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A class of generalized finite T -groups

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Dedicated to the memory of Professor Klaus Doerk (1939-2004)

Abstract

Let T denote the class of finite groups in which normality is a transitive relation. Let \mathfrak{F} be a formation of full characteristic such that any subgroup of any T -group in \mathfrak{F} belongs to \mathfrak{F} . A subgroup M of a group G is said to be \mathfrak{F} -normal in G if $G/Core_G(M)$ belongs to \mathfrak{F} . Named after Kegel, a subgroup U of a group G is called a $K - \mathfrak{F}$ -subnormal subgroup of G if either $U = G$ or there is a chain of subgroups $U = U_0 \leq U_1 \leq \dots \leq U_n = G$ such that U_{i-1} is either normal in U_i or U_{i-1} is \mathfrak{F} -normal in U_i , for $i = 1, 2, \dots, n$. We call a finite group G a $T_{\mathfrak{F}}$ -group if every $K - \mathfrak{F}$ -subnormal subgroup of G is normal in G . When \mathfrak{F} is the class of all finite nilpotent groups, the $T_{\mathfrak{F}}$ -groups are precisely the T -groups. The aim of this paper is to analyse the structure of the $T_{\mathfrak{F}}$ -groups and show that in many cases $T_{\mathfrak{F}}$ is much more restrictive than T .

Keywords: T-group, Formation, F-subnormal Subgroup, Subnormal Subgroup, Pronormal Subgroup

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1. Introduction and Statements of Results

All groups considered in this paper are finite.

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1.1. T -groups

A group G is said to be a T -group if every subnormal subgroup of G is normal in G . The study of this class of groups has constituted a fruitful topic in group theory. The classical works by Gaschütz [5] and Robinson [8], for instance, reveal a very detailed picture of such groups. It is clear from the definition that a nilpotent group G is a T -group if and only if every subgroup of G is normal in G ; that is, G is a Dedekind group. More generally, Gaschütz proved the following result:

Theorem A (Gaschütz [5]). *Let G be a finite group with $G^{\mathfrak{n}}$ the nilpotent residual of G . Then G is a soluble T -group if and only if the following conditions hold:*

- (i) $G^{\mathfrak{n}}$ is a normal abelian Hall subgroup of G with odd order;
- (ii) $G/G^{\mathfrak{n}}$ is a Dedekind group;
- (iii) Every subgroup of $G^{\mathfrak{n}}$ is normal in G .

The following result due to Robinson [9, Theorem 4.1] characterises the non-soluble T -groups. Note that in the below result D can be taken to be the soluble residual of the group G . Also, a group G satisfies the condition T_p if, for all soluble normal subgroups N of G , the elements of G induce power automorphisms in every G -invariant p -subgroup of G/N of nilpotent class less than or equal to two.

Theorem B (Robinson [9]). *A group G is a T -group if and only if it has a perfect normal subgroup D such that:*

- (i) G/D is a soluble T -group;
- (ii) $D/Z(D) = U_1/Z(D) \times \dots \times U_k/Z(D)$ where $U_i/Z(D)$ is non-abelian simple and U_i is normal in G ;
- (iii) if $\{i_1, i_2, \dots, i_r\} \subseteq \{1, 2, \dots, k\}$, where $0 \leq r < k$, the group $G/U'_{i_1} \dots U'_{i_r}$ satisfies T_p for all primes p dividing the order of $Z(D)$.

Another characterisation of the soluble T -groups, due independently to Peng [7] and Robinson [8], characterises them in terms of the subgroup embedding property of pronormality.

Theorem C (Peng [7], Robinson [8]). *A group G is a soluble T -group if and only if every subgroup of prime power order is pronormal in G .*

1.2. Extensions of subnormality and pronormality

It is abundantly clear that subgroup embedding properties, such as pronormality and subnormality, play an important role in elucidating the structure of a group. Hall considered the subnormal subgroups to be the bare bones of a group, since they are precisely those subgroups which occur as terms of composition series, whose factors are crucial in describing the structure of a group.

Pronormality is important when families of conjugate subgroups which remain conjugate in intermediate subgroups are considered, and was introduced by Hall (see [2, I; 6.1]). In some sense these properties are diametrically opposite. In fact the normal subgroups are exactly the subgroups which are pronormal and subnormal.

Working within the framework of formation theory, and motivated by the theory of \mathfrak{F} -normalizers, \mathfrak{F} a saturated formation of full characteristic, developed by Carter and Hawkes (see [1] and [2]), one may extend many classical embedding properties for a subgroup in a group such as subnormality or pronormality to an arbitrary saturated formation, and most of the results concerning these embedding properties can be read off by specializing to the case where \mathfrak{F} is the class of nilpotent groups. Let us now introduce extensions of subnormality and pronormality.

Let \mathfrak{F} be a formation. A subgroup M of a group G is said to be \mathfrak{F} -normal in G if $G/Core_G(M)$ belongs to \mathfrak{F} .¹ It is clear that M is \mathfrak{F} -normal if and only if $G^{\mathfrak{F}}$, the \mathfrak{F} -residual of G , is contained in M . Kegel (see [1, 6; 6.1.4]) introduced an extension of subnormality which has come to be known as $K - \mathfrak{F}$ -subnormality.

