# The division problem for tempered distributions of one variable

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**Abstract**

We give a characterization, in one variable case, of those $C^\infty$ multipliers $F$ such that the division problem is solvable in $S'(\mathbb{R})$. For these functions $F \in O_M(\mathbb{R})$ we even prove that the multiplication operator $M_F(G) = FG$ has a continuous linear right inverse on $S'(\mathbb{R})$, in contrast to what happens in the several variables case, as was shown by Langenbruch.

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## 1 Introduction and preliminaries

Hörmander [2] and Lojasiewicz [6] proved that for each polynomial $P$ and each (tempered) distribution $T$ there exists a (tempered) distribution $S$ such that $T = PS$. The division problem in the space $S'(\mathbb{R}^n)$ of tempered distributions on $\mathbb{R}^n$ can be stated as follows: Let $F$ be a multiplier of the space $S(\mathbb{R}^n)$ of rapidly decreasing functions, i.e. a smooth function $F$ satisfying $FS(\mathbb{R}^n) \subset S(\mathbb{R}^n)$. Find conditions on $F$ to ensure that for each tempered distribution $T \in S'(\mathbb{R}^n)$ there is a tempered distribution $S$ such that $T = FS$. It is known that $F \in \mathcal{E}(\mathbb{R}^n)$ is a multiplier in $S(\mathbb{R}^n)$ if and only if for each $k \in \mathbb{N}$ there exist $C > 0$ and $j \in \mathbb{N}$ such that $|F^{(\alpha)}(x)| \leq C(1+|x|^2)^j$ for each multiindex $\alpha$ with $|\alpha| \leq k$. Here $|x|$ denotes the Euclidean norm on $\mathbb{R}^n$. The space of multipliers on $S(\mathbb{R}^n)$ is denoted by $O_M(\mathbb{R}^n)$ and $O_M$ in case $n = 1$. See [4] and [10].

A multiplier $F \in O_M(\mathbb{R}^n), F \neq 0$, gives a positive solution for the division problem in $S'(\mathbb{R}^n)$ if and only if the multiplication operator $M_F : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$, $f \to Ff$, has closed range $\mathrm{rg}(M_F)$. Indeed, observe first that the transpose $M_F^t$ of the operator $M_F : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ coincides with the multiplication operator $S \to FS$ on $S'(\mathbb{R}^n)$. Now, necessity is an immediate consequence of the Closed Range Theorem [9, Theorem 26.3] since $M_F^t$ is surjective. The sufficiency follows since the operator $M_F$ is injective between Fréchet spaces whenever it has closed range, hence it is an isomorphism onto its image when it has closed range. The injectivity of $M_F$ when it has closed range can be concluded as follows: If $M_F$ is not injective, $F$ must have a zero in which all the derivatives of $F$ vanish. We may take this point in the boundary of the zero set of $F$ and may assume without loss of generality that this point is 0. By a result of Whitney [11, Corollaire V.1.6], the function $f := \varphi F/|x|^2$, with $\varphi \in \mathcal{D}(\mathbb{R}^n), \varphi(x) = 1, |x| \leq 1$, is in the closure of the range of $M_F$. Since the range is closed, we can find $G \in S(\mathbb{R}^n)$ such that $GF = f$. Now we select a sequence $(x_k)_k$ in $\mathbb{R}^n$ tending to 0 such that $F(x_k) \neq 0$. This implies $G(x_k) = 1/|x_k|^2$ for each $k$ and $G(0)$ is not well defined.

If $F \in \mathcal{E}(\mathbb{R}^n)$ is an arbitrary smooth function, the division problem for distributions is also equivalent to the fact that the multiplication operator $M_F : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ has closed range. The characterization for arbitrary dimension seems to be still open. However, in the one variable case, it was already known by Schwartz [10, Chap. V] that a smooth function $F \in \mathcal{E}(\mathbb{R})$ satisfies that $M_F : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})$ has closed range if and only if $F$ has only isolated zeros of bounded order. Although there is a close relation between the two cases and they are equivalent in case $F$ is a polynomial (cf. [10]), there is no analog characterization of those multipliers $F \in O_M$ such that the range of $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ is closed. This is the question we consider in this paper.

