Research Article

Wave Front Sets with respect to the Iterates of an Operator with Constant Coefficients

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H.1. Introduction

In the 1960s Komatsu characterized in [1] analytic functions \( f \) in terms of the behaviour not of the derivatives \( D^a f \), but of successive iterates \( P(D)^j f \) of a partial differential elliptic operator \( P(D) \) with constant coefficients, proving that a \( C^\infty \) function \( f \) is real analytic in \( \Omega \) if and only if for every compact set \( K \subset \Omega \) there is a constant \( C > 0 \) such that

\[
\| P(D)^j f \|_{2,K} \leq C^{j+1} (j!)^m ,
\]

where \( m \) is the order of the operator and \( \| \cdot \|_{2,K} \) is the \( L^2 \) norm on \( K \).

This result was generalized for elliptic operators with variable analytic coefficients by Kotake and Narasimhan [2, Theorem 1]. Later, this result was extended to the setting of Gevrey functions by Newberger and Zielezny [3] and completely characterized by Métivier [4] (see also [5]). Spaces of Gevrey type given by the iterates of a differential operator are called generalized Gevrey classes and were used by Langenbruch [6–9] for different purposes. We mention modern contributions like [10–13] also. More recently, Juan-Huguet [14] extended the results of Komatsu [1], Newberger and Zielezny [3], and Métivier [4] to the setting of nonquasianalytic classes in the sense of Braun et al. [15]. In [14], Juan-Huguet introduced the generalized spaces of ultradifferentiable functions \( \mathcal{E}_p^\omega (\Omega) \) on an open subset \( \Omega \) of \( \mathbb{R}^n \) for a fixed linear partial differential operator \( P \) with constant coefficients and proved that these spaces are complete if and only if \( P \) is hypoelliptic. Moreover, Juan-Huguet showed that, in this case, the spaces are nuclear. Later, the same author in [16] established a Paley-Wiener theorem for the classes \( \mathcal{E}_p^\omega (\Omega) \) again under the hypothesis of the hypoellipticity of \( P \).

The microlocal version of the problem of iterates was considered by Bolley et al. [17] to extend the microlocal regularity theorem of Hörmander [18, Theorem 5.4]. Bolley and Camus [19] generalized the microlocal version of the problem of iterates in [17] for some classes of hypoelliptic operators with analytic coefficients. We mention [20, 21] for investigations of the same problem for anisotropic and multianisotropic Gevrey classes. On the other hand, a version of the microlocal regularity theorem of Hörmander in the setting of [15] can be found in [22, 23] by one of the authors, which continues the study begun in [24].

Here, we continue in a natural way the previous work in [14] and study the microlocal version of the problem of iterates for generalized ultradifferentiable classes in the sense of Braun et al. [15]. We begin in Section 2 with some notation and preliminaries. In Section 3, we fix a hypoelliptic linear
partial differential operator with constant coefficients \( P \) and introduce the wave front set \( \text{WF}^P_\mathcal{F}(u) \) with respect to the iterates of \( P \) of a distribution \( u \in \mathcal{D}'(\Omega) \) (Definition 7). To do this, we describe carefully the singular support in this setting (Proposition 6). We also prove that the new wave front set gives a more precise information for the study of the propagation of singularities than previous ones in Proposition 9, Theorem 13, and Example 15 (improving the previous works [22, 23] by one of the authors for operators with constant coefficients). More precisely, we clarify in Theorem 13 the necessity of the hypoellipticity of \( P \) with a new version of the microlocal regularity theorem of Hörmander for an operator with constant coefficients. In Section 4, we prove that the product of a function in a suitable Gevrey class and a function in \( \mathcal{E}^0(\Omega) \) is still in \( \mathcal{E}^0(\Omega) \) (Proposition 17). This fact is used to give a more involved example, inspired in [25, Theorem 8.1.4], in which we construct a classical distribution with prescribed wave front set (Theorem 18). Finally, we mention that, as far as we know, this is the first time that a result like Proposition 17 is discussed.

2. Notation and Preliminaries

Let us recall from [15] the definitions of weight functions \( \omega \) and of the spaces of ultradifferentiable functions of Beurling and Roumieu type.

**Definition 1.** A nonquasianalytic weight function is a continuous increasing function \( \omega : [0, +\infty) \to [0, +\infty] \) with the following properties:

\( \alpha \) \( L > 0 \) s.t. \( \omega(2t) \leq L(\omega(t) + 1) \) \( \forall t \geq 0 \),

\( \beta \) \( \int_1^{\omega(\epsilon)} \frac{\omega(t)}{t^\beta} dt < +\infty \),

\( \gamma \) \( \log(t) = o(\omega(t)) \) as \( t \to +\infty \),

\( \delta \) \( \varphi_\omega \) : \( t \mapsto \omega(\epsilon) \) is convex.

Normally, we will denote \( \varphi_\omega \) simply by \( \varphi \).

For a weight function \( \omega \), we define \( \overline{\omega} : \mathbb{C}^n \to [0, +\infty] \) by \( \overline{\omega}(z) := \omega(|z|) \) and again we denote this function by \( \omega \).