Definition 1. *A subgroup U of a group G is called a $K - \mathfrak{F}$ -subnormal subgroup of G if either $U = G$ or there is a chain of subgroups*

$$U = U_0 \leq U_1 \leq \dots \leq U_n = G$$

such that either U_{i-1} is normal in U_i or U_{i-1} is \mathfrak{F} -normal in U_i , for $i = 1, 2, \dots, n$.

It is rather clear that the subnormal subgroups of a group G are exactly the $K - \mathfrak{N}$ -subnormal subgroups. $K - \mathfrak{F}$ -subnormality has been extensively studied with many results obtained (see [1, chapter 6]).

On the other hand, the second author ([3]) and, independently, Müller, in his Diplomarbeit supervised by Doerk ([6]), extended the property of pronormality of a subgroup of a soluble group to \mathfrak{F} -pronormality, where \mathfrak{F} is a subgroup-closed saturated formation of full characteristic, and explored some connections between \mathfrak{F} -pronormality and \mathfrak{F} -subnormality. (Because \mathfrak{F} -subnormality coincides with $K - \mathfrak{F}$ -subnormality in the soluble universe and we use $K - \mathfrak{F}$ -subnormality for nonsoluble groups here, we will present our results in terms of $K - \mathfrak{F}$ -subnormality.) \mathfrak{F} -bases, which are an extension of the Hall systems of soluble groups, play an important role in their approach. Outside of solubility, it is not possible to use \mathfrak{F} -bases and so an alternative definition, first put forward by Müller, is required.

Definition 2. *Let G be a group and U a subgroup of G . Then U is said to be \mathfrak{F} -pronormal in G if, for each $g \in G$, there exists $x \in \langle U, U^g \rangle^{\mathfrak{F}}$ such that $U^x = U^g$.*

¹Note that in other references, \mathfrak{F} -normality is defined in a slightly different way.

Assume that U is a pronormal subgroup of G ; then $UX^{\mathcal{N}}$ is a normal subgroup of $X = \langle U, U^g \rangle$ for all $g \in G$. Hence $X = UX^{\mathcal{N}}$ and so U and U^g are conjugate in $X^{\mathcal{N}}$. Hence U is pronormal in G if and only if for all $g \in G$, U and U^g are conjugate in $\langle U, U^g \rangle^{\mathcal{N}}$. Thus the above definition is a natural one as the \mathfrak{N} -pronormal subgroups are precisely the pronormal ones. Note that the \mathfrak{F} -projectors associated to saturated formations are typical examples of \mathfrak{F} -pronormal subgroups in the category of all soluble groups.

1.3. Statements of results

The theory of $K - \mathfrak{F}$ -subnormality and \mathfrak{F} -pronormality is usually defined only for subgroup-closed saturated formations. However, we can make do with a somewhat weaker assumption. Suppose \mathfrak{F} is a formation that contains the class of all nilpotent groups, and has the property that if $H \leq G \in T \cap \mathfrak{F}$, then $H \in \mathfrak{F}$. Then we say \mathfrak{F} possesses *Property ***. Of course any subgroup-closed saturated formation that contains the class of nilpotent groups will have Property **.

Bearing in mind the above discussion on T -groups and $K - \mathfrak{F}$ -subnormality, the following class of groups naturally arises:

Definition 3. *A group G is said to be a $T_{\mathfrak{F}}$ -group if every $K - \mathfrak{F}$ -subnormal subgroup of G is normal in G .*

It is clear that every subnormal subgroup is $K - \mathfrak{F}$ -subnormal and so $T_{\mathfrak{F}}$ is a class of T -groups. Moreover $T_{\mathfrak{N}} = T$. However, in many cases $T_{\mathfrak{F}}$ is much more restrictive. For instance every soluble T -group is supersoluble, whereas a soluble $T_{\mathfrak{U}}$ -group, where \mathfrak{U} is the formation of all supersoluble groups, is a Dedekind group. To describe each $T_{\mathfrak{F}}$ fully, we need first to analyse some properties of $T_{\mathfrak{F}}$ -groups. We begin to do this by taking the characterisation of soluble T -groups due to Peng and Robinson into account – it seems natural to look for a similar characterisation of $T_{\mathfrak{F}}$ -groups using \mathfrak{F} -pronormal subgroups. This was already done by the second author when \mathfrak{F} is soluble and subgroup-closed ([4, Theorem 1]). Our first major result analyses a more general case.

Theorem 1. *Let G be a group and \mathfrak{F} be a formation with Property **. The following statements are pairwise equivalent:*

- (i) G is a soluble $T_{\mathfrak{F}}$ -group.
- (ii) Every subgroup of G is \mathfrak{F} -pronormal in G .
- (iii) Every subgroup of G of prime power order is \mathfrak{F} -pronormal in G .

Corollary 1. *For each \mathfrak{F} with Property **, the class of soluble $T_{\mathfrak{F}}$ -groups is subgroup-closed.*

Now let p and q be primes, and let $\mathfrak{X}_{\mathfrak{F}}$ be the class of non-abelian groups of order pq that are elements of \mathfrak{F} , and let $\mathfrak{Y}_{\mathfrak{F}}$ be the class of non-abelian simple groups that are elements of \mathfrak{F} .

Definition 4. A group G is said to be an $R_{\mathfrak{F}}$ -group if G is a T -group and

- (i) No section of $G/G^{\mathfrak{S}}$ is isomorphic to an element of $\mathfrak{X}_{\mathfrak{F}}$.
- (ii) No chief factor of $G^{\mathfrak{S}}$ is isomorphic to an element of $\mathfrak{Y}_{\mathfrak{F}}$.