A fundamental system of seminorms of the Fréchet space $S(\mathbb{R}^n)$ of rapidly decreasing functions of Schwartz is given by

$$||f||_s := \max_{|\alpha|\leq s} \max_{x\in\mathbb{R}^n} (1+|x|^2)^s |f^{(\alpha)}(x)|, \quad s \in \mathbb{N}.$$

In particular $||f||_0 := \max_{x\in\mathbb{R}^n} |f(x)|$. Our notation for the theory of distribution and functional analysis is standard. We refer the reader to [4], [9], [3] and [10]. An excellent survey about the division of distributions is due to Malgrange [8].

Given a function $F$, we denote by $Z_F$ the set of zeros of $F$. An element $x \in Z_F$ is said to have finite order whenever there exists $\alpha \in \mathbb{N}_0^n$ such that $F^{(\alpha)}(x) \neq 0$. If $x \in Z_F$, the order $o_F(x)$ of $x$ in $F$ is the minimum of the natural numbers $|\alpha|$ satisfying this condition. If $x$ does not belong to $Z_F$ then the order of $x$ is defined as $o_F(x) = 0$.

Our purpose is to characterize the multipliers $F \in O_M(\mathbb{R})$, $F \neq 0$, such that the operator $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ has closed range in terms of the zeros of $F$ in the one variable case. This is obtained in Theorem 2.1. Clearly it is possible to use Fourier transform to formulate our result in terms of a characterization of surjective convolution operators on $S'(\mathbb{R})$. We will not state these consequences explicitly in this paper. It is important to remark that in the present setting it is not possible to use complex analytic methods and weighted spaces of entire functions as in the treatment of surjectivity of convolution operators on other spaces of distributions, see [3]. Our characterization in Theorem 2.1 permits us to show in Theorem 2.3 that if $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ has closed range, then it admits a continuous linear left inverse. This result does not longer hold in the case of several variables even for polynomials $F$, as follows from results due to Langenbruch [5]. We conclude our paper with an example showing that two conditions stated by Hörmander in the remark after Theorem 1 in [2] are not necessary for a multiplier $F$ to satisfy that $M_F$ has closed range.

The following lemma is well-known, we include the proof for the sake of completeness.

**Lemma 1.1** *If $F \in O_M(\mathbb{R}^n)$, $F \neq 0$, satisfies that $M_F : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ has closed range then there exists $m \in \mathbb{N}$ such that $o_F(x) \leq m$ for all $x \in \mathbb{R}^n$.*

**Proof.** For simplicity, during all the proof $C > 0$ denotes a constant not depending on $\varepsilon > 0$ which can change at every step. Since $F \neq 0$, the operator $M_F : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is injective; see the argument indicated above. Since the range of $M_F$ is closed, the operator is an isomorphism into and there exist $s \in \mathbb{N}, C > 0$ such that

$$\left\| \frac{f}{F} \right\|_0 \leq C \|f\|_s \tag{1.1}$$

for each $f \in \mathrm{rg}(M_F)$. We fix $x \in Z_F$ and we consider a test function $\phi$ with compact support in the ball $B(0,1)$ of center 0 and radius 1 and such that $\phi(0) = 1$. For $0 < \varepsilon < 1$ we define $f_\varepsilon(y) := \phi\left(\frac{y-x}{\varepsilon}\right) F(y)$. Assume $o_F(x) > s$, i.e. $F^{(\alpha)}(x) = 0$ for each $|\alpha| \leq s$. The Taylor polynomials of each $F^{(\beta)}$ up to order $s - |\beta|$ centered at $x$ are null for each $|\beta| \leq s$. We apply Taylor's theorem to get that, for each $|\beta| \leq s$ and $y \in B(x, \varepsilon)$, there exists $\xi$ in the segment linking $x$ and $y$ such that

$$F^{(\beta)}(y) = \sum_{\beta \leq \gamma, |\gamma| = s+1} C_{\gamma,\beta} F^{(\gamma)}(\xi)(y-x)^{\gamma-\beta}.$$

