollary 4.16 we do not assume \( 2w_i \leq d \) as in [9]. This assumption cannot be eliminated from the main result of [9], as the following example shows. The result in 4.16 holds for any number of variables, but the assumptions are also restrictive, since we are assuming that the weights \( w_i \) divide \( d \).

Example 4.17. Let us consider the polynomial \( f \) of \( \mathcal{O}_3 \) given by \( f = x_1 x_3 + x_2^2 + x_1^2 x_2 \). We observe that \( f \) is weighted homogeneous of degree 4 with respect to the vector of weights \( w = (1, 2, 3) \). The Jacobian ideal is \( \langle x_1, x_2, x_3 \rangle \) so that \( \mathcal{L}_0(\nabla f) = 1 \neq 3 \). We remark that it is easy to check that \( f \) does not admit a \( w \)-matching.
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