A BISHOP-PHELPS-BOLLOBÁS TYPE THEOREM FOR UNIFORM ALGEBRAS

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ABSTRACT. This paper is devoted to showing that Asplund operators with range in a uniform Banach algebra have the Bishop-Phelps-Bollobás property, *i.e.*, they are approximated by norm attaining Asplund operators at the same time that a point where the approximated operator almost attains its norm is approximated by a point at which the approximating operator attains it. To prove this result we use the weak*-to-norm fragmentability of weak*-compact subsets of dual of Asplund spaces and we need to observe a Urysohn type result producing peak complex-valued functions in uniform algebras that are small outside a given open set and whose image is inside a Stolz region.

1. INTRODUCTION

Mathematical optimization is associated to maximizing or minimizing real functions. James's compactness theorem [16] and Bishop-Pehlps's theorem [5] are two landmark results along this line in functional analysis. The former characterizes reflexive Banach spaces X as those for which continuous linear functionals $x^* \in X^*$ attain their norm in the unit sphere S_X. The latter establishes that for any Banach space X every continuous linear functional $x^* \in X^*$ can be approximated (in norm) by linear functionals that attain the norm in S_X. This paper is concerned with the study of a strengthening of Bishop-Phelps's theorem that mixes ideas of Bollobás [6] –see Theorem 3.1 here– and Lindenstrauss [19] –who initiated the study of the Bishop-Phelps property for bounded operators between Banach spaces. Our starting point is the following definition brought in by Acosta, Aron, García and Maestre in 2008:

Definition 1 ([1]). A pair of Banach spaces (X, Y) is said to have the Bishop-Phelps-Bollobás property (BPBp for short) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$, such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that $\|T(x_0)\| > 1 - \delta(\varepsilon)$, then there exist $u_0 \in S_X$ and $\widetilde{T} \in S_{L(X,Y)}$ satisfying

$$\left\|\widetilde{T}(u_0)\right\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \left\|T - \widetilde{T}\right\| < \varepsilon.$$

A good number of papers regarding BPBp have been written during the last years, as for instance [3, 7, 8]. Very recently, a general result has been proved

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in [2], that in particular says that pairs of the form $(X, C(K))$ do have the BPBp whenever X is an Asplund space and $C(K)$ is the space of continuous functions defined on a compact Hausdorff space K: this result provided the first examples of pairs of the kind (c_0, Y) with BPBp for Y infinite dimensional Banach space. Our aim here is to extend and sharpen the results of [2] and prove the following:

Theorem 3.6. *Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T\colon X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that*

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \textit{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

For $\mathfrak{A} = C(K)$ the above result was proved in [2, Theorem 2.4] with worse estimates. The key points for the known proof when $\mathfrak{A} = C(K)$ were, on one hand, the asplundness of T hidden in Lemma 2.3 of [2] that led to a suitable open set $U \subset K$ and, on the other hand, the Urysohn's lemma that applied to an arbitrary $t_0 \in U$ produces a function $f \in C(K)$ satisfying

$$f(t_0) = \|f\|_\infty = 1,\ f(K) \subset [0, 1] \text{ and } \operatorname{supp}(f) \subset U.$$

With all this setting, $\widetilde{T}$ was explicitly defined by

$$\widetilde{T}(x)(t) = f(t) \cdot y^*(x) + (1 - f(t)) \cdot T(x)(t),\ x \in X,\ t \in K, \tag{1.1}$$

where $y^* \in S_{X^*}$ was chosen satisfying, amongst other things, satisfying $1 = |y^*(u_0)| = \|u_0\|$ and $\|x_0 - u_0\| < \varepsilon$. The provisos about y^* and f were used then to prove that T and $\widetilde{T}$ were close and that $1 = \|\widetilde{T}\| = \|\widetilde{T}u_0\|$. With just the details above the reader should be able to prove indeed that $1 = \|\widetilde{T}\| = \|\widetilde{T}u_0\|$, but he or she will have to make use of the fact that $f(K) \subset [0, 1]$. Once this is said, it becomes clear that the arguments above cannot work for a proof of Theorem 3.6 for a general uniform algebra $\mathfrak{A} \subset C(K)$. Certainly, $\mathfrak{A}$ could be too rigid (for instance the disk algebra) to allow the construction of $f \in \mathfrak{A}$ peaking at t_0 and with $f(K) \subset [0, 1]$. To overcome these difficulties we observe in Lemma 2.5 below an easy but useful statement about the existence of peak functions $f \in \mathfrak{A}$ that are small outside an open set and with $f(K)$ contained in the Stolz's region

$$\mathrm{St}_\varepsilon = \{z \in \overline{\mathbb{D}}\colon |z| + (1 - \varepsilon)|1 - z| \leq 1\},$$

see Figure 1.