We use this definition to characterise $T_{\mathfrak{F}}$ in our second main result.

Theorem 2. If G is a group and \mathfrak{F} has Property **, then $G \in T_{\mathfrak{F}}$ if and only if $G \in R_{\mathfrak{F}}$.

Corollary 2. Assume \mathfrak{F}_1 and \mathfrak{F}_2 are two formations with Property **. Then $T_{\mathfrak{F}_2}$ is contained in $T_{\mathfrak{F}_1}$ if and only if $\mathfrak{X}_{\mathfrak{F}_1}$ is contained in $\mathfrak{X}_{\mathfrak{F}_2}$ and $\mathfrak{Y}_{\mathfrak{F}_1}$ is contained in $\mathfrak{Y}_{\mathfrak{F}_2}$.

Corollary 3. If \mathfrak{F} has Property **, $T_{\mathfrak{F}} = T_{\mathfrak{N}}$ if and only if $\mathfrak{X}_{\mathfrak{F}}$ is empty.

With Theorem A in mind it is only natural to ask if a natural extension of this result for $T_{\mathfrak{F}}$ -groups would be:

Let G be a finite group with $G^{\mathfrak{S}}$ the \mathfrak{F} -residual of G . Then G is a soluble $T_{\mathfrak{F}}$ -group if and only if $G^{\mathfrak{S}}$ is a normal abelian Hall subgroup of G with odd order, $G/G^{\mathfrak{S}}$ is a Dedekind group, and every subgroup of $G^{\mathfrak{S}}$ is normal in G .

However it is not true in general as the following example shows:

Example 1. Let \mathfrak{F} be a saturated formation with the following canonical local definition: $F(7) = \mathfrak{S}_{2,7}$, the class of all soluble $\{2,7\}$ -groups and $F(p) = \mathfrak{S}$ the class of all soluble groups for all primes $p \neq 7$. Then it is clear that $\mathfrak{N} \subseteq \mathfrak{F}$. The cyclic group of order 6 has an irreducible and faithful module V over $GF(7)$ of dimension 1. Let G be the corresponding semidirect product. It is clear that G is a T -group. Note that G is not an \mathfrak{F} -group and $G^{\mathfrak{S}} = V$. Denote by H a Sylow 2-subgroup of G . It is easy to check that H is a $K - \mathfrak{F}$ -subnormal subgroup of G which is not normal. Hence G satisfies the above conditions and it is not a $T_{\mathfrak{F}}$ -group.

Modifying the conditions slightly, we obtain a Gaschütz type characterisation of soluble $T_{\mathfrak{F}}$ -groups.

Theorem 3. Let G be a group and \mathfrak{F} have Property **. Then G is a soluble $T_{\mathfrak{F}}$ -group if and only if the following conditions hold:

- (i) $G^{\mathfrak{S}}$ is a normal abelian Hall subgroup of G with odd order;
- (ii) $X/X^{\mathfrak{S}}$ is a Dedekind group for every $X \leq G$;
- (iii) Every subgroup of $G^{\mathfrak{S}}$ is normal in G .

2. Preliminaries

The main properties of $K - \mathfrak{F}$ -subnormal subgroups are listed in the following result; the proofs if G is arbitrary and \mathfrak{F} is subgroup-closed are contained in the results cited. Hence here we supply a proof only of the last result, in the case where $G \in T$ and we assume only that \mathfrak{F} has Property **. This is the only place that the proofs differ from those of the original results.

Lemma 1. [1, 6; 6.1.6, 6.1.7 and 6.1.9] *Let G be a group and \mathfrak{F} be a formation containing the class of nilpotent groups such that either \mathfrak{F} is subgroup-closed or G is in T and \mathfrak{F} has Property **.*

- (i) *If H is $K - \mathfrak{F}$ -subnormal in L and L is $K - \mathfrak{F}$ -subnormal in G , then H is $K - \mathfrak{F}$ -subnormal in G .*
- (ii) *If N is a normal subgroup of G and U/N is $K - \mathfrak{F}$ -subnormal in G/N , then U is $K - \mathfrak{F}$ -subnormal in G .*
- (iii) *If H is $K - \mathfrak{F}$ -subnormal in G and N is a normal subgroup of G , then HN/N is $K - \mathfrak{F}$ -subnormal in G/N .*
- (iv) *If H is a subgroup of G with $G^{\mathfrak{F}} \leq H$, then H is $K - \mathfrak{F}$ -subnormal in G . In particular, if G is an \mathfrak{F} -group, then every subgroup of G is $K - \mathfrak{F}$ -subnormal in G . Hence if G is in \mathfrak{F} and $T_{\mathfrak{F}}$, it is Dedekind.*

Proof. We establish (iv), assuming G is a T -group and \mathfrak{F} has Property **. Suppose $G \in \mathfrak{F}$. Then for any subgroup V of G , by assumption $V \in \mathfrak{F}$, so $V^{\mathfrak{F}} = 1$. Hence if $U \leq V \leq G$, U is \mathfrak{F} -normal in V . This implies that if $H \leq G$, H is $K - \mathfrak{F}$ -subnormal in G , establishing the particular case. More generally, because $G \in T$, $G/G^{\mathfrak{F}} \in T \cap \mathfrak{F}$, so if $G^{\mathfrak{F}} \leq H$, then $H/G^{\mathfrak{F}}$ is $K - \mathfrak{F}$ -subnormal in the \mathfrak{F} -group $G/G^{\mathfrak{F}}$. The result follows from Assertion (ii). \square

Note that by applying Lemma 1(ii), we have that $T_{\mathfrak{F}}$ is closed under epimorphic images.

