Upper set monoids and length preserving morphisms*

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Abstract

Length preserving morphisms and inverse of substitutions are two well-studied operations on regular languages. Their connection with varieties generated by power monoids was established independently by Reutenauer and Straubing in 1979. More recently, an ordered version of this theory was proposed by Polák and by the authors. In this paper, we present an improved version of these results and obtain the following consequence of pure semigroup theory. Given a variety of finite ordered monoids $\mathbf{V}$, let $\mathbf{P}^{\uparrow}\mathbf{V}$ be the variety of finite ordered monoids generated by the upper set monoids of members of $\mathbf{V}$. Then $\mathbf{P}^{\uparrow}(\mathbf{P}^{\uparrow}\mathbf{V}) = \mathbf{P}^{\uparrow}\mathbf{V}$. This contrasts with the known results for the unordered case: the operator $\mathbf{PV}$ corresponding to power monoids satisfies $\mathbf{P}^3\mathbf{V} = \mathbf{P}^4\mathbf{V}$, but the varieties $\mathbf{V}$, $\mathbf{PV}$, $\mathbf{P}^2\mathbf{V}$ and $\mathbf{P}^3\mathbf{V}$ can be distinct. We also present some examples of varieties satisfying $\mathbf{P}^{\uparrow}\mathbf{V} = \mathbf{V}$.

All semigroups considered in this paper are either finite or free. In particular, we use the term *variety of monoids* for *variety of finite monoids*.

Warning. This paper introduces some change of terminology and notation, compared to the existing literature. We believe that this new terminology is an improvement over the previous one, but it is fair to warn the reader of these changes.

1 Introduction

Power monoids and power varieties (varieties of the form $\mathbf{PV}$) are the topic of numerous articles [1, 2, 3, 4, 6, 8, 11, 12, 10, 13, 14, 15, 16, 21, 22] and many other references can be found in Almeida's remarkable survey [3]. Initially, the

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study of power varieties was partly motivated by semigroup theoretic questions and partly by applications to language theory. By the way, several results on power varieties were first established by using arguments of language theory.

The key result in this direction, proved independently by Reutenauer [21] and Straubing [22] in 1979, establishes a surprising link between power varieties and two natural operations on regular languages: length preserving morphisms and inverse of substitutions. This result can be summarized as follows. Let $\mathbf{V}$ be a variety of monoids and let $\mathcal{V}$ be the corresponding variety of languages. Let also $\mathcal{PV}$ be the variety of languages corresponding to $\mathbf{PV}$. Then, for each alphabet A, $\mathcal{PV}(A^*)$ is the Boolean algebra generated by the set $\Lambda\mathcal{V}(A^*)$ of all languages of the form $\varphi(L)$, where φ is a length preserving morphism from B^* into A^* and L is a language of $\mathcal{V}(B^*)$. There is an analogous result for inverses of substitutions.

The extension of Eilenberg's variety theorem to ordered monoids [17] called for a generalization of Reutenauer's and Straubing's result to the ordered case. Such an extension was proposed by Polák [20, Theorem 4.2] and by the authors [5, 6]. First, as shown by Polák, the upper set monoid is the proper extension of the notion of power monoid to the ordered case. Let $\mathbf{P}^{\uparrow}$ be the extension of this operator to varieties of ordered monoids. Denoting by $\mathcal{P}^{\uparrow}\mathcal{V}$ the positive variety of languages corresponding to $\mathbf{P}^{\uparrow}\mathbf{V}$, it was no surprise to see that $\mathcal{P}^{\uparrow}\mathcal{V}(A^*)$ is the closure under union and intersection of $\Lambda\mathcal{V}(A^*)$.

The main result of this paper, Theorem 5.1, is an improvement of this result. It shows that $\mathcal{P}^{\uparrow}\mathcal{V}(A^*)$ is actually equal to $\Lambda\mathcal{V}(A^*)$. In particular, although $\mathcal{P}^{\uparrow}\mathcal{V}$ is closed under intersection, its description does not require this operation. As it stands, this result looks like a rather minor improvement, but this is not the case. Indeed, a major consequence of our result is that the operator $\mathbf{P}^{\uparrow}$ is idempotent. This contrasts with the corresponding result for the operator $\mathbf{P}$, which satisfies $\mathbf{P}^3 = \mathbf{P}^4$, but $\mathbf{P}^2 \neq \mathbf{P}^3$.

In Section 6, we present a list of varieties of ordered monoids satisfying $\mathbf{P}^{\uparrow}\mathbf{V} = \mathbf{V}$. It is probably a difficult challenge to find all varieties satisfying this condition, but any partial result in this direction may help to describe the varieties of the form $\mathbf{PV}$, a widely open problem. Proving that the operator $\mathbf{P}^{\uparrow}$ is idempotent by purely algebraic arguments might be an easier task, that we leave to the reader.

2 Notation and background

In this section, we briefly recall some basic facts about ordered monoids and profinite words. More details can be found in [19] for ordered monoids and in [1, 18] for profinite words.

2.1 Ordered monoids

An *ordered monoid* is a monoid M equipped with a partial order $\leqslant$ compatible with the product on M: for all $x, y, z \in M$, if $x \leqslant y$ then $zx \leqslant zy$ and $xz \leqslant yz$. Given two ordered monoids M and N, a *morphism of ordered monoids* $\varphi\colon M \to N$ is an order-preserving monoid morphism from M into N.

An ordered monoid M is a *quotient* of an ordered monoid R if there exists a surjective morphism of ordered monoids from R onto M. An *ordered submonoid* of M is a submonoid of M, equipped with the restriction of the order on M. Let M and N be ordered monoids. Then M *divides* N if M is a quotient of an ordered submonoid of N.

The *product* of a family $(M_i)_{i\in I}$ of ordered monoids, is the product monoid $\prod_{i\in I} M_i$ equipped with the product order given by

$$(s_i)_{i\in I} \leqslant (s'_i)_{i\in I} \text{ if and only if, for all } i \in I,\ s_i \leqslant s'_i.$$

A *variety of monoids* is a class of monoids closed under taking submonoids, quotients and finite direct products [7]. Equivalently, a variety of monoids is a class of monoids closed under division and finite direct products. *Varieties of ordered monoids* are defined analogously [17].

2.2 Profinite words

Let A be a finite alphabet. The set of profinite words is defined as the completion of the free monoid A^* for a certain metric.

A finite monoid M *separates* two words u and v of A^* if there is a monoid morphism $\varphi : A^* \to M$ such that $\varphi(u) \neq \varphi(v)$. One can show that any pair of distinct words of A^* can be separated by a finite monoid.

Given two words $u, v \in A^*$, we set

$$r(u, v) = \min \{|M| \mid M \text{ is a monoid that separates } u \text{ and } v\}$$

We also set $d(u, v) = 2^{-r(u,v)}$, with the usual conventions $\min \emptyset = +\infty$ and $2^{-\infty} = 0$. Then d is a metric and the completion of the metric space (A^*, d) is the set of *profinite words* on the alphabet A. Since the product of two words is a uniformly continuous function from $A^* \times A^*$ to A^*, it can be extended by continuity (in a unique way) to profinite words. The resulting topological monoid, denoted $\widehat{A^*}$, is called the *free profinite monoid* on A. It is a compact monoid. It is a well known fact that, in a compact monoid, the smallest closed subsemigroup containing a given element s has a unique idempotent, denoted s^ω.

One can show that every morphism φ from A^* onto a (discrete) finite monoid M extends uniquely to a uniformly continuous morphism from $\widehat{A^*}$ onto M. It follows that if x is a profinite word and $s = \varphi(x)$, then $\varphi(x^\omega) = s^\omega$.

