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A comparative study to the numerical approximation of random Airy differential equation

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Abstract

The aim of this paper is twofold. Firstly, we deal with the extension to the random framework of piecewise Fröbenius method to solve the Airy differential equation. This extension is based on the mean square stochastic calculus. Secondly, we want to explore its capability to provide not only reliable approximations, for both, the average and the standard deviation functions associated to the solution stochastic process, but also to save computational time as it occurs in the deterministic scenario. This includes a comparison of the numerical results with respect to those obtained by other operational commonly used methods such as polynomial chaos and Monte Carlo. To conduct this comparative study, we have chosen the Airy random differential equation because it has highly oscillatory solutions. This feature allows us to emphasize differences between all the considered approaches.

Key words: Piecewise random Fröbenius method, Polynomial chaos, Monte Carlo simulation, Random Airy-type differential equations

1 Introduction

In the deterministic scenario Airy differential equation appear in a variety of applications to mathematical physics such as the description of the solution of Schrödinger equation for a particle confined within a triangular potential;

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in the solution for one-dimensional motion of a quantum particle affected by a constant force or in the theory of diffraction of radio waves around the earth's surface [1,2]. From these few examples, it seems to be quite natural the introduction of randomness in the Airy differential equation. This can be straightforward justified from two perspectives, first, because in practice the calibration of both, the initial conditions and coefficient, require exhaustive measurements that usually contain some sort of error. Secondly, the inherent complexity of the phenomena under study, justifies that it be more coherent to consider the information that determines the model as random variables rather than deterministic values. These types of arguments lead to consider the Airy random differential equation of great interest in Physics and its related areas.

Solutions to deterministic Airy differential equations are highly oscillatory and it has claimed the attention of numerical analysts to compare the effectiveness of different computational methods [3]. Thus, it seems to be a good example to check the numerical capacity of different techniques to compute the corresponding solution stochastic process of random Airy differential equation. Recently, in reference [4] some of the authors have studied the random Airy differential equation

$$\ddot{X}(t) + AtX(t) = 0,\ -\infty < t < \infty, \quad X(0) = Y_0, \quad \dot{X}(0) = Y_1, \qquad (1)$$

where A, Y_0 and Y_1 are random variables. This study is based on an extension of the deterministic Fröbenius method to the random framework by applying the mean square calculus, see [5]. In that paper, it is assumed that statistical absolute moments with respect to the origin of random input A grow at the most exponentially, i.e, there exist a nonnegative integer n_0 and positive constants H and M such that

$$\mathrm{E}\left[|A|^n\right] \leq H\, M^n < +\infty, \qquad \forall\, n \geq n_0. \qquad (2)$$

This allows to obtain an approximate solution stochastic process of the Airy model as well as its main statistical functions such as average and variance by truncating a random power series solution. In [4], it is shown that every random variable A whose codomain or support is bounded satisfy such condition, otherwise (as for example, it happens in the case that A is a Gaussian random variable), we proposed to truncate the support to take advantage of that approach. The truncation can be done in such a way that the censured support contains most of the values of the random variable (for instance, if A is a Gaussian random variable with mean μ_A and standard deviation σ_A, then the interval $[\mu_A - 3\sigma_A, \mu_A + 3\sigma_A]$ contains in average the 99.7% of its values). Although a priori this truncation may mean a loss of accuracy that could affect computations related to relevant statistical information about the solution stochastic process, such as its average and standard deviation functions, this

potential inconvenient can be easily overcome by enlarging the length of the censured interval. Even more, in practice this strategy does not entail any significant increase of the computational cost.

In the deterministic scenario a modification of the Fröbenius method has been successfully developed by some of the authors in order to save computational time when dealing with problems like (1) [6,7]. This motivates the two goals of this paper, firstly, to explore whether this modification of the Fröbenius method works in the random framework and also speeds-up computations and, secondly, to compare results obtained by this new approach with respect to those provided by other available methods including polynomial chaos technique.