To prove Theorems 1 and 2, we will use the following result.

Lemma 2. *Suppose \mathfrak{F} is a formation that contains all nilpotent groups and has Property **.*

- (i) *If $G \in R_{\mathfrak{F}}$ and N is normal in G , then $G/N \in R_{\mathfrak{F}}$; i.e., $R_{\mathfrak{F}}$ is closed under epimorphic images.*
- (ii) *If $G \in R_{\mathfrak{F}}$ and $G \in \mathfrak{F}$, then G is Dedekind.*
- (iii) *If $G \in \mathfrak{S}$ and $G \in T_{\mathfrak{F}}$, then $G \in R_{\mathfrak{F}}$.*

Proof. For (i), let $G \in R_{\mathfrak{F}}$, and let K be a normal subgroup of G . Then $(G/K)^{\mathfrak{S}} = G^{\mathfrak{S}}K/K$, so $(G/K)/(G/K)^{\mathfrak{S}} = (G/K)/(G^{\mathfrak{S}}K/K)$ is isomorphic to $G/G^{\mathfrak{S}}K$, which is an epimorphic image of $G/G^{\mathfrak{S}}$, and therefore has no section that is an element of $\mathfrak{X}_{\mathfrak{F}}$. Similarly, $G^{\mathfrak{S}}K/K$ is isomorphic to an epimorphic image of $G^{\mathfrak{S}}$, so it has no chief factor that is an element of $\mathfrak{Y}_{\mathfrak{F}}$.

For (ii), we use induction on $|G|$. Let G be a non-Dedekind group of minimal order in $\mathfrak{F} \cap R_{\mathfrak{F}}$. Suppose N is a minimal normal subgroup of G . Hence $G/N \in \mathfrak{F}$, and $G/N \in R_{\mathfrak{F}}$ by (i) above. Thus G/N is Dedekind by minimality of G . Thus if $N \cap G^{\mathfrak{S}} = 1$, then N is isomorphic to a subgroup of $G/G^{\mathfrak{S}}$ and must be abelian. This means that if N is non-abelian, $N \leq G^{\mathfrak{S}}$. Furthermore, since G is a T -group, N is non-abelian simple, so it is a chief factor of $G^{\mathfrak{S}}$. However, $G \in T \cap \mathfrak{F}$, so $N \in \mathfrak{F}$ by Property **. This contradicts the assumption that $G \in R_{\mathfrak{F}}$. Thus N must be abelian; since G is a T -group, N is of prime order, p , and also G is soluble, so $G^{\mathfrak{S}} = 1$. If N is central in G , then G is nilpotent and therefore Dedekind since it is a T -group, so we may assume N is not central. Being of prime order and normal in G , N will be central in any Sylow p -subgroup of G , so there exists a prime q different from p and a Sylow q -subgroup Q of G that does not centralise N . Because the automorphism group of N is cyclic, $QN/C_Q(N)$ is a non-abelian group of order pq . Note that $QN \in \mathfrak{F}$ by Property **, so $QN/C_Q(N)$ is also, contradicting the assumption that $G \in R_{\mathfrak{F}}$ and establishing the result.

For (iii), we again use induction on $|G|$, beginning with a soluble G of minimal order in $T_{\mathfrak{F}} \setminus R_{\mathfrak{F}}$. Let N be minimal normal in G . Then $G/N \in T_{\mathfrak{F}}$, so $G/N \in R_{\mathfrak{F}}$ by minimality of G . Because G is a T -group, N is of prime order. There must be a section A/B of G that is non-abelian of order pq with normal subgroup of order p , where p and q are primes, with $A/B \in \mathfrak{F}$. Note that the following quotient groups are isomorphic: AN/BN , $A/A \cap BN$, and $A/B(A \cap N)$. Since $G/N \in R_{\mathfrak{F}}$, $A \cap N > 1$. Hence $N \leq A$. But if $N \leq B$, $AN/BN = A/B$, a contradiction. Thus $B \cap N = 1$, BN/B is the unique nontrivial proper normal subgroup in A/B , and $|N| = p$. Hence every minimal normal subgroup of G is of order p . Note that if G is nilpotent, it is in $R_{\mathfrak{F}}$; hence $G^{\mathfrak{N}} > 1$, and since G is a soluble T -group, every subgroup of $G^{\mathfrak{N}}$ is normal in G by Theorem A(iii), and $G^{\mathfrak{N}}$ is an abelian Hall subgroup of G by Theorem A(i). This implies that $G^{\mathfrak{N}} = P$ is a Sylow p -subgroup of G . Now if $B \cap P > 1$, since every subgroup of $P = G^{\mathfrak{N}}$ is normal in G , $B \cap P$, and therefore B , contains some minimal normal subgroup of G , a contradiction. Hence B and P are of relatively prime order. Then since $|A/B| = pq$, p^2 does not divide $|A|$, so A cannot contain more than one normal subgroup of order p . Hence G has a unique minimal normal subgroup N , so P is cyclic. Note that AP is normal in G because G/P is Dedekind, so AP is in $T_{\mathfrak{F}}$, and A/B is a section of AP , so by minimality of G , $G = AP$. Note that if $P = N$, then $G = A$, so A/B is in $T_{\mathfrak{F}}$ and in \mathfrak{F} ; hence A/B is Dedekind, a contradiction. Therefore, $P > N$ and $\Phi(P) > 1$. Now let Q be a Sylow p -subgroup of A . Then QP is normal in G because it contains P , so $QP \in T_{\mathfrak{F}}$, and therefore $QP/\Phi(P) \in T_{\mathfrak{F}}$. Then by minimality of G , $QP/\Phi(P) \in R_{\mathfrak{F}}$, implying $((Q\Phi(P)/\Phi(P))(P/\Phi(P)))/C_{(Q\Phi(P)/\Phi(P))}(P/\Phi(P))$ is not of order pq^i for $i > 0$. Hence Q centralises $P/\Phi(P)$, so that Q centralises P , and Q centralises N . But $A = BQN$, so A/B is nilpotent, a contradiction establishing the result. □

