GENERIC LINEAR SECTIONS OF COMPLEX HYPERSURFACES AND MONOMIAL IDEALS

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ABSTRACT. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ. Under the hypothesis that f is Newton non-degenerate, we compute the μ^*-sequence of f in terms of the Newton polyhedron of f. This sequence was defined by Teissier in order to characterize the Whitney equisingularity of deformations of complex hypersurfaces.

1. INTRODUCTION

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ with an isolated singularity at the origin and let us denote by $\mu(f)$ the Milnor number of f. Teissier proved in [21, p. 299] that, given an integer $i \in \{0, 1, \dots, n\}$, the Milnor number of the restriction of f to a generic plane in $\mathbb{C}^n$ of dimension i only depends on f and i. Then, Teissier defined in [21] the analytic invariant

$$\mu^*(f) = \big(\mu^{(n)}(f), \mu^{(n-1)}(f), \dots, \mu^{(1)}(f), \mu^{(0)}(f)\big), \tag{1}$$

where $\mu^{(i)}(f)$ denotes the Milnor number of the restriction of f to a generic plane of dimension i passing through the origin in $\mathbb{C}^n$, for $i = 0, 1, \dots, n$. The vector given in (1) is also known as the μ^*-sequence of f. We observe that $\mu^{(n)}(f) = \mu(f)$, $\mu^{(1)}(f) = \mathrm{ord}(f) - 1$ and $\mu^{(0)}(f) = 1$, where $\mathrm{ord}(f)$ denotes the *order* or *multiplicity of f at the origin*, that is, the maximum of those $r \geqslant 1$ such that $f \in m^r$.

It was initially conjectured by Teissier [21] that the topological triviality of a given analytic deformation $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ forces the sequence $\mu^*(f_t)$ to be constant. But Briançon and Speder [5] found an example of a topologically trivial deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that $\mu^{(2)}(f_t)$ is not constant. By the results of Teissier [21] and Briançon-Speder [6], the constancy of $\mu^*(f_t)$ is equivalent to the Whitney equisingularity of the deformation.

Let us denote by $\mathcal{O}_n$ the ring of analytic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$ and by m_n, or simply by m if no confusion arises, the maximal ideal of $\mathcal{O}_n$. Let $J(f)$ be the Jacobian ideal of f, that is $J(f)$ is the ideal of $\mathcal{O}_n$ generated by $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$. If $I_1, \dots, I_n$ are ideals of finite colength of $\mathcal{O}_n$, then we denote by $e(I_1, \dots, I_n)$ the mixed multiplicity of $I_1, \dots, I_n$ in the sense of Teissier and Risler (we refer to [11, §17], [18], [20] or [21, p. 302] for definitions and

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basic results about mixed multiplicities of ideals). Teissier showed in [21] that

$$\mu^{(i)}(f) = e\big(\underbrace{m_n, \dots, m_n}_{n-i}, \underbrace{J(f), \dots, J(f)}_{i}\big), \tag{2}$$

for all $i = 0, 1, \dots, n$. Therefore, the μ^*-sequence admits also an algebraic approach.

Kouchnirenko obtained in [12, Théorème I] a formula for the Milnor number of any function f with an isolated singularity at the origin in terms of the Newton polyhedron $\Gamma_+(f)$ of f, when f is Newton non-degenerate. As pointed out by Mima [16] (see also [17]), the main difficulty encountered in the attempt of computing $\mu^*(f)$ using Kouchnirenko's result is that the restriction of a Newton non-degenerate function f to a generic i-plane in $\mathbb{C}^n$ passing through the origin is not Newton non-degenerate in general, for $i \geqslant 2$. Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a Newton non-degenerate function with an isolated singularity at the origin and let $g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be the function given by $g(x, y) = f(x, y, ax + by)$, for generic $a, b \in \mathbb{C}$. Then Mima proved in [16] a formula expressed in terms of Newton numbers for the difference $\mu^{(2)} - \nu^{(2)}$, where $\nu^{(2)}$ is the Newton number of $\Gamma_+(g)$ and $\mu^{(2)} = \mu^{(2)}(f)$ (see [16] for details).