The Young conjugate \( \varphi^* : [0, +\infty) \to [0, +\infty] \) is defined by

\[
\varphi^*(s) := \sup_{t \geq 0} \{ st - \varphi(t) \}.
\]

There is no loss of generality to assume that \( \omega \) vanishes on \( [0, 1] \). Then \( \varphi^* \) has only nonnegative values, it is convex, \( \varphi^*(t)/t \) is increasing and tends to \( \infty \) as \( t \to \infty \), and \( \varphi^{**} = \varphi \).

**Example 2.** The following functions are, after a change in some interval \([0, M]\), examples of weight functions:

\( \alpha \) \( \omega(t) = t^d \) for \( 0 < d < 1 \),

\( \beta \) \( \omega(t) = (\log(1 + t))^s \), \( s > 1 \),

\( \gamma \) \( \omega(t) = t(\log(e + t))^{\beta} \), \( \beta > 1 \),

\( \delta \) \( \omega(t) = \exp(\beta(\log(1 + t))^\alpha), 0 < \alpha < 1 \).

In what follows, \( \Omega \) denotes an arbitrary subset of \( \mathbb{R}^n \) and \( K \subset \subset \Omega \) means that \( K \) is a compact subset in \( \Omega \).

**Definition 3.** Let \( \omega \) be a weight function.

\( \alpha \) For a compact subset \( K \) in \( \mathbb{R}^n \) which coincides with the closure of its interior and \( \lambda > 0 \), we define the seminorm

\[
P_{K,\lambda}(f) := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^n} \left| f^{(\alpha)}(x) \right| \exp \left( -\lambda \varphi^* \left( \frac{|\alpha|}{\lambda} \right) \right),
\]

where \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and set

\[
\mathcal{E}^\omega_{\lambda}(K) := \{ f \in C^\infty(\Omega) : P_{K,\lambda}(f) < \infty \},
\]

which is a Banach space endowed with the \( p_{K,\lambda}(-) \)-topology.

\( \beta \) For an open subset \( \Omega \) in \( \mathbb{R}^n \), the class of \( \omega \)-ultradifferentiable functions of Beurling type is defined by

\[
\mathcal{E}_{\omega}(\Omega) := \{ f \in C^\infty(\Omega) : P_{K,\lambda}(f) < \infty, \text{ for every } K \subset \subset \Omega \text{ and every } \lambda > 0 \}.
\]

The topology of this space is

\[
\mathcal{E}_{\omega}(\Omega) = \text{proj} \lim_{\lambda \to \infty} \mathcal{E}_{\omega_{\lambda}}(K).
\]

and one can show that \( \mathcal{E}_{\omega}(\Omega) \) is a Fréchet space.

\( \gamma \) For a compact subset \( K \) in \( \mathbb{R}^n \) which coincides with the closure of its interior and \( \lambda > 0 \), set

\[
\mathcal{E}_{\omega}(K) := \{ f \in C^\infty(\Omega) : \text{there exists } m \in \mathbb{N} \text{ such that } P_{K,1/m}(f) < \infty \}. \tag{7}
\]

This space is the strong dual of a nuclear Fréchet space (i.e., a (DFN) space) if it is endowed with its natural inductive limit topology; that is,

\[
\mathcal{E}_{\omega}(K) = \text{ind lim}_{m \to \infty} \mathcal{E}^\omega_{\lambda/m}(K). \tag{8}
\]

\( \delta \) For an open subset \( \Omega \) in \( \mathbb{R}^n \), the class of \( \omega \)-ultradifferentiable functions of Roumieu type is defined by

\[
\mathcal{E}_{\omega}(\Omega) := \{ f \in C^\infty(\Omega) : \forall K \subset \subset \Omega \exists \lambda > 0 \text{ such that } P_{K,\lambda}(f) < \infty \}. \tag{9}
\]

Its topology is the following:

\[
\mathcal{E}_{\omega}(\Omega) = \text{proj} \lim_{\lambda \to \infty} \mathcal{E}^\omega_{\lambda}(K); \tag{10}
\]

that is, it is endowed with the topology of the projective limit of the spaces \( \mathcal{E}^\omega_{\lambda}(K) \) when \( K \) runs the compact subsets of \( \Omega \). This is a complete PLS-space, that is, a complete space which is a projective limit of LB-spaces (i.e., a countable inductive limit of Banach spaces) with compact linking maps in the (LB) steps. Moreover, \( \mathcal{E}_{\omega}(\Omega) \) is also a nuclear and reflexive locally convex space. In particular, \( \mathcal{E}_{\omega}(\Omega) \) is an ultrabornological (hence barrelled and bornological) space.