Lemma 2.5. *Let $\mathfrak{A} \subset C(K)$ be a unital uniform algebra and Γ_0 its Choquet boundary. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $0 < \varepsilon < 1$, there exist $f \in \mathfrak{A}$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \|f\|_\infty = 1$, $|f(t)| < \varepsilon$ for every $t \in K \setminus U$ and $f(K) \subset \mathrm{St}_\varepsilon$,* i.e.

$$|f(t)| + (1 - \varepsilon)|1 - f(t)| \leq 1, \textit{ for all } t \in K. \tag{2.2}$$

With Lemma 2.5 in mind we can appeal at the full power of Lemma 2.3 of [2], that is also suited for a boundary instead of K, to produce U and then modify the definition of $\widetilde{T}$ in (1.1) with an auxiliary ε' as

$$\widetilde{T}(x)(t) = f(t) \cdot y^*(x) + (1 - \varepsilon')(1 - f(t)) \cdot T(x)(t),\ x \in X,\ t \in K. \tag{1.2}$$

Here f is linked to ε' and U via Lemma 2.5. Inequality (2.2) allows us to prove again $1 = \|\widetilde{T}\| = \|\widetilde{T}u_0\|$ and the other thesis in Lemma 2.5 imply $\|T - \widetilde{T}\| < 2\varepsilon$.

The explanations above cover the relevant results of this paper and isolate the difficulties we have had to overcome to prove them. We should stress that our results are proved for unital and non unital uniform algebras, and that to the best of our knowledge these results are not known even for the Bishop-Phelps property.

The paper is divided as follows. This introduction finishes with a subsection devoted to *Notation and Terminology.* Then, Section 2 is devoted to prove the existence of peak functions for uniform algebras with values in St_ε: this is what we observe as our Urysohn type lemmas, see Lemma 2.5 and Lemma 2.7, that are needed to establish our main result in this paper, Theorem 3.6. The difficulty to prove the existence of peak functions in uniform algebras with values in our needed St_ε is the same that when St_ε is replaced by the closure of any bounded simply connected region with simple boundary points: for this reason we have observed these general facts too in Proposition 2.8. Section 3 is devoted to prove Theorem 3.6, its preparatives and its consequences.

Notation and terminology. By letters X and Y we always denote Banach spaces. Unless otherwise stated our Banach spaces can be real or complex. B_X and S_X are the closed unit ball and the unit sphere of X. By X^* –respectively X^{**}– we denote the topological dual –respectively bidual– of X. Given a complex Banach space X we will write $X_{\mathbb{R}}$ to denote X but with its subjacent real Banach structure. The weak topology in X is denoted by w, and w^* is the weak* topology in X^*. $L(X,Y)$ stands for the space of norm bounded linear operators from X into Y endowed with its usual norm of uniform convergence on bounded sets of X. A subset B of the dual unit ball B_{X^*} is said to be 1-norming if for every $x \in X$ we have $\|x\| = \sup\{|x^*(x)| \colon x^* \in B\}$. Given a convex subset $C \subset X$ we denote by $\mathrm{ext}(C)$ the set of extreme points of C, *i.e.*, those points in C that are not midpoints of non-degenerate segments in C. Given $C \subset X$, $x^* \in X^*$ and $\alpha > 0$ we write

$$S(x^*, C, \alpha) := \{y \in C \colon \operatorname{Re} x^*(y) > \sup_{z \in C} \operatorname{Re} x^*(z) - \alpha\}.$$

$S(x^*, C, \alpha)$ is called a slice of C. In particular, if $C \subset X^*$ and $x^* = x$ is taken in the predual X we say that the slice $S(x, C, \alpha)$ is a w^*-slice of C. A classical Choquet's lemma says that for a convex and w^*-compact set $C \subset X^*$, given a point $x^* \in \mathrm{ext}(C)$, the family of w^*-slices

$$\{S(x, C, \alpha) \colon \alpha > 0, x \in X, x^* \in S(x, C, \alpha)\}$$

forms a neighborhood base of x^* in the relative w^*-topology of C – see [9, Proposition 25.13].

The letters K and L are reserved to denote compact and locally compact Hausdorff spaces respectively. $C(K)$ stands for the space of complex-valued continuous functions defined on K and $\|\cdot\|_\infty$ denotes the supremum norm on $C(K)$. A uniform algebra is a $\|\cdot\|_\infty$-closed subalgebra $A \subset C(K)$ equipped with the supremum norm, that separates the points of K (that is, for every $x \neq y$ in K there exists $f \in A$ such that $f(x) \neq f(y)$). Given $x \in K$, we denote by $\delta_x \colon A \to \mathbb{C}$ the evaluation functional at x given by $\delta_x(f) = f(x)$, for $f \in A$. The natural injection $\imath \colon K \to A^*$ defined by $\imath(t) = \delta_t$ for $t \in K$ is a homeomorphism from K onto $(\imath(K), w^*)$. A set $S \subset K$ is said to be a boundary for the uniform algebra A if for every $f \in A$ there exists $x \in S$ such that $|f(x)| = \|f\|_\infty$. We say that the uniform algebra $A \subset C(K)$ is unital if the constant function $\mathbf{1}$ belongs to A. Given $x \in K$ we denote by $\mathcal{N}_x$ the family of the open sets in K containing x.