Hence there exists $C > 0$ such that $|F^{(\beta)}(y)| \leq C\varepsilon^{s+1-|\beta|}$ for $y \in B(0, \varepsilon)$. We apply Leibniz formula to $f_\varepsilon$ to get that there exists $C$ such that for each $|\alpha| \leq s$, and $y \in B(x, \varepsilon)$

$$|f_\varepsilon^{(\alpha)}(y)| = \left| \sum_{\beta \leq \alpha} C_{\alpha,\beta} \varepsilon^{-(|\alpha|-|\beta|)} \phi^{(\alpha-\beta)}((y-x)/\varepsilon) F^{(\beta)}(y) \right| \leq C\varepsilon^{s+1-|\alpha|} \leq C\varepsilon.$$

Finally, if we put in (1.1) $f_\varepsilon$ we get

$$1 \leq C \left| \sup_{|\alpha| \leq s, y \in B(x,\varepsilon)} (1 + |y|^2)^s f_\varepsilon^{(\alpha)}(y) \right| \leq C\varepsilon,$$

a contradiction. □

From Rolle's theorem it easily follows that if $F$ is a $C^\infty$ one variable function then every $z$ in the accumulation of $Z_F$ belongs to $Z_F$ and has infinite order. Hence from Lemma 1.1 we get that, in the one variable case, *if* $F \in O_M, F \neq 0$, *satisfies that* $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ *has closed range, then* $Z_F$ *is discrete.*

# 2 Closed range multipliers in $O_M(\mathbb{R})$

In our next Theorem we use the following notation. If $T \in \mathbb{N}$ and $x \in \mathbb{R}$, we set $I_{x,T} := [x - 1/(1+|x|^2)^T, x + 1/(1+|x|^2)^T]$.

**Theorem 2.1** *A multiplier* $F \in O_M, F \neq 0$, *satisfies that* $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ *has closed range if and only if there exists an* $N \in \mathbb{N}$ *and* $T, c > 0$ *such that* $F$ *satisfies the following two conditions for each* $x \in \mathbb{R}$:

*(a) The cardinality of the set* $Z_F \cap I_{x,T}$, *the zeros counted with their multiplicities, is smaller than* $N$.

*(b)* $(1+|x|^2)^T |F(x)| \geq c \prod_{i=1}^k |x - x_i|$, $(x_i)_{i=1}^k$ *being the zeros of* $F$ *in* $I_{x,T}$ *counting multiplicities.*

**Proof. 1)** Assume first that $M_F$ has closed range. We can find $s \in \mathbb{N}$ and $C > 0$ be such that

$$\left\| \frac{f}{F} \right\|_0 \leq C \|f\|_s \tag{2.1}$$

for every $f \in rg(M_F)$ (with the corresponding extensions at $Z_F$). If we suppose that $(a)$ is not satisfied, we can take in (2.1) $s$ big enough such that there is a sequence $(y_n)$ tending to infinity in absolute value, a sequence $(\varepsilon_n)_{n=1}^\infty$ such that $(1+|y_n|)^T \varepsilon_n$ tends to zero for each $T > 0$ and, if we denote $I_n := [y_n - \varepsilon_n, y_n + \varepsilon_n]$, then $k_n := \sum_{x \in I_n} o_F(x) > s$. By Lemma 1.1, shrinking the intervals $I_n$ if necessary, we can assume the sequence $(k_n)_n$ bounded.

Now we consider $Z_F \cap I_n = \{x_1, \ldots, x_{k_n}\}$, the zeros counted with their multiplicities, and we define the functions $g_n(x) := \prod_{1 \leq i \leq k_n} (x - x_i)$. We take the polynomials $H_n$ in $I_n$ that are the solution of the Hermite interpolation at $Z_F \cap I_n$ of the function $F$ and its derivatives up to the order of each zero (cf.[1, Chapter 4, section 7]), i.e. we have that these $H_n$ are the identically null polynomial for each $n \in \mathbb{N}$. Hence, by the remainder theorem for these approximation polynomials [1, Chapter 4 (7.3),(7.4)], we have that for each $y \in I_n$ there is $\xi \in I_n$ such that $F(y) = F(y) - H_n(y) = \frac{1}{k_n!} g_n(y) F^{(k_n)}(\xi)$. Now we apply that $F \in O_M$ and $(k_n)_n$ is bounded to get $C > 0$ and $l \in \mathbb{N}$ such that

$$\sup_{y \in I_n} \left| \frac{F(y)}{g_n(y)} \right| \leq C \sup_{y \in I_n} |F^{(k_n)}(y)| \leq \sup_{y \in I_n} C(1+|y|)^l \leq C(1+|y_n|^2)^l. \tag{2.2}$$