For instance, the set of subsets of A is a monoid under union and the function which maps a word u onto the set of letters occurring in u is a continuous morphism, which can be extended by continuity to profinite words. The resulting map is called the *content* mapping and is denoted by c. For instance, $c(abab) = \{a, b\} = c(((ab)^\omega(ba)^\omega)^\omega)$.

3 Upper set monoids

Let $(M, \leqslant)$ be an ordered monoid. A *lower set* of M is a subset E of M such that if $x \in E$ and $y \leqslant x$ then $y \in E$. An *upper set* of M is a subset F of M such that if $x \in F$ and $x \leqslant y$ then $y \in F$. Note that a subset of M is an upper set if and only if its complement is a lower set. Given an element s of M, the set

$$\uparrow s = \{t \in M \mid s \leqslant t\}$$

is an upper set, called the *upper set generated by s*. More generally, if X is a subset of M, *the upper set generated by X* is the set

$$\uparrow X = \bigcup_{s \in X} \uparrow s$$

The *product of two upper sets* X and Y is the upper set

$$XY = \{z \in M \mid \text{there exist } x \in X \text{ and } y \in Y \text{ such that } xy \leqslant z\}$$

This operation makes the set of upper sets of M a monoid, denoted by $\mathcal{P}^{\uparrow}(M)$ and called the *upper set monoid* of M. The identity element is $\uparrow 1$ and the empty set is a zero of $\mathcal{P}^{\uparrow}(M)$. If we omit this zero, we get the submonoid $\mathcal{P}^{\uparrow}_{\star}(M)$ of *nonempty upper sets* of M.

Let us define a relation $\leqslant$ on $\mathcal{P}^{\uparrow}(M)$ by setting $X \leqslant Y$ if and only if $Y \subseteq X$. In particular, one gets $X \leqslant \emptyset$ for any upper set X. We just mention for the record that the other natural way to define an order on $\mathcal{P}^{\uparrow}(M)$ yields exactly the same definition.

Proposition 3.1 *One has $X \leqslant Y$ if and only if, for each $y \in Y$, there exists $x \in X$ such that $x \leqslant y$.*

Proof. Suppose that $Y \subseteq X$. Then the condition of the statement is clearly satisfied by taking $x = y$. Conversely, suppose that this condition is satisfied and let $y \in Y$. Then there exists an element $x \in X$ such that $x \leqslant y$. Since X is an upper set, y is also in X and thus Y is a subset of X. $\square$

Therefore, the monoids $\mathcal{P}^{\uparrow}(M)$ and $\mathcal{P}^{\uparrow}_{\star}(M)$ are ordered monoids and they will be considered as such in the remainder of this paper.

Example 3.1 Let U_1 be the monoid $\{0, 1\}$ under the usual multiplication of integers. We denote by U_1^+ the ordered monoid defined by the order $0 \leqslant 1$ and by U_1^- the ordered monoid defined by the order $1 \leqslant 0$. Then $\mathcal{P}^{\uparrow}_{\star}(U_1^-)$ has two elements ($\{0, 1\}$ and $\{0\}$) and is isomorphic to U_1^-. Similarly, $\mathcal{P}^{\uparrow}_{\star}(U_1^+)$ has two elements ($\{0, 1\}$ and $\{1\}$) and is isomorphic to U_1^+.

Example 3.2 Let 1 be the trivial monoid. Then $\mathcal{P}^{\uparrow}_{\star}(1) = 1$ and $\mathcal{P}^{\uparrow}(1) = U_1^-$.

The next three propositions were proved in [6].

Proposition 3.2 *Let M be an ordered monoid. Then M is a submonoid of $\mathcal{P}^{\uparrow}_{\star}(M)$, $\mathcal{P}^{\uparrow}_{\star}(M)$ is a submonoid of $\mathcal{P}^{\uparrow}(M)$ and $\mathcal{P}^{\uparrow}(M)$ is a quotient of $U_1^- \times \mathcal{P}^{\uparrow}_{\star}(M)$.*

Proposition 3.3 *Let M_1 and M_2 be two ordered monoids. Then the ordered monoid $\mathcal{P}^{\uparrow}_{\star}(M_1) \times \mathcal{P}^{\uparrow}_{\star}(M_2)$ is an ordered submonoid of $\mathcal{P}^{\uparrow}_{\star}(M_1 \times M_2)$.*

Note that the corresponding result for $\mathcal{P}^{\uparrow}(M)$ does not hold. Indeed, if M_1 and M_2 are the trivial monoid, $M_1 \times M_2$ is also the trivial monoid, but $\mathcal{P}^{\uparrow}(M_1) = \mathcal{P}^{\uparrow}(M_2) = \mathcal{P}^{\uparrow}_{\star}(M_1 \times M_2) = U_1^-$ and thus $\mathcal{P}^{\uparrow}(M_1) \times \mathcal{P}^{\uparrow}(M_2)$ is not an ordered submonoid of $\mathcal{P}^{\uparrow}(M_1 \times M_2)$.

Proposition 3.4 *Let M be an ordered monoid and let X be a nonempty upper set of M. Then the upper subset X^ω is a semigroup and for each $x \in X^\omega$, there is an idempotent e of X^ω such that $x \leqslant_{\mathcal{J}} e$ in X^ω.*

We also need two elementary facts on lower sets.

Proposition 3.5 *Let M be an ordered monoid, let S be a lower set of M and let E be a subset of M. Then the conditions $E \cap S \neq \emptyset$ and $\uparrow E \cap S \neq \emptyset$ are equivalent.*

Proof. Since E is contained in $\uparrow E$, it suffices to prove that if S meets $\uparrow E$, then it also meets E. Let $x \in \uparrow E \cap S$. Then there exists an element $y \in E$ such that $y \leqslant x$. Since S is a lower set, one gets $y \in S$ and thus S meets E. □

Proposition 3.6 *Let M be an ordered monoid and let S be a lower set of M. Then the set $\{X \in \mathcal{P}^\uparrow(M) \mid X \cap S \neq \emptyset\}$ is a lower set of $\mathcal{P}^\uparrow(M)$.*

Proof. Suppose that $X \cap S \neq \emptyset$ and $Y \leqslant X$. Then $X \subseteq Y$ and thus $Y \cap S \neq \emptyset$. □

Given a variety of ordered monoids $\mathbf{V}$, we denote by $\mathbf{P}^\uparrow\mathbf{V}$ [$\mathbf{P}^\uparrow_\star\mathbf{V}$] the variety of ordered monoids generated by the monoids of the form $\mathcal{P}^\uparrow(M)$ [$\mathcal{P}^\uparrow_\star(M)$], where $M \in \mathbf{V}$. A slightly more precise definition of $\mathbf{P}^\uparrow_\star\mathbf{V}$ is given in the next proposition, which is an immediate consequence of Proposition 3.3.

Proposition 3.7 *Let $\mathbf{V}$ be a variety of ordered monoids. An ordered monoid belongs to $\mathbf{P}^\uparrow_\star\mathbf{V}$ if and only if it divides a monoid $\mathcal{P}^\uparrow_\star(M)$, with $M \in \mathbf{V}$.*

Proof. Recall that an ordered monoid belongs to the variety generated by a class $\mathcal{C}$ of ordered monoids if and only if it divides a product of members of $\mathcal{C}$. The statement is therefore a consequence of Proposition 3.3. □

The varieties $\mathbf{P}^\uparrow\mathbf{V}$ and $\mathbf{P}^\uparrow_\star\mathbf{V}$ are related as follows.