We collect now some properties of \mathfrak{F} -pronormal subgroups which are par-

ticularly useful when inductive arguments are applied. All but (iv) appear in [6].

Lemma 3. *Let U be a subgroup of a group G and let \mathfrak{F} be a formation.*

- (i) *If $U \leq H$ and U is \mathfrak{F} -pronormal in G , then U is \mathfrak{F} -pronormal in H .*
- (ii) *If N is a normal subgroup of G and U is \mathfrak{F} -pronormal in G , then UN/N is \mathfrak{F} -pronormal in G/N .*
- (iii) *If N is a normal subgroup of G and U/N is \mathfrak{F} -pronormal in G/N , then U is \mathfrak{F} -pronormal in G .*
- (iv) *If \mathfrak{F} has Property **, $G \in T$ is an \mathfrak{F} -group, and U is \mathfrak{F} -pronormal in G , then U is a normal subgroup of G .*

Proof. (i), (ii), and (iii) follow easily from the definition. Moreover if U is \mathfrak{F} -pronormal in G , and $g \in G$, there exists $x \in \langle U, U^g \rangle^{\mathfrak{F}}$ such that $U^x = U^g$. Since G is a $T \cap \mathfrak{F}$ -group, $\langle U, U^g \rangle \in \mathfrak{F}$, so $\langle U, U^g \rangle^{\mathfrak{F}} = 1$ and $U^g = U$ for each $g \in G$; that is, U is a normal subgroup of G , which yields (iv). \square

Proposition 1. *Let U be a subgroup of a group G and let N be a normal subgroup of G such that $U \leq N \leq G$. Then if \mathfrak{F} is a formation, the following conditions are equivalent:*

- (i) *U is \mathfrak{F} -pronormal in G .*
- (ii) *U is \mathfrak{F} -pronormal in N and $G = N_G(U)N$.*

Proof. Assume that U is an \mathfrak{F} -pronormal subgroup of G contained in a normal subgroup N of G . If $g \in G$, there exists $x \in \langle U, U^g \rangle^{\mathfrak{F}} \leq N$ such that $U^g = U^x$. Therefore $gx^{-1} \in N_G(U)$ and hence $G = N_G(U)N$.

To see that (ii) implies (i) consider $g = xn \in G = N_G(U)N$ with $x \in N_G(U)$ and $n \in N$. Then $U^g = U^{xn} = U^n$. On the other hand, since U is \mathfrak{F} -pronormal in N , there exists $m \in \langle U, U^n \rangle^{\mathfrak{F}}$ such that $U^n = U^m$. Consequently $U^g = U^m$ and $m \in \langle U, U^n \rangle^{\mathfrak{F}} = \langle U, U^g \rangle^{\mathfrak{F}}$; that is, U is \mathfrak{F} -pronormal in G . \square

3. Proofs of the main results

Proof of Theorem 1. It is clear that (ii) implies (iii). Next we show that (i) implies (ii). Assume the contrary, and that G is a counterexample of minimum possible order. Observe first that $G \in T$, so by Lemma 1(iv), since $G/G^{\mathfrak{F}}$ is a $T_{\mathfrak{F}}$ -group, it is Dedekind. Thus $1 \neq G^{\mathfrak{F}}$ contains $G^{\mathfrak{N}}$. Since it is clear that $G^{\mathfrak{F}} \leq G^{\mathfrak{N}}$, it follows at once that $G^{\mathfrak{F}} = G^{\mathfrak{N}}$. By Gaschütz's characterisation of soluble T -groups, $G^{\mathfrak{F}}$ is an abelian Hall subgroup of G and every subgroup of $G^{\mathfrak{F}}$ is normal in G .

Let H be a non- \mathfrak{F} -pronormal subgroup of G . Suppose $R = \text{Core}_G(H) > 1$. Then G/R is a $T_{\mathfrak{F}}$ -group, so by induction, H/R is \mathfrak{F} -pronormal in G/R , so H is \mathfrak{F} -pronormal in G by Lemma 3(iii), a contradiction. Thus $R = 1$ and

$H \cap G^{\mathfrak{F}} = 1$. Therefore H is a Hall subgroup of the normal subgroup $HG^{\mathfrak{F}}$ of G . Suppose that H is \mathfrak{F} -pronormal in $HG^{\mathfrak{F}}$. Then, since $G = G^{\mathfrak{F}}N_G(H)$ by the Frattini argument, we obtain that H is \mathfrak{F} -pronormal in G by Proposition 1. This contradiction shows that H is not \mathfrak{F} -pronormal in the $T_{\mathfrak{F}}$ -group $HG^{\mathfrak{F}}$ and so $G = HG^{\mathfrak{F}} = HG^{\mathfrak{N}}$ by the minimal choice of G . Applying [2, IV; 5.18], H is an \mathfrak{N} -projector of G , i.e. a Carter subgroup of G , and therefore pronormal in G .