In what follows $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ is the open unit disk of the complex plane, $\overline{\mathbb{D}} = \{z \in \mathbb{C}\colon |z| \leq 1\}$ is the the closed unit disk and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. By $A(\mathbb{D})$ we denote the disk algebra, *i.e.*, the uniform subalgebra of $C\big(\overline{\mathbb{D}}\big)$ made of functions whose restrictions to $\mathbb{D}$ are analytic. Given $z \in \mathbb{C}$ and $r > 0$, we write $D(z;r)$ –respectively $D[z;r]$– to denote the open disk $z + r\mathbb{D}$ –respectively the closed disk $z + r\overline{\mathbb{D}}$.

Our standard references are: [15] for Banach space theory, [10] for Banach algebras, [23] for complex analysis and [17] for harmonic analysis.

2. A Urysohn type lemma for uniform algebras

As we mentioned in the introduction our main goal in this paper is to extend [2, Theorem 2.4] to any uniform algebra. As noted, this result in [2] depends on Urysohn's lemma, that for a compact K allows us to find for a given $x \in K$ and $U \in \mathcal{N}_x$, a continuous real valued function of norm one, taking value 1 at x and vanishing on $K \setminus U$. We can not use this lemma in the setting of a general uniform algebra A, because the resulting function does not necessarily belong to A. Therefore, our first task here is to prove a Urysohn type lemma for uniform algebras on which we can rely on.

2.1. **Unital algebras and Stolz regions.** Throughout this subsection A is a unital uniform algebra on K. If

$$S := \{x^* \in A^*\colon \ \|x^*\| = 1, x^*(\mathbf{1}) = 1\}, \tag{2.1}$$

then $\Gamma_0 = \{t \in K\colon \delta_t \in \operatorname{ext}(S)\}$ is a boundary for A that is called the Choquet boundary of A, see [10, Lemma 4.3.2 and Proposition 4.3.4].

A stronger version of Lemma 2.1 below can be proved taking into account that in unital uniform algebras the Choquet boundary consists exactly of the strong boundary points of K for the algebra, see [10, Theorem 4.3.5] (see also Proposition 2.8 in this paper where this result is applied). Nonetheless, we prefer to state Lemma 2.1 as follows because this is exactly what is needed to prove our main result in Section 3. On the other hand the proof that we provide makes this part self-contained and our arguments will be later adapted when proving the corresponding result for non-unital algebras, see Lemma 2.6.

Lemma 2.1. *Let $A \subset C(K)$ be as above. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $\delta > 0$, there exists $f = f_\delta \in A$ and $t_0 \in U \cap \Gamma_0$ such that $\|f\|_\infty = f(t_0) = 1$ and $|f(t)| < \delta$ for every $t \in K \setminus U$.*

Proof. Observe first that $\imath(U)$ is a w^*-open set in $\imath(K)$. Therefore, there exists a w^*-open set $V \subset S$ such that $\imath(U) = V \cap \imath(K)$. Fix $x \in U \cap \Gamma_0$. Since δ_x is an extreme point of the w^*-compact set S and δ_x belongs to $V \subset S$, Choquet's lemma ensures the existence of $f_0 \in A$ and $r \in \mathbb{R}$ such that the w^*-slice of S, $\{x^* \in S\colon \operatorname{Re} x^*(f_0) > r\}$, is included into $V \cap S$ and contains δ_x. In particular, $\operatorname{Re} f_0(x) > r$ and $\operatorname{Re} f_0(t) \leq r$ for all $t \in K \setminus U$.

Note that $\max_{t\in K} \operatorname{Re} f_0(t) =: m > r$ and consider $g(t) := e^{f_0(t)}$ for $t \in K$. It is clear that $g \in A$ –see Lemma 2.2–, $g(K) \subset e^m\overline{\mathbb{D}}$ and that g maps $K \setminus U$

into $e^r\overline{\mathbb{D}}$, *i.e.*, strictly inside of $e^m\overline{\mathbb{D}}$. Since Γ_0 is a boundary for A, there exists $t_0 \in U \cap \Gamma_0$ such that $|g(t_0)| = e^m$. Now, take $n \in \mathbb{N}$ such that $e^{n(r-m)} < \delta$. Then, the function defined by

$$f(t) = \left(\frac{g(t)}{g(t_0)}\right)^n, \text{ for } t \in K,$$

is the one that we need. $\square$

We also need the following two lemmas that gather some basic and known results about uniform algebras. Lemma 2.3 that we write down without a proof can be proved in several different easy ways; it also appears as a very particular and straightforward consequence of some other much stronger result, see for instance Mergelyan's theorem [23, Theorem 20.5].