In the rest of the proof the constant $C > 0$ can change from step to step but it is independent on $n$. Take a function $\phi \in \mathcal{D}(]-1,1[)$ such that $\phi|_{]-1/2,1/2[} \equiv 1$ and define $\phi_n := \phi((x - y_n)/\varepsilon_n)$, and $h_n := g_n \phi_n$. Observe that $h_n$ has more zeros than $F$ and $o_F(z) \leq o_{h_n}(z)$ for each $z \in Z_F$. Hence, as a consequence of Taylor theorem we get that the quotient $h_n/F$ can be extended to $Z_F \cap I_n$ as a $C^\infty$ function. Thus $h_n/F \in \mathcal{D}(\mathbb{R}) \subset S(\mathbb{R})$, and $h_n$ is in the range of $M_F$. We evaluate the inequality (2.2) at the interval $J_n := ]y_n - \varepsilon_n/2, y_n + \varepsilon_n/2[$, in which $h_n = g_n$, to get

$$\inf_{y \in J_n} \left| \frac{h_n(y)}{F(y)} \right| \geq \frac{1}{C(1 + |y_n|^2)^l}. \tag{2.3}$$

Using Leibniz formula we have, for each $k \in \mathbb{N}$,

$$h_n^{(k)}(y) = \sum_{0 \leq j \leq k} C_{j,k} \varepsilon_n^{-j} \phi^{(j)}((y - x)/\varepsilon_n) g_n^{(k-j)}(y).$$

At this point, recall (a consequence of) the Markov inequality. Given $k \in \mathbb{N}$, there exists $C > 0$ such that for each polynomial $P$ with degree less or equal than $k$, for each interval $I$ and for each $0 \leq l \leq k$

$$\max_{x \in I} |P^{(l)}(x)| \leq C \frac{\max_{x \in I} |P(x)|}{\text{length}(I)^l}.$$

Since $g_n(y)$ are polynomials and $I_n$ are intervals with diameter $\varepsilon_n$, we can apply Markov's inequality above to obtain $C > 0$, which is independent of $n$ since the degree $k_n$ of the polynomials is bounded, such that

$$\sup_{y \in I_n} |g_n^{(l)}(y)| \leq \frac{C}{\varepsilon_n^l} \sup_{y \in I_n} |g_n(y)| = \frac{C}{\varepsilon_n^l} \sup_{y \in I_n} \prod_{i=1}^{k_n} |y - x_i| \leq \frac{C}{\varepsilon_n^l} \varepsilon_n^{k_n} = C \varepsilon_n^{k_n - l}. \tag{2.4}$$

Hence we can estimate

$$\sup_{0 \leq k \leq s, y \in I_n} |h_n^{(k)}(y)| \leq C \varepsilon_n^{-j} \varepsilon_n^{k_n - k + j} \leq C \varepsilon_n^{k_n - s} \leq C \varepsilon_n \tag{2.5}$$

since $k_n > s$ for all $n$. Thus, as $(k_n)_n$ is bounded, putting $h_n$ in (2.1), since $h_n \in \text{rg} M_F$, we have

$$\left\| \frac{h_n}{F} \right\|_0 \leq C \sup_{y \in I_n, 0 \leq j \leq s} (1 + |y|^2)^s |h_n^{(j)}(y)| \leq C(1 + |y_n|^2)^s \varepsilon_n. \tag{2.6}$$

Taking $C$ big enough such that (2.3) and (2.6) hold simultaneously and evaluating at an arbitrary $y \in J_n = [y_n - \varepsilon_n/2, y_n + \varepsilon_n/2]$ we obtain

$$\frac{1}{C^2} \leq (1 + |y_n|^2)^{s+l} \varepsilon_n,$$

which contradicts the choice of $(\varepsilon_n)_n$.