The application of homogeneous and generalized polynomial chaos method to the solution of random differential equations have already been tested successfully by some authors [8–10], although other contributions have highlighted its current limitations in dealing with random differential models appearing in some engineering applications [11]. As well we emphasize that interesting alternative methods to deal with random inputs are those based on the Wiener-Hermite expansion, which can be regarded as its continuous counterpart [12].

The paper is organized as follows. Based on the deterministic approach showed in [7], Section 2 is devoted to present a modification of random Fröbenius method developed in paper [4]. Section 3 is addressed to introduce the polynomial chaos method including its application to model (1). In Section 4, we compare through an illustrative example the numerical results obtained by the modified random Fröbenius method to approximate the average and standard deviation functions with respect to the corresponding ones computed by polynomial chaos, Monte Carlo simulations and random Fröbenius method presented in [4]. Conclusions are presented in Section 5.

2 Developing a piecewise random Fröbenius method

In the recent paper [4] an extension of the deterministic Fröbenius method to deal with the random Airy differential equation (1) is presented. The method is based on the construction of a mean square convergent random infinite power series solution centered at the origin $t = 0$ which is truncated in order to obtain approximations of the average and variance of the solution stochastic process to (1). To apply the method, condition (2) is assumed to be satisfied by random input A. Although computation time required in numerical experiments presented in [4] showed to be competitive with respect to other approaches, as we have pointed out in the Introduction, in this paper we are

also interested in comparing it with respect to other strategies. More precisely, in this section we want to adapt to the random framework a piecewise version of the Fröbenius method that some of the authors have tested to be more advantageous in the deterministic scenario [7].

The method consists on divide the t-interval where we want to construct the approximate solution, say $[0,T]$, into K subintervals of length $a = T/K$, denoted respectively by $[(j-1)a, ja]$, $1 \leq j \leq K$, where $K = [T/a]$ being $[\cdot]$ the integer part function. Then, following the reference [4], in the first step we construct the solution $X_K^1(t)$ on the interval $[0,a]$ using a random power series centered at $t_0 = 0$ and considering the random initial data $X_K^1(0) = Y_0$, $\dot{X}_K^1(0) = Y_1$. Taking as initial conditions $X_K^2(a) = X_K^1(a)$ and $\dot{X}_K^2(a) = \dot{X}_K^1(a)$, in the second step, now we construct an approximate random power series solution $X_K^2(t)$ centered at the point $t_1 = a$ on the interval $[a, 2a]$. In general, in the j-th step, we construct an approximate random power series $X_K^j(t)$ centered at the point $t_{j-1} = (j-1)a$ on the interval $[(j-1)a, ja]$ taking as initial conditions $X_K^j((j-1)a) = X_K^{j-1}((j-1)a)$ and $\dot{X}_K^j((j-1)a) = \dot{X}_K^{j-1}((j-1)a)$. The procedure continues until K approximate random power series solutions have been defined on each subinterval, respectively, covering the total domain $[0,T]$. Then a piecewise random power series solution $X_K(t)$ is defined on interval $[0,T]$ through $\left\{X_K^j(t) : t \in [(j-1)a, ja],\, 1 \leq j \leq K\right\}$.

Based on the previous exposition and following an analogous development as it is shown in [4], the approximate random power series solution centered at the point t_{j-1} is constructed on the interval $[(j-1)a, ja]$ as follows

$$X_K^j(t) = \sum_{n \geq 0} X_n^j (t - t_{j-1})^n, \qquad t_{j-1} = (j-1)a, \quad 1 \leq j \leq K, \tag{3}$$

where coefficients X_n^j satisfy the following recurrence relationship:

$$\begin{aligned} X_2^j &= -\frac{A t_{j-1} X_0^j}{2}, \\ X_{n+2}^j &= -\frac{A\left(X_{n-1}^j + t_{j-1} X_n^j\right)}{(n+2)(n+1)}, \qquad n \geq 1, \quad 1 \leq j \leq K, \end{aligned} \tag{4}$$