The elements of \( \mathcal{E}_{\omega}(\Omega) \) (resp., \( \mathcal{E}_{\omega}(\Omega) \)) are called ultradifferentiable functions of Beurling type (resp., Roumieu type) in \( \Omega \).
In the case that \( \omega(t) := t^d \ (0 < d < 1) \), the corresponding Roumieu class is the Gevrey class with exponent \( 1/d \). In the limit case \( d = 1 \), not included in our setting, the corresponding Roumieu class \( \mathcal{E}_\omega(\Omega) \) is the space of real analytic functions on \( \Omega \), whereas the Beurling class \( \mathcal{E}_\omega(\mathbb{R}^n) \) gives the entire functions.

If a statement holds in the Beurling and the Roumieu case, then we will use the notation \( \mathcal{E}_\omega(\Omega) \). It means that in all cases, * can be replaced either by \( \omega \) or \( |\omega| \).

For a compact set \( K \) in \( \mathbb{R}^n \), define
\[
\mathcal{D}_\omega(K) := \{ f \in \mathcal{E}_\omega(\mathbb{R}^n) : \text{supp } f \subset K \},
\]
endowed with the induced topology. For an open set \( \Omega \) in \( \mathbb{R}^n \), define
\[
\mathcal{D}_\omega(\Omega) := \text{ind} \mathcal{D}_\omega(K).
\]

Following [14], we consider smooth functions in an open set \( \Omega \) such that there exists \( C > 0 \) verifying for each \( j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \),
\[
\| P^j(D)f \|_{2, K} \leq C \exp \left( \lambda \Phi^* \left( \frac{jm}{\lambda} \right) \right),
\]
where \( K \) is a compact subset in \( \Omega \), \( \| \cdot \|_{2, K} \) denotes the \( L^2 \)-norm on \( K \), and \( P^j(D) \) is the \( j \)th iterate of the partial differential operator \( P(D) \) of order \( m \); that is,
\[
P^j(D) = P(D) \ast \cdots \ast P(D).
\]

If \( j = 0 \), then \( P^0(D)f = f \).

Given a polynomial \( P \in C[z_1, \ldots, z_n] \) with degree \( m \), \( P(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha \), the partial differential operator \( P(D) \) is the following: \( P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha \), where \( D = (1/i) \theta \).

The spaces of ultradifferentiable functions with respect to the successive iterates of \( P \) are defined as follows.

Let \( \omega \) be a weight function. Given a polynomial \( P \), an open set \( \Omega \) of \( \mathbb{R}^n \), a compact subset \( K \subset \subset \Omega \), and \( \lambda > 0 \), we define the seminorm
\[
\| f \|_{K, \lambda} := \sup_{j \in \mathbb{N}_0} \| P^j(D)f \|_{2, K} \exp \left( -\lambda \Phi^* \left( \frac{jm}{\lambda} \right) \right)
\]
and set
\[
\mathcal{E}^\lambda_{P, \omega}(K) = \{ f \in C^\omega(\Omega) : \| f \|_{K, \lambda} < +\infty \}.
\]

It is a Banach space endowed with the \( \| \cdot \|_{K, \lambda} \)-norm (it can be proved by the same arguments used for the standard class \( \mathcal{E}^\lambda_\omega(K) \) in the sense of Braun et al.; see [15]).

The space of ultradifferentiable functions of Beurling type with respect to the iterates of \( P \) is
\[
\mathcal{E}^\lambda_{P, \omega}(\Omega) = \{ f \in C^\omega(\Omega) : \| f \|_{K, \lambda} < +\infty \}
\]
for each \( K \subset \subset \Omega, \lambda > 0 \).
3. Wave Front Sets with respect to the Iterates of an Operator

Now, we assume that $A$ is a bounded open set in $\mathbb{R}^n$ and we use the following notation:

$$A_s := \{x \in A : d(x, \partial A) > s\},$$

where $d(x, \partial A)$ is the distance of $x$ to the boundary of $A$. Given a linear partial differential operator $P(D)$, we denote by $P^{(l)}(D)$ the operator corresponding to the polynomial $P^{(l)}(\xi)$. If $P(D)$ is hypoelliptic, by [27, Theorem 4.1] and the argument used in the proof of [3, Theorem 1], there are constants $C > 0$ and $\gamma > 0$ such that for every $s \geq 0$ and $t > 0$ we have

$$\left\| P^{(l)}(D) f \right\|_{2, A_s} \leq C \left( t^{\gamma/2} \left\| P(D) f \right\|_{2, A_s} + t^{3\gamma/2} \left\| f \right\|_{2, A_s} \right),$$

$$f \in C^\infty(A).$$

We observe also that if $P(D)$ has constant coefficients, its formal adjoint is $P(-D)$ and, if $P(D)$ is hypoelliptic, $P(-D)$ is also hypoelliptic (because of the behavior of the associated polynomial $P(-\xi)$). Moreover, any power $P(D)^{t\ell}$ or $P(-D)^{t\ell}$, with $\ell \in \mathbb{N}$, of $P(D)$ or $P(-D)$, is also hypoelliptic.

We now want to generalize the notion of $*-$singular support of Proposition 4, using the iterates of a hypoelliptic linear partial differential operator $P$ with constant coefficients. The idea is to substitute the sequence $u_N$ which satisfies an estimate of the form (23) or (24) by the sequence $f_N = P(D)^N u$ whose Fourier transform satisfies the following estimates (29) or (30).