To prove (b) we consider $\phi$ as the same test function as in the proof of the necessity of (a) and we denote $J_{x,T} := [x - 1/(2(1 + |x|^2)^T)), x + 1/(2(1 + |x|^2)^T))]$ and $g_x(y) := \prod_{i=1}^k (y - x_i)$, where $T > 0$ satisfies (a) and $\{x_1, \ldots, x_k\}$ are the zeros of $F$ in $I_{x,T}$ counting multiplicities with cardinality $k$ depending on $x$ (but bounded by $N$ since we are assuming (a)). Consider now the function $f_x(y) := \phi((y - x)(1 + |x|^2)^T)) g_x(y)$. We have that $f_x = g_x$ in $J_{x,T}$ and $\text{supp} f_x \subset I_{x,T}$. Hence $f_x$ is in $rg(M_F)$ because $f_x$ is a compactly supported $C^\infty$ function which vanishes at $Z_F$ at least of the same order as $F$. Moreover, for each $l \in \mathbb{N}$, since the polynomials $g_x$ are of bounded degree, we proceed as in (2.4) to get $C > 0$ (which may change during the rest of the proof but is always independent of $x$) such that

$$\sup_{y\in I_{x,T},0\le l\le k} |g_x^{(l)}(y)| \le C(1+|x|^2)^{-T(k-l)} \le C. \tag{2.7}$$

Applying Leibniz formula as in (2.5) we get $C>0$ such that

$$\sup_{y\in I_{x,T},0\le l\le s} |f_x^{(l)}(y)| \le (1+|x|^2)^{Ts}C. \tag{2.8}$$

Therefore

$$\left|\frac{\prod_{i=1}^k (x-x_i)}{F(x)}\right| \le \sup_{y\in J_{x,T}} \left|\frac{g_x(y)}{F(y)}\right| \le \sup_{y\in I_{x,T}} \left|\frac{f_x(y)}{F(y)}\right| \le$$
$$\le C \sup_{y\in I_{x,T},0\le i\le s} (1+|y|^2)^s |f_x^{(i)}(y)| \le C(1+|x|^2)^M$$

for $M=(T+1)s$, and we have (b).

**2)** Assume that $F\in O_M$ satisfies conditions (a) and (b). We first show that there exist $C>0$ and $s\in\mathbb{N}$ such that if $f\in S(\mathbb{R})$ satisfies that $Z_F\subseteq Z_f$ and the order of $f$ at $x$ is not smaller than the corresponding order of $F$ for each $x\in Z_F$, then

$$\left\|\frac{f}{F}\right\|_0 \le C\|f\|_s. \tag{2.9}$$

To prove (2.9) take $x$ in $\mathbb{R}\setminus Z_F$ and let $J_{x,T}$ as in the proof of the necessity of (b). Again we consider $Z_F\cap I_{x,T}=\{x_1,\cdots,x_k\}$, the zeros counted with multiplicities, and $k$ depending on $x$ but bounded for a fixed $N\in\mathbb{N}$. If we interpolate $f$ and its derivatives up to their order in $F$ at $Z_F\cap I_{x,T}$ with the corresponding Hermite polynomial, since the order in $f$ is at least equal, we can use the remainder formula [1, chapter 4] to get that for each $y\in I_{x,T}$ there exists $\xi\in I_{x,T}$ such that $f(y)=\prod(y-x_i)f^{(k)}(\xi)/k!$. Now we use (b) to compute

$$\left|\frac{f(x)}{F(x)}\right| \le \sup_{y\in I_x}\left|\frac{f(y)}{F(y)}\right| = \sup_{y\in I_x}\left|\frac{\prod_{i=1}^k (y-x_i)f^{(k)}(\xi)/k!}{F(y)}\right| \le$$
$$\le \sup_{y\in I_x} C(1+|y|^2)^T|f^{(k)}(\xi)| \le C(1+|\xi|^2)^T|f^{(k)}(\xi)| \le C\|f\|_T.$$