for given initial conditions X_0^j and X_1^j. These coefficients become those given in expression (17) in [4] when $K = 1$ (and so $j = K = 1$). Note that fixed a value j: $1 \leq j \leq K$, recurrence (4) starts from $X_0^j = X_K^{j-1}(t_{j-1})$ and $X_1^j = \dot{X}_K^{j-1}(t_{j-1})$. Setting a truncation order of series (3), say N, one obtains the following approximate random power series solution on the interval $[(j-1)a, ja]$ which is centered at the point t_{j-1}:

$$X_{K,N}^j(t) = \sum_{n=0}^{N} X_n^j (t - t_{j-1})^n, \qquad t_{j-1} = (j-1)a, \quad 1 \leq j \leq K.$$

This allows us to define the following approximation for the average of the approximate solution stochastic process on the interval $[(j-1)a, ja]$:

$$\mathrm{E}\left[X_{K,N}^{j}(t)\right] = \sum_{n=0}^{N} \mathrm{E}\left[X_n^j\right] (t - t_{j-1})^n,\ t_{j-1} = (j-1)a,\ 1 \leq j \leq K. \tag{5}$$

In order to compute an approximation of the variance, we must take into account that

$$\mathrm{Var}\left[X_{K,N}^{j}(t)\right] = \mathrm{E}\left[\left(X_{K,N}^{j}(t)\right)^2\right] - \left(\mathrm{E}\left[X_{K,N}^{j}(t)\right]\right)^2, \tag{6}$$

together with

$$\begin{aligned}\mathrm{E}\left[\left(X_{K,N}^{j}(t)\right)^2\right] &= \sum_{n=0}^{N} \mathrm{E}\left[\left(X_n^j\right)^2\right] (t - t_{j-1})^{2n} \\ &\quad + 2\sum_{n=1}^{N}\sum_{m=0}^{n-1} \mathrm{E}\left[X_n^j X_m^j\right] (t - t_{j-1})^{n+m}, \qquad 1 \leq j \leq K.\end{aligned}$$

In this way, approximate average and standard deviation functions of piecewise random power series $X_K(t)$ are defined. In the following these approximations will be denoted by $\mu^{\mathrm{F}}_{X_{K,N}}(t)$ and $\sigma^{\mathrm{F}}_{X_{K,N}}(t)$, respectively.

3 Applying the polynomial chaos method

This section is concerned to introduce the polynomial chaos method including its application to construct an approximate solution stochastic process to problem (1). Henceforth we shall assume that coefficient A is a random variable (r.v.) defined on a sample space Ω of certain probability space $(\Omega, \mathcal{F}, P)$ [13, part I]. Thus, r.v. A depends on an outcome $\omega \in \Omega$, i.e., $A = A(\omega)$. As a consequence the solution $X(t) = X(t;\omega)$ to problem (1) becomes a stochastic process (s.p.).

Polynomial chaos method was firstly introduced by N. Wiener who called it *the homogeneous chaos* [14]. In this context, if L_2 denotes the set of all r.v.'s χ whose statistical second-order moments with respect to the origin are finite, i.e., r.v.'s such that $\langle \chi^2 \rangle < +\infty$ (and as a consequence its variance is also finite) then every $\chi \in L_2$ can be represented in the form

$$\begin{aligned}\chi(\omega) &= \chi_0 H_0 + \sum_{i_1=1}^{\infty} \chi_{i_1} H_1(\xi_{i_1}(\omega)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} \chi_{i_1 i_2} H_2(\xi_{i_1}(\omega), \xi_{i_2}(\omega)) \\ &\quad + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} \chi_{i_1 i_2 i_3} H_3(\xi_{i_1}(\omega), \xi_{i_2}(\omega), \xi_{i_3}(\omega)) + \cdots .\end{aligned}$$