The main result of this paper shows an expression for the whole sequence $\mu^*(f)$ in terms of $\Gamma_+(f)$ under the condition that f is Newton non-degenerate. This result is based on the formula proven by the author in [3] for the Milnor number of an isolated complete intersection singularity $(f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ via the Newton polyhedra of the component functions f_i. We also deduce some consequences that lead to find examples of deformations $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with constant Milnor number such that $\mu^{(2)}(f_t)$ is not constant. These examples may contribute to the better understanding of classification problems in metric singularity theory (see [2, §4]) and questions like the Zariski's multiplicity conjecture (see [8]).

2. Main result

If I is an ideal of $\mathcal{O}_n$ of finite colength then we denote by $e(I)$ the Samuel multiplicity of I (see [7, p. 278] or [11, §11]) and by $\overline{I}$ the integral closure of I. We recall that if I is generated by n elements, say $g_1, \dots, g_n$, then $e(I) = \dim_{\mathbb{C}} \mathcal{O}_n/I$ and in turn this number is equal to the geometric degree of the map $(g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ (see [15, p. 258]). As mentioned in the introduction, the mixed multiplicity of n ideals $I_1, \dots, I_n$ of finite colength in $\mathcal{O}_n$ is denoted by $e(I_1, \dots, I_n)$.

Let us suppose that the residue field $k = R/m$ is infinite. Let $I_1, \dots, I_n$ be ideals of R and let $a_{i1}, \dots, a_{is_i}$ be a generating system of I_i, where $s_i \geqslant 1$, for $i = 1, \dots, n$. We say that a property holds for *sufficiently general* elements of $I_1 \oplus \cdots \oplus I_n$ if there exists a non-empty Zariski-open set U in k^s, where $s = s_1 + \cdots + s_n$, such that the said property holds for all elements $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$ for which $g_i = \sum_j u_{ij} a_{ij}$, $i = 1, \dots, n$, with $(u_{11}, \dots, u_{1s_1}, \dots, u_{n1}, \dots, u_{ns_n})$ belonging to U.

We recall that $e(I_1, \dots, I_n) = e(g_1, \dots, g_n)$, where $(g_1, \dots, g_n)$ is a sufficiently general element of $I_1 \oplus \cdots \oplus I_n$, by virtue of a result of Rees (see [11, §17] or [18]).

Let $I_1, \dots, I_r$ be ideals of $\mathcal{O}_n$ of finite colength, for some $r \leqslant n$. Let $(i_1, \dots, i_r) \in \mathbb{Z}^r_{\geqslant 0}$ such that $i_1 + \cdots + i_r = n$. Then $e_{i_1,\dots,i_r}(I_1, \dots, I_r)$ will denote the mixed multiplicity $e(I_1, \dots, I_1, \dots, I_r, \dots, I_r)$ where I_j is repeated i_j times, for all $j = 1, \dots, r$. If I, J are two ideals of finite colength of $\mathcal{O}_n$, then we denote by $e_i(I, J)$ the mixed multiplicity $e_{n-i,i}(I, J)$, for all $i \in \{0, 1, \dots, n\}$. Then we can restate relation (2) by $\mu^{(i)}(f) = e_i(m_n, J(f))$, for all $i = 0, 1, \dots, n$.

Let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{C}^n$ and let $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$. We denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ by x^k. If $h \in \mathcal{O}_n$ and $h = \sum_k a_k x^k$ is the Taylor expansion of h around the origin, then we denote by $\mathrm{supp}(h)$ the *support of* h, that is, $\mathrm{supp}(h) = \{k : a_k \neq 0\}$. If $h = 0$, then we set $\mathrm{supp}(h) = \emptyset$. The *Newton polyhedron of* h, denoted by $\Gamma_+(h)$, is the convex hull of the set $\{k + v : k \in \mathrm{supp}(h), v \in \mathbb{R}^n_+\}$.