This completes the proof of (2.9). We have to use what we have proved so far to show that if $F\in O_M$ satisfies (a) and (b), then $M_F$ has closed range. We must show that the estimates for the zero norm (2.9) suffice to conclude that $M_F$ has closed range. To see this observe that $F^{2^n}$ also is in $O_M$, it satisfies conditions (a) and (b), possibly for other $N$, $T$ and $c>0$, and, for $1\le j\le n$,

$$(1+|x|^2)^n\left(\frac{f}{F}\right)^{(j)}(x) = (1+|x|^2)^n\frac{g_j}{F^{2^j}}(x). \tag{2.10}$$

Moreover, $x\mapsto(1+|x|^2)^n g_j(x)$ is a rapidly decreasing function which vanishes at $Z_{F^{2^j}}$ with at least the same orders as $F^{2^j}$. Here each $g_j$ is a linear combination of products of derivatives of $f$ of order at most $j$ and powers of $F$ and its derivatives, which are polynomially bounded, and thus for each $k \in \mathbb{N}$ there are $C > 0$, $l \in \mathbb{N}$ such that $\|g_j\|_k \leq C\|f\|_l$ for each $f$ satisfying the required conditions. This fact together with (2.10) imply that $M_F$ is an isomorphism into.

□

**Corollary 2.2** *If $F \in O_M$ satisfies that $M_F$ has closed range and there is $t > 0$ such that $|x-y| > t$ for each $x, y \in Z_F$, $x \neq y$, then there exists $C, T > 0$ such that $(1+x^2)^T|F^{(o(x))}(x)| \geq C$ for each $x \in Z_F$.*

**Theorem 2.3** *Let $F \in O_M$ be not identically zero. The operator $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ possesses a continuous linear left inverse if and only if its range is closed in $S(\mathbb{R})$.*

**Proof.** We only have to show that if $M_F$ has closed range, then the range is complemented. We take a test function $\psi \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{supp}\psi = [-3/4, 3/4]$, $\psi = 1$ on $[-1/4, 1/4]$ and $\psi$ is symmetric and positive. We can also assume that $\psi(x) + \psi(1-x) = 1, x \in [0,1]$. We define $\psi_k(x) := \psi(x-k)$, $k \in \mathbb{Z}$, and $(\psi_k)_{k\in\mathbb{Z}}$ is a $C^\infty$ partition of unity. Let $\nu : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism such that $\nu(x) = x^S$ for $|x| \geq 1/4$, with $S$ odd. Setting $\tilde{\psi}_k := \psi_k \circ \nu$ we get that $(\tilde{\psi}_k)_{k\in\mathbb{Z}}$ is a new $C^\infty$ partition of unity and each $x \in \mathbb{R}$ belongs at most to two supports. We show that $S$ can be taken big enough such that the number of zeros of $F$ counting multiplicities contained in each $I_k := \operatorname{supp}\tilde{\psi}_k$, $k \in \mathbb{Z}$, is uniformly bounded. Since $M_F$ has closed range, by Theorem 2.1 there exists $T > 0$ such that for each $|x| \geq 1$ the number of zeros counting multiplicities of $F$ in $[x, x + 1/|x|^T]$ is bounded by $T$, and the zeros in [-1,1] are also bounded by $T$. We prove that for $S > T + 1$ there exists $k_0$ such that for $k > k_0$ there are no more than $T$ zeros in $I_k$. For $k < -k_0$ the computation is analogous. For $k > 1$ the support of $\tilde{\psi}_k$ is

$$[x_k, y_k] := \nu^{-1}([k - 3/4, k + 3/4]) = [(k - 3/4)^{1/S}, (k + 3/4)^{1/S}].$$

We want to show that, if $S > T + 1$ then there is $k_0$ such that $x_k + \frac{1}{|x_k|^T} > y_k$, which implies the bound for the number of zeros counting mulpiplicities of $F$ in $I_k$ for $k \geq k_0$. The desired inequality holds if and only if $(k+3/4)^{1/S} - (k-3/4)^{1/S} < (k-3/4)^{-T/S}$. This is satisfied whenever $3/2 \leq (k-3/4)^{\frac{S-1-T}{S}}$, which is true for large $k$.