Proposition 3.8 *Let $\mathbf{V}$ be a variety of ordered monoids. Then $\mathbf{P}^\uparrow_\star\mathbf{V}$ is a subvariety of $\mathbf{P}^\uparrow\mathbf{V}$. Further, if $\mathbf{V}$ contains U_1^-, then $\mathbf{P}^\uparrow\mathbf{V} = \mathbf{P}^\uparrow_\star\mathbf{V}$.*

Proof. This is an immediate consequence of Proposition 3.2. □

4 Two operations on languages

Let L be a regular language of A^*. An ordered monoid M *recognizes* L if there exists a morphism $\eta : A^* \to M$ and a lower set S of M such that $L = \eta^{-1}(S)$.

In this section, we show how two operations on languages, length preserving morphisms and inverses of substitutions, are related to upper set monoids. These results are well known in the unordered case [21, 22] and were proved in [5, 6, 20]) in a slightly different way. Let us first define our two operations.

A *length preserving morphism* is a morphism φ from A^* into B^*, such that, for each word u, the words u and $\varphi(u)$ have the same length. It is equivalent to require that, for each letter a, $\varphi(a)$ is also a letter, that is, $\varphi(A) \subseteq B$.

A *substitution* σ from A^* into B^* is a morphism from A^* into the monoid $\mathcal{P}(B^*)$ of subsets of B^*. Note that the languages $\sigma(a)$, for $a \in A$, completely determines σ. Indeed, one has $\sigma(1) = \{1\}$ and for all nonempty word $a_1 \cdots a_n$, $\sigma(a_1 \cdots a_n) = \sigma(a_1) \cdots \sigma(a_n)$.

Considered as a relation, σ has an inverse which maps a language K of A^* to the language $\sigma^{-1}(K)$ of B^* defined by

$$\sigma^{-1}(K) = \{u \in A^* \mid \sigma(u) \cap K \neq \emptyset\}$$

We shall also consider two restrictions of these operations: *surjective length preserving morphisms* and *nonempty substitutions*.

There is an obvious connection between length preserving morphisms and substitutions. If $\varphi : A^* \to B^*$ is a [surjective] length preserving morphism, then the relation $\varphi^{-1} : B^* \to A^*$ is a [nonempty] substitution such that $\sigma^{-1} = \varphi$. We shall see in Section 5 that there is an even tighter connection between these operations. For now, we establish a first link with upper set monoids.

Proposition 4.1 *Let L be a language of B^* recognized by an ordered monoid M and let $\sigma : A^* \to B^*$ be a substitution [nonempty substitution]. Then $\sigma^{-1}(L)$ is recognized by $\mathcal{P}^{\uparrow}(M)$ [$\mathcal{P}^{\uparrow}_{\star}(M)$].*

Proof. Since M recognizes L, there is a monoid morphism $\eta : B^* \to M$ and a lower set P of M such that $L = \eta^{-1}(P)$. Define a map $\psi : A^* \to \mathcal{P}^{\uparrow}(M)$ by setting $\psi(u) = {\uparrow}\,\eta(\sigma(u))$. The definition of the product of two upper sets implies that ψ is a morphism. By Proposition 3.6, the set

$$\mathcal{X} = \{X \in \mathcal{P}^{\uparrow}(M) \mid X \cap P \neq \emptyset\}$$

is a lower set of $\mathcal{P}^{\uparrow}(M)$. Furthermore, one has

$$\begin{aligned}
\psi^{-1}(\mathcal{X}) &= \{u \in A^* \mid \psi(u) \cap P \neq \emptyset\} \\
&= \{u \in A^* \mid {\uparrow}\,\eta(\sigma(u)) \cap P \neq \emptyset\} \\
&= \{u \in A^* \mid \eta(\sigma(u)) \cap P \neq \emptyset\} \quad \text{(by Proposition 3.5)} \\
&= \{u \in A^* \mid \sigma(u) \cap \eta^{-1}(P) \neq \emptyset\} \\
&= \{u \in A^* \mid \sigma(u) \cap L \neq \emptyset\} \\
&= \sigma^{-1}(L).
\end{aligned}$$

Thus $\sigma^{-1}(L)$ is recognized by $\mathcal{P}^{\uparrow}(M)$.

If σ is a nonempty substitution, it suffices to replace every occurrence of $\mathcal{P}^{\uparrow}(M)$ by $\mathcal{P}^{\uparrow}_{\star}(M)$ to get the proof. $\square$

Since a length preserving morphism is a special case of substitution, we get as a corollary:

Corollary 4.2 *Let L be a language of A^* recognized by an ordered monoid M and let $\varphi : A^* \to B^*$ be a [surjective] length preserving morphism. Then $\varphi(L)$ is recognized by $\mathcal{P}^{\uparrow}(M)$ [$\mathcal{P}^{\uparrow}_{\star}(M)$].*

5 Main result

Let us first recall a few definitions. A *class of languages* is a correspondence $\mathcal{C}$ which associates with each alphabet A a set $\mathcal{C}(A^*)$ of regular languages of A^*. A *positive variety of languages* is a class of regular languages $\mathcal{V}$ such that:

(1) for every alphabet A, $\mathcal{V}(A^*)$ is closed under union and intersection,

(2) if $\varphi\colon A^* \to B^*$ is a morphism, $L \in \mathcal{V}(B^*)$ implies $\varphi^{-1}(L) \in \mathcal{V}(A^*)$,

(3) if $L \in \mathcal{V}(A^*)$ and $u \in A^*$, then $u^{-1}L$ and Lu^{-1} are in $\mathcal{V}(A^*)$.

A *variety of languages* is a positive variety closed under complement.

There is a one to one correspondence between varieties of finite monoids (resp. varieties of finite ordered monoids) and varieties of languages (resp. positive varieties of languages) [7, 17].

Let $\mathcal{V}$ be a positive variety of languages. For each alphabet A, we denote by $\Lambda\mathcal{V}(A^*)$ $[\Lambda'\mathcal{V}(A^*)]$ the set of all languages of A^* of the form $\varphi(K)$, where φ is a [surjective] length preserving morphism from B^* to A^* and K is a language of $\mathcal{V}(B^*)$.

Similarly, we denote by $\Sigma\mathcal{V}(A^*)$ $[\Sigma'\mathcal{V}(A^*)]$ the set of all languages of A^* of the form $\sigma^{-1}(K)$, where σ is a [nonempty] substitution from A^* into B^* and K is a language of $\mathcal{V}(B^*)$.

Let $\mathbf{V}$ be a variety of ordered monoids. A description of the variety of languages corresponding to $\mathbf{P}^{\uparrow}\mathbf{V}$ was given by Polák [20, Theorem 4.2] and by the authors [5] and [6, Proposition 6.3]. Our main theorem gives a stronger form of these results. Indeed, contrary to the previous results, our description does not require intersection.

Theorem 5.1 *Let $\mathbf{V}$ be a variety of ordered monoid, and let $\mathcal{V}$ by the corresponding positive variety of languages. Then the positive variety of languages corresponding to $\mathbf{P}^{\uparrow}\mathbf{V}$ is equal to $\Lambda\mathcal{V}$ and to $\Sigma\mathcal{V}$.*

The proof relies on an improvement of a result of [5, 6, 20], which itself extends to positive varieties an argument of [21, 22]. However, the algebraic encoding of intersection and union makes our proof more technical than that of the weaker versions.