Proposition 6. Let $P(D)$ be a linear partial differential operator of order $m$ with constant coefficients which is hypoelliptic. Let $\Omega$ be an open subset of $\mathbb{R}^n$, $u \in \mathcal{D}'(\Omega)$, $x_0 \in \Omega$ and consider the following three conditions:

(i) $f^N = P(D)^N u$  

$$\left| \hat{f}_N(\xi) \right| \leq C_{M,N} (|\xi|^{1/k}\rho^m)(1 + |\xi|)^M,$$  

(ii) $(Roumieu)$  

$$\forall k \in \mathbb{N}, \forall M \in \mathbb{R}, \exists C_{M,N} > 0, \forall N \in \mathbb{N}, \text{ and } \xi \in \mathbb{R}^n, \text{ we have}$$

$$\left| \hat{f}_N(\xi) \right| \leq C_{M,N} (|\xi|^{1/k}\rho^m)(1 + |\xi|)^M.$$

(iii) $(Beurling)$  

$$\forall k \in \mathbb{N} \text{ and } M \in \mathbb{R}, \exists C_{M,N} > 0, \forall N \in \mathbb{N}, \text{ and } \xi \in \mathbb{R}^n, \text{ we have}$$

$$\left| \hat{f}_N(\xi) \right| \leq C_{M,N} (|\xi|^{1/k}\rho^m)(1 + |\xi|)^M.$$

Then, the distribution $u \in \mathcal{E}^P_\rho(U)$ ($u \in \mathcal{E}^P_\rho(U)$), where $U$ is some neighborhood of $x_0$, if and only if there exist a neighborhood $V$ of $x_0$ and a sequence $(f_N)$ in $\mathcal{E}^P(U)$ that satisfies (i) and (ii) in $V$ (that satisfies (i) and (iii) in $V$).

Proof.

Sufficiency (Roumieu Case). Let $u \in \mathcal{E}^P_\rho(U)$ with $U = B_\rho(x_0)$, the ball in $\mathbb{R}^n$ of center $x_0$ and radius $\rho$, $r > 0$. We choose $\chi \in \mathcal{D}(\Omega)$ such that $\chi = 1$ in $B_r(x_0)$ and $\chi = 0$ in $(B_2(x_0))^c$. We set $f_N = \chi P(D)^N u$. Then, $f_N \in \mathcal{E}^P(U)$ and $f_N = P(D)^N u$ in $B_r(x_0)$.

Now, fix $\ell \in \mathbb{N}$. From the hypoellipticity of $P(D)$, there are constants $\tilde{D}, \tilde{d} > 0$ such that, for $|\xi|$ large enough, $|P(\xi)| \geq \tilde{D}|\xi|^\tilde{d}$. Then, from the definition of $f_N$ we obtain, for $|\xi|$ large enough,

$$\left| D^\ell \left| \hat{f}_N(\xi) \right| \right| \leq \left| P(\xi) \right|^\ell \left| \hat{f}_N(\xi) \right|$$

$$\leq \left| P(\xi) \right|^\ell \left| \int_{\mathbb{R}^n} \chi(x) \, P(D)^N u(x) \, e^{-ix\cdot\xi} \, dx \right|$$

$$\leq \left| \int_{\mathbb{R}^n} \chi(x) \, P(D)^N u(x) \, P(-D)^\ell \left( e^{-ix\cdot\xi} \right) \, dx \right|.$$

We integrate by parts in the integral above, which will be equal to

$$\left| \int_{\mathbb{R}^n} P(D)^\ell \left( \chi(x) \cdot P(D)^N u(x) \right) e^{-ix\cdot\xi} \, dx \right|.$$

From the generalized Leibniz rule, we can write (here $m$ is the order of $P(D)$)

$$P(D)^\ell \left( \chi(x) \cdot P(D)^N u(x) \right)$$

$$= \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha \chi(x) \cdot \left( P(D)^{N \ell} \right)(D) \left( P(D)^N u(x) \right).$$

Since $P(D)^\ell$ is hypoelliptic and $P(D)^N u$ is a $C^\infty$-function in the bounded set $B_{\rho^m}(x_0)$, we can apply formula (28) to the
operator $P(D)^{t}$ with $t = \epsilon$, for $0 < \varepsilon < r$, $A_{\varepsilon+t} = B_{2\varepsilon}(x_{0})$, and $f = P(D)^{N}u$ (and $A_{\varepsilon} = B_{2\varepsilon}(x_{0})$) to obtain constants $C_{\varepsilon}, \gamma > 0$ (which do not depend on $N$) such that

\[
\left\| (P(D)^{t})^{(a)} \left( P(D)^{N}u \right) \right\|_{2, B_{2\varepsilon}(x_{0})} \leq C_{\varepsilon} \left( \varepsilon^{k1} \left\| P(D)^{N+\varepsilon}u \right\|_{2, B_{2\varepsilon}(x_{0})} + \varepsilon^{(a-\gamma)} \left\| P(D)^{N}u \right\|_{2, B_{2\varepsilon}(x_{0})} \right).
\]