Now we consider the polynomial $H_k = H_k(f)$ that is the solution of the Hermite interpolation of $f$ at $Z_F \cap I_k$ up to the corresponding multiplicities of each zero. The degrees of the polynomials $H_k$ are bounded. Define $P(f) = \sum_k \tilde{\psi}_k(x)H_k(x)$. We claim that $id - P : S(\mathbb{R}) \to S(\mathbb{R})$ is a projector onto $rg(M_F)$. Since $rg(M_F)$ is closed, it coincides with the ideal (see [11, Corollaire V.1.6] for $\mathcal{E}(\mathbb{R})$ and use a standard argument for $S(\mathbb{R})$).

$$I := \{f \in S(\mathbb{R}) : f^{(j)}(z) = 0 \text{ for all } z \in Z_F, 0 \leq j < o_F(z)\}. \tag{2.11}$$

We prove that $P(S(\mathbb{R})) \subset S(\mathbb{R})$. We proceed to check this inclusion. We have to show that given $N \in \mathbb{N}$ there is $M \in \mathbb{N}$ and $C > 0$ such that

$$\max_{0\leq j\leq N} \sup_{x\in\mathbb{R}} (1+|x|)^N |P(f)^{(j)}(x)| \leq C \max_{0\leq j\leq M} \sup_{x\in\mathbb{R}} (1+|x|)^M |f^{(j)}(x)|.$$

Denote by $n_0$ the smallest integer bound of the sequence of the degrees $(\deg(H_k))_{k\in\mathbb{Z}}$. In the following computations $C$ will be a constant which depends on $n_0$ but not on $k$ and may change at every step. First we obtain estimates on the polynomials $H_k$. By Markov inequality, there is $C > 0$ such that

$$\sup_{x\in I_k} \max_{0\le j\le n_0} |H_k^{(j)}(x)| \le \frac{C}{\text{length}(I_k)^{n_0}} \sup_{x\in I_k} |H_k(x)|.$$

If $k \ge 1$, then $y_k = (x_k^S + 3/2)^{1/S}$ and there is $C$ such that $Cx_k \ge y_k$ for all $k \ge 1$. Hence we can apply the equality $y_k^S - x_k^S = (y_k - x_k)(y_k^{S-1} + y_k^{S-2} x_k + \dots + x_k^{S-1})$ to get that there is a constant $C$ such that

$$\text{length}(I_k) \ge \frac{1}{C(x_k)^{S-1}}.$$

By the symmetry of the supports, if we denote $I_k := [y_k, x_k]$ for $k < 0$, the constant $C > 0$ can be taken such that

$$\frac{1}{\text{length}(I_k)} \le C(1 + |x_k|)^{S-1},$$

for each $k \in \mathbb{Z}$. Therefore

$$\sup_{x\in I_k} \max_{0\le j\le n_0} |H_k^{(j)}(x)| \le C(1 + |x_k|)^{n_0(S-1)} \sup_{x\in I_k} |H_k(x)|$$

for each $k \in \mathbb{Z}$. By the formula for the terms in the Hermite interpolation polynomials [1, Chapter 4, Section 7], there exists $C > 0$ such that

$$\sup_{x\in I_k} |H_k(x)| \le C(1 + |x_k|)^{n_0} \max_{0\le j\le n_0} \sup_{x\in I_k} |f^{(j)}(x)|.$$

Hence

$$\sup_{x\in I_k} \max_{0\le j\le n_0} |H_k^{(j)}(x)| \le C(1 + |x_k|)^{n_0(S-1)} \sup_{x\in I_k} |H_k(x)| \le$$

$$\le C(1 + |x_k|)^{n_0 S} \max_{0\le j\le n_0} \max_{z\in I_k} |f^{(j)}(z)|.$$

For $|k| \ge 1$, $\tilde{\psi}_k^{(j)}(x) = \psi^{(j)}(x^S - k)$. Therefore, for $N$ fixed we can choose $C$ in such a way that there exists $L > N$ such that