In this representation, $H_n = H_n(\boldsymbol{\xi})$ are Hermite polynomials in terms of vector $\boldsymbol{\xi}^{\mathrm{T}} = (\xi_{i_1}, \ldots, \xi_{i_n})$ whose components are n independent standard Gaussian r.v.'s. An explicit formula to generate these polynomials is given by

$$H_n(\xi_{i_1}(\omega), \ldots, \xi_{i_n}(\omega)) = \exp\left(\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}\right)(-1)^n \frac{\partial^n}{\partial\xi_{i_1}\cdots\partial\xi_{i_n}}\left(-\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}\right).$$

$H_n(\cdot)$ is usually referred to as the *n-th order homogeneous chaos*. As a consequence, the two first terms in the representation (7) related to H_0 and H_1 can be interpreted as the Gaussian part of r.v. χ. For convenience, this representation can be arranged through certain polynomials basis $\{\Phi_i\}$ as

$$\chi(\omega) = \sum_{i=0}^{\infty} \chi_i \Phi_i(\boldsymbol{\xi}(\omega)), \tag{7}$$

since there is a one-to-one correspondence between $\Phi_i(\cdot)$ and $H_i(\cdot)$. The number of the r.v.'s in $\boldsymbol{\xi}$ represents the *dimension of the chaos*. $\{\Phi_i\}$ constitutes a complete set of statistically orthogonal s.p.'s of the Hilbert space L_2 with respect to the inner product, i.e., $\langle \Phi_i, \Phi_j \rangle = \delta_{ij} \langle \Phi_i, \Phi_i \rangle$, where $\langle\cdot\rangle$ denotes the following average

$$\begin{aligned} \langle f(\boldsymbol{\xi}), g(\boldsymbol{\xi})\rangle &= \int_{\mathbb{R}^n} f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) W(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}, \\ W(\boldsymbol{\xi}) &= \frac{1}{\sqrt{(2\pi)^n}} \exp\left(-\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}\right), \end{aligned} \tag{8}$$

and δ_{ij} is Kronecker delta function. In addition, for $i \geq 1$ these polynomials are centered at the origin, i.e., $\langle \Phi_i \rangle = 0$, $i \geq 1$, and $\Phi_0 = 1$. As a consequence, from (7) expectation and variance of r.v. χ can be computed in terms of coefficients χ_i in the following way

$$\langle \chi(\omega) \rangle = \chi_0, \qquad \mathrm{Var}\left[\chi(\omega)\right] = \sum_{i=1}^{\infty} (\chi_i)^2 \left\langle \left(\Phi_i(\boldsymbol{\xi}(\omega))\right)^2 \right\rangle, \tag{9}$$

see [15] for further details.

In the operational practice, the infinite summation (7) needs to be truncated at a finite term, say P. In our case, this leads to the following expansion of both, input r.v. $A(\omega)$ and solution s.p. $X(t;\omega)$

$$A(\omega) = \sum_{i=0}^{P} A_i \Phi_i(\boldsymbol{\xi}(\omega)), \qquad X(t;\omega) = \sum_{i=0}^{P} X_i(t) \Phi_i(\boldsymbol{\xi}(\omega)). \tag{10}$$

In these expansions, the total number of terms is $P+1$. This value is fixed by the dimension of the chaos, i.e., n (the number of components of vector

$\boldsymbol{\xi}$) and the highest order p of the polynomial basis $\{\Phi_i\}$ in the following way: $P+1 = (n+p)!/(n!p!)$. Since we are going to consider A as the only input r.v. in problem (1), we will take $n = 1$, so $p = P$. In practice, the value of truncation index P is obtained after observing the stabilization of numerical solution. Only as an illustrative example, if we fix $p = 4$, then this implies that r.v. A is going to be expanded by means of the one-dimensional polynomial chaos whose functionals $\{\Phi_i\}$ are just the Hermite polynomials of degree $0, 1, \ldots, 4$ which depend on r.v. ξ_1 (see Table 1). In this particular case $P = 4$ in the spectral representation given by (10).

i	p, order of the polynomial chaos	i-th polynomial chaos Φ_i	$\left\langle (\Phi_i)^2 \right\rangle$
0	$p = 0$	1	1
1	$p = 1$	ξ_1	1
2	$p = 2$	$(\xi_1)^2 - 1$	2
3	$p = 3$	$(\xi_1)^3 - 3\xi_1$	6
4	$p = 4$	$(\xi_1)^4 - 6(\xi_1)^2 + 3$	24

Table 1
One-dimensional polynomial chaoses and their variances for $n = 1$ (excerpted from table 2.1. [15, p.52]).