Proposition 5.2 *Let M be an ordered monoid of $\mathbf{V}$ and L be a language of A^* recognized by $\mathcal{P}^{\uparrow}(M)$. Then L belongs to $\Lambda\mathcal{V}(A^*)$.*

Proof. Let M be an ordered monoid of $\mathbf{V}$ and let L be a language of A^* recognized by $\mathcal{P}^{\uparrow}(M)$. Then there exists a morphism $\psi : A^* \to \mathcal{P}^{\uparrow}(M)$ and a lower set $\mathcal{S}$ of $\mathcal{P}^{\uparrow}(M)$ such that $L = \psi^{-1}(\mathcal{S})$. Since $\mathcal{S}$ is a lower set, one has $\mathcal{S} = \bigcup_{Z\in\mathcal{S}} {\downarrow} Z$ and hence

$$L = \psi^{-1}(\mathcal{S}) = \bigcup_{Z\in\mathcal{S}} \psi^{-1}({\downarrow} Z)$$

Further, one has

$$\begin{aligned}
\psi^{-1}({\downarrow} Z) &= \{w \in A^* \mid \psi(w) \leqslant Z\} \\
&= \{w \in A^* \mid \text{ for every } z \in Z \text{ there exists } t \in \psi(w) \text{ such that } t \leqslant z\} \\
&= \{w \in A^* \mid \text{ for every } z \in Z, \psi(w) \cap {\downarrow}\, z \neq \emptyset\}
\end{aligned}$$

Setting $X_z = \{w \in A^* \mid \psi(w) \cap {\downarrow} z \neq \emptyset\}$ for each $z \in Z$, we get

$$\psi^{-1}({\downarrow} Z) = \bigcap_{z \in Z} X_z$$

and finally

$$L = \bigcup_{Z \in \mathcal{S}} \bigcap_{z \in Z} X_z \tag{1}$$

Let U be the disjoint union of the sets Z, for $Z \in \mathcal{S}$ and let $N = M^U$. An element of N is a family $(m_z)_{z \in U}$ which can also be written as $\big((m_z)_{z \in Z}\big)_{Z \in \mathcal{S}}$.

Consider the lower set $\mathcal{J}$ of N defined by

$$\mathcal{J} = \{\big((m_z)_{z \in Z}\big)_{Z \in \mathcal{S}} \in N \mid \text{ for some } Z_0 \in \mathcal{S}, \text{ for all } z \in Z_0,\ m_z \leqslant z\}$$

and the alphabet

$$B = \{(a, n) \in A \times N \mid n = \big((m_z)_{z \in Z}\big)_{Z \in \mathcal{S}} \text{ where each } m_z \text{ is a minimal element of } \psi(a)\}$$

Let us define a length preserving morphism $\varphi : B^* \to A^*$ by $\varphi(a, n) = a$ and a morphism $\eta : B^* \to N$ by $\eta(a, n) = n$. Let $K = \eta^{-1}(\mathcal{J})$. Since N belongs to $\mathbf{V}$ by construction, K is a language of $\mathcal{V}(B^*)$.

We claim that $\varphi(K) = L$. First, if $a_1 \ldots a_k \in L$, there exists by (1) an element $Z_0 \in \mathcal{S}$ such that

$$a_1 \cdots a_k \in \bigcap_{z \in Z_0} X_z$$

Thus, for every $z \in Z_0$, one has $\psi(a_1 \cdots a_k) \cap {\downarrow} z \neq \emptyset$ and there are some elements $y_{1,z} \in \psi(a_1), \ldots, y_{k,z} \in \psi(a_k)$ such that $y_{1,z} \cdots y_{k,z} \leqslant z$. For every $z \in Z_0$, let us choose a minimal element $x_{j,z} \in \psi(a_j)$ such that $x_{j,z} \leqslant y_{j,z}$. Let $b_j = (a_j, n_j)$ be the letter of B defined by $n_j = \big((x_{j,z})_{z \in Z}\big)_{Z \in \mathcal{S}}$. Setting $v = b_1 \cdots b_k$, we get $\varphi(v) = a_1 \cdots a_k$ by construction. Furthermore, $\eta(v) = n_1 \cdots n_k$. We are only interested in the component in M^{Z_0}, whose value is $(x_{1,z})_{z \in Z_0} \cdots (x_{k,z})_{z \in Z_0}$. Now, for $z \in Z_0$, we have by definition

$$(x_{1,z})_{z \in Z_0} \cdots (x_{k,z})_{z \in Z_0} \leqslant (y_{1,z})_{z \in Z_0} \cdots (y_{k,z})_{z \in Z_0}$$

Since $y_{1,z} \cdots y_{k,z} \leqslant z$, we get $\eta(v) \in \mathcal{S}$, which shows that $v \in K$ and that $L \subseteq \varphi(K)$.

To prove the opposite inclusion, consider a word $v = b_1 \cdots b_k$ of K. Let us set, for $1 \leqslant j \leqslant k$, $b_j = (a_j, n_j)$, with $n_j = \big((m_{j,z})_{z \in Z}\big)_{Z \in \mathcal{S}}$. Since $\eta(v) \in \mathcal{J}$, there exists a set $Z_0 \in \mathcal{S}$, such that, for all $z \in Z_0$, $m_{1,z} \cdots m_{k,z} \leqslant z$. By definition of B, each element $m_{j,z}$ is a minimal element of $\psi(a_j)$. Therefore, one gets $\psi(a_1 \cdots a_k) \cap {\downarrow} z \neq \emptyset$ for every $z \in Z_0$. It follows that $\varphi(v) = a_1 \cdots a_k \in \bigcap_{z \in Z_0} X_z$. Consequently, $a_1 \cdots a_k \in L$ and thus $\varphi(K) \subseteq L$. This proves the claim and concludes the proof of the proposition. $\square$

Note that if ψ is recognised by $\mathcal{P}^{\uparrow}_{\star}(M)$, then $\psi(a)$ is never empty and the length preserving morphism $\varphi : B^* \to A^*$ is surjective. Therefore we get the following corollary.

Corollary 5.3 *Let M be an ordered monoid of* $\mathbf{V}$ *and L be a language of A^* recognised by $\mathcal{P}^{\uparrow}_{\star}(M)$. Then L belongs to $\Lambda'\mathcal{V}(A^*)$.*

Let us now complete the proof of our main theorem.

Proof of Theorem 5.1. Let $\mathcal{P}^{\uparrow}\mathcal{V}$ be the positive variety of languages corresponding to $\mathbf{P}^{\uparrow}\mathbf{V}$. We prove successively the inclusions $\Lambda\mathcal{V} \subseteq \Sigma\mathcal{V} \subseteq \mathcal{P}^{\uparrow}\mathcal{V} \subseteq \Lambda\mathcal{V}$.

The inclusion $\Lambda\mathcal{V} \subseteq \Sigma\mathcal{V}$ stems from the fact that a length preserving morphism is a special case of inverse of substitution.

Let $L \in \mathcal{V}(B^*)$ and let $\sigma : A^* \to B^*$ be a substitution. If L is recognised by M, then, by Proposition 4.1, $\sigma^{-1}(L)$ is recognised by $\mathcal{P}^{\uparrow}(M)$ and thus $\sigma^{-1}(L) \in \mathcal{P}^{\uparrow}\mathcal{V}(A^*)$. This proves the inclusion $\Sigma\mathcal{V} \subseteq \mathcal{P}^{\uparrow}\mathcal{V}$.