(34)

Now, as $u \in \mathcal{F}_{(\omega)}^{\mathcal{C}}(U)$, there are constants $k \in \mathbb{N}$ and $C > 0$ such that (we use the convexity of $\varphi^{\ast}$)

\[
\left\| P(D)^{N+\varepsilon}u \right\|_{2, B_{2\varepsilon}(x_{0})} \leq C_{\varepsilon} (1/k)^\varphi^\ast (kmN+\varepsilon)^{\gamma}, \quad \ell, N \in \mathbb{N}.
\]

Therefore, we can estimate, by Hölder’s inequality, the Fourier transform $\mathfrak{F}_{\xi}(\xi)$ for $|\xi|$ big enough in the following way (at the end, we use the fact that $\varphi^\ast(x)/x$ is an increasing function):

\[
D^\beta |\xi|^{d\beta} \left| \mathfrak{F}_{\xi}(\xi) \right| \leq C_{\varepsilon} \sum_{|\alpha| \leq (k+1)\alpha1} \frac{1}{|\alpha|!} \left| D^{\alpha} \chi \right|_{2, B_{2\varepsilon}(x_{0})} + \varepsilon^{(a-\gamma)} \left| D^{\alpha} \chi \right|_{2, B_{2\varepsilon}(x_{0})} \leq D_{\varepsilon, \{0\}} \mathfrak{F}_{\xi}(\xi) \left( 1/(k)^{\varphi^\ast} (kmN+\varepsilon) + e^{(1/k)^{\varphi^\ast} (kmN)} \right) \leq E_{\varepsilon, \{0\}} e^{(1/2)^{\varphi^\ast} (2kmN)}.
\]

On the other hand, if $|\xi|$ is bounded, we put $D_{\varepsilon} = \left\| \chi \right\|_{2, B_{2\varepsilon}(x_{0})}$ and, by Hölder’s inequality, we have

\[
\left| \mathfrak{F}_{\xi}(\xi) \right| \leq \left| \int_{\mathbb{R}^{n}} \chi(x) P(D)^{N}u(x) e^{-i(x, \xi)} \, dx \right| \leq D_{\varepsilon} \left\| P(D)^{N}u \right\|_{2, B_{2\varepsilon}(x_{0})} \leq C_{\varepsilon} e^{(1/k)^{\varphi^\ast} (2kmN)},
\]

(37)

which finishes this implication.

The Beurling case is similar.

Necessity (Beurling Case). Let $\{ f_{\xi} \}_{\xi \in \mathbb{R}^{n}} \subset \mathcal{F}(\Omega)$ satisfying (i) in some neighborhood $U$ of $x_{0}$ and (ii). We fix a compact set $K \subset U$ and take $M > (n+1)/2$. Now, by (ii), there is $k \in \mathbb{N}$ and a constant $C > 0$ that depends on $n$ and $P(D)$ such that, by Parseval’s formula,

\[
\left\| P(D)^{N}u \right\|_{L^{2}(K)} = \left\| f_{\xi} \right\|_{L^{2}(\mathbb{R}^{n})} \leq \left\| f_{\xi} \right\|_{L^{2}(\mathbb{R}^{n})} \frac{1}{(2\pi)^{n}} \left( \int_{\mathbb{R}^{n}} (1 + |\xi|)^{2M} \right)^{1/2} \leq e^{0} \left( 1 + |\xi| \right)^{2M} \left( \int_{\mathbb{R}^{n}} (1 + |\xi|)^{2M} \, dx \right)^{1/2}.
\]

(39)

In a similar way, using the Fourier transform, we can see that the distributions $D^{\alpha}u$ satisfy analogous estimates for each multi-index $\alpha$ on $K$. By the hypoellipticity of $P(D)$ we conclude that $u \in C^{\infty}(U)$, and this finishes the proof in the Roumieu case.

As above, in the Beurling case we can argue in a similar way.

\[\square\]

In the rest of the paper, it is assumed that the operator $P(D)$ is hypoelliptic, but not elliptic. Hypoellipticity is not only useful for Proposition 6, but also because it gives some good properties of the space $\mathcal{F}_{(\omega)}^{\mathcal{C}}(\Omega)$, such as completeness (cf. [14]). On the contrary, the elliptic case is not really interesting here since $\mathcal{F}_{(\omega)}^{\mathcal{C}}(\Omega) = \mathcal{F}_{(\omega)}(\Omega)$ if and only if $P$ is elliptic, as we have already mentioned at the end of Section 2.

Proposition 6 leads us to define the wave front set with respect to the iterates of an operator.