Now, we are ready to explain how the polynomial chaos operational methodology works in model (1). Firstly, we impose that truncated polynomial chaos series given by (10) satisfies random Airy differential equation (1)

$$\sum_{i=0}^{P} \ddot{X}_i(t)\Phi_i(\xi_1(\omega)) + t\sum_{i=0}^{P}\sum_{j=0}^{P} A_i X_j(t)\Phi_i(\xi_1(\omega))\Phi_j(\xi_1(\omega)) = 0.$$

A Galerkin projection of previous equation onto each polynomial basis $\{\Phi_i\}$ is then conducted in order to ensure the error is orthogonal to the functional space spanned by the finite-dimensional basis $\{\Phi_i\}$

$$\sum_{i=0}^{P} \ddot{X}_i(t)\left\langle \Phi_i(\xi_1(\omega)), \Phi_l(\xi_1(\omega))\right\rangle$$
$$+t\sum_{i=0}^{P}\sum_{j=0}^{P} A_i X_j(t)\left\langle \Phi_i(\xi_1(\omega))\Phi_j(\xi_1(\omega)), \Phi_l(\xi_1(\omega))\right\rangle = 0, \qquad l = 0, 1, \ldots, P.$$

Now taking advantage of orthogonality properties of polynomial basis $\{\Phi_i\}$, one obtains the following coupled second-order system of deterministic differ-

ential equations

$$\ddot{X}_l(t) = -\frac{t}{e_l}\sum_{i=0}^{P}\sum_{j=0}^{P} e_{ijl}A_iX_j(t), \qquad l = 0, 1, \ldots, P, \tag{11}$$

where

$$e_{ijl} = \langle \Phi_i(\xi_1(\omega))\Phi_j(\xi_1(\omega)), \Phi_l(\xi_1(\omega))\rangle\,, \qquad 0 \leq i, j, l \leq P,$$

$$e_l = \left\langle \left(\Phi_l(\xi_1(\omega))\right)^2 \right\rangle, \quad A_i = \frac{\langle A, \Phi_i(\xi_1(\omega))\rangle}{\left\langle \left(\Phi_i(\xi_1(\omega))\right)^2 \right\rangle}, \quad l, i = 0, 1, \ldots, P. \tag{12}$$

Note that coefficients e_l and e_{ijl} can be computed directly from expression (8). More precisely, in the illustrative case previously introduced where $n = 1$, $p = 4$, to compute coefficients e_l and e_{ijl}, we just need to use expression (8) and the two last columns of Table 1. In the significant case where A is also a Gaussian r.v., coefficients A_i can still be computed in the same way that e_l and e_{ijl}. Whereas if A is a non-Gaussian r.v., the computation of the numerator of coefficients A_i requires that both involved r.v.'s, A and ξ_1, to be transformed to the same uniformly distributed r.v. u using the inverse transformation method. This can be made as follows

$$\langle A, \Phi_i(\xi_1(\omega))\rangle = \int_0^1 F_A^{-1}(u)\Phi_i(F_{\xi_1}^{-1}(u))\,\mathrm{d}u, \qquad i = 0, 1, \ldots, P, \tag{13}$$

where $F_A^{-1}(\cdot)$ and $F_{\xi_1}^{-1}(\cdot)$ denote the inverse probability distribution functions of r.v.'s A and ξ_1, respectively.