Let L be a language of $\mathcal{P}^{\uparrow}\mathcal{V}(A^*)$. By definition, L is recognised by a monoid of $\mathbf{P}^{\uparrow}\mathbf{V}$. It follows from Proposition 3.3 that every ordered monoid of $\mathbf{P}^{\uparrow}\mathbf{V}$ divides an ordered monoid of the form $\mathcal{P}^{\uparrow}(M)$, with $M \in \mathbf{V}$. Therefore, L is recognised by a monoid of the form $\mathcal{P}^{\uparrow}(M)$, with $M \in \mathbf{V}$. Proposition 5.2 now shows that L belongs to $\Lambda\mathcal{V}(A^*)$. □

One can use Corollary 5.3 to obtain the following description of the positive variety of languages corresponding to $\mathbf{P}^{\uparrow}_{\star}\mathbf{V}$.

Theorem 5.4 *Let* $\mathbf{V}$ *be a variety of ordered monoid, and let $\mathcal{V}$ by the corresponding positive variety of languages. Then the positive variety of languages corresponding to* $\mathbf{P}^{\uparrow}_{\star}\mathbf{V}$ *is equal to $\Lambda'\mathcal{V}$ and to $\Sigma'\mathcal{V}$.*

Theorems 5.1 and 5.4 have an important consequence.

Corollary 5.5 *For every variety of ordered monoids* $\mathbf{V}$, $\mathbf{P}^{\uparrow}(\mathbf{P}^{\uparrow}\mathbf{V}) = \mathbf{P}^{\uparrow}\mathbf{V}$ *and* $\mathbf{P}^{\uparrow}_{\star}(\mathbf{P}^{\uparrow}_{\star}\mathbf{V}) = \mathbf{P}^{\uparrow}_{\star}\mathbf{V}$.

Proof. Let $\mathcal{V}$ be the positive variety of languages corresponding to $\mathcal{V}$. Since the composition of two length preserving morphisms is again length preserving, the equality $\mathcal{P}^{\uparrow}(\mathcal{P}^{\uparrow}\mathcal{V}) = \mathcal{P}^{\uparrow}\mathcal{V}$ holds for each positive variety of languages $\mathcal{V}$. It follows that $\mathbf{P}^{\uparrow}(\mathbf{P}^{\uparrow}\mathbf{V}) = \mathbf{P}^{\uparrow}\mathbf{V}$. The equality $\mathbf{P}^{\uparrow}_{\star}(\mathbf{P}^{\uparrow}_{\star}\mathbf{V}) = \mathbf{P}^{\uparrow}_{\star}\mathbf{V}$ is obtained in a similar way. □

6 Examples of varieties closed under $\mathbf{P}^{\uparrow}$

In this section, we give examples of varieties of ordered monoids $\mathbf{V}$ such that $\mathbf{P}^{\uparrow}\mathbf{V} = \mathbf{V}$.

Let us start with a trivial observation. If $\mathbf{V}$ is a variety of monoids, the ordered monoids of the form $(M, \leqslant)$, with $M \in \mathbf{V}$, constitute a variety of ordered monoids, also denoted $\mathbf{V}$. Now, since $\mathbf{V} \subseteq \mathbf{P}^{\uparrow}\mathbf{V} \subseteq \mathbf{PV}$, the condition $\mathbf{V} = \mathbf{PV}$ implies $\mathbf{V} = \mathbf{P}^{\uparrow}\mathbf{V}$. A complete classification of the varieties of monoids satisfying the equality $\mathbf{V} = \mathbf{PV}$ has been achieved. Perrot [13] showed that the commutative varieties satisfying this condition are the varieties of commutative monoids whose groups belong to a given variety of commutative groups. And Esik and Simon [9] showed that the only noncommutative variety of monoids satisfying $\mathbf{V} = \mathbf{PV}$ is the variety of all finite monoids.

For ordered monoids, the situation is more involved, even in the commutative case. Let $\mathbf{J_1^-} = [\![xy = yx, x^2 = x, 1 \leqslant x]\!]$ and $\mathbf{J_1^+} = [\![xy = yx, x^2 = x, x \leqslant 1]\!]$ be the varieties of ordered monoids generated by U_1^- and by U_1^+, respectively.

Proposition 6.1 *The following equalities hold:* $\mathbf{P}_\star^\uparrow\mathbf{I} = \mathbf{I}$, $\mathbf{P}^\uparrow\mathbf{I} = \mathbf{J_1^-}$ *and* $\mathbf{P}_\star^\uparrow\mathbf{J_1^-} = \mathbf{P}^\uparrow\mathbf{J_1^-} = \mathbf{J_1^-}$.

Proof. Let 1 be the trivial monoid. Since $\mathcal{P}_\star^\uparrow(1) = 1$ and $\mathcal{P}^\uparrow(1) = U_1^-$, one has $\mathbf{P}_\star^\uparrow\mathbf{I} = \mathbf{I}$ and $\mathbf{P}^\uparrow\mathbf{I} = \mathbf{J_1^-}$. The last two formulas are a consequence of Proposition 3.8 and Corollary 5.5. □

Proposition 6.2 *The following equalities hold:* $\mathbf{P}_\star^\uparrow\mathbf{J_1^+} = [\![xy = yx, x \leqslant 1]\!]$ *and* $\mathbf{P}^\uparrow\mathbf{J_1^+} = [\![xy = yx, x^2 \leqslant x]\!]$.

Proof. (1) Let $M \in \mathbf{J_1^+}$. Since M satisfies the identity $x \leqslant 1$, the set $\{1\}$ is an upperset of M. Further, if X is a nonempty upper set of M and if $x \in X$, one has $x \leqslant 1$, which proves that $X \leqslant \{1\}$. It follows that $\mathcal{P}_\star^\uparrow(M)$ satisfies the identity $x \leqslant 1$ and is commutative. Consequently, $\mathbf{P}_\star^\uparrow\mathbf{J_1^+}$ is contained in $[\![xy = yx, x \leqslant 1]\!]$.

To prove the opposite inclusion, consider an ordered monoid M of the variety $[\![xy = yx, x \leqslant 1]\!]$. Since M is commutative, it divides the product of its monogenic submonoids. These monogenic submonoids are group-free: indeed, the only possible order relation in a group is the equality relation and thus $x \leqslant 1$ implies $x = 1$ in a group. Further, $x \leqslant 1$ implies $x^n \leqslant x^{n-1} \leqslant \cdots \leqslant x \leqslant 1$ for every $n > 0$. It follows that a monogenic submonoid of M is a submonoid of an ordered monoid of the form $T_n = \{1, a, a^2, \ldots, a^n\}$, where $a^{n+1} = a^n < a^{n-1} < \cdots < a < 1$. It suffices now to prove that T_n belongs to $\mathbf{P}^\uparrow\mathbf{J_1^+}$. Consider the following upperset of the ordered monoid $(U_1^+)^n$:

$$X = \{(u_1, \ldots, u_n) \mid \text{at most one of the } u_i \text{ is equal to } 0\}$$

Then, for $0 \leqslant k \leqslant n$, one gets

$$X^k = \{(u_1, \ldots, u_n) \mid \text{at most } k \text{ of the } u_i \text{ are equal to } 0\}$$

It follows immediately that the submonoid of $\mathcal{P}_\star^\uparrow(U_1^+)^n$ generated by X is equal to T_n. Consequently, T_n belongs to $\mathbf{P}_\star^\uparrow\mathbf{J_1^+}$ and $\mathbf{P}_\star^\uparrow\mathbf{J_1^+} = [\![xy = yx, x \leqslant 1]\!]$.