Lemma 2.2. *Let $A \subset C(K)$ be a uniform algebra, $M \subset \mathbb{C}$ and $g\colon M \to \mathbb{C}$ a function that is the uniform limit of a sequence of complex polynomials restricted to M. For every $f \in A$ with $f(K) \subset M$ the following statements hold true:*

(i) *If A is unital, then $g \circ f \in A$.*
(ii) *If A is non-unital, $0 \in M$ and $g(0) = 0$, then $g \circ f \in A$.*

Proof. Let us fix a sequence $p_n\colon \mathbb{C} \to \mathbb{C}$ of polynomials that converges uniformly to g on M. In case (i), $p_n \circ f \in A$ for $n \in \mathbb{N}$ and $g \circ f$ is the uniform limit on K of $(p_n \circ f)_n$, and therefore $g \circ f \in A$. In case (ii), we define $q_n := p_n - p_n(0)$ for every $n \in \mathbb{N}$. Now, $q_n \circ f \in A$ for $n \in \mathbb{N}$ and $g \circ f$ is the uniform limit on K of $(q_n \circ f)_n$, and therefore $g \circ f \in A$. $\square$

Lemma 2.3. *Every $\phi \in A(\mathbb{D})$ is the uniform limit of a sequence of complex polynomials on $\overline{\mathbb{D}}$.*

As already recalled in the Introduction for $0 < \varepsilon < 1$ the Stolz region is defined by

$$\mathrm{St}_\varepsilon := \{z \in \mathbb{C}\colon |z| + (1-\varepsilon)|1-z| \le 1\}.$$

Let us note that St_ε is convex, $\mathrm{St}_\varepsilon \subset \overline{\mathbb{D}}$ and 1 is the only point of the unit circle $\mathbb{T}$ that belongs to (the boundary of) St_ε. Note also that $\varepsilon^2\overline{\mathbb{D}} \subset \mathrm{St}_\varepsilon$ and therefore 0 is an interior point of St_ε. Indeed, for every $z \in \varepsilon^2\overline{\mathbb{D}}$ we have that

$$|z| + (1-\varepsilon)|1-z| \le \varepsilon^2 + (1-\varepsilon)(1+\varepsilon^2) < \varepsilon^2 + (1-\varepsilon)(1+\varepsilon) = 1.$$

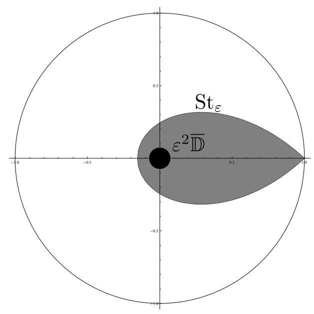

FIGURE 1. Stolz's region

Theorem 14.19 of [23] implies that the Stolz region has the following property.

Remark 2.4. *There exists a homeomorphism* $\phi : \overline{\mathbb{D}} \to \mathrm{St}_\varepsilon$ *such that:*

(i) ϕ *restricted to* $\mathbb{D}$ *is a conformal mapping onto the interior* $\mathrm{int}(\mathrm{St}_\varepsilon)$ *of* St_ε*;*
(i) $\phi(1) = 1$*;*
(ii) $\phi(0) = 0$.

Finally we can prove the auxiliary lemma, announced in the Introduction:

Lemma 2.5. *Let* $A \subset C(K)$ *be a unital uniform algebra. Then, for every open set* $U \subset K$ *with* $U \cap \Gamma_0 \neq \emptyset$ *and* $0 < \varepsilon < 1$*, there exist* $f \in A$ *and* $t_0 \in U \cap \Gamma_0$ *such that* $f(t_0) = \|f\|_\infty = 1$*,* $|f(t)| < \varepsilon$ *for every* $t \in K \setminus U$ *and* $f(K) \subset \mathrm{St}_\varepsilon$*, i.e.*

$$|f(t)| + (1 - \varepsilon)|1 - f(t)| \leq 1, \text{ for all } t \in K. \tag{2.2}$$

Proof. Let $\phi \in A(\mathbb{D})$ be the function from Remark 2.4. The set $\phi^{-1}(\varepsilon^2\mathbb{D}) \subset \mathbb{D}$ is an open neighbourhood of 0. Let $\delta > 0$ be such that $\delta\mathbb{D} \subset \phi^{-1}(\varepsilon^2\mathbb{D})$ and let f_δ be the function of norm one and t_0 the corresponding point in $U \cap \Gamma_0$ provided by Lemma 2.1. Then the function $f = \phi \circ f_\delta$ is the one that we need. Indeed, on one hand Lemmas 2.2 and 2.3 assure us that $f \in A$. On the other hand, we have that $f(K) \subset \mathrm{St}_\varepsilon$ that gives us inequality (2.2), and also $f(t_0) = \phi(f_\delta(t_0)) = 1 = \|f\|_\infty$. Finally we have that,

$$f(K \setminus U) = \phi(f_\delta(K \setminus U)) \subset \phi(\delta\mathbb{D}) \subset \varepsilon^2\mathbb{D} \subset \varepsilon\mathbb{D}.$$

Thus, $|f(z)| < \varepsilon$ for every $t \in K \setminus U$ and the proof is finished. $\square$

2.2. **Non-unital algebras and Stolz regions.** Throughout this subsection B is a non-unital uniform algebra, that is, a closed subalgebra of $C(K)$, separating points and with $\mathbf{1} \notin B$. Denote by $A := \{c\mathbf{1} + f \colon c \in \mathbb{C}, f \in B\}$ the $\|\cdot\|_\infty$-closed subalgebra generated by $B \cup \{\mathbf{1}\}$. Since the natural embedding of A into the space of continuous functions on the set of characters of A is an isometry, we can assume without loss of generality that K is the Gelfand compactum –*i.e.* set of characters– of A. Consider the Choquet boundary of A, $\Gamma_0(A) \subset K$. Since B is a maximal ideal of A (note that it is 1-codimensional), Gelfand-Mazur theorem assures us that there exists $\nu \in K$ such that $B = \{f \in A \colon \delta_\nu(f) = 0\}$. Denote $\Gamma_0 = \Gamma_0(A)\setminus\{\nu\}$. Observe that Γ_0 is a boundary for B. For general background on Gelfand representation theory we refer to [13].