4 Comparing the modified random Fröbenius method with respect to other techniques: An illustrative example

This section is addressed to compare the modified random Fröbenius method presented in Section 2 with respect to other available methods. On the one hand, as we have explained in Section 1, we have selected the random Airy differential equation to conduct this comparative study because it has highly oscillating solutions, so it is expected to be an adequate model capable to show discrepancies between the different methods. On the other hand, a random variable with unbounded domain is going to be considered to play the role of coefficient A in model (1) in order to require the truncation of its codomain to deal with random Fröbenius method. This will allow us to show better the differences between the considered approaches including polynomial chaos and Monte Carlo methods [16]. Specifically, this numerical comparative study will be made computing approximations of the average and standard deviation functions of the solution s.p. to problem (1).

From (11), note that we first need to compute coefficients e_l, e_{ijl} and A_i. As we have already pointed out, coefficients e_l and e_{ijl} do not depend on r.v. A, therefore these computations can be stored for reusability with independence of the involved r.v. A. Note also that coefficients A_i given by (12) neither depend on the form of the random differential equation to be solved nor the initial conditions.

Let us assume the frequent case where A is a Gaussian r.v. with mean μ_A and standard deviation $\sigma_A > 0$, i.e., $A \sim \mathrm{N}(\mu_A; \sigma_A)$. Taking into account that A can be written as $A = \mu_A + \sigma_A \xi_1$, $\xi_1 \sim \mathrm{N}(0;1)$, then from (12) it is straightforward to see that

$$A_0 = \mu_A,\ A_1 = \sigma_A, A_i = 0,\ i = 2, 3, 4, \ldots, P.$$

In the context of the illustrative example above introduced, that is, for $n = 1$, $p = P = 4$, the deterministic coupled linear differential system (11) becomes

$$\left.\begin{aligned}
\ddot{X}_0(t) &= -t\left(\mu_A X_0(t) + \sigma_A X_1(t)\right), \\
\ddot{X}_1(t) &= -t\left(\sigma_A X_0(t) + \mu_A X_1(t) + 2\sigma_A X_2(t)\right), \\
\ddot{X}_2(t) &= -\frac{t}{2}\left(2\sigma_A X_1(t) + 2\mu_A X_2(t) + 6\sigma_A X_3(t)\right), \\
\ddot{X}_3(t) &= -\frac{t}{6}\left(6\sigma_A X_2(t) + 6\mu_A X_3(t) + 24\sigma_A X_4(t)\right), \\
\ddot{X}_4(t) &= -\frac{t}{24}\left(24\sigma_A X_3(t) + 24\mu_A X_4(t)\right).
\end{aligned}\right\} \tag{14}$$

In order to establish the corresponding initial conditions, let us assume that $Y_0 \sim \mathrm{N}(\mu_{Y_0}; \sigma_{Y_0})$ and $Y_1 \sim \mathrm{N}(\mu_{Y_1}; \sigma_{Y_1})$, hence $Y_0 = \mu_{Y_0} + \sigma_{Y_0}\xi_1$, $Y_1 = \mu_{Y_1} + \sigma_{Y_1}\xi_1$. We now multiply by Φ_i, $0 \leq i \leq P$ expression (10) for $X(t;\omega)$ and $\dot{X}(t;\omega)$ with $t = 0$. Then we apply the expectation operator and finally we take advantage of orthogonality of polynomial basis $\{\Phi_i\}$, thus we obtain:

$$\left.\begin{aligned}
X_0(0) = \mu_{Y_0}; \quad X_1(0) = \sigma_{Y_0}; \quad X_i(0) = 0,\ i = 2, 3, \ldots, P, \\
\dot{X}_0(0) = \mu_{Y_1}; \quad \dot{X}_1(0) = \sigma_{Y_1}; \quad \dot{X}_i(0) = 0,\ i = 2, 3, \ldots, P.
\end{aligned}\right\} \tag{15}$$

Since we are just interested in comparing piecewise random Fröbenius method with respect to other approaches considering the case that random input A has codomain unbounded, for readability, we have taken deterministic initial conditions: $Y_0 = 3$ and $Y_1 = 1$ (then $\mu_{Y_0} = 3$, $\mu_{Y_1} = 1$, $\sigma_{Y_0} = \sigma_{Y_1} = 0$) and $A \sim \mathrm{N}(\mu_A = 2; \sigma_A = 0.5)$. For computations, we have taken $[-6, 10]$ as the censured interval for random variable A, in order to apply random Fröbenius method.