Given a subset $\mathtt{I} \subseteq \{1, \dots, n\}$, we set $\mathbb{R}^n_{\mathtt{I}} = \{x \in \mathbb{R}^n : x_i = 0, \text{ for all } i \notin \mathtt{I}\}$. We denote by $h^{\mathtt{I}}$ the series obtained as the sum of all terms $a_k x^k$ with $k \in \mathbb{R}^n_{\mathtt{I}}$; if no such terms exist, then we set $h^{\mathtt{I}} = 0$. We denote by $\mathcal{O}_{n,\mathtt{I}}$, or by $\mathcal{O}_{\mathtt{I}}$, the subring of $\mathcal{O}_n$ formed by the functions $h \in \mathcal{O}_n$ depending only on the variables x_i such that $i \in \mathtt{I}$. If J is an ideal of $\mathcal{O}_n$, then we denote by $J^{\mathtt{I}}$ the ideal of $\mathcal{O}_{n,\mathtt{I}}$ generated by all the elements $h^{\mathtt{I}}$, where h varies in J.

Let J be an ideal of $\mathcal{O}_n$ and let $g_1, \dots, g_s$ be a generating system of J. Then the *Newton polyhedron of* J, that we denote by $\Gamma_+(J)$, is defined as the convex hull of $\Gamma_+(g_1) \cup \cdots \cup \Gamma_+(g_s)$. It is easy to check that this definition does not depend on the given generating system of J. Moreover, we denote by $\Gamma(J)$ the union of the compact faces of $\Gamma_+(J)$. Let $\mathcal{P}(J)$ denote the vector space of all polynomial functions $h \in \mathbb{C}[x_1, \dots, x_n]$ such that $\mathrm{supp}(h) \subseteq \Gamma(J)$. We remark that $\mathcal{P}(J)$ is a finite-dimensional complex vector space.

If V is a finite-dimensional complex vector space, then we say that a given property is *generic* in V when there exists a Zariski-open set $U \subseteq V$ such that any element $u \in U$ satisfies the said property.

Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic map. Let us denote by f_t the map $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that $f_t(x) = F(t, x)$. Let us suppose that f_t has an isolated singularity at the origin, for all small enough t. We say that F is a μ^**-constant deformation* when $\mu^*(f_t)$ does not depend on t, for all small enough t. Maybe the following result is well-known for the specialists, however we include a proof of it.

Lemma 2.1. *Under the above setup, let us assume that the function f_t has an isolated singularity at the origin and that $\Gamma_+(f_t)$ does not depend on t, for all small enough t. If f_0 is Newton non-degenerate, then F is μ^*-constant.*

Proof. By [22, Theorem 3] it is known that F is μ^*-constant if and only if

$$\frac{\partial F}{\partial t} \in \overline{m_n \left\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right\rangle}, \tag{3}$$

where the bar denotes integral closure in $\mathcal{O}_{n+1}$ and in this case m_n denotes the ideal of $\mathcal{O}_{n+1}$ generated by $x_1, \dots, x_n$.

Let us denote by Γ_+ the common Newton polyhedron of the functions f_t and let I denote the ideal of $\mathcal{O}_{n+1}$ generated by $x_1\frac{\partial F}{\partial x_1}, \dots, x_n\frac{\partial F}{\partial x_n}$. We observe that $\Gamma_+(F) = \Gamma_+(I) = \mathbb{R}_+ \times \Gamma_+$, since the functions f_t have the same Newton polyhedron. We remark that, in order to construct the Newton polyhedron $\Gamma_+(I)$, we represent the exponent of a monomial $t^\alpha x_1^{k_1}\cdots x_n^{k_n}$ of $\mathcal{O}_{n+1}$ by $(\alpha, k_1, \dots, k_n)$. Therefore the set of compact faces of $\Gamma_+(I)$ is equal to the set of compact faces of $\{0\} \times \Gamma_+ \subseteq \mathbb{R}_+ \times \mathbb{R}^n_+$. Then the Newton non-degeneracy of F only depends on the monomials of the support of F belonging to the compact faces of $\{0\} \times \Gamma_+ \subseteq \mathbb{R} \times \mathbb{R}^n$. In particular, if Δ is a compact face of Γ_+ and $i \in \{1, \dots, n\}$, then

$$\left(x_i\frac{\partial F}{\partial x_i}\right)_{\{0\}\times\Delta} = \left(x_i\frac{\partial f}{\partial x_i}\right)_{\Delta}.$$

Thus the function F is Newton non-degenerate and consequently $\overline{I}$ is equal to the monomial ideal generated by all monomials in $\mathcal{O}_{n+1}$ whose support belongs to $\Gamma_+(F)$ (see [25], or [19] for a more general result). Since $\Gamma_+(f_t) = \Gamma_+$, for all t, the support of $\frac{\partial F}{\partial t}$ is contained in $\Gamma_+(F)$. In particular, we have $\frac{\partial F}{\partial t} \in \overline{I}$ and hence relation (3) holds. $\square$

If J is a monomial ideal of $\mathcal{O}_n$, the we denote by $\mathcal{O}(J)$ the set of all analytic function germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin such that $\Gamma_+(f) = \Gamma_+(J)$.