Suppose $g \in G$ and let $X = \langle H, H^g \rangle$. The pronormality of H in G implies that $H^g = H^x$ for some $x \in X$. Now $X^{\mathfrak{N}} \leq G^{\mathfrak{N}}$ is abelian and H is an \mathfrak{N} -projector of X , so by [2, IV; 5.18], $X = HX^{\mathfrak{N}}$. Note that Definition 4 easily implies that any subgroup of a soluble member of $R_{\mathfrak{F}}$ is also in $R_{\mathfrak{F}}$. Now by Lemma 2(iii), $G \in R_{\mathfrak{F}}$, so $X \in R_{\mathfrak{F}}$. Thus by Lemma 2(i) and (ii), $X/X^{\mathfrak{F}}$ is Dedekind, implying $X^{\mathfrak{F}} = X^{\mathfrak{N}}$. Hence $H^x = H^y$, where $y \in X^{\mathfrak{F}}$. Thus H is \mathfrak{F} -pronormal in G , the final contradiction.

Finally we prove that (iii) implies (i), arguing by induction on the order of G . It is clear that every subgroup of prime power order of G is pronormal in G and so G is a soluble T -group by Theorem C. Suppose that $1 \neq G^{\mathfrak{F}}$. Then all the subgroups of p -power order in $G/G^{\mathfrak{F}}$ are epimorphic images of such subgroups in G , so they are \mathfrak{F} -pronormal by Lemma 3(ii). The induction hypothesis leads to the conclusion that $G/G^{\mathfrak{F}}$ is a $T_{\mathfrak{F}}$ -group. It is in \mathfrak{F} , so it is Dedekind. If $H < G$ and H is K - \mathfrak{F} -subnormal in G , then H is K - \mathfrak{F} -subnormal in a proper subgroup M of G , where M is either an \mathfrak{F} -normal subgroup of G or M is normal in G . In the former case, $G^{\mathfrak{F}} \leq M$. But then $M/G^{\mathfrak{F}}$ is normal in the Dedekind group $G/G^{\mathfrak{F}}$, so M is normal in G .

Hence H is K - \mathfrak{F} -subnormal in a proper normal subgroup M of G . Now induction, via Lemma 3(i), yields M is a $T_{\mathfrak{F}}$ -group. Thus H is normal in M , which is normal in G , so because G is a T -group, H is normal in G . We may suppose then that $G^{\mathfrak{F}} = 1$ and so every subgroup of prime power order is normal in G by Lemma 3(iv). Therefore G is a Dedekind group and then $G \in T_{\mathfrak{F}}$. \square

Proof of Corollary 1. Applying Theorem 1 and Lemma 3(i), if H denotes a subgroup of G , we see that every subgroup of H is \mathfrak{F} -pronormal in G and therefore in H ; hence H is a $T_{\mathfrak{F}}$ -group. \square

Proof of Theorem 2. Suppose first that $G \in T_{\mathfrak{F}}$. Then $G/G^{\mathfrak{S}}$ is a soluble $T_{\mathfrak{F}}$ -group, so by Lemma 2(iii), $G/G^{\mathfrak{S}} \in R_{\mathfrak{F}}$, and Condition (i) of Definition 4 is satisfied. Now each normal subgroup and chief factor of a $T_{\mathfrak{F}}$ -group is a $T_{\mathfrak{F}}$ -group, so $G^{\mathfrak{S}}$ and all its chief factors are in $T_{\mathfrak{F}}$. Hence if any chief factor of $G^{\mathfrak{S}}$ is in $\mathfrak{Y}_{\mathfrak{F}}$, then it is a Dedekind group, which is impossible. Thus Condition (ii) of Definition 4 is satisfied, and $G \in R_{\mathfrak{F}}$.

Hence we need to prove that if $G \in R_{\mathfrak{F}}$, then $G \in T_{\mathfrak{F}}$. Suppose not, choose a group G of minimal possible order in $R_{\mathfrak{F}} \setminus T_{\mathfrak{F}}$, and let H be a subgroup of minimal order in G such that H is K - \mathfrak{F} -subnormal in G but not normal in G . Write $R = Core_G(H)$. Suppose that $R > 1$. It follows that $G/R \in R_{\mathfrak{F}}$ by Lemma 2(i), and $G/R \in T_{\mathfrak{F}}$ by the choice of G . As H/R is a K - \mathfrak{F} -subnormal

subgroup of G/R by Lemma 1(iii), we have that H/R is normal in G/R , and therefore H is normal in G , contrary to assumption. Therefore $R = 1$.