$$\sup_{x\in I_k} |\tilde{\psi}_k^{(j)}(x)| \le C(1 + |x_k|)^L$$

for $0 \le j \le N$. Obviously $C$ can be taken to satisfy such inequality also for $I_0$.
As each $x \in \mathbb{R}$ may belong at most to two supports, we can take $C > 0$ such that

$$\max_{0\le j\le N} \sup_{x\in\mathbb{R}} (1 + |x|)^N |P^{(j)}(f)(x)| \le C \sum_{k\in\mathbb{Z}} (1 + |x_k|)^N \max_{0\le j\le N} \max_{0\le i\le j} \sup_{x\in I_k} |\tilde{\psi}_k^{(i)}(x) H_k^{(j-i)}(f)(x)|.$$

Consequently, for $M := N + L + n_0 S$, we get $\xi_k \in I_k$ and $0 \le j_0 \le n_0$ such that

$$\max_{0\le j\le N} \sup_{x\in I_k} (1 + |x|)^N |P^{(j)}(f)(x)| \le C(1 + |x_k|)^M \max_{0\le j\le N} \sup_{z\in I_k} |f^{(j)}(z)| =$$

$$= C(1 + |x_k|)^M |f^{(j_0)}(\xi_k)| \le C \max_{0\le j\le M} \sup_{\xi\in\mathbb{R}} (1 + |\xi|)^M |f^{(j)}(\xi)|,$$

with $C > 0$ is independent from $k$. These estimates also show that $P : S(\mathbb{R}) \to S(\mathbb{R})$ is continuous.

We see now that $(id - P)(I) = I$. In fact, since $(\tilde{\psi}_k)_{k\in\mathbb{Z}}$ is a $C^\infty$ partition of unity and each $\tilde{\psi}_k(x)$ is identically zero outside $I_k$, we get, for $f \in S(\mathbb{R})$, $x \in Z_F$ and $0 \le j < o_F(x)$,

$$P(f)^{(j)}(x) = \sum_{k=-\infty}^{\infty} \sum_{i=0}^{j} \tilde{\psi}_k(x) f^{(j)}(x) = f^{(j)}(x).$$

Hence $(id - P)(S(\mathbb{R})) \subseteq I$. Moreover for each $f \in I$ each $H_k$ is the null polynomial, hence $(id - P)$ coincides with $id$ on $I$. □

We conclude this article with the following example. Hörmander stated in the remark after Theorem 1 in [2] that each $C^\infty$ multiplier $F$ having all the zeros of uniform bounded order (condition (4.2) in [2]) and satisfying certain inequalities (condition (4.10) in [2]) satisfies that the range of $M_F$ is closed. In case all the zeros are of order one, this condition (4.10) consists of the following inequalities (see the comments before introducing (4.10) and Lemma 2 in [2]): There exist $\mu_1, \mu_2, \mu_3 \in \mathbb{N}$ and $C > 0$ such that

$$|F(x)| \ge C \frac{d(x, Z_F)^{\mu_1}}{(1 + |x|^2)^{\mu_2}} \quad \text{for all } x \in \mathbb{R}^N \tag{2.12}$$

and

$$|F'(x)| \ge C \frac{1}{(1 + |x|^2)^{\mu_3}} \quad \text{for all } x \in Z_F \tag{2.13}$$

Let $F_1(x) := \sin(x)$ and $F_2(x) := \sin(x + \alpha e^{-x^2})$, $\alpha$ being a positive number small enough to ensure that the map $x \mapsto x + \alpha e^{-x^2}$ a diffeomorphism. We consider the multiplier $F = F_1 F_2$. Since $F_1$ and $F_2$ satisfy conditions (a) and (b) in our Theorem 2.1, as it is easily checked, both $M_{F_1}$ and $M_{F_2}$ are isomorphisms into, hence the multiplier $M_F = M_{F_1} \circ M_{F_2}$ has closed range, too. However, as a consequence of Taylor's theorem we have

$$F'(n\pi) \simeq \alpha e^{-(n\pi)^2},$$

thus (2.13) does not hold. Accordingly, Hormander's conditions 2.12 and 2.13 are not necessary for a multiplier $F$ to satisfy that $M_F$ has closed range.

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