Definition 2.2. Let J be an ideal of finite colength of $\mathcal{O}_n$. Let $i \in \{1, \dots, n-1\}$, then we define

$$a_i(J, m) = \sum_{j=i}^{n-1} \binom{j-1}{i-1} e_j(J, m). \tag{4}$$

We also set $a_0(J, m) = e(J)$. We observe that $a_{n-1}(J, m) = e_{n-1}(J, m) = \operatorname{ord}(J)$ (see Lemma 3.1) and that $a_1(J, m) = e_1(J, m) + \cdots + e_{n-1}(J, m)$.

For $i = 1, \dots, n$, we define the *i-th Newton number of J*, that we denote by $\nu^{(i)}(J)$, as

$$\nu^{(i)}(J) = \sum_{s=n-i+1}^{n} (-1)^{n-s} \Bigg(\sum_{\substack{\mathtt{I}\subseteq\{1,\dots,n\} \\ |\mathtt{I}|=s}} a_{n-i}\big(J^{\mathtt{I}}, m^{\mathtt{I}}\big) \Bigg) + (-1)^i.$$

Then we define

$$\nu^*(J) = \big(\nu^{(n)}(J), \nu^{(n-1)}(J), \dots, \nu^{(1)}(J), \nu^{(0)}(J)\big),$$

where we set $\nu^{(0)}(J) = 1$.

Theorem 2.3. *Let J be a monomial ideal of finite colength, let $f \in \mathcal{O}(J)$ and let $i \in \{0, 1, \dots, n\}$. Then*

$$\mu^{(i)}(f) \geqslant \nu^{(i)}(J) \tag{5}$$

and equality holds if f is Newton non-degenerate.

Proof. It is well known that $e(J) = n!\mathrm{V}_n\big(\mathbb{R}^n \smallsetminus \Gamma_+(J)\big)$, where V_n denotes the n-dimensional volume (see for instance [23]). Then the case $i = n$ arises directly from this equality and the main theorem of Kouchnirenko in [12].

Let us fix an index $i \in \{1, \dots, n-1\}$. If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an isolated complete intersection singularity, then we denote the Milnor number of g (in the sense of Hamm [10] and Lê [13]) by $\mu(g)$. By the definition of $\mu^*(f)$ and the definition of Milnor number of an isolated complete intersection singularity [13] we have

$$\mu^{(n-i)}(f) = \mu(f, \ell_1, \dots, \ell_i), \tag{6}$$

where ℓ_j denotes a generic $\mathbb{C}$-linear form, for all $j = 1, \dots, i$. We observe that $\Gamma_+(\ell_i) = \Gamma_+(m)$ and therefore, by [3, Theorem 3.9] we conclude the inequality

$$\mu^{(n-i)}(f) \geqslant \nu, \tag{7}$$

where ν stands for the number

$$\nu = \sum_{s=i+1}^{n} (-1)^{n-s} \Bigg(\sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=s}} \sum_{\substack{r_1+\cdots+r_{i+1}=s \\ r_1,\dots,r_{i+1} \geqslant 1}} e_{r_1,\dots,r_{i+1}}(J^{\mathtt{I}}, \underbrace{m^{\mathtt{I}}, \dots, m^{\mathtt{I}}}_{i}) \Bigg) + (-1)^{n-i}.$$