(2) Since U_1^- satisfies the identities $xy = yx$ and $x^2 = x$, it also satisfies $x^2 \leqslant x$. Further, the first part of the proof shows that $\mathcal{P}_\star^\uparrow(M)$ satisfies the same identities. It follows now by Proposition 3.2 that $\mathcal{P}^\uparrow(M)$ is a quotient of $U_1^- \times \mathcal{P}_\star^\uparrow(M)$ and thus satisfies the identities $xy = yx$ and $x^2 \leqslant x$. Therefore $\mathbf{P}^\uparrow\mathbf{J_1^+}$ is contained in $[\![xy = yx, x^2 \leqslant x]\!]$.

To prove the opposite inclusion, one can mimic the proof of (1) and it suffices to prove that $\mathbf{P}^\uparrow\mathbf{J_1^+}$ contains the ordered monoid of the form $T'_n = \{1, a, a^2, \ldots, a^n\}$, where $a^{n+1} = a^n < a^{n-1} < \cdots < a$. But T'_n divides $T_n \times U_1$ and U_1 divides $U_1^+ \times U_1^-$. Since $\mathbf{P}^\uparrow\mathbf{J_1^+}$ contains U_1^+ and U_1^-, it also contains U_1. Further, $\mathbf{P}^\uparrow\mathbf{J_1^+}$ contains T_n by (1). Thus $\mathbf{P}^\uparrow\mathbf{J_1^+}$ contains T'_n for every n, which concludes the proof. □

We turn now to noncommutative varieties. Let us start with a general result.

Proposition 6.3 *Let u be a profinite word on the alphabet $\{x_1, \ldots, x_n\}$.*

(1) *If $\mathbf{V} = [\![u \leqslant x_1 \cdots x_n]\!]$, then $\mathbf{P}^{\uparrow}_{\star}\mathbf{V} = \mathbf{V}$.*

(2) *If $\mathbf{V} = [\![u \leqslant x_1 \cdots x_n]\!]$ with $c(u) \subseteq c(x_1 \cdots x_n)$, then $\mathbf{P}^{\uparrow}\mathbf{V} = \mathbf{V}$.*

Proof. Let $A = \{x_1, \ldots, x_n\}$ and let $\widehat{A^*}$ be the free profinite monoid on A. Let $M \in \mathbf{V}$ and let X be a nonempty upper set of M. Let $\pi : \widehat{A^*} \to \mathcal{P}^{\uparrow}_{\star}(M)$ be a continuous morphism. We claim that $\pi(u) \leqslant \pi(x_1 \cdots x_n)$, that is, $\pi(x_1 \cdots x_n) \subseteq \pi(u)$. Let z be an element of $\pi(x_1 \cdots x_n)$. Since $\pi(x_1 \cdots x_n) = \pi(x_1) \cdots \pi(x_n)$, there is, for $1 \leqslant i \leqslant n$, an element s_i of $\pi(x_i)$ such that $s_1 \cdots s_n \leqslant z$. The map σ from A into M defined by $\sigma(x_i) = s_i$ extends uniquely to a continuous morphism $\sigma : \widehat{A^*} \to M$. Since $s_i \in \pi(x_i)$, this morphism satisfies $\sigma(v) \in \pi(v)$ for every $v \in \widehat{A^*}$. Further, M satisfies the identity $u \leqslant x_1 \cdots x_n$ and consequently $\sigma(u) \leqslant \sigma(x_1 \cdots x_n) = s_1 \cdots s_n \leqslant z$. Thus $\sigma(u) \leqslant z$ and since $\pi(u)$ is an upper set containing $\sigma(u)$, it also contains z, which proves the claim. Therefore $\mathcal{P}^{\uparrow}_{\star}(M)$ belongs to $\mathbf{V}$.

(2) Let $A = \{x_1, \ldots, x_n\}$ and let $\widehat{A^*}$ be the free profinite monoid on A. Let $M \in \mathbf{V}$. Let $\pi : \widehat{A^*} \to \mathcal{P}^{\uparrow}(M)$ be a continuous morphism. We claim that $\pi(u) \leqslant \pi(x_1 \cdots x_n)$. Two cases can arise: some $\pi(x_i)$ is empty or every $\pi(x_i)$ is nonempty. In the first case, one gets $\pi(x_1 \cdots x_n) = \emptyset$ and the relation $\pi(u) \leqslant \pi(x_1 \cdots x_n)$ holds trivially. In the latter case, $\pi(x_1 \cdots x_n)$ is nonempty and since $c(u) \subseteq c(x_1 \cdots x_n)$, $\pi(u)$ is also nonempty. One can now apply the argument used in (1) to conclude. □

Proposition 6.4 *Let $n \geqslant 0$.*

(1) *If $\mathbf{V} = [\![x^n \leqslant 1]\!]$, then $\mathbf{P}^{\uparrow}_{\star}\mathbf{V} = \mathbf{V}$.*

(2) *If $\mathbf{V} = [\![x^n \leqslant x]\!]$, then $\mathbf{P}^{\uparrow}\mathbf{V} = \mathbf{P}^{\uparrow}_{\star}\mathbf{V} = \mathbf{V}$.*

Proof. (1) If $n = 0$, $\mathbf{V}$ is the variety of all ordered monoids and the result is trivial. Suppose that $n > 0$. Let $M \in \mathbf{V}$ and let X be a nonempty upper set of M. We claim that $X^n \leqslant {\uparrow}1$. Indeed, let $z \in {\uparrow}1$ and let $x \in X$. Since $M \in \mathbf{V}$, one has $x^n \leqslant 1$, and hence $x^n \leqslant z$. Since $x^n \in X^n$, this proves the claim and shows that $\mathbf{P}^{\uparrow}_{\star}(M)$ satisfies the identity $x^n \leqslant 1$. Therefore $\mathbf{P}^{\uparrow}_{\star}\mathbf{V} = \mathbf{V}$.

(2) is a consequence of Proposition 6.3 (2). □

A similar argument (omitted) would prove the following results. Note that the variety $[\![x^\omega \leqslant 1]\!]$, also known as $\mathbf{BG}^+$, plays an important role in semigroup theory [16, 19].

Proposition 6.5

(1) *If $\mathbf{V} = [\![x^\omega \leqslant 1]\!]$, then $\mathbf{P}^{\uparrow}_{\star}\mathbf{V} = \mathbf{V}$.*

(2) *If $\mathbf{V} = [\![x^\omega \leqslant x]\!]$, then $\mathbf{P}^{\uparrow}\mathbf{V} = \mathbf{P}^{\uparrow}_{\star}\mathbf{V} = \mathbf{V}$.*

Of course, similar results hold for the varieties $[\![xy = yx]\!] \cap \mathbf{V}$, where $\mathbf{V}$ is one of the varieties considered in Propositions 6.3, 6.4 and 6.5. Let us mention some other examples.

Proposition 6.6

(1) *If $\mathbf{V} = [\![x^\omega \leqslant x, x^\omega y = yx^\omega]\!]$, then $\mathbf{P}^{\uparrow}\mathbf{V} = \mathbf{P}^{\uparrow}_{\star}\mathbf{V} = \mathbf{V}$.*

(2) *If* $\mathbf{V} = [\![x^\omega \leqslant x, x^\omega y^\omega = y^\omega x^\omega]\!]$, *then* $\mathbf{P}^\uparrow\mathbf{V} = \mathbf{P}^\uparrow_\star\mathbf{V} = \mathbf{V}$.

Proof. Since U_1^- satisfies the identities $x^\omega \leqslant x$, $x^\omega y = yx^\omega$ and $x^\omega y^\omega = y^\omega x^\omega$, Proposition 3.8 shows that $\mathbf{P}^\uparrow\mathbf{V} = \mathbf{P}^\uparrow_\star\mathbf{V}$, both in case (1) and (2). Thus, it suffices to show that $\mathbf{P}^\uparrow_\star\mathbf{V} \subseteq \mathbf{V}$.

