An infinity family of one-step iterators for solving non-linear equations to increase the order of convergence and a new algorithm of global convergence

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Abstract

In this paper we present an infinity family of one-step iterative formulas for solving non-linear equations (Present Method One), from now on PMI, that can be expressed as $x_{n+1} = F_m(x_n)$, with $1 \leq m < \infty$, integer, F_m being functions to be built later, in such a way that the velocity of convergence of such iterations increases more and more as m goes to infinity; in other words: given an arbitrary integer $m_0 \geq 1$, we will proof that the corresponding iteration formula of the family, $x_{n+1} = F_{m_0}(x_n)$, has order of convergence $m_0 + 1$.

The increment of the velocity of convergence of the sequence of the iterator family $x_{n+1} = F_{m+1}(x_n)$ with respect to the previous one $x_{n+1} = F_m(x_n)$ is attained at the expense of one derivative evaluation more.

Besides, we introduce a new algorithm (Present Method Two), from now on PMII, that plays the role of *seeker* for an initial value to guarantee the local convergence of the PMI.

Both of them can be composed as an only algorithm of global convergence, included the case of singular roots, that does not depend on the chosen initial value, and that allows to find all the roots in a feasible interval in a general and complete way, these are, in my opinion, the main results of this work.

Keywords Nonlinear equations; root-solver; iterative methods; convergence order; global convergence.

1 Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis, with a great field of applications in engineering. Within this area, this paper concerns iterative methods to find a simple or singular root of $f(x) = 0$, f being a real function.

In recent years, a lot of root-finding methods have been published, with the aim of improving the order of convergence of the well known classical methods such as Newton's (NM) of order of convergence two, Euler's (EM), Ostrowski's (OM), Chebyshev's (CHM) and Hally's (HM) of order of convergence three, Jarratt's (JM) of order of convergence four, etc, most frequently by composing two or more of them (multi-point or multi-step iterative methods), and using adequate approximations for the derivatives, as it can be seen in numerous references therein (see [1]-[12]). The increment of the velocity of convergence is usually attained at the expense of the number of function and derivative evaluations to accomplish each iteration, what might affect its computational efficiency. This problem is usually solved by Traub's formula, that says: the computational efficiency of an iterative method (IM) of order p and λ function evaluations is given by the so-called efficiency index:

$$E(IM) = p^{1/\lambda} \tag{1}$$

As an alternative, we present a family of one-step iterative methods to increase the velocity of convergence, introducing convenient modifications in the NM. Next, we are going to motivate the main ideas of our proposal. NM, EM, OM, CHM, HM, JM and other classical one-step iterations can be expressed under a common structure, as the reader can easily observe:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} G(x_n) \tag{2}$$

where $G(x)$ is a different function for each of them. Indeed:

For Newton's $G(x_n)$ takes the form:

$$G_N(x_n) = 1; \forall\, n \tag{3}$$

For Euler's $G(x_n)$ takes the form:

$$G_E(x_n) = \frac{2}{1+\sqrt{1-4\frac{f(x_n-\xi(x_n))}{f(x_n)}}}, \text{ with } \xi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \tag{4}$$

For Ostrowski's $G(x_n)$ takes the form:

$$G_0(x_n) = 1 + \frac{f(\xi(x_n))}{f(x_n) - 2\, f(\xi(x_n))} \tag{5}$$

For Chebyshev's $G(x_n)$ takes the form:

$$G_{CH}(x_n) = 1 + \frac{f''(x_n) f(x_n)}{2f'(x_n)^2} \tag{6}$$

For Hally's $G(x_n)$ takes the form:

$$G_H(x_n) = 1 + \frac{M(x_n)}{1 - 1/2M(x_n)}; \text{ with } M(x_n) = \frac{f''(x_n) f(x_n)}{f'(x_n)^2} \tag{7}$$

For Jarratt's $G(x_n)$ takes the form:

$$G_J(x_n) = 1 - \frac{3}{2}\frac{f'(\xi_1(x_n)) - f'(x_n)}{3f'(\xi_1(x_n)) - f'(x_n)}, \text{ with } \xi_1(x_n) = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \tag{8}$$

Besides, if r is a root of f, the values of each $G(x)$ and its respective derivatives at $x = r$ bear a relation to the velocity of convergence, as illustrated in Table 1 where, for the sake of the clarity, b_k, k=2, 3, ..., is defined as:

$$b_k = \frac{f^{(k)}(r)}{k! f'(r)} \tag{9}$$

As the reader can easily check the following regularities hold: All the iterations of Table 1 of order of convergence two or more satisfy that $G(r) = 1$; the ones of order of convergence three or more satisfy that $G(r) = 1$ and $G'(r) = b_2$; and, the ones of order of convergence four or more satisfy that $G(r) = 1$, $G'(r) = b_2$ and $G^{(2)}(r) = -2b_2^2 + 4b_3$.

At sight of such regularities one might ask whether it is possible to find the conditions for G to increase the convergence order of (2) more and more, even unlimitedly; and whether, once known such conditions, such a function G could be built in an explicit way.

In order to response these questions, in this paper we are going to build a family of formulas of iteration given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} G_m(x_n),\ 1 \le m < \infty \tag{10}$$

Table 1: Regularities of $G(x)$

	$G(r)$	$G'(r)$	$G^{(2)}(r)$	Order of convergence
Jarrat	$G_J(r) = 1$	$G'_J(r) = b_2$	$G_J^{(2)}(r) = -2b_2^2 + 4b_3$	4
Euler	$G_E(r) = 1$	$G'_E(r) = b_2$	$G_E^{(2)}(r) \neq -2b_2^2 + 4b_3$	3
Ostrowski	$G_O(r) = 1$	$G'_O(r) = b_2$	$G_O^{(2)}(r) \neq -2b_2^2 + 4b_3$	3
Hally	$G_H(r) = 1$	$G'_H(r) = b_2$	$G_H^{(2)}(r) \neq -2b_2^2 + 4b_3$	3
Newton	$G_N(r) = 1$	$G'_N(r) \neq b_2$	$G_N^{(2)}(r) \neq -2b_2^2 + 4b_3$	2

where G_m are functions to be determine later, in such a way that the $m - th$ formula of iteration has order of convergence $(m+1)$ at the expense of only one derivative evaluation more in relation to the previous one.

On the other hand, all the papers in the background literature about this subject, for solving either a single or a system of nonlinear equations, they all start with a guessed initial approximation, but no global procedure is provided in order to find such a convenient approximation for guaranteeing the convergence of the iteration process.

In other to improve this problem we are going to introduce a new algorithm with the role of root-seeker in an interval as great as needed, that can be composed with iterations (10) as an only algorithm of global convergence.

This paper is organized as follows. In Section 2, we collect some recent results which the following are based on. In the next Sections 3 and 4 we introduce the PMI and analyze its order of convergence and its local convergence. In Section 5 we derive the PMII and analyze its convergence. In Section 6 we compose PMI and PMII as a new algorithm of global convergence, for solving all the roots of the scalar functions $f(x)$ in a feasible interval. In Section 7 we carry out a comparison with other methods, finishing with the conclusions of the last section.

2 Some recent results

Some recent results published by the author in [13] are resumed, that will be needed throughout this writing.

Note 1 *Let $P(x)$ be a polynomial function of m degree given by:*

$$y = P(x) = a_0 + a_1 x + \cdots + a_m x^m \quad (a_m \neq 0) \tag{11}$$

If the inequality:

$$\frac{m^2}{m-1}\left|\frac{a_0 a_2}{a_1^2}\right| + \frac{m^3}{(m-1)^2}\left|\frac{a_0^2 a_3}{a_1^3}\right| + \cdots + \frac{m^m}{(m-1)^{m-1}}\left|\frac{a_0^{m-1} a_m}{a_1^m}\right| < 1 \tag{12}$$

holds, then $P(x)$ has a real root, r, given by the absolutely convergent series:

$$r = \frac{a_0}{-a_1}\sum_{n=0}^{\infty}\sum_{q_2+\ldots+q_m=n} d(q_2,...,q_m)\left(\frac{a_0 a_2}{(-a_1)^2}\right)^{q_2}\cdots\left(\frac{a_0^{m-1} a_m}{(-a_1)^m}\right)^{q_m} \tag{13}$$

$d(q_2,...,q_m)$ being:

$$d(q_2,...,q_m) = \frac{(2q_2 + 3q_3 + \cdots + mq_m)!}{(q_2 + 2q_3 + \cdots + (m-1)q_m + 1)!\, q_2! q_3! \cdots q_m!} \tag{14}$$

with $q_2,...,q_m$, non-negative integers.

Note 2 *Besides, r is either the smallest positive root, if $a_0/(-a_1) > 0$, or the greatest negative root, if $a_0/(-a_1) < 0$.*

On the contrary, if $P(x)$ has a real simple root, r, and the series (13) is not convergent, we provided a solution to this question by shifting the polynomial, $P(x)$, throughout the X-axis, according to the following Note.