We observe that

$$\sum_{\substack{r_1+\cdots+r_{i+1}=n \\ r_1,\dots,r_{i+1} \geqslant 1}} e_{r_1,\dots,r_{i+1}}(J, \underbrace{m, \dots, m}_{i}) = \sum_{r=1}^{n-i} \Bigg(\sum_{\substack{r_2+\cdots+r_{i+1}=n-r \\ r_2,\dots,r_{i+1} \geqslant 1}} e_{r,r_2,\dots,r_{i+1}}(J, \underbrace{m, \dots, m}_{i}) \Bigg)$$

$$= \sum_{r=1}^{n-i} \left(e_{n-r}(J, m) \cdot \# \left\{ (r_2, \dots, r_{i+1}) \in \mathbb{Z}^i_{\geqslant 1} : r_2 + \cdots + r_{i+1} = n - r \right\} \right)$$

$$= \sum_{r=1}^{n-i} \binom{n-r-1}{i-1} e_{n-r}(J, m) = \sum_{j=i}^{n-1} \binom{j-1}{i-1} e_j(J, m) = a_i(J, m).$$

Given a subset $\mathtt{I} \subseteq \{1, \dots, n\}$, $|\mathtt{I}| = s$, a similar computation leads to the equality

$$\sum_{\substack{r_1+\cdots+r_{i+1}=s \\ r_1,\dots,r_{i+1} \geqslant 1}} e_{r_1,\dots,r_{i+1}}(J^{\mathtt{I}}, \underbrace{m^{\mathtt{I}}, \dots, m^{\mathtt{I}}}_{i}) = a_i(J^{\mathtt{I}}, m^{\mathtt{I}}).$$

Hence

$$\nu = \sum_{s=i+1}^{n} (-1)^{n-s} \Bigg(\sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=s}} a_i(J^{\mathtt{I}}, m^{\mathtt{I}}) \Bigg) + (-1)^{n-i} = \nu^{(n-i)}(J)$$

and the inequality (5) is proven. By [3, Theorem 3.9], equality holds in (7) if the map $(f, \ell_1, \dots, \ell_i)$ is Newton non-degenerate in the sense of [3, Definition 3.8]. Concerning the property of Newton non-degeneracy of maps, in this proof we will only use the genericity of this condition (see [3, Lemma 6.11]).

For $i = 1, \dots, n-1$, we denote by $\mathcal{P}_i$ the product vector space $\mathcal{P}(J) \times \mathcal{P}(m) \times \cdots \times \mathcal{P}(m)$, where $\mathcal{P}(m)$ is repeated i times. Then we denote by A_i the set of Newton non-degenerate maps of $\mathcal{P}_i$ and by A'_i the projection of A_i onto $\mathcal{P}(J)$.

The Newton non-degeneracy condition of a map belonging to $\mathcal{P}_i$ is a generic condition, by [3, Lemma 6.11], for all $i = 1, \dots, n-1$. Then, there exists a Zariski open set U of $\mathcal{P}(J)$ such that $U \subseteq A'_1 \cap \cdots \cap A'_{n-1}$ and h has an isolated singularity at the origin, for all $h \in U$. Let us consider an analytic deformation $P : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that, if $p_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ denotes the map given by $p_t(x) = P(t, x)$, then

(1) $p_t \in U$, for all $t \neq 0$;
(2) $p_0 = p(f)$,

where $p(f)$ is the principal part of f. That is, if $f = \sum_k a_k x^k$ is the Taylor expansion of f around the origin, then $p(f)$ is the sum of all terms $a_k x^k$ such that $k \in \Gamma(J)$.

Let us assume that f is Newton non-degenerate. Therefore P is a μ^*-constant deformation, by Lemma 2.1. If $t \neq 0$, the polynomial p_t belongs to $A'_1 \cap \cdots \cap A'_{n-1}$, which implies that $\mu^*(p_t) = \nu^*(J)$, by [3, Theorem 3.9] applied to (7). Hence $\mu^*(p_0) = \mu^*(p(f)) = \nu^*(J)$.

Let $f' = f - p(f)$. By the definition of $p(f)$, the support of f' is contained in $\Gamma_+(J)$ and $\mathrm{supp}(f') \cap \Gamma(J) = \emptyset$. Let us consider the homotopy $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ given by $F_t = p(f) + tf'$. This deformation is μ^*-constant, by Lemma 2.1. Then

$$\mu^*(f) = \mu^*(p(f))$$

and the result follows. □

3. Some particular cases and examples

If J is an ideal of a local ring (R, m) of dimension n, then we denote by $\mathrm{ord}(J)$ the *order* of J, that is, the maximum of those $r \geqslant 0$ such that $J \subseteq m^r$. In particular, if J is an ideal of $\mathcal{O}_n$ and $\mathtt{I} \subseteq \{1, \dots, n\}$, $\mathtt{I} \neq \emptyset$, then $\mathrm{ord}(J^{\mathtt{I}})$ denotes the order of $J^{\mathtt{I}}$ as an ideal of $\mathcal{O}_{n,\mathtt{I}}$.