Tables 2 and 3 collect, respectively, the numerical approximations at several points for the average (μ) and standard deviation (σ) by using the following techniques: random Fröbenius approach and its piecewise modification introduced in Section 2, Monte Carlo simulations and polynomial chaos method. Columns $\mu^{\mathrm{F}}_{X_N}(t)$ and $\sigma^{\mathrm{F}}_{X_N}(t)$ have been computed by applying the random Fröbenius method with truncation order N. Columns $\mu^{\mathrm{F}}_{X_{K,N}}(t)$ and $\sigma^{\mathrm{F}}_{X_{K,N}}(t)$ have been computed by applying the piecewise random Fröbenius method with truncation order N and splitting interval $[0,5]$ into K subintervals of the same length. Note that expressions (5)–(7) have been used to deal with computations. Columns $\mu^{\mathrm{MC}}_{X_m}(t)$ and $\sigma^{\mathrm{MC}}_{X_m}(t)$ have been obtained by applying Monte Carlo technique with m simulations and, finally, columns $\mu^{\mathrm{pc}}_{X_P}(t)$, $\sigma^{\mathrm{pc}}_{X_P}(t)$ have been calculated by formulae (9), respectively, following the polynomial chaos approach previously presented. In the column 2, for both, random Fröbenius methods of order N and $\{N,K\}$, respectively, and also for polynomial chaos of degree P (last column), these numerical computations have been performed until they have reached the stabilization of six significative digits of the numerical values with respect to N and P. *Mathematica*© instruction NDSolve has been used to obtain a numerical solution of the corresponding analogous systems to (14) together with the initial conditions (15). In accordance with Tables 2 and 3, at this point, we stress that in order to obtain the numerical stabilization of average, a nonlinear coupled system like (14) but with seven equations have been solved while the corresponding one for the standard deviation has thirteen equations.

t	$\mu^{\mathrm{F}}_{X_{69}}(t), \mu^{\mathrm{F}}_{X_{2,35}}(t), \mu^{\mathrm{F}}_{X_{5,17}}(t)$	$\mu^{\mathrm{F}}_{X_{200,4}}(t)$	$\mu^{\mathrm{MC}}_{X_m}(t)$ $m = 100\,000$	$\mu^{\mathrm{pc}}_{X_6}(t)$
0.00	3.00000	3.00000	3.00000	3.00000
1.00	2.91023	2.91023	2.90932	2.91023
2.00	−1.22508	−1.22508	−1.228	−1.22508
3.00	−0.759985	−0.759962	−0.755933	−0.759985
4.00	1.07227	1.07223	1.06919	1.07227
5.00	−0.705977	−0.705934	−0.702016	−0.705977

Table 2
Comparison of the average by using random Fröbenius method, piecewise random Fröbenius method, Monte Carlo simulations and polynomial chaos approach for $A \sim \mathrm{N}(\mu_A = 2; \sigma_A = 0.5)$, $Y_0 = 3$ and $Y_1 = 1$.

From the previous example, we have observed that both versions of random Fröbenius method as well as polynomial chaos technique achieve stabilization of the numerical results. We underline that by increasing the number K of subintervals associated to the piecewise random Fröbenius method, we can reduce the degree N of the approximate finite series, which is just a polynomial. In practice, the smaller is the degree N of the polynomial, the lower will be

t	$\sigma^F_{X_{63}}(t), \sigma^F_{X_{2,32}}(t), \sigma^F_{X_{5,17}}(t)$	$\sigma^F_{X_{200,4}}(t)$	$\sigma^{MC}_{X_m}(t)$ $m = 100000$	$\sigma^{pc}_{X_{12}}(t)$
0.00	0	0	0	0
1.00	0.256018	0.25602	0.255866	0.256018
2.00	0.816923	0.81692	0.815421	0.816923
3.00	1.19504	1.19504	1.19562	1.19504
4.00	1.18406	1.18404	1.18305	1.18406
5.00	1.39934	1.39927	1.38065	1.39934

Table 3
Comparison of the standard deviation by using random Fröbenius method, piecewise random Fröbenius method, Monte Carlo simulations and polynomial chaos approach for $A \sim \mathrm{N}(\mu_A = 2; \sigma_A = 0.5)$, $Y_0 = 3$ and $Y_1 = 1$.

the computational cost. However, this cost will increase as the number K of subintervals raise. Comments in this issue are added later.