Note 3 *We define the function:*

$$C_P(x) = \frac{m^2}{m-1}\left\|\frac{P^{(2)}(x)P(x)}{2!P'^2(x)}\right\| + \frac{m^3}{(m-1)^2}\left\|\frac{P^{(3)}(x)P^2(x)}{3!P'^3(x)}\right\| + ... + \frac{m^m}{(m-1)^{m-1}}\left\|\frac{P^{(m)}(x)P^{m-1}(x)}{m!P'^m(x)}\right\| \quad (15)$$

Consider $\overline{x} \in \mathcal{R}$ sufficiently close to r to satisfy the inequality:

$$C_P(\overline{x}) < 1 \quad (16)$$

Let us write the Taylor's formula of $P(x)$ around $\overline{x}$:

$$P(y) = P(\overline{x}) + P'(\overline{x})(y-\overline{x}) + \frac{P''(\overline{x})}{2!}(y-\overline{x})^2 + ... + \frac{P^{(m)}(\overline{x})}{m!}(y-\overline{x})^m \quad (17)$$

Changing $(y-\overline{x})$ by x in (17) leads to:

$$P_{\overline{x}}(x) = P(\overline{x}+x) = P(\overline{x}) + P'(\overline{x})x + \frac{P''(\overline{x})}{2!}x^2 + ... + \frac{P^{(m)}(\overline{x})}{m!}x^m \quad (18)$$

From (16) $P_{\overline{x}}$ satisfies (12), and we arrive at:

$$r = \overline{x} + \frac{P(\overline{x})}{-P'(\overline{x})}\sum_{p=0}^{\infty}\sum_{q_2+\cdots+q_m=p} d(q_2\cdots q_m)\left(\frac{P(\overline{x})P''(\overline{x})}{2!(-P'(\overline{x}))^2}\right)^{q_2}\left(\frac{P(\overline{x})^2P^{(3)}(\overline{x})}{3!(-P'(\overline{x}))^3}\right)^{q_3}\cdots\left(\frac{P(\overline{x})^{m-1}P^{(m)}(\overline{x})}{m!(-P'(\overline{x}))^m}\right)^{q_m} \quad (19)$$

Note 4 *As in Note 2, r is either the closest root to $\overline{x}$ on the right ($\overline{x} < r$) if $P(\overline{x})/(-P'(\overline{x})) > 0$ or, on the left ($\overline{x} > r$), if $P(\overline{x})/(-P'(\overline{x})) < 0$.*

Note 5 *If (12) or (16) hold, then in consonance with Proposition 1 of [13], the inequalities:*

$$\left|\frac{a_0}{-a_1}\right|\sum_{n=0}^{\infty}\sum_{q_2+...+q_m=n} d(q_2,...,q_m)\left|\frac{a_0a_2}{(-a_1)^2}\right|^{q_2}...\left|\frac{a_0^{m-1}a_m}{(-a_1)^m}\right|^{q_m} \le \frac{|a_0|}{|a_1|}\frac{m}{m-1}; \quad or \quad (20)$$

$$\left|\frac{P(x)}{P'(x)}\right|\sum_{p=0}^{\infty}\sum_{q_2+\cdots+q_m=p} d(q_2\cdots q_m)\left|\frac{P(x)P''(x)}{2!P'^2(x)}\right|^{q_2}\left|\frac{P(x)^2P^{(3)}(x)}{3!P'^3(x)}\right|^{q_3}...\left|\frac{P(x)^{m-1}P^{(m)}(x)}{m!P'^m(x)}\right|^{q_m} \le \left|\frac{P(x)}{P'(x)}\right|\frac{m}{m-1} \quad (21)$$

are respectively verified.

Having done this, hereafter, we introduce the new results of this work.

3 Convergence velocity study of the PMI

For the clarity of the exposition, given a sufficiently differentiable function f or a polynomial P, let us denominate:

$$a_k(x) = \frac{f^{(k)}(x)}{k!f'(x)} \text{ and } b_k(x) = \frac{P^{(k)}(x)}{k!P'(x)} \quad (22)$$

respectively.

Lemma 1 *If polynomial (11) has a simple real zero, r, then there exists a neighborhood of r, U^r, where r is the only root of (11), $P'(x)$ preserves sign and such that the function given by:*

$$g(x) = x + \frac{P(x)}{-P'(x)} \sum_{p=0}^{\infty} \sum_{q_2+\cdots+q_m=p} d(q_2 \cdots q_m) \left(\frac{P(x)P''(x)}{2!(-P'(x))^2}\right)^{q_2} \left(\frac{P(x)^2 P^{(3)}(x)}{3!(-P'(x))^3}\right)^{q_3} \cdots \left(\frac{P(x)^{m-1} P^{(m)}(x)}{m!(-P'(x))^m}\right)^{q_m} \quad (23)$$

satisfies that $g(x) = r$, $\forall\, x \in U^r$.

Proof.

(16) holds at $\overline{x} = r$ since $C_P(r) = 0 < 1$, therefore there exists a neighborhood of r, U^r, where (16) is verified; what implies that r is the only root of f in U^r. Indeed, let $R_1 \in U^r$ be the closest one to r, then due to Rolle's Theorem there would be one point $\alpha \in U^r$ such that $C_P(\alpha) = \infty$, what is a contradiction. Consequently $P'(x)$ preserves sign in U^r.

Now, take any $x \in U^r$. As $C_P(x) < 1$, then $g(x)$ converges to a root of $P(x)$, say R_2, but $R_2 \in U^r$, because if $P(x)/(-P'(x)) > 0$ r and R_2 are greater than x (take into account that $P'(x)$ preserves sign in U^r) and, in agreement with Note 4, $x < R_2 \leq r$, and as a consequence $R_2 \in U^r$; on the contrary, if $P(x)/(-P'(x)) < 0$ r and R_2 are lower than x and, in agreement with Note 4, $x > R_2 \geq r$, and $R_2 \in U^r$. Then we can conclude that $r = R_2$ and that $g(x) = r$ for all $x \in U^r$.

Lemma 2 *In agreement with (22), series (23) can be rearranged as:*

$$g(x) = x - \frac{P(x)}{P'(x)} \sum_{p=0}^{\infty} A_p(x) \left(\frac{P(x)}{P'(x)}\right)^p \quad (24)$$

where

$$A_p(x) = \sum_{q_2+2q_3+\ldots+(m-1)q_m=p} (-1)^{2q_2+\ldots+mq_m} d(q_2, \ldots, q_m) b_2(x)^{q_2} b_3(x)^{q_3} \ldots b_m(x)^{q_m};\ p = 0,\ 1,\ 2,\ \ldots \quad (25)$$

Before proving it, for the sake of clarity in the exposition, we introduce the following Note to this Lemma.

Note 6

$$A_0(x) = \sum_{q_2+2q_3+\ldots+(m-1)q_m=0} (-1)^{2q_2+\ldots+mq_m} d(q_2, \ldots, q_m) b_2(x)^{q_2} b_3(x)^{q_3} \ldots b_m(x)^{q_m} \quad (26)$$

in such a way that $q_2 = q_3 = \cdots = q_m = 0$, and $A_0(x)$ is left as:

$$A_0(x) = 1 \quad (27)$$

Regarding to A_1:

$$A_1(x) = \sum_{q_2+2q_3+\ldots+(m-1)q_m=1} (-1)^{2q_2+\ldots+mq_m} d(q_2, \ldots, q_m) b_2(x)^{q_2} b_3(x)^{q_3} \ldots b_m(x)^{q_m} \quad (28)$$

therefore $q_2 = 1$, $q_3 = \cdots = q_m = 0$ and $A_1(x)$ is left as:

$$A_1(x) = (-1)^2 d(1, 0\ldots, 0) b_2(x) \quad (29)$$

Observe that the maximum order of differentiation that appears in $A_1(x)$ is $P^{(2)}(x)$.

With respect to $A_2(x)$:

$$A_2(x) = \sum_{q_2+2q_3+\ldots+(m-1)q_m=2} (-1)^{2q_2+\ldots+mq_m} d(q_2, \ldots, q_m) b_2(x)^{q_2} b_3(x)^{q_3} \ldots b_m(x)^{q_m} \quad (30)$$

what means that $q_4 = \cdots = q_m = 0$ *and* $A_2(x)$ *is left as:*

$$A_2(x) = \sum_{q_2+2q_3=2} (-1)^{2q_2+3q_3} d(q_2, q_3, 0, ..., 0) b_2(x)^{q_2} b_3(x)^{q_3} \tag{31}$$

Observe that the maximum order of differentiation that appears in $A_2(x)$ *is* $P^{(3)}(x)$*.*

Continuing this process so far $A_{m-2}(x)$*:*

$$A_{m-2}(x) = \sum_{q_2+...+(m-1)q_m=m-2} (-1)^{2q_2+...+mq_m} d(q_2, ..., q_m) b_2(x)^{q_2} b_3(x)^{q_3} ... b_m(x)^{q_m} \tag{32}$$

in other words q_m *must be equal to zero, and* $A_{m-2}(x)$ *is left as:*

$$A_{m-2}(x) = \sum_{q_2+...+(m-2)q_{m-1}=m-2} (-1)^{2q_2+...+(m-1)q_{m-1}} d(q_2, ..., q_{m-1}, 0) b_2(x)^{q_2} b_3(x)^{q_3} ... b_{m-1}(x)^{q_{m-1}} \tag{33}$$

Observe that the maximum order of differentiation that appears in A_{m-2} *is* $P^{(m-1)}(x)$*.*

Finally, $A_{m-1}(x)$ *is the first term of the series where* $P^{(m)}(x)$ *appears:*

$$A_{m-1}(x) = \sum_{q_2+...+(m-1)q_m=m-1} (-1)^{2q_2+...+mq_m} d(q_2, ..., q_m) b_2(x)^{q_2} b_3(x)^{q_3} ... b_m(x)^{q_m} \tag{34}$$

Next, we accomplish the proof of the Lemma.

Proof.

According to (22), series (23) turns out:

$$g(x) = x + \frac{P(x)}{-P'(x)} \sum_{p=0}^{\infty} \sum_{q_2+\cdots+q_m=p} (-1)^{2q_2+3q_3+...+mq_m} d(q_2, ..., q_m) (b_2(x))^{q_2} \cdots (b_m(x))^{q_m} \left(\frac{P(x)}{P'(x)}\right)^{q_2+2q_3+\cdots+(m-1)q_m} \tag{35}$$

making $q_2 + 2q_3 + \cdots + (m-1)q_m = n$ the terms of the series can be rearranged in the way:

$$g(x) = x + \frac{P(x)}{-P'(x)} \sum_{n=0}^{\infty} \sum_{i_2+2i_3+...+(m-1)i_m=n} (-1)^{2i_2+3i_3+...+mi_m} d(i_2, ..., i_m) (b_2(x))^{i_2} \cdots (b_m(x))^{i_m} \left(\frac{P(x)}{P'(x)}\right)^n \tag{36}$$

In fact, each term of the series (35) given by:

$$A(q_2, ..., q_m) = (-1)^{2q_2+3q_3+...+mq_m} d(q_2, ..., q_m) (b_2(x))^{q_2} \cdots (b_m(x))^{q_m} \left(\frac{P(x)}{P'(x)}\right)^{q_2+2q_3+\cdots+(m-1)q_m} \tag{37}$$

with $q_2 + \cdots + q_m = p$ matches the term of the series (36) given by:

$$B(i_2, ..., i_m) = (-1)^{2i_2+3i_3+...+mi_m} d(i_2, ..., i_m) (b_2(x))^{i_2} \cdots (b_m(x))^{i_m} \left(\frac{P(x)}{P'(x)}\right)^n \tag{38}$$

with $i_2 = q_2$, ..., $i_m = q_m$ and $i_2 + 2i_3 + ... + (m-1)i_m = n$.