Lemma 3.1. *Let (R, m) be a regular local ring of dimension n such that the residue field R/m is infinite. Let J be an ideal of R of finite colength. Then*

$$e_{n-1}(J, m) = \mathrm{ord}(J).$$

Proof. By [18] we have $e_{n-1}(J, m) = e(f, \ell_1, \dots, \ell_{n-1})$, where $(f, \ell_1, \dots, \ell_{n-1})$ is a sufficiently general element of $J \oplus m \oplus \cdots \oplus m$ (see also [11, §17]). Therefore

$$e_{n-1}(J, m) = e(f, \ell_1, \dots, \ell_{n-1}) = \ell(R'/\langle \overline{f} \rangle),$$

where R' denotes the quotient ring $R/\langle \ell_1, \dots, \ell_{n-1} \rangle$ and $\overline{f}$ is denotes the image of f in R'. But $\ell(R'/\langle \overline{f} \rangle) = \mathrm{ord}(f)$, since R' is regular and 1-dimensional. □

As an immediate application of Theorem 2.3 and Lemma 3.1 we obtain the following result.

Corollary 3.2. *Let J be a monomial ideal of $\mathcal{O}_n$ of finite colength and let $f \in \mathcal{O}(J)$. Then*

$$\mu^{(2)}(f) \geqslant e_{n-2}(J, m) + (n-2)\mathrm{ord}(J) - \Bigg(\sum_{\substack{\mathtt{I} \subseteq \{1, \dots, n\} \\ |\mathtt{I}| = n-1}} \mathrm{ord}(J^{\mathtt{I}}) \Bigg) + 1 \tag{8}$$

$$\mu^{(n-1)}(f) \geqslant \sum_{s=2}^{n} (-1)^{n-s} \Bigg(\sum_{\substack{\mathtt{I} \subseteq \{1, \dots, n\} \\ |\mathtt{I}| = s}} \big(e_1(J^{\mathtt{I}}, m^{\mathtt{I}}) + \cdots + e_{s-1}(J^{\mathtt{I}}, m^{\mathtt{I}}) \big) \Bigg) + (-1)^{n-1} \tag{9}$$

and equality holds in the above inequalities if f is Newton non-degenerate.

We remark that $\mu^{(n-1)}(f)$ has an important geometrical content in general via polar curves (see [24, p. 270]) and the notion of Euler obstruction (see [14, p. 486]).

In the case $n = 3$, the right hand side of relations (8) and (9) coincide with $\nu^{(2)}(J)$ and in this case we have

$$\nu^{(2)}(J) = -\mathrm{ord}(J^{\{1,2\}}) - \mathrm{ord}(J^{\{1,3\}}) - \mathrm{ord}(J^{\{2,3\}}) + \mathrm{ord}(J) + e(m, J, J) + 1. \tag{10}$$

The above expression leads to the following result, which helps in the task of finding examples of μ-constant deformations $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ that are not $\mu^{(2)}$-constant.

Corollary 3.3. *Let J_0 and J_1 be monomial ideals of finite colength of $\mathcal{O}_3$ such that*

$$\mathrm{ord}(J_0) - \sum_{1\leqslant i<j\leqslant 3} \mathrm{ord}(J_0^{\{i,j\}}) = \mathrm{ord}(J_1) - \sum_{1\leqslant i<j\leqslant 3} \mathrm{ord}(J_1^{\{i,j\}}).$$

Let us consider an analytic deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that

(1) *f_t is Newton non-degenerate, for all t;*
(2) *$\Gamma_+(f_0) = \Gamma_+(J_0)$;*
(3) *$\Gamma_+(f_t) = \Gamma_+(J_1)$, for all $t \neq 0$.*

Then the deformation f_t is not $\mu^{(2)}$-constant if and only if $e(m, J_0, J_0) > e(m, J_1, J_1)$.