Definition 7. Let $\Omega$ be an open subset of $\mathbb{R}^{n}$, $u \in \mathcal{D}(\Omega)$, and $P(D)$ a linear partial differential hypoelliptic operator of order $m$ with constant coefficients. We say that a point $(x_{0}, \xi_{0}) \in \Omega \times (\mathbb{R}^{n} \setminus \{0\})$ is not in the $\{\omega\}$-wave front set with respect to the iterates of $P$, $WF_{\omega}(u)$ ($\{\omega\}$-wave front set with respect to the iterates of $P$, $WF_{\omega}(u)$), if there are a neighborhood $U$ of $x_{0}$, an open conic neighborhood $\Gamma$ of $\xi_{0}$, and a sequence $\{ f_{N} \}_{N \in \mathbb{N}} \subset \mathcal{F}(\Omega)$ such that (i) and (ii) of the following conditions hold ((i) and (iii) of the following conditions hold):

(i) For every $N \in \mathbb{N}$, $f_{N} = P(D)^{N}u$ in $U$.

(ii) Roumieu:

(a) there are constants $k \in \mathbb{N}, M > 0$, and $C > 0$, such that

\[
\left| \mathfrak{F}_{\xi}(\xi) \right| \leq C_{\varepsilon} e^{(1/k)^{\varphi^\ast} (Nm)} \left( 1 + |\xi| \right)^{M},
\]

(40)

(b) there is a constant $k \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}_{0}$, there is $C_{\varepsilon} > 0$ with the property

\[
\left| \mathfrak{F}_{\xi}(\xi) \right| \leq C_{\varepsilon} e^{(1/k)^{\varphi^\ast} (KnM)} \left( 1 + |\xi| \right)^{-\ell},
\]

(41)

(iii) Beurling:
(a) there are $M, C > 0$ such that for all $k \in \mathbb{N}$, there is $C_k > 0$ such that
\[
\left| \tilde{f}_N (\xi) \right| \leq C_k C^N \left( e^{(k/N)m} \phi^* (N^m/k) \right)^N (1 + |\xi|)^M, \quad N \in \mathbb{N}, \ \xi \in \mathbb{R}^n;
\]
(b) for all $\ell \in \mathbb{N}_0$ and $k \in \mathbb{N}$ there is $C_{k, \ell} > 0$ such that
\[
\left| \tilde{f}_N (\xi) \right| \leq C_{k, \ell} e^{k \rho^* (N^m/k)} (1 + |\xi|)^\ell, \quad N \in \mathbb{N}, \ \xi \in \Gamma.
\]

If we compare the last definition with Definition 5 we can deduce, as Proposition 9 will show, that the new wave front set gives more precise information about the propagation of singularities of a distribution than the $\ast$-wave front set of a classical distribution ($\ast = \{a\}$ or $\{a\}$). We first recall the following result that we state as a lemma (see [19, Proposition 1.8]).

**Lemma 8.** Let $\Omega$ be an open subset of $\mathbb{R}^n$, $u \in D' (\Omega)$, and $P(D)$ a linear partial differential operator with analytic coefficients in $\Omega$ of order $m$. Let $\chi_N \in D (\Omega)$ such that
\[
|D^\alpha \chi_N| \leq C (CN)^{|\alpha|}, \quad |\alpha| \leq N,
\]
where $C > 0$ does not depend on $N = 0, 1, 2, \ldots$ Then the sequence $f_N = \chi_{pmn^2} P(D)^N u$, for $p \in \mathbb{N}$ large enough independent of $N$ satisfies
\[
\left| \tilde{f}_N (\xi) \right| \leq C^N (mN + |\xi|)^{mN} (1 + |\xi|)^M, \quad \xi \in \mathbb{R}^n, \quad N = 0, 1, 2, \ldots,
\]
for some constants $C > 0$ and $M > 0$.

**Proposition 9.** Let $\Omega$ be an open subset of $\mathbb{R}^n$, $u \in D' (\Omega)$, $\omega$ a weight function, and $P(D)$ a hypoelliptic linear partial differential operator of order $m$ with constant coefficients. Then, the following inclusions hold:
\[
WF^p (u) \subset WF (u), \quad WF^p (\omega u) \subset WF (\omega u).
\]

**Proof.**

**Roumieu Case.** Let $x_0, \xi_0 \in \mho$. From Definition 5, there exist a neighborhood $U$ of $x_0$, an open conic neighborhood $F$ of $\xi_0$, and a bounded sequence $u_N \in \Phi (\Omega)$ such that $u_N = u$ in $U$ for every $N \in \mathbb{N}$ and for some constants $C > 0, k \in \mathbb{N}$
\[
|\xi_0|^N \left| \tilde{u}_N (\xi) \right| \leq C e^{(1/k) \rho^* (kN)}, \quad \xi \in F, \quad N \in \mathbb{N}.
\]

By (18, Lemma 2.2), we can find a sequence $\chi_N \in D(U)$ such that $\chi_N = 1$ in a neighborhood $V$ of $x_0$ and
\[
|D^\alpha \chi_N | \leq C_\alpha (C_\alpha N)^{|\beta|}, \quad \beta \in \mathbb{N}_0^{n}, \quad |\beta| \leq N.
\]