From a computational standpoint, unlike what happens in the deterministic scenario, piecewise random Fröbenius method carries out more computational time than random Fröbenius method. This is expected because splitting of the whole domain into subintervals entails a considerable increase of the involved algebraic expressions to be handled in order to deals with average and standard deviation functions.

In practice, a balance between K and N must be sought. In our case, we found that timing until numerical stabilization was less by applying random Fröbenius method (corresponding to $N = 69$), while computations by polynomial chaos ($P = 6$) and piecewise random Fröbenius method (with $K = 5, N = 17$) were similar.

As the numerical approximations obtained by the differential approaches practically coincide, for the sake of clarity in the presentation, Figure 1 shows average (given by $X_0(t)$) and standard deviation (denoted by $\sigma_X(t)$) approximations on the interval $[0, 5]$. Note that the standard deviation shape is justified by the oscillatory behavior of the average.

Fig. 1. Representation of mean approximation $X_0(t)$ (left) and standard deviation approximation $\sigma_X(t)$ (right) on the interval $[0, 5]$ in the case that $A \sim \mathrm{N}(\mu_A = 2; \sigma_A = 0.5)$, $Y_0 = 3$ and $Y_1 = 1$.

5 Conclusions

Based on the results obtained by some of the authors through a piecewise Fröbenius method in the deterministic case [6,7], in this paper we have explored whether we can also take advantage of this approach by extending it to the random scenario. For this study, we have chosen the Airy differential equation because in the deterministic framework their solutions are highly oscillatory. Thus, it seems to be a good example to test the quality of this piecewise random Fröbenius method, and compare it with respect to other techniques commonly used in the study of random differential equations, including the polynomial chaos method.

The formulation in the random framework of the piecewise random Fröbenius method has taken as a starting point a previous paper by some of the authors [4]. In that former contribution, the input random variable A was assumed to satisfy condition (2). Note that a flexible and wide family of random variables satisfying this condition is defined by random variables having bounded codomain. Otherwise, truncation method allows us to deal with unbounded random variables, although in this case a loss of the quality of the approximations is expected. As we want also to consider this feature, we have chosen a Gaussian random variable to play the role of the random input A.

As a remarkable difference with respect to it occurs in the deterministic case, and based on the illustrative example that we have considered, we realized that piecewise random Fröbenius method requires more computational time than its former version developed in [4]. With respect to piecewise random Fröbenius method and polynomial chaos are concerned, we have seen that both require similar computational time whenever an appropriate balance between parameters $\{N, K\}$ is kept. Although being one of the most popular methods to deal with random differential equations, in this example Monte Carlo technique is not competitive with both random Fröbenius and polynomial chaos approaches. This feature is highlighted in our test example likely due to the highly oscillatory behavior of the solutions.

Although it is not the case for the study of random Airy differential equation (1), the mean square analyticity of the coefficients is, in general, demanded in order to apply random Fröbenius method. Nevertheless, we want to stress that useful characterizations of m.s. analyticity can be found in terms of correlation function [5, p.99]. We point out that another significant advantage of Fröbenius method with respect to polynomial chaos, is that Fröbenius method provides us a series representation of the solution stochastic process *directly* in terms of the random input, say A, rather than in function of standard Gaussian random variables which constitutes the cornerstone of the homogeneous polynomial chaos type-representation. This feature can become of prime importance in

order to deduce, for instance, the true statistical distribution of the solution stochastic process.

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