Conversely, each term of the series (36), $B(i_2, ..., i_m)$, with $i_2 + 2i_3 + ... + (m-1)i_m = v$, matches the term of the series $A(q_2, ..., q_m)$, with $q_2 = i_2$, ..., $q_m = i_m$ and $q_2 + \cdots + q_m = t$. And the result follows.

Lemma 3 *Under the same hypothesis as Lemma 1, the function:*

$$h_m(x) = x - \sum_{p=0}^{m-1} A_p(x) \left(\frac{P(x)}{P'(x)}\right)^{p+1} \tag{39}$$

satisfies the equalities:

$$h_m^{(i)}(r) = 0; \text{ for } i = 1,\ 2,\ ...,\ m \tag{40}$$

Proof.

Consider the function:

$$h(x) = \sum_{p=m}^{\infty} A_p(x) \left(\frac{P(x)}{P'(x)} \right)^{p+1} \tag{41}$$

then, (24) becomes:

$$g(x) = h_m(x) + h(x) \tag{42}$$

In agreement with Lemma 1, $g(x)$ is constant for all $x \in U^r$, therefore:

$$g^{(i)}(x) = h_m^{(i)}(x) + h^{(i)}(x) = 0 \tag{43}$$

for all $i > 0$. As

$$\left. \frac{d^i}{dx} \left(\frac{P(x)}{P'(x)} \right)^{p+1} \right|_{x=r} = 0, \ \forall \ (p+1) > i \tag{44}$$

As in the function $h(x)$, $p+1 \geq m+1 > m$, then $h^{(i)}(r) = 0$ for all $i \leq m$, and from (43) the result follows.

Definition 1 *Let f be a sufficient differentiable real function, then we introduce the infinity family of functions F_m as:*

$$F_m(x) = x - \frac{f(x)}{f'(x)} \sum_{p=0}^{m-1} B_p(x) \left(\frac{f(x)}{f'(x)} \right)^p = x - \sum_{p=0}^{m-1} B_p(x) \left(\frac{f(x)}{f'(x)} \right)^{p+1} \tag{45}$$

with m=1, 2, $\cdots$

$$B_p(x) = \sum_{q_2+2q_3+...+(m-1)q_m=p} (-1)^{2q_2+...+mq_m} d(q_2, ..., q_m) a_2(x)^{q_2} a_3(x)^{q_3} ... a_m(x)^{q_m} \tag{46}$$

and $a_k(x)$, $2 \leq k \leq m$, introduced in (22).

Theorem 1 *Given an integer $m \geq 1$ and $F_m(x)$, introduced in (45), let r be a simple real root of a sufficiently differentiable function, f, then each i-th derivative, $1 \leq i \leq m$, of function (45), valued at r, equals zero.*

Proof.

The Taylor polynomial of m degree of the function f around r is:

$$P_1(y) = f(r) + f'(r)(y-r) + \frac{f^{(2)}(r)}{2!}(y-r)^2 + ... + \frac{f^{(m)}(r)}{m!}(y-r)^{m+1} \tag{47}$$

Obviously, r is also a simple root of P_1 and consequently Polynomial (47) holds the hypothesis of Lemma 1 and Lemma 3. In order words:

$$h_m(x) = x - \sum_{p=0}^{m-1} A_p(x) \left(\frac{P_1(x)}{P_1'(x)} \right)^{p+1} \tag{48}$$

satisfies the equalities:

$$h_m^{(i)}(r) = 0; \text{ for } i = 1, \ 2, \ ..., \ m \tag{49}$$

$F_m(x)$ and $h_m(x)$ have the same rational structure, the one in the variables $f(x)$, $f'(x)$, ..., $f^{(m)}(x)$ and in the variables $P_1(x)$, $P_1'(x)$, ..., $P_1^{(m)}(x)$, the other one. Therefore, due to derivation rules, its derivatives $F_m'(x)$

and $h'_m(x)$, $F_m^{(2)}(x)$ and $h_m^{(2)}(x)$, ..., $F_m^{(m)}(x)$ and $h_m^{(m)}(x)$ have also the same rational structure respectively: R_1, R_2,..., R_m, concretely at $x = r$ and taking into account that:

$$\left.\frac{d^j\left(\dfrac{f(x)}{f'(x)}\right)^i}{dx^j}\right|_{x=r} = 0; \text{ with } i > j \tag{50}$$

$$\begin{aligned}
F_m^{(j)}(x)\Big|_{x=r} &= \frac{d^j}{dx^j}\left(x - \sum_{p=0}^{m-1} B_p(x)\left(\frac{f(x)}{f'(x)}\right)^{p+1}\right)\Bigg|_{x=r} \\
&= \frac{d^j}{dx^j}(x) - \sum_{p=0}^{m-1}\frac{d^j}{dx^j}\left(B_p(x)\left(\frac{f(x)}{f'(x)}\right)^{p+1}\right)\Bigg|_{x=r} \\
&= \frac{d^j}{dx^j}(x) - \sum_{p=0}^{m-1}\sum_{k=0}^{j}\binom{j}{k}\frac{d^{j-k}}{dx^{j-k}}\left[B_p(x)\right]\frac{d^k}{dx^k}\left[\left(\frac{f(x)}{f'(x)}\right)^{p+1}\right]\Bigg|_{x=r} \\
&= \frac{d^j}{dx^j}(x) - \sum_{p=0}^{j-1}\sum_{k=p+1}^{j}\binom{j}{k}\frac{d^{j-k}}{dx^{j-k}}\left[B_p(x)\right]\frac{d^k}{dx^k}\left[\left(\frac{f(x)}{f'(x)}\right)^{p+1}\right]\Bigg|_{x=r}
\end{aligned} \tag{51}$$

Each term of sum (51) is in the form:

$$\binom{j}{k}\frac{d^{j-k}}{dx^{j-k}}\left[B_p(x)\right]\frac{d^k}{dx^k}\left[\left(\frac{f(x)}{f'(x)}\right)^{p+1}\right]\Bigg|_{x=r}; \text{ with } 0 \le p \le j-1; \quad p+1 \le k \le j \tag{52}$$

which, in agreement with Note 6, either they are equal to zero, or they have j as maximum order of derivation of f. Therefore:

$$F_m^{(2)}(x)\Big|_{x=r} = R_2\left(f(r), f'(r), f^{(2)}(r)\right) = R_2\left(P_1(r), P_1'(r), P_1^{(2)}(r)\right) = h_m^{(2)}(r) = 0 \tag{53}$$

since $f(r)$=$P_1(r)$, $f'(r)$=$P_1'(r)$ and $f^{(2)}(r)$=$P_1^{(2)}(r)$. And in the same way it is proven that:

$$F_m^{(m)}(x)\Big|_{x=r} = R_m\left(f(r), f'(r), ..., f^{(m)}(r)\right) = R_m\left(P_1(r), P_1'(r), ..., P_1^{(m)}(r)\right) = h_m^{(m)}(r) = 0 \tag{54}$$

since $f(r)$=$P_1(r)$, $f'(r)$=$P_1'(r)$, ..., $f^{(m)}(r)$=$P_1^{(m)}(r)$. And the result follows.

Corollary 1 *Given an integer number $m > 0$, let r be a simple zero of a sufficiently differentiable function f. If there exits a close enough α to r, in such a way that the sequence:*

$$x_0 = \alpha; \; x_{n+1} = F_m(x_n), \forall\, n > 0 \tag{55}$$

is convergent, where F_m was defined in (45), then it has convergence order $m + 1$.

Proof.

The result follows from Taylor's Formula:

$$\begin{aligned}
F(x_n) &= F(r) + F'(r)(x_n - r) + \cdots + \frac{F^{(m)}(r)}{(m+1)!}(x_n - r)^m + \quad \frac{F^{(m+1)}(\gamma)}{(m+1)!}(x_n - r)^{m+1} \\
&\Rightarrow x_{n+1} = r + \frac{F^{(m+1)}(\gamma)}{(m+1)!}(x_n - r)^{m+1} \\
&\Rightarrow \frac{x_{n+1} - r}{(x_n - r)^{m+1}} = \frac{F^{(m+1)}(\gamma)}{(m+2)!}
\end{aligned}$$

4 Study of the local convergence of the PMI

Lemma 4 *Fixed an integer $m > 0$, let r be a simple real root of f, then there exit an $1 > \epsilon > 0$ and a neighborhood of r, $V^r = [r-\epsilon, r+\epsilon]$, such that for all $x \in [r-\epsilon, r+\epsilon]$ the inequalities:*

$$f'(x) \neq 0 \tag{56}$$

$$\frac{m}{m-1}\frac{|f(x)|}{|f'(x)|} \leq \epsilon \tag{57}$$

$$\begin{aligned} C_f(x) &= \frac{m^2}{m-1}\left|\frac{f^{(2)}(x)f(x)}{2!f'^2(x)}\right| + \frac{m^3}{(m-1)^2}\left|\frac{f^{(3)}(x)f^2(x)}{3!f'^3(x)}\right| + \dots + \frac{m^m}{(m-1)^{m-1}}\left|\frac{f^{(m)}(x)f^{m-1}(x)}{m!f'^m(x)}\right| \\ &= |c_2(x)|\left|\frac{f(x)}{f'(x)}\frac{m}{m-1}\right| + |c_3(x)|\left|\frac{f(x)}{f'(x)}\frac{m}{m-1}\right|^2 + \cdots |c_m(x)|\left|\frac{f(x)}{f'(x)}\frac{m}{m-1}\right|^m < \frac{1}{m} \end{aligned} \tag{58}$$

$$|F'_m(x)| < \epsilon \tag{59}$$

hold.