We select $p \in \mathbb{N}$ as in Lemma 8 (or bigger if necessary) and set $f_N = \chi_{pmn^2} P(D)^N u$. We first observe that, as $u = u_N$ in $U$ for all $N \in \mathbb{N}$ and $\chi_N \in D(U)$, we have $f_N = \chi_{pmn^2} P(D)^N u$, for all $s \in \mathbb{N}$. We want to prove (i), (ii)(a), and (ii)(b) in Definition 7. By the choice of $\chi_N$, condition (i) is fulfilled in the neighborhood $V$. To see (ii)(a), we observe that from Lemma 8 there is $\bar{C} > 0$ such that
\[
\left| \tilde{f}_N (\xi) \right| \leq C^N (mN + |\xi|)^{mN} (1 + |\xi|)^M, \quad \xi \in \mathbb{R}^n, \quad N = 0, 1, 2, \ldots,
\]
for some constant $M > 0$. Since the weight function $\omega(t) = o(t)$ as $t$ tends to infinity, from [22, Remark 2.4(b)], for every $k \in \mathbb{N}$ there is $C_k > 0$ such that
\[
N^m \leq C_k^{(1/Nm) (k/Nm)^{mN}/k}, \quad N \in \mathbb{N}.
\]
In particular, for $k = 1$, we obtain
\[
\left| \tilde{f}_N (\xi) \right| \leq C_1 C^N \left( e^{(1/Nm) \rho^* (mN)} + |\xi| \right)^{mN} (1 + |\xi|)^M, \quad \xi \in \mathbb{R}^n, \quad N = 0, 1, 2, \ldots,
\]
which proves (ii)(a).

We prove now (ii)(b). We fix $\ell \in \mathbb{N}$ and set, for $f_N = \chi_{pmn^2} P(D)^N u_{N^m \ell}$,
\[
(1 + |\xi|)^\ell \left| \tilde{f}_N (\xi) \right| \leq (1 + |\xi|)^\ell \int \left| \tilde{\chi}_{pmn^2} P (\xi - \eta) \right|^N \times |\hat{u}_{N^m \ell} (\xi - \eta)| \, d\eta
\]
\[
= : J_1 (\xi) + J_2 (\xi),
\]
where $J_1 (\xi)$ is the integral when $|\eta| \leq c |\xi|$, for $c > 0$ to be chosen, and $J_2 (\xi)$ is the integral when $|\eta| \geq c |\xi|$, both considered with the factor $(1 + |\xi|)^\ell$. In $J_2 (\xi)$, we have
\[
|\xi - \eta| \leq |\xi| + |\eta| \leq (1 + c^{-1}) |\eta|.
\]
Since $u_{N^m \ell}$ is a bounded sequence in $\Phi (\Omega)$, there is $M > 0$ such that $|\hat{u}_{N^m \ell} (\xi)| \leq C_2 \xi^M$ for all $\xi \in \mathbb{R}^n$ and $N \in \mathbb{N}$.

From (48), we can differentiate $\chi_{pmn^2}$ up to the order $N^m$ to obtain constants $C_2 > 0, C_2$ that depend on $n, \ell, M$ such that (see [22, Lemma 3.5])
\[
\left| \tilde{\chi}_{pmn^2} P (\xi) \right| \leq C_2 C_2^{N^{m+1}} e^{(1/k) \rho^* (kN^m)} \times \left( |\eta| + e^{(1/Nm) \rho^* (N^m)} \right)^{N^m (1 + |\eta|)^{N^m - 1 - M^\ell}} \eta \in \mathbb{R}^n.
\]

As $P(D)$ has order $m$, we also have $|P (\xi)|^N \leq C (1 + |\xi|)^{N^m}$ for some constant $C > 0$ and each $\xi \in \mathbb{R}^n$ and $N \in \mathbb{N}$.
Moreover, in $J_2(\xi), (1 + |\xi|)^\ell \leq (1 + c^{-1})^\ell (1 + |\eta|)^\ell$ and
\[(1 + |\xi - \eta|)^{Nm+M} \leq (1 + c^{-1})^{Nm+M} (1 + |\eta|)^{Nm+M}. \tag{55}\]
Therefore, from (54), we obtain
\[
|J_2(\xi)| \\
\leq DC_\ell (1 + c^{-1})^{M+Nm+\ell \epsilon} \\
\times \int_{|\eta|\leq |\xi|} (1 + |\eta|)^{Nm+\ell} (1 + |\eta|)^{M} |\bar{X}_{Nmp}(\eta)| \, d\eta \\
\leq D' C_\ell C_2^{Nm+1} (1 + c^{-1})^{M+Nm+\ell \epsilon} e^{(1/k)p^*(Nm/k)}\tag{56}
\]
for some $D', D' > 0$.