If $G \in \mathfrak{F}$, by Lemma 2(ii) G is Dedekind, and in $T_{\mathfrak{F}}$. Thus we may assume G is not in \mathfrak{F} , so $G^{\mathfrak{F}} > 1$ and by induction, $G/G^{\mathfrak{F}}$ is a $T_{\mathfrak{F}}$ -group. Then $G/G^{\mathfrak{F}}$ is Dedekind, and any subgroup of G containing $G^{\mathfrak{F}}$ is normal in G . Also as above, H is \mathfrak{K} - \mathfrak{F} -subnormal in a proper normal subgroup M of G . Thus if $M \in R_{\mathfrak{F}}$, by induction $M \in T_{\mathfrak{F}}$, so H is normal in M and therefore normal in the T -group G , establishing the Theorem.

Suppose first that M is not soluble. Then $M^{\mathfrak{C}} > 1$, and $M^{\mathfrak{C}}$, being a characteristic subgroup of the normal subgroup M in G , is normal in G . Hence $G/M^{\mathfrak{C}}$, which is in $R_{\mathfrak{F}}$ by Lemma 2(i), is in $T_{\mathfrak{F}}$ by assumption. Thus the normal subgroup $M/M^{\mathfrak{C}}$ is also in $T_{\mathfrak{F}}$, and therefore in $R_{\mathfrak{F}}$. Now $(M/M^{\mathfrak{C}})^{\mathfrak{C}}$ is trivial, so Condition (i) of the definition of $R_{\mathfrak{F}}$ is satisfied by M because it is satisfied by $M/M^{\mathfrak{C}}$. Also, $M^{\mathfrak{C}}$ is a normal subgroup of $G^{\mathfrak{C}}$, which is a T -group. Then by Theorem B, $G^{\mathfrak{C}}/Z(G^{\mathfrak{C}})$ is a direct product of non-abelian simple groups, each of which is a chief factor of $G^{\mathfrak{C}}$. Thus none of these simple groups is in $\mathfrak{Y}_{\mathfrak{F}}$ because $G \in R_{\mathfrak{F}}$. Now $M^{\mathfrak{C}}Z(G^{\mathfrak{C}})/Z(G^{\mathfrak{C}})$ is normal in $G^{\mathfrak{C}}/Z(G^{\mathfrak{C}})$, so it is a direct product of non-abelian simple groups that are not in $\mathfrak{Y}_{\mathfrak{F}}$; these groups are its chief factors. Hence a chief series of $M^{\mathfrak{C}}/M^{\mathfrak{C}} \cap Z(G^{\mathfrak{C}})$ will have the same chief factors, and a chief series of $M^{\mathfrak{C}}$ passing through the abelian group $M^{\mathfrak{C}} \cap Z(G^{\mathfrak{C}})$ will have the same non-abelian chief factors. Hence M satisfies condition (ii) of the definition of $R_{\mathfrak{F}}$, so $M \in R_{\mathfrak{F}}$.

Now suppose M is soluble, so $M^{\mathfrak{C}} = 1$ and Condition (ii) of the definition of $R_{\mathfrak{F}}$ is satisfied trivially. We may assume that M is maximal with respect to being proper and normal in G . If $M \geq G^{\mathfrak{C}}$, then $G^{\mathfrak{C}}$ is soluble and therefore trivial, so G is soluble and $M \in R_{\mathfrak{F}}$ because it is a subgroup of a soluble member of $R_{\mathfrak{F}}$, and we get a contradiction as above. Thus $G = MG^{\mathfrak{C}}$. Also, because G is not soluble, $G > MZ(G^{\mathfrak{C}})$, so $Z(G^{\mathfrak{C}}) \leq M$. Hence $M \cap G^{\mathfrak{C}} \geq Z(G^{\mathfrak{C}})$. But $M \cap G^{\mathfrak{C}}/Z(G^{\mathfrak{C}})$ is a soluble normal subgroup of $G^{\mathfrak{C}}/Z(G^{\mathfrak{C}})$ and is therefore trivial, so $M \cap G^{\mathfrak{C}} = Z(G^{\mathfrak{C}})$.

Then $G/G^{\mathfrak{C}} = MG^{\mathfrak{C}}/G^{\mathfrak{C}}$ is isomorphic to $M/M \cap G^{\mathfrak{C}} = M/Z(G^{\mathfrak{C}})$. Now $[G^{\mathfrak{C}}, G^{\mathfrak{C}}] = G^{\mathfrak{C}}$ by Theorem B, while $[M, G^{\mathfrak{C}}] \leq M \cap G^{\mathfrak{C}} = Z(G^{\mathfrak{C}})$. Thus $[M, G^{\mathfrak{C}}, G^{\mathfrak{C}}] = 1$, so by the Three Subgroups Lemma, $[G^{\mathfrak{C}}, G^{\mathfrak{C}}, M] = [G^{\mathfrak{C}}, M] = 1$. Hence $Z(G^{\mathfrak{C}}) \leq Z(G)$. Now consider a section A/B of M . Suppose $A \cap Z(G^{\mathfrak{C}}) = B \cap Z(G^{\mathfrak{C}})$, and consider the group $AZ(G^{\mathfrak{C}})/BZ(G^{\mathfrak{C}})$, which is a section of $M/Z(G^{\mathfrak{C}})$, which is isomorphic to $G/G^{\mathfrak{C}}$. Then each of the groups in the following sequence is isomorphic: A/B , $(A/A \cap Z(G^{\mathfrak{C}}))/(B/B \cap Z(G^{\mathfrak{C}}))$, $(AZ(G^{\mathfrak{C}})/Z(G^{\mathfrak{C}}))/(BZ(G^{\mathfrak{C}})/Z(G^{\mathfrak{C}}))$, and $AZ(G^{\mathfrak{C}})/BZ(G^{\mathfrak{C}})$. Thus A/B is not in $\mathfrak{X}_{\mathfrak{F}}$. And if $A \cap Z(G^{\mathfrak{C}}) > B \cap Z(G^{\mathfrak{C}})$, then $A \cap (BZ(G^{\mathfrak{C}})) > B$, so $B(A \cap Z(G^{\mathfrak{C}})) > B$. But this implies that A/B has a nontrivial central subgroup $B(A \cap Z(G^{\mathfrak{C}}))/B$, and again A/B is not in $\mathfrak{X}_{\mathfrak{F}}$. Thus $M \in R_{\mathfrak{F}}$, and the theorem is proved. \square