Proof. It is an immediate consequence of relation (10) and Corollary 3.2. □

Example 3.4. Let $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be the analytic family of functions given by

$$f_t(x, y, z) = x^{15} + y^8 + z^5 + xy^7 + ty^6 z. \tag{11}$$

Let us consider the ideals of $\mathcal{O}_3$ given by $J_0 = \langle x^{15}, y^8, z^5, xy^7\rangle$ and $J_1 = J_0 + \langle y^6 z\rangle$. We have $\Gamma_+(f_0) = \Gamma_+(J_0)$ and $\Gamma_+(f_t) = \Gamma_+(J_1)$, for all $t \neq 0$. The family f_t given in (11) is a modification of the Briançon-Speder example [5], that is, we have added the term y^8 to this example in order to have that the ideals J_0 and J_1 have finite colength in $\mathcal{O}_3$.

It is clear that $\mathrm{ord}(J_0) = \mathrm{ord}(J_1) = 5$ and

$$\begin{aligned} \mathrm{ord}(J_0^{\{1,2\}}) &= \mathrm{ord}(J_1^{\{1,2\}}) = 8 \\ \mathrm{ord}(J_0^{\{1,3\}}) &= \mathrm{ord}(J_1^{\{1,3\}}) = 5 \\ \mathrm{ord}(J_0^{\{2,3\}}) &= \mathrm{ord}(J_1^{\{2,3\}}) = 5. \end{aligned}$$

The numbers $e(m, J_0, J_0)$ and $e(m, J_1, J_1)$ can be computed effectively using the procedure described in [4, p. 405] and the aid of Singular [9]. Thus we obtain that

$$e(m, J_0, J_0) = 40 \qquad e(m, J_1, J_1) = 38.$$

Therefore, relation (10) gives

$$\nu^{(2)}(J_0) = 28 \qquad \nu^{(2)}(J_1) = 26.$$

and then $\mu^{(2)}(f_0) = 28$ and $\mu^{(2)}(f_t) = 26$, since f_t is Newton non-degenerate, for all t.

Example 3.5. [1, §4] Let us consider the analytic family $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by

$$f_t(x, y, z) = x^6 + y^5 + z^{12} + xy^3z + tx^3y^2.$$

This family is μ-constant but not $\mu^{(2)}$-constant. As indicated in [1], we have $\mu(f_t) = 166$, for all t, and

$$\mu^{(2)}(f_0) = 18 \qquad \mu^{(2)}(f_t) = 17, \quad \text{for all } t \neq 0.$$

Let us consider the monomial ideals of $\mathcal{O}_3$ given by

$$J_0 = \langle x^6, y^5, z^{12}, xy^3z\rangle \qquad J_1 = J_0 + \langle x^3y^2\rangle.$$

We observe that $\Gamma_+(f_0) = \Gamma_+(J_0)$, $\Gamma_+(f_1) = \Gamma_+(J_1)$ and $\nu(J_0) = \nu(J_1) = 166$. Moreover $e(m, J_0, J_0) = 28$ and $e(m, J_1, J_1) = 27$, which imply that $\nu^{(2)}(J_0) = 18$ and $\nu^{(2)}(J_1) = 17$, by (10).

Example 3.6. Let us consider the monomial ideals J_0 and J_1 of $\mathcal{O}_3$ given by

$$J_0 = \langle x^5, y^7, z^{15}, x^2y^2z\rangle \qquad J_1 = J_0 + \langle xy^4\rangle.$$

We observe that $\nu(J_0) = \nu(J_1) = 206$ and

$$e(m, J_0, J_0) = 29 \qquad e(m, J_1, J_1) = 27.$$

Therefore $\nu^{(2)}(J_0) = 18$ and $\nu^{(2)}(J_1) = 16$, by (10). This means that any deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that $\Gamma_+(f_0) = \Gamma_+(J_0)$, $\Gamma_+(f_t) = \Gamma_+(J_1)$, for all $t \neq 0$, and f_t is Newton non-degenerate, for all t, verifies that f_t is μ-constant but not $\mu^{(2)}$-constant.

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