With a bit of extra work in the proof of Lemma 2.1, its non-unital version is proved below.

Lemma 2.6. *Let* $B \subset C(K)$ *be as above. Then, for every open set* $U \subset K$ *with* $U \cap \Gamma_0 \neq \emptyset$ *and* $\delta > 0$*, there is* $f \in B$ *and* $t_0 \in U \cap \Gamma_0$ *such that* $\|f\|_\infty = f(t_0) = 1$ *and* $|f(t)| < \delta$ *for every* $t \in K \setminus U$*.*

Proof. Without loss of generality we can assume that $\nu \notin U$. We use the natural identification of K with $\imath(K)$ as we did in the proof of Lemma 2.1. Let us fix $x \in U \cap \Gamma_0$. Since x is an extreme point of S as defined in (2.1), by Choquet's lemma, there exists a w^*-slice of S that contains x and lies inside U. This slice that can be assumed generated by an element $f_0 \in B$ –note that $\mathbf{1}$ is constant on S– is of the form $\{y^* \in S \colon \mathrm{Re}\, y^*(f_0) > r\}$ for some $r \in \mathbb{R}$. So, $\mathrm{Re}\, f_0(x) > r$, and for every $t \in K \setminus U$ we have $\mathrm{Re}\, f_0(t) \leq r$ and in particular $0 = \mathrm{Re}\, f_0(\nu) \leq r$.

Note that $\max_{t \in K} \mathrm{Re}\, f_0(t) =: m > r$. Since Γ_0 is a boundary for B, there exists a $t_0 \in \Gamma_0 \cap U$ such that $\mathrm{Re}\, f_0(t_0) = m$. Define $g(t) = e^{f_0(t)} - 1$, $t \in K$. Then we have that $g \in B$ after Lemma 2.2, $g(K) \subset e^m\overline{\mathbb{D}} - 1$, and $g(K \setminus U) \subset$

$e^r\overline{\mathbb{D}}-1$, *i.e.*, strictly inside of $e^m\overline{\mathbb{D}}-1$. Observe that $0 \in e^m\overline{\mathbb{D}}-1$ because $m > r \geq \operatorname{Re} f_0(\nu) = 0$. Now, consider a Möbius transformation $h(z) = \frac{az+b}{cz+d}$ that conformally maps $e^m\mathbb{D}-1$ onto $\mathbb{D}$, the boundary of $e^m\mathbb{D}-1$ onto the boundary of $\mathbb{D}$ and such that $h(0)=0$. Since $g(t_0) = e^{f_0(t_0)}-1$ belongs to the boundary of $e^m\overline{\mathbb{D}}-1$, its image $h(g(t_0))$ belongs to the boundary of $\overline{\mathbb{D}}$. Then

$$f(t) := \left(\frac{(h\circ g)(t)}{(h\circ g)(t_0)}\right)^n, t \in K,$$

for suitable $n \in \mathbb{N}$, is the function that we need. □

The main result of this subsection reads as follows:

Lemma 2.7. *Let $B \subset C(K)$ be as in the previous lemmas. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $0<\varepsilon<1$, there exist $f \in B$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \|f\|_\infty = 1$, $|f(t)| < \varepsilon$ for every $t \in K \setminus U$ and*

$$|f(t)| + (1-\varepsilon)|1-f(t)| \leq 1, \textit{ for all } t \in K.$$

Proof. The proof that is left to the reader is the same as for the analogous Lemma 2.5 for unital algebras: the idea now is to combine again Lemmas 2.2, 2.3, 2.6 and Remark 2.4 taking into account that since $\phi(0)=0$ our arguments work for the non-unital case as well. □

2.3. **General case: simply connected regions.** For our applications in this paper to the Bishop-Phelps-Bollobás property included in Section 3 we just need the Lemmas 2.5 and 2.7 as presented already. Nonetheless the reader might have realized that our previous arguments work for arbitrary bounded simply connected region with simple boundary points. Although we do not need it we complete this section with a few comments about this general case.

Recall that a boundary point β of a simply connected region Ω of $\mathbb{C}$ is said to be a *simple boundary point* of Ω if β has the following property: to every sequence $(z_n)_n$ in Ω such that $z_n \to \beta$ there corresponds a curve $\gamma : [0,1] \to \mathbb{C}$ and a sequence $(t_n)_n$,

$$0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \ldots \text{ with } t_n \to 1,$$

such that $\gamma(t_n) = z_n$ for every $n \in \mathbb{N}$ and $\gamma\big([0,1]\big) \subset \Omega$, see [23, p. 289]. All points in the boundary of $\overline{\mathbb{D}}$ and St_ε are simple boundary points.