Lemma 5 *Under the same hypothesis as Lemma 4, for all $\alpha \in V^r$ the inequality:*

$$\left|B_1(\alpha)\left(\frac{f(\alpha)}{f'(\alpha)}\right)\right| + \left|B_2(\alpha)\left(\frac{f(\alpha)}{f'(\alpha)}\right)^2\right| + \dots + \left|B_{m-1}(\alpha)\left(\frac{f(\alpha)}{f'(\alpha)}\right)^{m-1}\right| \leq \frac{1}{m-1} \tag{60}$$

holds.

Proof.
Let P be the Taylor polynomial of order m of f around α:

$$P(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \dots + \frac{f^{(m)}(\alpha)}{m!}(x-\alpha)^m \tag{61}$$

As $C_f(\alpha) < 1$, from Notes 3, 5 and Lemma 2 the following series is well defined and satisfies:

$$\begin{aligned} &\left\|\frac{f(\alpha)}{f'(\alpha)}\right\|\sum_{p=0}^{\infty}\left\|B_p(\alpha)\left(\frac{f(\alpha)}{f'(\alpha)}\right)^p\right\| \\ &= \left\|\frac{f(\alpha)}{f'(\alpha)}\right\|\sum_{p=0}^{\infty}\sum_{q_2+\cdots+q_m=f} d(q_2\cdots q_m)\left\|\frac{f(\alpha)f''(\alpha)}{2!f'^2(\alpha)}\right\|^{q_2}\left\|\frac{f(\alpha)^2f^{(3)}(\alpha)}{3!f'^3(\alpha)}\right\|^{q_3}\cdots\left\|\frac{f(\alpha)^{m-1}f^{(m)}(\alpha)}{m!f'^m(\alpha)}\right\|^{q_m} \\ &\leq \left\|\frac{f(\alpha)}{f'(\alpha)}\right\|\frac{m}{m-1} \end{aligned} \tag{62}$$

Therefore:

$$\sum_{p=1}^{m-1}\left\|B_p(\alpha)\left(\frac{f(\alpha)}{f'(\alpha)}\right)^p\right\| \leq \sum_{p=1}^{\infty}\left\|B_p(\alpha)\left(\frac{f(\alpha)}{f'(\alpha)}\right)^p\right\| \leq \frac{1}{m-1} \tag{63}$$

Lemma 6 *Under the same hypothesis as Lemma 4*

$$F_m(V^r) \subset V^r \tag{64}$$

is verified.

Proof.
Suppose, without loss of generality, that $\forall\, x \in V^r$ such that $x < r$ is $f(x) > 0$ and, on the contrary, $\forall\, x \in V^r$ such that $x > r$ is $f(x) < 0$. We take:

$$F_m(x) = x - \frac{f(x)}{f'(x)} \sum_{p=0}^{m-1} B_p(x) \left(\frac{f(x)}{f'(x)}\right)^p = x - \frac{f(x)}{f'(x)} - \frac{f(x)}{f'(x)} \sum_{p=1}^{m-1} B_p(x) \left(\frac{f(x)}{f'(x)}\right)^p \tag{65}$$

First. Assume that $x < r$ then, on the one hand, (65) becomes:

$$F_m(x) = x - \frac{f(x)}{f'(x)} \sum_{p=0}^{m-1} B_p(x) \left(\frac{f(x)}{f'(x)}\right)^p \leq x + \left|\frac{f(x)}{f'(x)}\right| \sum_{p=0}^{m-1} \left|B_p(x) \frac{f(x)}{f'(x)}\right|^p \leq x + \left|\frac{f(x)}{f'(x)}\right| \frac{m}{m-1} \leq r + \epsilon \tag{66}$$

and on the other hand:

$$\begin{aligned} F_m(x) &= x - \frac{f(x)}{f'(x)} - \frac{f(x)}{f'(x)} \sum_{p=1}^{m-1} B_p(x) \left(\frac{f(x)}{f'(x)}\right)^p = x + \left|\frac{f(x)}{f'(x)}\right| + \left|\frac{f(x)}{f'(x)}\right| \sum_{p=1}^{m-1} B_p(x) \left(-\left|\frac{f(x)}{f'(x)}\right|\right)^p \\ &\geq x + \left|\frac{f(x)}{f'(x)}\right| - \left|\frac{f(x)}{f'(x)}\right| \sum_{p=1}^{m-1} \left|B_p(x) \frac{f(x)}{f'(x)}\right|^p \geq x + \left|\frac{f(x)}{f'(x)}\right| - \left|\frac{f(x)}{f'(x)}\right| \frac{1}{m-1} \geq x \end{aligned} \tag{67}$$

Second. Assume that $x > r$ then, on the one hand, (65) becomes:

$$\begin{aligned} F_m(x) &= x - \frac{f(x)}{f'(x)} - \frac{f(x)}{f'(x)} \sum_{p=1}^{m-1} B_p(x) \left(\frac{f(x)}{f'(x)}\right)^p = x - \left|\frac{f(x)}{f'(x)}\right| - \left|\frac{f(x)}{f'(x)}\right| \sum_{p=1}^{m-1} B_p(x) \left|\frac{f(x)}{f'(x)}\right|^p \\ &\leq x - \left|\frac{f(x)}{f'(x)}\right| + \left|\frac{f(x)}{f'(x)}\right| \sum_{p=1}^{m-1} \left|B_p(x) \frac{f(x)}{f'(x)}\right|^p \leq x - \left|\frac{f(x)}{f'(x)}\right| + \left|\frac{f(x)}{f'(x)}\right| \frac{1}{m-1} \leq x \end{aligned} \tag{68}$$

and on the other hand:

$$\begin{aligned} F_m(x) &= x - \frac{f(x)}{f'(x)} \sum_{p=0}^{m-1} B_p(x) \left(\frac{f(x)}{f'(x)}\right)^p = x - \left|\frac{f(x)}{f'(x)}\right| \sum_{p=0}^{m-1} B_p(x) \left|\frac{f(x)}{f'(x)}\right|^p \\ &\geq x - \left|\frac{f(x)}{f'(x)}\right| \sum_{p=0}^{m-1} \left|B_p(x) \frac{f(x)}{f'(x)}\right|^p \geq x - \left|\frac{f(x)}{f'(x)}\right| \frac{m}{m-1} \geq r - \epsilon \end{aligned} \tag{69}$$

Theorem 2 *The sequence:*

$$x_{n+1} = F_m(x_n) \tag{70}$$

converges to r*,* x_0 *being any* $x \in V^r$*.*

Proof.
In agreement with Lemma 6 $x_n \in V^r$, $\forall\, n \geq 0$, and the result follows from Lipschitz condition, since $\forall\, n \geq 0$ sequence (70) satisfies:

$$|x_n - r| = |F_m(x_{n-1}) - F_m(r)| = |F'_m(\beta_n)|\, |x_{n-1} - r| \leq L\, |x_{n-1} - r| \tag{71}$$

with $\beta_n \in V^r$ and, therefore, with $L < 1$

As a consequence of all these results, we introduce the following note, which only aims to introduce a reflection to be developed in future works:

Note 7 *Take* $m = \infty$ *in (45) and consider* x_0 *and* f *in such a way that the series:*

$$r = F_\infty(x_0) = x_0 - \sum_{p=0}^{\infty} B_p(x_0) \left(\frac{f(x_0)}{f'(x_0)}\right)^{p+1} \tag{72}$$

makes sense, then is r *a root of* f*, as in the case of polynomial functions, according to (23) and (24)?*

5 The PMII and the analyze of its convergence

As well known, one of the main problems dealing with iterative methods for solving nonlinear equations is the initialization of the iteration. In this section we introduce the PMII to initialize the search of solutions in a given interval, in such a way that the convergence does not depend on the chosen initial value. The PMII can be used not only for starting the process, but also as a root-solver by its own.

Definition 2 *Let f be a function in $\mathcal{C}^2((c,d))$, and $[a,b] \subset (c,d)$, with $f(a),\ f(b) \neq 0$; let $C_1,\ C_2 \in \mathcal{R}$ be real numbers different to zero and such that the inequalities $C_1 \leq f''(x) \leq C_2$ hold for all $x \in [a,b]$; and let $x_0 \in [a,b]$ be such that $f(x_0),\ f'(x_0) \neq 0$.*

Then we define:

1. *the real number m:*

$$m = \min\left\{\frac{C_1 f(x_0)}{2!(-f'(x_0))^2},\ \frac{C_2 f(x_0)}{2!(-f'(x_0))^2}\right\} \neq 0; \tag{73}$$

2. *the parabola:*

$$L : \mathcal{R} \to \mathcal{R};\ L(z) = mz^2 - z + 1; \tag{74}$$

3. *and the open interval:*

$$J_{x_0} = \begin{cases} \left(a-\epsilon,\ x_0 - \frac{f(x_0)}{f'(x_0)} r_1\right);\ \textit{if}\ \frac{f(x_0)}{f'(x_0)} < 0\ \textit{and}\ 0 < m \leq 1/4 \\ \left(x_0 - \frac{f(x_0)}{f'(x_0)} r_1, b+\epsilon\right);\ \textit{if}\ \frac{f(x_0)}{f'(x_0)} > 0\ \textit{and}\ 0 < m \leq 1/4 \\ \left(x_0 - \frac{f(x_0)}{f'(x_0)} r_1,\ x_0 - \frac{f(x_0)}{f'(x_0)} r_2\right);\ \textit{if}\ \frac{f(x_0)}{f'(x_0)} < 0\ \textit{and}\ 0 > m \\ \left(x_0 - \frac{f(x_0)}{f'(x_0)} r_2, x_0 - \frac{f(x_0)}{f'(x_0)} r_1\right);\ \textit{if}\ \frac{f(x_0)}{f'(x_0)} > 0\ \textit{and}\ 0 > m \end{cases} \tag{75}$$

   where r_1 and r_2 are the real roots of L, when $m \leq 1/4$, with $r_1 \leq r_2$; and $\epsilon > 0$, small enough to guarantee that $f(x) \neq 0\ \forall\ x \in (a-\epsilon, a) \cup (b, b+\epsilon) \subset (c,d)$.