(1) Let $M \in \mathbf{V}$. Proposition 6.5 (2) shows that $\mathcal{P}^\uparrow_\star(M)$ satisfies $x^\omega \leqslant x$. Let X and Y be two nonempty upper sets of M. We claim that $X^\omega Y \leqslant YX^\omega$. If $z \in YX^\omega$, there exist two elements $y \in Y$ and $x \in X^\omega$ such that $yx \leqslant z$. Since X^ω is a semigroup, one has $x^\omega \in X^\omega$ and hence $x^\omega y \in X^\omega Y$. Further, $x^\omega y = yx^\omega \leqslant yx \leqslant z$, which proves the claim. A dual argument would prove that $YX^\omega \leqslant X^\omega Y$ and thus $\mathcal{P}^\uparrow_\star(M)$ satisfies the identity $x^\omega y = yx^\omega$. Therefore $\mathbf{P}^\uparrow_\star\mathbf{V} \subseteq \mathbf{V}$.

(2) Let $M \in \mathbf{V}$. Proposition 6.5 shows that $\mathcal{P}^\uparrow_\star(M)$ satisfies $x^\omega \leqslant x$. Let X and Y be two nonempty upper sets of M. We claim that $X^\omega Y^\omega \leqslant Y^\omega X^\omega$. If $z \in Y^\omega X^\omega$, there exist two elements $y \in Y^\omega$ and $x \in X^\omega$ such that $yx \leqslant z$. Since X^ω and Y^ω are semigroups, one has $x^\omega \in X^\omega$ and $y^\omega \in Y^\omega$, whence $x^\omega y^\omega \in X^\omega Y^\omega$. Further, $x^\omega y^\omega = y^\omega x^\omega \leqslant yx \leqslant z$, which proves the claim. A dual argument would prove that $Y^\omega X^\omega \leqslant X^\omega Y^\omega$ and thus $\mathcal{P}^\uparrow_\star(M)$ satisfies the identity $x^\omega y^\omega = y^\omega x^\omega$. Therefore $\mathbf{P}^\uparrow_\star\mathbf{V} \subseteq \mathbf{V}$. $\square$

Corollary 6.7

(1) *If* $\mathbf{V} = [\![x \leqslant 1, x^\omega y = yx^\omega]\!]$, *then* $\mathbf{P}^\uparrow_\star\mathbf{V} = \mathbf{V}$.

(2) *If* $\mathbf{V} = [\![x \leqslant 1, x^\omega y^\omega = y^\omega x^\omega]\!]$, *then* $\mathbf{P}^\uparrow_\star\mathbf{V} = \mathbf{V}$.

(3) *If* $\mathbf{V} = [\![x^2 \leqslant x, x^\omega y = yx^\omega]\!]$, *then* $\mathbf{P}^\uparrow\mathbf{V} = \mathbf{P}^\uparrow_\star\mathbf{V} = \mathbf{V}$.

(4) *If* $\mathbf{V} = [\![x^2 \leqslant x, x^\omega y^\omega = y^\omega x^\omega]\!]$, *then* $\mathbf{P}^\uparrow\mathbf{V} = \mathbf{P}^\uparrow_\star\mathbf{V} = \mathbf{V}$.

Proof. The identity $x \leqslant 1$ implies $x^2 \leqslant x$, which in turn implies $x^\omega \leqslant x$. It follows that

$$[\![x \leqslant 1, x^\omega y = yx^\omega]\!] = [\![x^\omega \leqslant x, x^\omega y = yx^\omega]\!] \cap [\![x \leqslant 1]\!]$$
$$[\![x \leqslant 1, x^\omega y^\omega = y^\omega x^\omega]\!] = [\![x^\omega \leqslant x, x^\omega y^\omega = y^\omega x^\omega]\!] \cap [\![x \leqslant 1]\!]$$
$$[\![x^2 \leqslant x, x^\omega y = yx^\omega]\!] = [\![x^\omega \leqslant x, x^\omega y = yx^\omega]\!] \cap [\![x^2 \leqslant x]\!]$$
$$[\![x^2 \leqslant x, x^\omega y^\omega = y^\omega x^\omega]\!] = [\![x^\omega \leqslant x, x^\omega y^\omega = y^\omega x^\omega]\!] \cap [\![x^2 \leqslant x]\!]$$

The corollary follows now from Propositions 6.4 and 6.6. $\square$

Proposition 6.8

(1) *If* $\mathbf{V} = [\![1 \leqslant x, x^\omega y = yx^\omega]\!]$, *then* $\mathbf{P}^\uparrow\mathbf{V} = \mathbf{P}^\uparrow_\star\mathbf{V} = \mathbf{V}$.

(2) *If* $\mathbf{V} = [\![1 \leqslant x, x^\omega y^\omega = y^\omega x^\omega]\!]$, *then* $\mathbf{P}^\uparrow\mathbf{V} = \mathbf{P}^\uparrow_\star\mathbf{V} = \mathbf{V}$.

Proof. Since U_1^- satisfies the identities $1 \leqslant x$, $x^\omega y = yx^\omega$ and $x^\omega y^\omega = y^\omega x^\omega$, Proposition 3.8 shows that $\mathbf{P}^\uparrow\mathbf{V} = \mathbf{P}^\uparrow_\star\mathbf{V}$, both in case (1) and (2). Thus, it suffices to show that $\mathbf{P}^\uparrow_\star\mathbf{V} \subseteq \mathbf{V}$.

(1) Let $M \in \mathbf{V}$. Proposition 6.5 shows that $\mathcal{P}^\uparrow_\star(M)$ satisfies $1 \leqslant x$. Let X and Y be two nonempty upper sets of M. We claim that $X^\omega Y \leqslant YX^\omega$. If $z \in YX^\omega$, there exist two elements $y \in Y$ and $x \in X^\omega$ such that $yx \leqslant z$. By Proposition 3.4, there is an idempotent $e \in X^\omega$ and two elements $x_1, x_2 \in X^\omega$ such that $x = x_1 e x_2$. Now the element ey belongs to $X^\omega Y$. Further, since M

satisfies the identities $1 \leqslant x$ and $x^\omega y = yx^\omega$, one gets $ey = ye \leqslant yx_1ex_2 = yx \leqslant z$, which proves the claim. A dual argument would prove that $YX^\omega \leqslant X^\omega Y$ and thus $\mathcal{P}_\star^\uparrow(M)$ satisfies the identity $x^\omega y = yx^\omega$. Therefore $\mathbf{P}_\star^\uparrow\mathbf{V} \subseteq \mathbf{V}$.

(2) Let $M \in \mathbf{V}$. Proposition 6.4 (with $n = 0$) shows that $\mathcal{P}_\star^\uparrow(M)$ satisfies $1 \leqslant x$. Let X and Y be two nonempty upper sets of M. We claim that $X^\omega Y^\omega \leqslant Y^\omega X^\omega$. If $z \in Y^\omega X^\omega$, there exist two elements $y \in Y^\omega$ and $x \in X^\omega$ such that $yx \leqslant z$. By Proposition 3.4, there are two idempotents $e \in X^\omega$, $f \in Y^\omega$ and some elements $x_1, x_2 \in X^\omega$, $y_1, y_2 \in Y^\omega$ such that $x = x_1ex_2$ and $y = y_1fy_2$. Now the element ef belongs to $X^\omega Y^\omega$. Further, since M satisfies the identities $1 \leqslant x$ and $x^\omega y^\omega = y^\omega x^\omega$, one gets $ef = fe \leqslant y_1fy_2x_1ex_2 = yx \leqslant z$, which proves the claim. A dual argument would prove that $Y^\omega X^\omega \leqslant X^\omega Y^\omega$ and thus $\mathcal{P}_\star^\uparrow(M)$ satisfies the identity $x^\omega y^\omega = y^\omega x^\omega$. Therefore $\mathbf{P}_\star^\uparrow\mathbf{V} \subseteq \mathbf{V}$. □

Let us mention another example.