Every bounded simply connected region Ω such that all points in its boundary $\partial\Omega$ are simple has the property that every conformal mapping of Ω onto $\mathbb{D}$ extends to a homeomorphism of $\overline{\Omega}$ onto $\overline{\mathbb{D}}$, see [23, Theorem 14.19].

Proposition 2.8. *Let $A \subset C(K)$ be a unital uniform algebra, $\Omega \subset \mathbb{C}$ a bounded simply connected region such that all points in its boundary $\partial\Omega$ are simple. Let us fix two different points a and b with $b \in \partial\Omega$, $a \in \overline{\Omega}$ and a neighbourhood $V_a \subset \overline{\Omega}$ of a. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and for every $t_0 \in U \cap \Gamma_0$, there exists $f \in A$ such that*

(i) *$f(K) \subset \overline{\Omega}$;*
(i) *$f(t_0) = b$;*
(i) *$f(K \setminus U) \subset V_a$.*

Proof. According to [10, Theorem 4.3.5] any point $t_0 \in \Gamma_0$ is a strong boundary point for A and therefore for every $\delta > 0$ there exists a function $g_\delta \in A$ such that $g_\delta(t_0) = 1 = \|g_\delta\|_\infty$ and $g_\delta(K \setminus U) \subset \delta\mathbb{D}$.

We distinguish two cases for the proof:

CASE 1: $a \in \Omega$. According to [23, Theorem 14.19] we can produce a homeomorphism $\phi : \overline{\mathbb{D}} \to \overline{\Omega}$ such that ϕ is a conformal mapping from $\mathbb{D}$ onto Ω with $\phi(1) = b$ and $\phi(0) = a$. Using and adequate g_δ as described above and ϕ the proof goes along the path that we followed in the proof of Lemma 2.5.

CASE 2: $a \in \partial\Omega$. Since $\operatorname{int}(V_a) \cap \Omega \neq \emptyset$ we can take $a' \in \Omega$ and $\delta' > 0$ such that $D(a', \delta') \subset V_a \cap \Omega$. Now, we apply *CASE 1* to a', its neighbourdhood $D(a', \delta')$ and b. The thesis follows. $\square$

Needless to say that in the non-unital case other results in the vein of the above proposition with the right hypothesis could be proved too.

3. Bishop-Phelps-Bollobás property

The result below that appears as Theorem 1 in [6] is known nowadays in the literature as the Bishop-Phelp-Bollobás theorem:

Theorem 3.1. *Let X be a Banach space, $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ such that $|1 - x_0^*(x_0)| \leq \varepsilon^2/2$ ($0 < \varepsilon < 1/2$). Then there exists $x^* \in S_{X^*}$ that attains the norm at some $x \in S_X$ such that*

$$\|x_0^* - x^*\| \leq \varepsilon \text{ and } \|x_0 - x\| < \varepsilon + \varepsilon^2.$$

It is easily seen that in the real case, if we assume that $x_0^*(x_0) \geq 1 - \varepsilon^2/4$ then the points x^* and x above can be taken satisfying $\|x_0^* - x^*\| \leq \varepsilon$ and $\|x_0 - x\| \leq \varepsilon$.

Note that a direct application of Brøndsted-Rockafellar variational principle, [22, Theorem 3.17], gives a better result:

Corollary 3.2. *Let X be a real Banach space, $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ such that $x_0^*(x_0) \geq 1 - \varepsilon^2/2$ ($0 < \varepsilon < \sqrt{2}$). Then there exists $x^* \in S_{X^*}$ that attains the norm at some $x \in S_X$ such that*

$$\|x_0^* - x^*\| \leq \varepsilon \text{ and } \|x_0 - x\| \leq \varepsilon. \tag{3.1}$$

We remark that in the previous corollary the hypothesis $x_0^*(x_0) \geq 1 - \varepsilon^2/2$ can not be weakened if we still wish to obtain the estimates (3.1), see [6, Remark].

Corollary 3.2 is easily extended to the complex case. Recall that given a complex Banach space X, the canonical map $\Re\colon X^* \to (X_{\mathbb{R}})^*$ defined by $\operatorname{Re}(x^*)(x) := \operatorname{Re} x^*(x)$, for $x^* \in X^*$ and $x \in X$, is an isometry and also an homeomorphism from (X^*, w^*) onto $((X_{\mathbb{R}})^*, w^*)$.