Theorem 3 *If the hypothesis of Definition 2 hold, then:*

1. *$J_{x_0} \neq \emptyset$;*
2. *if $m > \frac{1}{4}$, there is not any root of f in $[a,b]$;*
3. *if $\frac{1}{4} \geq m > 0$, there is not any root of f in the interval $J_{x_0} \cap [a,b] \neq \emptyset$;*
4. *if $m < 0$, there is not any root of f in the interval $J_{x_0} \cap [a,b] \neq \emptyset$.*

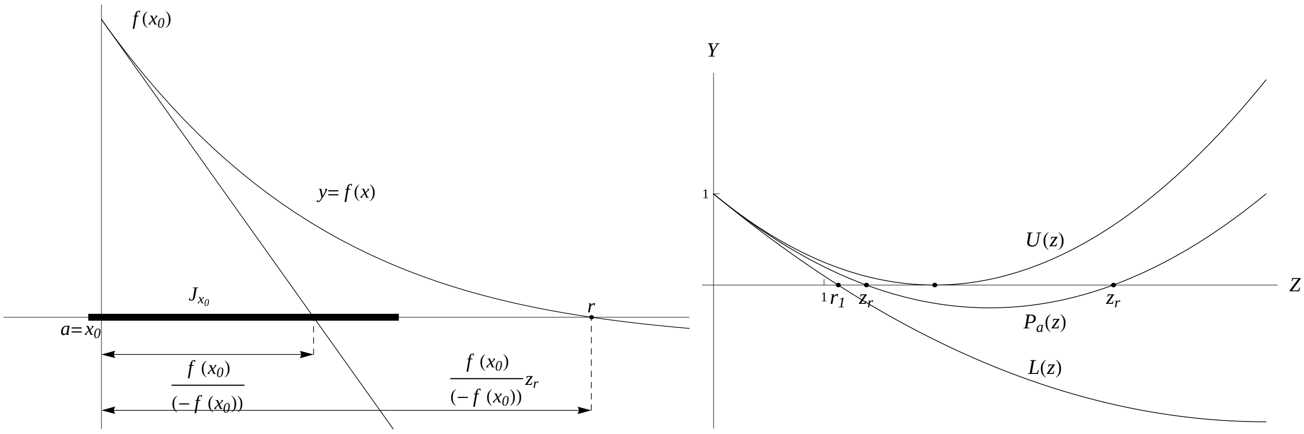
Proof.
First: It is obvious, since $r_1,\ r_2 \neq 0$, $x_0 \in [a,b]$ and $\frac{f(x_0)}{f'(x_0)} \neq 0$

Second: We proceed by absurd reduction method. Suppose that there is a root, r, in (a,b), then there exists a real number z_r that satisfies:

$$r = x_0 + \frac{f(x_0)}{-f'(x_0)} z_r \tag{76}$$

Using Taylor's Formula:

$$\begin{aligned} 0 = f(r) &= f(x_0) + f'(x_0)\frac{f(x_0)}{-f'(x_0)} z_r + \frac{f''(\beta)}{2!}\left(\frac{f(x_0)}{-f'(x_0)}\right)^2 z_r^2 \\ &= f(x_0)\left(1 - z_r + \frac{f''(\beta)}{2!}\frac{f(x_0)}{(-f'(x_0))^2} z_r^2\right) \end{aligned} \tag{77}$$



(a) J_{x_0} interval

(b) The three parabolas L, P_a, U

Figure 1: Case: $0 \leq m \leq \frac{1}{4}$

where $\beta \in (x_0, r)$, if $x_0 < r$ or $\beta \in (r, x_0)$ on the contrary. Therefore the parabola:

$$P_a(z) = 1 - z + az^2$$

with

$$a = \frac{f''(\beta)}{2!} \frac{f(x_0)}{(-f'(x_0)^2} \leq \frac{1}{4} \tag{78}$$

has a zero, z_r, and hence inequality (78) holds. Now, taking into account the parabola $L(z)$, introduced in (74), that satisfies the inequality:

$$L(z) = 1 - z + mz^2 \leq P_a(z) = 1 - z + az^2; \ \forall \ z \in \mathcal{R} \tag{79}$$

$m > 1/4$ being, we can deduce that $L(z) > 0$ and $P_a(z) > 0 \ \forall \ z \in \mathcal{R}$; contradiction that proves the second point of the theorem.

Third: First, we are going to prove that there is not any root in the subinterval of J_{x_0}:

$$\left(x_0, \ x_0 - \frac{f(x_0)}{f'(x_0)} r_1\right); \left(x_0 - \frac{f(x_0)}{f'(x_0)} r_1, x_0)\right) \tag{80}$$

If there is not any root the result is obviously true. On the contrary, suppose that there is one root of f in $[a, b]$ then, in agreement with (77), $P_a(z)$ has the root z_r, introduced in (76); and in consonance with inequality (78):

$$L(z) = 1 - z + mz^2 \leq P_a(z) = 1 - z + az^2 \leq U(z) = 1 - z + \frac{1}{4} z^2 \tag{81}$$

$U(z)$ has the double root $s_1 = s_2 = 2$, and as $0 < m \leq 1/4$, $L(z)$ has two positive roots that satisfy the inequalities $1 < r_1 \leq s_1 = 2 \leq r_2$; thus, from (81), $z_r \geq r_1 > 1$ and one arrives at:

$$\begin{aligned} &\text{if } \frac{f(x_0)}{f'(x_0)} < 0, \ r = x_0 - \frac{f(x_0)}{f'(x_0)} z_r \geq x_0 - \frac{f(x_0)}{f'(x_0)} r_1 > x_0 \to r \notin \left(x_0, x_0 - \frac{f(x_0)}{f'(x_0)} r_1\right) \neq \emptyset \\ &\text{if } \frac{f(x_0)}{f'(x_0)} > 0, \ r = x_0 - \frac{f(x_0)}{f'(x_0)} z_r \leq x_0 - \frac{f(x_0)}{f'(x_0)} r_1 < x_0 \to r \notin \left(x_0 - \frac{f(x_0)}{f'(x_0)} r_1, x_0\right) \neq \emptyset \end{aligned} \tag{82}$$

In order to show that there is not any root in the rest of the open interval J_{x_0}, we distinguish the following parts:

1. If $f(x_0)/f'(x_0) < 0$ and $f(x_0) > 0$, then $f(x) > 0, \forall\, x \in [a, x_0]$.

2. If $f(x_0)/f'(x_0) < 0$ and $f(x_0) < 0$, then $f(x) < 0, \forall\, x \in [a, x_0]$.

3. If $f(x_0)/f'(x_0) > 0$ and $f(x_0) > 0$, then $f(x) > 0, \forall\, x \in [x_0, b]$.

4. If $f(x_0)/f'(x_0) > 0$ and $f(x_0) < 0$, then $f(x) < 0, \forall\, x \in [x_0, b]$.

Regarding the first one, from Taylor's Formula, for any $x \in [a, x_0]$, we arrive at:

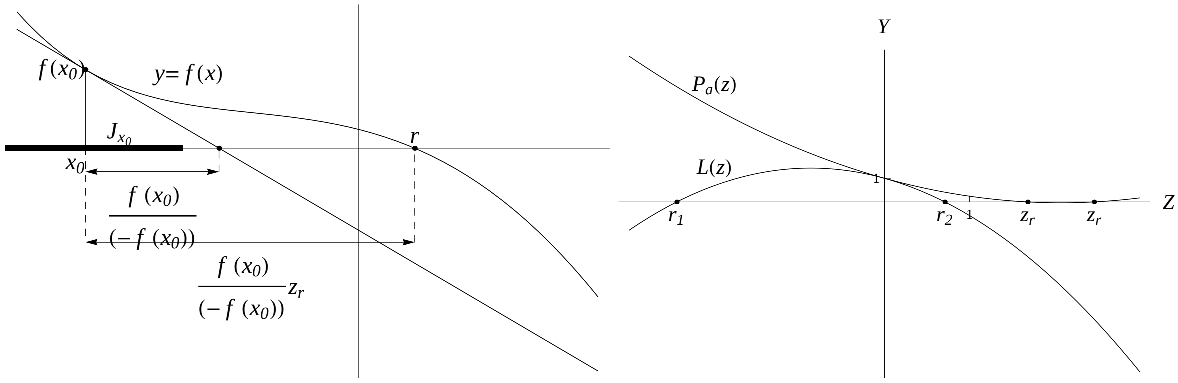
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\eta)}{2}(x - x_0)^2,\ \eta \in (x, x_0) \tag{83}$$

dividing by $(-f'(x_0)) > 0$:

$$\begin{aligned}\frac{f(x)}{-f'(x_0)} &= \frac{f(x_0)}{-f'(x_0)} + \frac{f'(x_0)}{-f'(x_0)}(x - x_0) + \frac{f''(\eta)}{2(-f'(x_0)}(x - x_0)^2 \\ &= \frac{f(x_0)}{-f'(x_0)} + x_0 - x + \frac{f''(\eta)}{2(-f'(x_0))}(x - x_0)^2\end{aligned} \tag{84}$$

From (73), as $m > 0$ and $f(x_0) > 0$, then $f''(x) > 0\ \forall\, x \in [a, b]$ and from (84) follows that $f(x) > 0$, $\forall\, x \in [a, x_0]$. The other three parts can be proven in a similar way, and to repeat the same reasoning is not worth.

Fourth: As in the previous case, if there is not any root of f in $[a, b]$, the result is obviously true. On the



(a) J_{x_0} interval (b) The two parabolas L, P_a

Figure 2: Case: $0 > m$

contrary, suppose that there is a root then, as $0 > m$, $L(z)$ has two real roots that satisfy the inequalities $r_1 < 0 < r_2 < 1$; thus $z_r \geq r_2$ or $z_r \leq r_1$ and one arrives at:

$$\begin{aligned} &r \notin \left(x_0 - \frac{f(x_0)}{f'(x_0)}r_1,\ x_0 - \frac{f(x_0)}{f'(x_0)}r_2\right) \neq \emptyset \text{ if } \frac{f(x_0)}{f'(x_0)} < 0, \text{ or} \\ &r \notin \left(x_0 - \frac{f(x_0)}{f'(x_0)}r_2, x_0 - \frac{f(x_0)}{f'(x_0)}r_1\right) \neq \emptyset \text{ if } \frac{f(x_0)}{f'(x_0)} > 0\end{aligned} \tag{85}$$

And the Theorem follows.