Theorem 6.9 *If* $\mathbf{V} = [\![(xy)^\omega(yx)^\omega(xy)^\omega \leqslant (xy)^\omega]\!]$, *then* $\mathbf{P}^\uparrow\mathbf{V} = \mathbf{V}$.

Proof. Since $\mathbf{V}$ contains U_1^-, it suffices to prove, by Proposition 3.8, that $\mathbf{P}_\star^\uparrow\mathbf{V} = \mathbf{V}$.

Recall that, in a monoid, two idempotents are $\mathcal{J}$-equivalent if and only if they are conjugate (see for instance [15, Proposition 1.12, p. 51]). Therefore, a monoid M belongs to $\mathbf{V}$ if and only if, for each pair (e, f) of idempotents of M, one has $efe \leqslant e$. We claim that $\mathcal{P}_\star^\uparrow(M)$ satisfies the equation $(xy)^\omega(yx)^\omega(xy)^\omega \leqslant (xy)^\omega$.

Let E and F be two $\mathcal{J}$-equivalent idempotents of $\mathcal{P}_\star^\uparrow(M)$. Then E and F are conjugate and there exist two elements X, Y of $\mathcal{P}_\star^\uparrow(M)$ such that $XY = E$ and $YX = F$. Let $z \in E$. Since $E = E^2$, E is a semigroup and by Proposition 3.4 one has $z = z_1ez_2$ for some elements $z_1, e, z_2 \in E$ such that e is idempotent. Since $E = XY$, we also get $xy \leqslant e$ for some $x \in X$ and $y \in Y$. Now, $(xy)^\omega \in (XY)^\omega = E^\omega = E$ and $(yx)^\omega \in F^\omega = F$. Therefore,

$$z_1(xy)^\omega(yx)^\omega(xy)^\omega z_2 \in EEFEE = EFE$$

Now since $M \in \mathbf{V}$, one has

$$z_1(xy)^\omega(yx)^\omega(xy)^\omega z_2 \leqslant z_1(xy)^\omega z_2 \leqslant z_1ez_2 = z$$

and hence $EFE \leqslant E$. It follows that $\mathcal{P}_\star^\uparrow(M)$ belongs to $\mathbf{V}$. Therefore $\mathbf{P}_\star^\uparrow\mathbf{V} = \mathbf{V}$. □

Finally, the authors proved in [6] the existence of a unique maximal proper variety of ordered monoids $\mathbf{V}$ satisfying $\mathbf{P}_\star^\uparrow\mathbf{V} = \mathbf{P}^\uparrow\mathbf{V} = \mathbf{V}$. This variety, denoted by $\mathbf{W}$, is defined as follows: an ordered monoid M belongs to $\mathbf{W}$ if, for every pair (a, b) of mutually inverse elements of M, and for every element z of the minimal ideal of the submonoid generated by a and b, $(abzab)^\omega \leqslant ab$.

References

[1] Jorge Almeida. *Finite semigroups and universal algebra*. World Scientific Publishing Co. Inc., River Edge, NJ, 1994. Translated from the 1992 Portuguese original and revised by the author.

[2] Jorge Almeida. Power exponents of aperiodic pseudovarieties. *Semigroup Forum*, 58:18–32, 1999.

[3] Jorge Almeida. Power semigroups: results and problems. In *Algebraic Engineering. Proceedings of the first international conference on semigroups and algebraic engineering held in Aizu, Japan, March 24-28, 1997.*, pages 399–415. M. Ito and C. Nehaniv (Eds.), World Scientific, Singapore, 1999.

[4] Karl Auinger and Benjamin Steinberg. On power groups and embedding theorems for relatively free profinite monoids. *Math. Proc. Cambridge Philos. Soc.*, 138(2):211–232, 2005.

[5] Antonio Cano Gómez. *Semigroupes ordonnés et opérations sur les langages rationnels*. PhD thesis, Université Paris 7 and Departamento de Sistemas Informáticos y Computación, Universidad Politécnica de Valencia, 2003.

[6] Antonio Cano Gómez and Jean-Éric Pin. Shuffle on positive varieties of languages. *Theoret. Comput. Sci.*, 312:433–461, 2004.

[7] S. Eilenberg. *Automata, Languages and Machines*, volume B. Academic Press, New York, 1976.

[8] Ana Paula Escada Colaço. *Contribuições para o estudo de operadores potência sobre pseudovariedades de semigrupos.* Tese para obtençao so grau de doutor em matemática, Facultade de ciências da Universidade do Porto, 1999.

[9] Zoltan Ésik and Imre Simon. Modeling literal morphisms by shuffle. *Semigroup Forum*, 56:225–227, 1998.

[10] Stuart W. Margolis and Benjamin Steinberg. Power semigroups and polynomial closure. In *Words, languages & combinatorics, III (Kyoto, 2000)*, pages 311–322. World Sci. Publ., River Edge, NJ, 2003.

[11] S.W. Margolis. On m-varieties generated by power monoids. *Semigroup Forum*, 22:339–353, 1981.

[12] S.W Margolis and J.-E. Pin. Minimal noncommutative varieties and power varieties. *Pacific J. Math.*, 111:125–135, 1984.

[13] Jean-François Perrot. Variétés de langages et operations. *Theoret. Comput. Sci.*, 7:197–210, 1978.

[14] Jean-Éric Pin. Variétés de langages et monoïde des parties. *Semigroup Forum*, 20:11–47, 1980.

[15] Jean-Éric Pin. *Varieties of formal languages*. North Oxford, London and Plenum, New-York, 1986. (Translation of *Variétés de langages formels*, Masson, 1984).

[16] Jean-Éric Pin. $bg = pg$, a success story. In J. Fountain, editor, *NATO Advanced y Institute Semigroups, Formal Languages and Groups*, pages 33–47. Kluwer Academix Publisher, 1995.

[17] Jean-Eric Pin. A variety theorem without complementation. *Russian Mathematics (Iz. VUZ)*, 39:80–90, 1995.

[18] Jean-Éric Pin. Profinite methods in automata theory. In Susanne Albers and Jean-Yves Marion, editors, *26th International Symposium on Theoretical Aspects of Computer Science (STACS 2009)*, pages 31–50. Internationales Begegnungs- und Forschungszentrum für Informatik (IBFI), Schloss Dagstuhl, Germany, 2009.

[19] Jean-Éric Pin and Pascal Weil. Semidirect products of ordered semigroups. *Communications in Algebra*, 30:149–169, 2002.

[20] Libor Polák. Operators on classes of regular languages. In G. Gomes, J.-É. Pin, and P.V. Silva, editors, *Semigroups, Algorithms, Automata and Languages*, pages 407–422. World Scientific, 2002.

[21] Christophe Reutenauer. Sur les variétés de langages et de monoïdes. In *Theoretical computer science (Fourth GI Conf., Aachen)*, volume 67 of *Lect. Notes Comp. Sci.*, pages 260–265. Springer, Berlin, 1979.

[22] Howard Straubing. Recognizable sets and power sets of finite semigroups. *Semigroup Forum*, 18:331–340, 1979.