Corollary 3.3. *Let X be a Banach space, $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ such that $|x_0^*(x_0)| \geq 1 - \varepsilon^2/2$ ($0 < \varepsilon < \sqrt{2}$). Then there exists $x^* \in S_{X^*}$ that attains the norm at some $x \in S_X$ such that*

$$\|x_0^* - x^*\| \leq \varepsilon \text{ and } \|x_0 - x\| \leq \varepsilon.$$

Proof. Let us take $\lambda \in \mathbb{C}$ such that $|x_0^*(x_0)| = \lambda x_0^*(x_0)$. Then, we can apply Corollary 3.2 to the norm one real functional $\Re(x_0^*)$ and the norm one vector λx_0, to obtain $u^* \in S_{(X_\mathbb{R})^*}$ and $u \in S_X$ with $u^*(u) = 1$ and such that

$$\|u^* - \Re(x_0^*)\| \leq \varepsilon \text{ and } \|u_0 - \lambda x_0\| \leq \varepsilon.$$

If we set $x^* = \Re^{-1}(u^*)$ and $x = \lambda^{-1}u$, then x^* is a norm one complex continuous functional on X that satisfies $|x^*(x)| = |\lambda^{-1}| = 1$. On the other hand $\|x_0 - x\| = \|\lambda x_0 - u\| \leq \varepsilon$. Since $\Re$ is an isometry, we deduce that $\|x_0^* - x^*\| = \|\mathrm{Re}(x_0^*) - u^*\| \leq \varepsilon$, and the proof is over. $\square$

A complex Banach space X is said to be an *Asplund space* if its underlying real space $X_\mathbb{R}$ is Asplund, that is, whenever ψ is a convex continuous real valued function defined on an open convex subset U of X, the set of all points of U where ψ is Fréchet differentiable is a dense G_δ-subset of U. This definition is due to Asplund [4] under the name *strong differentiability space*. Combined efforts of Namioka, Phelps and Stegall led to Theorem 3.4 below that is valid both for real and complex Banach spaces. This result already hints at the power of the concepts involved.

Theorem 3.4 ([21, 24, 25])**.** *Let X be a Banach space. Then the following conditions are equivalent:*

(i) *X is an Asplund space;*
(ii) *every w^*-compact subset of (X^*, w^*) is fragmented by the norm;*
(iii) *each separable subspace of X has separable dual;*
(iv) *X^* has the Radon-Nikodým property.*

For the notion of the Radon-Nikodým property we refer to [12] and for the concept of fragmentability we refer to [20].

An operator $T \in L(X, Y)$ is said to be an Asplund operator if it factors through an Asplund space,

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "T_1"'] & & Y \\ & Z \arrow[ur, "T_2"'] & \end{tikzcd}$$

i.e., there are an Asplund space Z and operators $T_1 \in L(X, Z)$, $T_2 \in L(Z, Y)$ such that $T = T_2 \circ T_1$, see [14, 26]. Note that every weakly compact operator $T \in \mathcal{W}(X, Y)$ factors through a reflexive Banach space, see [11], and hence T is an Asplund operator.

A careful reading of [2, Lemma 2.3] together with the fact that (i) $\Leftrightarrow$ (ii) in Theorem 3.4, for real and complex spaces, should give the reader the tools to establish the validity of the following lemma. As usual T^* denotes the adjoint of T.

Lemma 3.5. *Let $T\colon X \to Y$ be an Asplund operator with $\|T\| = 1$ and $x_0 \in S_X$ such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$ ($0 < \varepsilon < \sqrt{2}$). For any given 1-norming set $\Gamma \subset B_{Y^*}$ if we write $M = T^*(\Gamma)$ then, for every $r > 0$ there exist:*

(i) *a w^*-open set $U_r \subset X^*$ with $U_r \cap M \neq \emptyset$, and*
(ii) *points $y_r^* \in S_{X^*}$ and $u_r \in S_X$ with $|y_r^*(u_r)| = 1$ such that*

$$\|x_0 - u_r\| \leq \varepsilon \quad \text{and} \quad \|z^* - y_r^*\| \leq r + \frac{\varepsilon^2}{2} + \varepsilon \text{ for every } z^* \in U_r \cap M. \tag{3.2}$$

We can prove now our main result in this paper as application of all the above.

Theorem 3.6. *Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T\colon X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that*

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \le \varepsilon \quad \textit{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

Proof. Fix arbitrary $r > 0$ and $0 < \varepsilon' < 1$. If $\mathfrak{A} = A$ is unital then take $\Gamma_0 = \Gamma(A)$ the Choquet boundary of A. If $\mathfrak{A} = B$ is not unital then change K and take Γ_0 as we did at the beginning of subsection 2.2. In any case, we can assume that we are dealing with an Asplund operator $T\colon X \to \mathfrak{A} \subset (C(K), \|\cdot\|_\infty)$ for which we can apply Lemma 3.5 for $Y := \mathfrak{A}$, $\Gamma = \{\delta_s \in \mathfrak{A}^* : s \in \Gamma_0\}$, r and $\varepsilon > 0$. We produce the w^*-open set U_r, the point u_r and the functional $y_r^* \in S_{X^*}$ satisfying the properties in the aforementioned lemma. Since $U_r \cap M \neq \emptyset$ we can pick $s_0 \in \Gamma_0$ such that $T^*\delta_{s_0} \in U_r$. The w^*-continuity of T^* ensures that $U = \{s \in K\colon T^*\delta_s \in U_r\}$ is an open neighborhood of s_0. Using Lemma 2.5 –or Lemma 2.7 in the not unital case– for the open set U –that clearly satisfies $U \cap \Gamma_0 \neq \emptyset$– and ε' we obtain a function $f \in \mathfrak{A}$ and $t_0 \in U \cap \Gamma_0$ satisfying

$$f(t_0) = \|f\|_\infty = 1, \tag{3.3}$$

$$|f(t)| < \varepsilon' \text{ for every } t \in K \setminus U \tag{3.4}$$

and

$$|f(t)| + (1 - \varepsilon')|1 - f(t)| \le 1 \text{ for every } t \in K. \tag{3.5}$$