Definition 3 *Let f be a function in $\mathcal{C}^2((c, d))$, $[a, b] \subset (c, d)$, and C_1, C_2, $m \in \mathcal{R}$ introduced in Theorem 3. Consider a sequence of points $x_1, \cdots, x_n \in [a, b]$ such that $f(x_i)$, $f'(x_i) \neq 0$, $1 \leq i \leq n$, with its intervals $J_{x_i} \neq \emptyset$, $1 \leq i \leq n$, respectively, in agreement with (75).*

Then we define the sets J and A as follows:

$$J = \bigcup_{i=1}^{n} J_{x_i}; \quad A = [a, b] \setminus J \tag{86}$$

Theorem 4 *Consider the sequence $x_1, \cdots, x_n$ and the set J, introduced in Definition 3. Let us introduce $I_1, \ldots I_l$, with $1 \le l \le n$, as the convex and separate components of J, in other words:*

$$J = \bigcup_{i=1}^{l} I_i \text{ and } \bigcap_{i=1}^{l} I_i = \emptyset \tag{87}$$

Then there are M, $M \ge 1$, $l-1 \le M \le l+1$, convex and separate closed intervals: $[s^1, t^1], \cdots, [s^M, t^M]$ such that:

$$A = \bigcup_{i=1}^{M} [s^i, t^i] \subset [a, b] \subset \left(\bigcup_{i=1}^{M} [s^i, t^i] \right) \bigcup \left(\bigcup_{i=1}^{l} I_i \right), \tag{88}$$

with

$$\left(\bigcup_{i=1}^{M} [s^i, t^i] \right) \bigcap \left(\bigcup_{i=1}^{l} I_i \right) = \emptyset \tag{89}$$

Proof.

First of all, we introduced the sequence of real numbers: $p^1 < q^1 \le p^2 < q^2 \le \cdots \le p^l < q^l$ to satisfy: $I_i = (p^i, q^i)$, and that will be used next.

For the sake of clarity, we distinguish four cases:

1. $a \in I_1$ and $b \in I_l$;
2. $a \in I_1$ and $b \notin I_l$;
3. $a \notin I_1$ and $b \in I_l$;
4. $a \notin I_1$ and $b \notin I_l$.

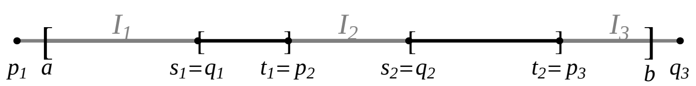


Figure 3: First case: $a \in I_1$ and $b \in I_l$; $l = 3$

First: a and b do not belong to A, so $a \in (p^1, q^1)$ and $b \in (p^l, q^l)$, then:

$$[a, b] \subset \left(\bigcup_{i=1}^{l-1} [s^i, t^i] \right) \bigcup \left(\bigcup_{i=1}^{l} I_i \right), \text{ with } s^1 = q^1,\ t^1 = p^2,\ \cdots,\ s^{l-1} = q^{l-1},\ t^{l-1} = p^l \tag{90}$$

A being:

$$A = [a, b] \setminus J = \bigcup_{i=1}^{l-1} [s^i, t^i] \subset [a, b] \tag{91}$$

Second: a does not belong to A, but b does, so $a \in (p^1, q^1)$ y $b \notin (p^l, q^l)$, then:

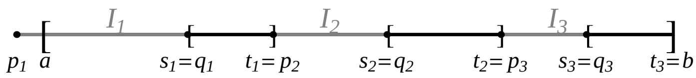


Figure 4: Second case: $a \in I_1$ and $b \notin I_l$; $l = 3$

$$[a, b] \subset \left(\bigcup_{i=1}^{l} [s^i, t^i] \right) \bigcup \left(\bigcup_{i=1}^{l} I_i \right), \text{ with } s^1 = q^1,\ t^1 = p^2,\ \cdots,\ s^l = q^l,\ t^l = b \tag{92}$$

A being:

$$A = [a, b] \setminus J = \bigcup_{i=1}^{l} [s^i, t^i] \subset [a, b] \tag{93}$$

Third: a belongs to A, but b does not, so $a \notin (p^1, q^1)$ y $b \in (p^l, q^l)$ then

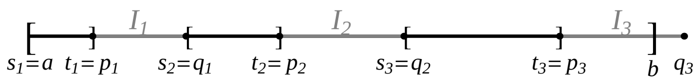


Figure 5: Third case: $a \notin I_1$ and $b \in I_l$; $l = 3$

$$[a, b] \subset \left(\bigcup_{i=1}^{l} [s^i, t^i] \right) \bigcup \left(\bigcup_{i=1}^{l} I_i \right), \text{ with } s^1 = a,\ t^1 = p^1,\ \cdots,\ s^l = q^{l-1},\ t^l = p^l \tag{94}$$

A being:

$$A = [a, b] \setminus J = \bigcup_{i=1}^{l} [s^i, t^i] \subset [a, b] \tag{95}$$

Fourth: a and b belong to A, so $a \notin \ (p^1, q^1)$ y $b \notin (p^l, q^l)$, then:

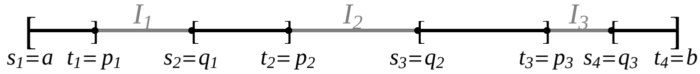


Figure 6: Fourth case: $a \notin I_1$ and $b \notin I_l$; $l = 3$

$$[a, b] \subset \left(\bigcup_{i=1}^{l+1} [s^i, t^i] \right) \bigcup \left(\bigcup_{i=1}^{l} I_i \right), \text{ with } s^1 = a,\ t^1 = p^1,\ \cdots,\ s^{l+1} = q^l,\ t^{l+1} = b \tag{96}$$

A being:

$$A = [a, b] \setminus J = \bigcup_{i=1}^{l+1} [s^i, t^i] \subset [a, b] \tag{97}$$

The result follows.

Definition 4 *Under the same hypothesis as Theorem 3 and Definition 2, we define the sequence A_n $n \geq 0$, proceeding as follows:*

First step

We apply Definition 3 and Theorem 4 to the sequence W_1, making $J = J_1$, $A = A_1$, and $M = M_1$, obtaining:

$$\begin{aligned}
&W_1 = \{w_1;\ \ w_1 \in (a,b);\ \textit{with } f(w_1);\ f'(w_1) \neq 0\} \\
&J_1 = J_{w_1},\ \textit{in agreement with (75)} \\
&A_1 = [a,b] \setminus J_1 = \bigcup_{k=1}^{M_1} [s_1^k, t_1^k],\ \textit{in agreement with (88)}
\end{aligned} \tag{98}$$

See Example 1 as a illustration.

Second step*:*

If $A_1 = \emptyset$ (stop criterion of the iteration), then $[a,b] \subset J_1$ and there is not any root in $[a,b]$. On the contrary, if $A_1 \neq \emptyset$, then we construct A_2 by applying again Definition 3 and Theorem 4 to the sequence W_2, making $J = J_2$, $A = A_2$, and $M = M_2$, obtaining:

$$\begin{aligned}
&W_2 = \left\{w_2^k;\ w_2^k \in [s_1^k, t_1^k];\quad f(w_2^k),\ f'(w_2^k) \neq 0;\quad 1 \leq k \leq M_1\right\} \\
&J_2 = \bigcup_{k=1}^{M_1} J_{w_2^k} \bigcup J_1,\ \textit{in agreement with (75)} \\
&A_2 = [a,b] \setminus J_2 = A_1 \setminus \bigcup_{k=1}^{M_1} J_{w_2^k} = \bigcup_{k=1}^{M_2} [s_2^k, t_2^k],\ \textit{in agreement with (88)}
\end{aligned} \tag{99}$$

If $A_2 \neq \emptyset$ we continue the process. See example 1 as an illustration.

Third step

Reasoning in the same way, given W_n, J_n and A_n, let us set A_{n+1} as follows:

$$\begin{aligned}
&W_{n+1} = \left\{w_{n+1}^k;\ w_{n+1}^k \in [s_n^k, t_n^k];\ f(w_{n+1}^k);\ f'(w_{n+1}^k) \neq 0;\ 1 \leq k \leq M_n\right\} \\
&J_{n+1} = \bigcup_{k=1}^{M_n} J_{w_{n+1}^k} \bigcup J_n,\ \textit{in agreement with (75)} \\
&A_{n+1} = [a,b] \setminus J_{n+1} = A_n \setminus \bigcup_{k=1}^{M_n} J_{w_{n+1}^k} = \bigcup_{k=1}^{M_{n+1}} [s_{n+1}^k, t_{n+1}^k],\ \textit{in agreement with (88)}
\end{aligned} \tag{100}$$

See example 1 as an illustration.

Note 8 *We recall the following statements:*

1. Given the sequence of sets M_n, $n \geq 0$, the upper (respectively lower) limit of M_n, denoted by $\overline{lim}$ (respectively $\underline{lim}$) is defined as:

$$\overline{\lim}_{n\to\infty} M_n = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} M_n,\ \underline{\lim}_{n\to\infty} M_n = \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} M_n \tag{101}$$

2. The sequence M_n *is convergent if* $\overline{\lim}_{n\to\infty} M_n = \underline{\lim}_{n\to\infty} M_n$

3. The sequence M_n *is no increasing if* $M_{n+1} \subset M_n$, $\forall\, n \geq 0$.

4. If the sequence M_n *is no increasing, then it is convergent, with* $\lim_{n\to\infty} M_n = \bigcap_{n=0}^{\infty} M_n$

The following Lemma is obvious due to the construction of the sequence A_n.

Lemma 7 *The sequence* A_n*, introduced in Definition 4, is strictly no increasing. In other words: If* $A_{n+1} \neq \emptyset$*, then* $A_{n+1} \subset A_n$ *and* $A_n \neq A_{n+1}$.

Theorem 5 *Let* $\lim_{n\to\infty} A_n = R$*, then or* $R = \emptyset$ *or* $R = \{\overline{x_k};\ \overline{x_k} \in [a,b];\ k = 1,2,...\}$.

Proof.

If there is n_0 such that $A_{n_0} = \emptyset$, obviously $R = \emptyset$.

On the contrary, If $A_n \neq \emptyset\ \forall\, n \geq 1$, then from Lemma 7 we can choose $x_0 \in [a,b] \setminus A_1$, $x_1 \in A_1 \setminus A_2$, with $x_0 \neq x_1$, ..., $x_n \in A_n \setminus A_{n+1}$.