Obviously the classes \mathfrak{U} and \mathfrak{N} satisfy Condition (ii) from Definition 4. However every non-abelian group of order pq , where p and q are primes, is an element of \mathfrak{U} , so no such group can be a $T_{\mathfrak{U}}$ -group. Then clearly not every T -group is

a $T_{\mathfrak{U}}$ -group. Hence Condition (i) is necessary to distinguish between different $T_{\mathfrak{F}}$'s. The following example shows that Condition (ii) cannot be dispensed with in Theorem 2 either.

Example 2. *Let S be a non-abelian simple group and let \mathfrak{G}_S denote the class of all groups G which are isomorphic to a section of a direct product of finitely many copies of S . It is clear that \mathfrak{G}_S is a subgroup-closed formation. Hence $\mathfrak{F}_S = \mathfrak{N} \circ \mathfrak{G}_S$ is a subgroup-closed saturated formation.*

Assume that S_1 and S_2 are two different non-abelian simple groups whose orders are divisible by the elements of the same set π of prime numbers. Let us consider the formations \mathfrak{F}_{S_1} and \mathfrak{F}_{S_2} . Then each of these formations will contain non-abelian groups of order pq , where p and q are primes and the non-normal Sylow subgroups are of order q , for exactly those q that are elements of π . Hence $\mathfrak{F}_{S_1} \setminus \mathfrak{F}_{S_2}$ contains no non-abelian subgroup of order pq . Assume now that $S_1 \notin \mathfrak{F}_{S_2}$ (for instance $S_1 = A_6$ and $S_2 = A_5$, the alternating groups of degrees 6 and 5 respectively). Then $S_1^{\mathfrak{F}_{S_2}} = S_1$ and so $S_1 \in T_{\mathfrak{F}_{S_2}} \setminus T_{\mathfrak{F}_{S_1}}$.

Proof of Corollary 3. We know $T_{\mathfrak{F}}$ is contained in $T_{\mathfrak{N}}$, and $\mathfrak{X}_{\mathfrak{N}}$ and $\mathfrak{Y}_{\mathfrak{N}}$ are empty, so by Corollary 2 we need only prove that if $\mathfrak{X}_{\mathfrak{F}}$ is empty, then so is $\mathfrak{Y}_{\mathfrak{F}}$. Let F be the canonical formation function defining \mathfrak{F} . If S is a non-abelian simple group in \mathfrak{F} , let p be an odd prime dividing $|S|$. Then by [2, IV; 4.2], $C_2 \in F(p)$, and $F(p)$, and therefore \mathfrak{F} , contains a non-abelian group of order $2p$. \square

Proof of Theorem 3. Assume that G is a soluble $T_{\mathfrak{F}}$ -group. Then G is a T -group and so $G^{\mathfrak{N}}$ is an abelian Hall subgroup of odd order and every subgroup of $G^{\mathfrak{N}}$ is normal in G by Gaschütz's Theorem. Since $G/G^{\mathfrak{F}}$ is a Dedekind group and $G^{\mathfrak{N}}$ contains $G^{\mathfrak{F}}$, it follows that $G^{\mathfrak{N}} = G^{\mathfrak{F}}$ and therefore G satisfies conditions (i) and (iii). Let X be a subgroup of G . By Theorem 1, X is a $T_{\mathfrak{F}}$ -group, so $X/X^{\mathfrak{F}}$ is Dedekind as seen above. Therefore Condition (ii) holds.

We prove that Conditions (i)-(iii) imply G is a soluble $T_{\mathfrak{F}}$ -group. We proceed by induction on $|G|$, noting that the conditions imply that G is a soluble T -group. Assume that $G^{\mathfrak{F}} \neq 1$ and let H be a proper $K - \mathfrak{F}$ -subnormal subgroup of G . Now H is $K - \mathfrak{F}$ -subnormal in a proper subgroup M of G such that M is either \mathfrak{F} -normal in G or normal in G . If M is \mathfrak{F} -normal in G , then M contains $G^{\mathfrak{F}}$, so we know that M is normal in G by Condition (ii). Thus in either case, we can choose M normal in G .

Now M is a T -group, and by Condition (ii), $M^{\mathfrak{F}} = M^{\mathfrak{N}}$. Hence M satisfies Conditions (i) and (iii) because it is a T -group, and M inherits Condition (ii) from G . Therefore M is a $T_{\mathfrak{F}}$ -group by minimality of G . Hence H is normal in M and subnormal in the T -group G , so H is normal in G . Consequently we may suppose that $G \in \mathfrak{F}$. In that case G is a Dedekind group and so $G \in T_{\mathfrak{F}}$. The proof of the theorem is now complete. \square

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