Define now the linear operator $\widetilde{T}\colon X \to \mathfrak{A}$ by the formula

$$\widetilde{T}(x)(t) = f(t)y_r^*(x) + (1 - \varepsilon')(1 - f(t))T(x)(t). \tag{3.6}$$

It is easily checked that $\widetilde{T}$ is well-defined. Bearing in mind (3.5) we prove that $\|\widetilde{T}\| \le 1$. On the other hand,

$$1 = |y_r^*(u_r)| \overset{(3.3)}{=} |\widetilde{T}(u_r)(t_0)| \le \|\widetilde{T}(u_r)\| \le 1$$

and therefore $\widetilde{T}$ attains the norm at the point $u_0 = u_r \in S_X$ for which we already had that $\|u_0 - x_0\| \le \varepsilon$.

Now, for every $x \in B_X$, since Γ_0 is a boundary for $\mathfrak{A}$, we have that

$$\begin{aligned}\left\|Tx - \widetilde{T}x\right\|_\infty &= \sup_{t\in\Gamma_0} \left|f(t)(y_r^*(x) - T(x)(t)) - \varepsilon'(1 - f(t))T(x)(t)\right| \\ &\le \sup_{t\in\Gamma_0} \left\{|f(t)|\,|y_r^*(x) - T^*\delta_t(x)| + \varepsilon'|1 - f(t)|\,|T(x)(t)|\right\} \\ &\overset{(3.3)}{\le} \sup_{t\in\Gamma_0} \{|f(t)|\,\|y_r^* - T^*\delta_t\|\} + 2\varepsilon'.\end{aligned}$$

On one hand, since $T^*\delta_t \in U_r \cap M$ for every $t \in U \cap \Gamma_0$, we deduce that

$$\sup_{t\in U\cap\Gamma_0} |f(t)|\,\|y_r^* - T^*\delta_t\| \overset{(3.2)}{\le} r + \frac{\varepsilon^2}{2} + \varepsilon.$$

On the other hand, since $t \in \Gamma_0 \setminus U$ implies $t \in K \setminus U$, we obtain that

$$\sup_{t\in\Gamma_0\setminus U} |f(t)|\|y_r^* - T^*\delta_t\| \overset{(3.4)}{\le} 2\varepsilon'.$$

Gathering the information of the last three inequalities we conclude that

$$\|T - \widetilde{T}\| \le \max\{4\varepsilon', 2\varepsilon' + r + \varepsilon^2/2 + \varepsilon\}.$$

Since $r > 0$ and $0 < \varepsilon' < 1$ are arbitrary, for suitable values

$$\max\{4\varepsilon', 2\varepsilon' + r + \varepsilon^2/2 + \varepsilon\} < 2\varepsilon.$$

To finish the proof we show that $\widetilde{T}$ is also an Asplund operator. To this end it suffices to observe that Asplund operators between Banach spaces form an operator ideal, and that $\widetilde{T}$ in (3.6) appears as a linear combination of a rank one operator, the operator T and the operator $x \mapsto f \cdot T(x)$. The latter is the composition of a bounded operator from $\mathfrak{A}$ into itself with T. Therefore $\widetilde{T}$ is an Asplund operator and the proof is over. □

We conclude the paper with a list of remarks concerning the peculiarities and scope of the results that we have proved here:

- **R1:** If we denote by $\mathcal{A}$ the ideal of Asplund operators between Banach spaces and $\mathcal{I} \subset \mathcal{A}$ is a sub-ideal, Theorem 3.6 naturally applies for any operator $T \in \mathcal{I}(X, \mathfrak{A})$ and the provided $\widetilde{T}$ belongs again to $\mathcal{I}(X, \mathfrak{A})$.
- **R2:** Theorem 3.6 applies in particular to the ideals of finite rank operators $\mathcal{F}$, compact operators $\mathcal{K}$, p-summing operators Π_p and of course to the weakly compact operators $\mathcal{W}$ themselves. To the best of our knowledge even in the case $\mathcal{W}(X, \mathfrak{A})$ the Bishop-Phelps property that follows from Theorem 3.6 is a brand new result.
- **R3:** Let L be a scattered and locally compact space. The space of continuous functions vanishing at infinity $C_0(L)$ on L endowed with its sup norm $\|\cdot\|_\infty$ is an Asplund space, see comments after Corollary 2.6 in [2]. Therefore $(C_0(L), \mathfrak{A})$ has the BPBp for any uniform algebra. More in particular, for any set Γ the pair, $(c_0(\Gamma), \mathfrak{A})$ has the BPBp. Note that the paper [2] provided the first example of an infinite dimensional Banach space Y such that (c_0, Y) has the Bishop-Phelps-Bollobás property, namely for any $Y = C_0(L)$ as before. In a different order of ideas, it has been established in the paper [18] that (c_0, Y) has the BPBp for every uniformly convex Banach space Y.

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