In this way we can construct an infinite sequence x_n, $n \geq 0$ in $[a,b]$, of different points, that has at least one accumulation point $\overline{x_1}$; as $[a,b]$ is a closed set, then $\overline{x_1} \in [a,b]$.

Consider now an arbitrary k_0, the sequence x_n, $n \geq k_0$ is in A_{k_0} and for the same reason $\overline{x_1} \in A_{k_0}$.

The result follows.

The following Corollary is a immediately consequence of the previous Theorem and Lemma.

Corollary 2 *If* $A_n \neq \emptyset\ \forall n \geq 1$*, then* $\lim_{n\to\infty} \|t_n^k - s_n^k\| = 0$, $\forall k$*, where*

$$A_n = \bigcup_{k=1}^{M_n} [s_n^k, t_n^k] \tag{102}$$

Theorem 6 *The following statements are true:*

1. *If* $R = \emptyset$*, then there is not any root of* f *in* $[a,b]$.
2. *If* $R = \{\overline{x_k};\ \overline{x_k} \in [a,b];\ k = 1,2,...\}$*, then* $f(\overline{x_k}) = 0$, $\forall \overline{x_k} \in R$.
3. *In* $[a,b] \setminus R$ *there is not any root of* f.

Proof.

First: There exists n_0 such that $A_{n_0} = \emptyset$, therefore $[a,b] \subset J_{n_0}$ and the result follows from Theorem 3.

Second:

Consider m, r_1 and r_2 as in Definition 2.

Suppose that $f(\overline{x_k}) \neq 0$, then there is $e_1 > 0$, such that $\|f(x)\| > 0\ \forall\, x \in [\overline{x_k} - \epsilon_1, \overline{x_k} + \epsilon_1]$.

Let r be a real number, given by $r = \max\{\|r_1\|, \|r_2\|\}$; let b_1, b_2 and M_1 be real numbers to satisfy: $(\|f(x)\|\ r) > b_1 > 0$, $\|f'(x)\| < b_2$, $\forall\, x \in [\overline{x_k} - \epsilon_1, \overline{x_k} + \epsilon_1]$, with $M_1 = b_1/b_2$; and finally let $\epsilon > 0$ be a real number to satisfy $\epsilon = \min\{\epsilon_1, M_1\}$. Then, we can conclude that:

$$\frac{\|f(x)\|\ r}{\|f'(x)\|} \geq \frac{b_1}{b_2} = M_1 \geq \epsilon > 0 \tag{103}$$

hods for all $x \in [\overline{x_k} - \epsilon, \overline{x_k} + \epsilon]$.

As $\overline{x_k} \in R$, $\overline{x_k} \in A_n$, $\forall\, n \geq 1$. In agreement with Corollary 2, consider $n_0, k_0 \in \mathcal{Z}^+$ such that:

$$A_{n_0} = \bigcup_{k=1}^{M_{n_0}} [s_{n_0}^k, t_{n_0}^k];\ \|t_{n_0}^k - s_{n_0}^k\| < \epsilon/2,\ \forall\, k;\ \overline{x_k} \in [s_{n_0}^{k_0}, t_{n_0}^{k_0}] \tag{104}$$

As a consequence of all of this, given any $z \in [s_{n_0}^{k_0}, t_{n_0}^{k_0}]$, such that $f'(z) \neq 0$, on the one hand, if $m < 0$:

$$[s_{n_0}^{k_0}, t_{n_0}^{k_0}] \subset [z - \epsilon/2, z + \epsilon/2] \subset J_z \tag{105}$$

since $[z - \epsilon/2, z + \epsilon/2] \subset [\overline{x_k} - \epsilon, \overline{x_k} + \epsilon]$ and inequality (103) holds and, in particular, if we take $z = w_{n_0+1}^{k_0}$, then:

$$[s_{n_0}^{k_0}, t_{n_0}^{k_0}] \subset J_{w_{n_0+1}^{k_0}} \Rightarrow [s_{n_0}^{k_0}, t_{n_0}^{k_0}] \cap A_{n_0+1} = \emptyset \Rightarrow \overline{x_k} \notin R \tag{106}$$

And, on the other hand, if $m > 0$:

$$\begin{array}{l} [s_{n_0}^{k_0}, t_{n_0}^{k_0}] \subset [a, z + \epsilon/2] \subset J_z \text{ or} \\ [s_{n_0}^{k_0}, t_{n_0}^{k_0}] \subset [z - \epsilon/2, b] \subset J_z \end{array} \tag{107}$$

obtaining the same conclusion: $\overline{x_k} \notin R$, contradiction that proves the result.

Third: It follows from the fact that there is not any root of f in J_n for all n.

Note 9 *Observe that the set R contains as much simple as singular roots of $f(x)$.*

6 Composed algorithm from PMI and PMII

Before presenting the algorithm itself, we introduce the following example as a illustration of the main ideas.

Example 1 *Find all the roots of $f(x) = 2\cos(x) - 0.5x = 0$ in the interval $[-2\pi, 2\pi]$, with five exact digits.*

According to Definition 4, we accomplish the solution in the following steps:

First step:

$$\begin{array}{l} W_1 = \{1\} \\ J_1 = (-1.68614, 1.18614) \\ A_1 = [-2\pi, -1.68614] \cup [1.18614, 2\pi] = [s_1^1, t_1^1] \cup [s_1^2, t_1^2] \end{array} \tag{108}$$

According to Theorem 3, in J_1 there is not any root, so if there are some roots, they are in A_1. Taking $x_0 \in [s_1^1, t_1^1]$ (for example, its half point) as initial value and applying the iteration formula (70), with $m = 1$:

$$\begin{array}{l} x_0 = -3.98466 \\ x_1 = F_1(x_0) = -3.61577 \\ x_2 = F_1(x_1) = -3.59531 \end{array} \tag{109}$$

the root of $f(x)$, $r_1 = -3.59531$, is obtained. We proceed in a similar way taking $y_0 \in [s_1^2, t_1^2]$ (for example, its half point):

$$\begin{array}{l} y_0 = 3.73466 \\ y_1 = F_1(y_0) = -34.2718 \end{array} \tag{110}$$

As $y_1 \notin [s_1^2, t_1^2]$ and A_1 has not any else subinterval, we go to the second step.

Second step:

$$\begin{aligned} W_2 &= \{-3.98466, 3.73466\} \\ J_2 &= (-6.26797, -3.69474) \cup (-1.68614, 1.18614) \cup (1.5228, 5.3287) \\ A_2 &= [-2\pi, -6.26797] \cup [-3.69474, -1.68614] \cup [1.18614, 1.5228] \cup [5.3287, 2\pi] \\ &= [s_2^1, t_2^1] \cup [s_2^2, t_2^2] \cup [s_2^3, t_2^3] \cup [s_2^4, t_2^4] \end{aligned} \tag{111}$$

W_2 is the set of the half points of the intervals of A_1. In J_2 there is not any root. If there are some else roots, there are in A_2. Reasoning as in the previous step, we take $x_0 \in [s_2^1, t_2^1]$ (for example, its half point):

$$\begin{aligned} x_0 &= -6.27558 \\ x_1 &= F_1(x_0) = -189.312 \end{aligned} \tag{112}$$

As $x_1 \notin [s_2^1, t_2^1]$ and $r_1 \in [s_2^2, t_2^2]$, we take $y_0 \in [s_2^3, t_2^3]$ (for example, its half point):

$$\begin{aligned} y_0 &= 1.35447 \\ y_1 &= F_1(y_0) = 1.25251 \\ y_2 &= F_1(y_1) = 1.25235 \end{aligned} \tag{113}$$

obtaining the second root of $f(x)$, $r_2 = 1.25235$. Next we take $z_0 \in [s_2^4, t_2^4]$ (for example, its half point):

$$\begin{aligned} z_0 &= 5.80594 \\ z_1 &= F_1(z_0) = 23.855 \end{aligned} \tag{114}$$

As $z_1 \notin [s_2^4, t_2^4]$ and A_2 has not any else subinterval, we go to the third step.

Third step:

$$\begin{aligned} W_3 &= \{-6.27558, -2.69044, 1.35447, 5.80594\} \\ J_3 &= (-8.81443, -3.69474) \cup (-3.57592, -2.17697) \cup (-1.68614, 1.18614) \cup (1.25726, 6.6784) \\ A_3 &= [-3.69474, -3.57592] \cup [-2.17697, -1.68614] \cup [1.18614, 1.25726] \\ &= [s_3^1, t_3^1] \cup [s_3^2, t_3^2] \cup [s_3^3, t_3^3] \end{aligned} \tag{115}$$

W_3 is the set of the half points of the intervals of A_2. In J_3 there is not any root. If there are some else roots, there are in A_3. Reasoning as in the previous step, as $r_1 \in [s_3^1, t_3^1]$, we take $x_0 \in [s_3^2, t_3^2]$ (for example, its half point):

$$\begin{aligned} x_0 &= -1.93155 \\ x_1 &= F_1(x_0) = -2.13026 \\ x_2 &= F_1(x_1) = -2.13333 \end{aligned} \tag{116}$$

and the third root of $f(x)$, $r_3 = -2.13333$, is reached. As $r_2 \in [s_3^3, t_3^3]$ we go to the fourth step.

Fourth step:

$$\begin{aligned} W_4 &= \{-3.63533, -1.93155, 1.2217\} \\ J_4 &= (-8.81443, -3.59729) \cup (-3.57592, -2.17697) \cup (-2.10027, 1.2521) \cup (1.25726, 6.6784) \\ A_4 &= [-3.59729, -3.57592] \cup [-2.17697, -2.10027] \cup [1.2521, 1.25726] \\ &= [s_4^1, t_4^1] \cup [s_4^2, t_4^2] \cup [s_4^3, t_4^3] \end{aligned} \tag{117}$$

As $r_1 \in [s_4^1, t_4^1]$, $r_3 \in [s_4^2, t_4^2]$ and $r_2 \in [s_4^3, t_4^3]$ we go to the fifth step.

Fifth step:

$$\begin{aligned} W_5 &= \{-3.5866, -2.13862, 1.25468\} \\ J_5 &= (-8.81443, -3.59729) \cup (-3.5953, -2.13334) \cup (-2.10027, 1.2521) \cup (1.25236, 6.6784) \\ A_5 &= [-3.59729, -3.5953] \cup [-2.13334, -2.10027] \cup [1.2521, 1.25236] \\ &= [s_5^1, t_5^1] \cup [s_5^2, t_5^2] \cup [s_5^3, t_5^3] \end{aligned} \tag{118}$$

As $r_1 \in [s_5^1, t_5^1]$, $r_3 \in [s_5^2, t_5^2]$ and $r_2 \in [s_5^3, t_5^3]$ we go to the sixth step.

Sixth step:

$$\begin{aligned} W_6 &= \{-3.59629, -2.1168, 1.25223\} \\ J_6 &= (-8.81443, -3.59531) \cup (-3.5953, -2.13334) \cup (-2.133, 1.25235) \cup (1.25236, 6.6784) \\ A_6 &= [-3.59531, -3.5953] \cup [-2.13334, -2.1333] \cup [1.25235, 1.25236] \\ &= [s_6^1, t_6^1] \cup [s_6^2, t_6^2] \cup [s_6^3, t_6^3] \end{aligned} \tag{119}$$

As $r_1 \in [s_6^1, t_6^1]$, $r_3 \in [s_6^2, t_6^2]$ and $r_2 \in [s_6^3, t_6^3]$ we stop because of $s_6^1 = t_6^1$, $s_6^2 = t_6^2$ and $s_6^3 = t_6^3$ with five equal digits as required, and in J_6 there are not roots anymore.

The procedure of Example 1 can be generalized by the following algorithm, whose proof is based on the previous results.

Algorithm 1 *Algorithm for finding all the roots of the scalar equation $f(x) = 0$ in an arbitrary interval $[a, b]$, with f sufficiently differentiable to apply the iteration $x_{n+1} = F_m(x_n)$, introduced in (70).*

First Step

Introduce the interval extremes a and b, the constants C_1 and C_2 (see Definition 2) and the iteration function F_m. Besides define the iterators k, i, q and n and continue.

Second Step

Take $k = 1$ and continue.

Third step

Compute W_1 and J_1 (see Definition 4)

1. **If** $[a, b] \subset J_1$
   Then, end the Algorithm (there is not any root in [a,b])
   Else, compute A_1 (see Definition 4) and j_1=number of subintervals of A_1 and go to the following step
   end **If**

Fourth Step

Take $i = 1$ and continue.

Fifth Step

1. **If** $i \leq j_1$
   take $x_0 \in [s_1^i, t_1^i]$
   Then, compute $x_1 = F_m(x_0)$ and go to the point 2 of the fifth step
   Else, go to the sixth step
   end **If**

2. Take n=2 and continue

3. Compute $x_n = F_m(x_{n-1})$
   If $x_n \in [s_1^i, t_1^i]$ and there is not $x_p = x_n$, with $n - p > 1$ (there is not a loop)
   Then, compute $x_{n+1} = F_m(x_n)$

   If $x_{n+1} = x_n$, with the required number of digits
   Then, take q=1, compute $r_q = x_n$, take i=i+1 and go to the point 1 of the fifth step
   Else, take $n = n + 1$ and go to initialize again the point 3 of the fifth step
   end **If**

   Else, take i=i+1 and go to the point 1 of the fifth step
   end **If**

Sixth step

Take $k = k + 1$ and continue.

Compute W_k and J_k

1. **If** $[a, b] \subset J_k$
   Then, end the Algorithm (there is not any root in [a,b])
   Else, compute A_k and j_k=number of subintervals of A_k

   If $s_k^i = t_k^i$ with the number of required digits, $1 \leq i \leq j_k$
   Then, end the Algorithm
   Else, go to the seventh step

   end **If**

Seventh Step

Take $i = 1$ again and continue.

Eighth Step

1. **If** $i \leq j_k$

   If $r_h \in [s_k^i, t_k^i]$, $1 \leq h \leq q$
   Then, $i = i + 1$ and go to initialize again the point 1 of the eighth step
   Else, take $x_0 \in [s_k^i, t_k^i]$, compute $x_1 = F_m(x_0)$ and go to the point 2 of the eighth step
   end **If**

   Else, go to the sixth step
   end **If**

2. Take n=2

3. Compute $x_n = F_m(x_{n-1})$
   If $x_n \in [s_k^i, t_k^i]$ and there is not $x_p = x_n$, with $n - p > 1$ (there is not a loop)
   Then compute $x_{n+1} = F_m(x_n)$

   If $x_{n+1} = x_n$, with the required number of digits
   Then, take $q = q + 1$, compute $r_q = x_n$, take i=i+1 and go to the point 1 of the eighth step
   Else, take n=n+1 and go to initialize the point 3 of the eighth step again
   end **If**

   Else take i=i+1 and go to the point 1 of the eighth step
   end **If**

7 Comparison with other methods

Find all the roots of the function:

$$f(x) = e^{3x} - 12e^x + 16 \tag{120}$$

in the feasible region $-10 \leq x \leq 2$, with five decimal digits. How many real roots are there in this region?

First step:

$$\begin{aligned} C_1 &= -6 \\ c_2 &= 3543 \\ a &= -10 \\ b &= 2 \\ W_1 &= \{-4\} \\ J_1 &= (-6.33041, -1.74285) \\ A_1 &= [-10, -6.33041] \cup [-1.74285, 2] = [s_1^1, t_1^1] \cup [s_1^2, t_1^2] \end{aligned} \tag{121}$$

According to Theorem 3, in J_1 there is not any root, so if there are some roots, they are in A_1. Taking $x_0 \in [s_1^1, t_1^1]$ (for example, its half point) as initial value and applying the iteration formula (70), with $m = 2$:

$$\begin{aligned} x_0 &== -8.1652 \\ x_1 &= F_1(x_0) = 3.4323\ 10^{10} \end{aligned} \tag{122}$$

As $x_1 \notin [s_1^2, t_1^2]$, we take $y_0 \in [s_1^2, t_1^2]$ (for example, its half point):

$$\begin{aligned} y_0 &= 0.128575 \\ y_1 &= F_2(y_0) = 0.572365 \\ y_2 &= F_2(y_1) = 0.658323 \\ y_3 &= F_2(y_2) = 0.682514 \\ y_4 &= F_2(y_3) = 0.689848 \\ y_5 &= F_2(y_4) = 0.692118 \\ y_6 &= F_2(y_5) = 0.692826 \\ y_7 &= F_2(y_6) = 0.693047 \\ y_8 &= F_2(y_7) = 0.693116 \\ y_9 &= F_2(y_8) = 0.693137 \\ y_{10} &= F_2(y_9) = 0.693144 \\ y_{11} &= F_2(y_{10}) = 0.693146 \\ y_{12} &= F_2(y_{11}) = 0.693147 \end{aligned} \tag{123}$$

obtaining the root of $f(x)$, $r_1 = 0.693147$. As A_1 has not any else subinterval, we go to the following step.

Second step:

$$\begin{aligned} W_2 &= \{1.24915\} \\ J_2 &= (-10.4749, 0.498292) \cup (1.05647, 29.9046) \\ A_2 &= [0.498292, 1.05647] = [s_2^1, t_2^1] \end{aligned} \tag{124}$$

W_2 is the set of the half points of the intervals of A_1. In J_2 there is not any root. If there are some else roots, there are in A_2. As $r_1 \in [s_2^1, t_2^1]$, we go to the following step:

Third step:

$$\begin{aligned} W_3 &= \{0.777379\} \\ J_3 &= (-10.4749, 0.498292) \cup (0.738502, 29.9046) \\ A_3 &= [0.498292, 0.738502] = [s_3^1, t_3^1] \end{aligned} \tag{125}$$

W_3 is the set of the half points of the intervals of A_2. In J_3 there is not any root. If there are some else roots, there are in A_3. As $r_1 \in [s_3^1, t_3^1]$, we go to the following step. And going on, we arrive at:

Fourth step:

$$\begin{aligned}W_{16} &= \{0, 693147\}\\ J_{16} &= (-10.4749, 0.693147) \cup (0.693147, 29.9046)\\ A_{16} &= [0.693147, 0.693147]\end{aligned} \tag{126}$$

As $r_1 \in A_{16}$, we stop because of in J_{16} there are not roots anymore. Therefore in the feasible region there is only one root.

Taking the same initial value $x_0 = -4$ as in the previous algorithm, the midpoint of the feasible region, we have:

1. Bisection method fails as much to locate smaller intervals as to calculate the roots (observe it is a singular root).
2. Chebishev's method diverges.
3. Euler's method, after 20 iterations preserves the initial value $x_20 = x_0 = -4.00000$
4. Halley's method diverges.
5. Jarrat's method diverges.
6. Ostrowski's method, after 20 iterations, takes the value $x_20 = 96{,}0297$

Similar results would be obtained using the multi-steps algorithms based on them.

Observe that PMII had calculated the roots, using any other initial value in the feasible region.

Regarding the number of roots in the feasible region, nowadays there is not a general procedure to determine it, except in the case of polynomials (Sturm's method).

8 Conclusions

As known, given a scalar equation or a system of nonlinear equations, there is not a general procedure to locate and calculate all its real roots, including the singular ones. The existent algorithms need a guess initial value to initialize the process for each of their roots, without providing how to calculate them in the general case. As proven throughout this article, Algorithm 1 solves this problem in the case of the scalar equations, locating and solving all their real roots, included the singular ones, without needing initial values.

Most of the times in engineering fields you need to solve $f(x) = 0$ within a feasible region $a \leq x \leq b$, and to know the total number of roots in such a region, then the presented algorithm provides a general and complete procedure to do it, as proven and illustrated throughout this work.

Furthermore, we are obtaining hopeful results to generalize this method in order to locate and solve all the real roots of nonlinear systems, by improving and generalizing the ideas expressed in [13] for polynomial systems.

As also proven, the PMI improves the velocity of convergence in an unlimited way, but besides, obviously, its computational efficiency can also be improved by using the well known technics of acceleration of the multi-step methods, nevertheless this issue is left for future works, since this paper is already extensive enough.

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