On the other hand, if we consider the estimate $(1 + |\xi|)^\ell \leq (1 + |\xi - \eta|)^\ell (1 + |\eta|)^\ell$, we obtain
\[
|J_1(\xi)| \leq \left( \int (1 + |\eta|)^{Nm} |\bar{X}_{Nmp}(\eta)| \, d\eta \right) \\
\times \sup_{|\eta|\leq |\xi|} |\bar{u}_{Nm+\ell}(\xi - \eta)| \\
\cdot (1 + |\xi - \eta|)^\ell \cdot |P(\xi - \eta)|^N. \tag{57}
\]

We observe that the integral is less than or equal to $C_\ell A^N$ for some constant $C_\ell > 0$ that depends on $\ell$ and the support of $\bar{X}_{Nmp}$ and some constant $A > 0$. Now, we write $\xi = \xi - \eta$. If $\Gamma$ is a conic neighborhood of $\xi_0$ such that $\Gamma \subset F$, we can select $0 < c < 1$ such that for $\xi \in \Gamma$ and $|\xi - \xi_0| \leq c|\xi_0|$, we have $\xi \in F$. Consequently, we obtain, by assumption on $\bar{u}_{Nm+\ell}$ (see (47)), and by the estimate $|P(\xi)|^N \leq C^{-N}(1 + |\xi|)^{-Nm}$ for some positive constant $C$, for $\xi \in \Gamma$,
\[
|J_1(\xi)| \leq C_\ell A^N \cdot \sup_{|\xi - \xi_0| \leq |\xi|} |\bar{u}_{Nm+\ell}(\xi)| \\
\cdot (1 + |\xi|)^\ell \cdot |P(\xi)|^N \\
\leq C_\ell C^{Nm+1 + (1/k)p^*(Nm/k)} \cdot (1 + |\xi|)^\ell \cdot |P(\xi)|^N \tag{58}
\]
for some $C > 0$. We conclude, using the convexity of $\varphi^*$, that there are constants $D_\ell > 0$ and $E > 0$ such that
\[
(1 + |\xi|)^\ell |\tilde{f}_N(\xi)| \leq |J_1(\xi)| + |J_2(\xi)| \\
\leq D_\ell E^{Nm+1} e^{(1/k) p^*(2kNm)}, \quad \xi \in \Gamma. \tag{59}
\]

Beurling Case. Let us assume now that $(x_0, \xi_0) \notin WF_{\phi(u)}$. From Definition 5, there exist a neighborhood $U$ of $x_0$, an open conic neighborhood $F$ of $\xi_0$, and a bounded sequence $\{u_N\}_{N \in \mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that $u_N = u$ in $U$ for every $N \in \mathbb{N}$ and for every $k \in \mathbb{N}$ there is $C_k > 0$, such that
\[
|\xi|^N |u_N(\xi)| \leq C_k e^{p^*(Nm/k)}, \quad \xi \in F, \ N \in \mathbb{N}. \tag{60}
\]
We take $\chi_N$ and $f_N$ as in the Roumieu case. From (50), for any $k \in \mathbb{N}$, there is $D_k > 0$ satisfying
\[
|\tilde{f}_N(\xi)| \leq D_k C_N e^{(1/k)p^*(Nm/k)} (1 + |\xi|)^{Nm}(1 + |\eta|)^M, \quad \xi \in \mathbb{R}^n, \quad N = 0, 1, 2, \ldots, \tag{61}
\]
which proves (iii)(a).

To prove (iii)(b), fix $\ell \in \mathbb{N}$ and consider now the estimate (use (48) and (50))
\[
|\tilde{X}_{Nmp}(\eta)| \leq C_k C_e^{Nm+1} \left| |\eta| + e^{(k/Nm)p^*(Nm/k)} + Nm \right| (1 + |\eta|)^{Nm} \times (1 + |\eta|)^{-n-1-M-\ell}, \quad \eta \in \mathbb{R}^n. \tag{62}
\]
Here,
\[
(1 + |\xi|)^\ell |\tilde{f}_N(\xi)| \leq (1 + |\xi|)^\ell \\
\times \left| \int |\tilde{X}_{Nmp}(\eta)| |P(\xi - \eta)|^N \right| \\
\times |\tilde{u}_{Nm+\ell}(\xi - \eta)| \, d\eta \\
=: J_1(\xi) + J_2(\xi), \tag{63}
\]
where $J_1(\xi)$ is the integral when $|\eta| \leq c|\xi|$, for $c > 0$ to be chosen, and $J_2(\xi)$ is the integral when $|\eta| \geq c|\xi|$. In this case, we use (60) and obtain a constant $C' \geq 0$ which depends on $\ell$ (and $M, n$) and a constant $E > 0$ with the property that for every $k \in \mathbb{N}$ there is a constant $C_k > 0$ such that for any $\xi \in \Gamma$ and $N \in \mathbb{N}$,
\[
(1 + |\xi|^N |\tilde{f}_N(\xi)| \leq C_k e^{Nm+1} C_k e^{p^*(Nm/k)}, \quad \xi \in \Gamma, \ N \in \mathbb{N}. \tag{64}
\]
This concludes the Beurling case.

\[\square\]

Corollary 10. Let $u \in \mathcal{D}'(\Omega)$, and let $K$ be a compact subset of $\Omega$ and $F$ a closed cone in $\mathbb{R}^n$. Let $\omega$ be a weight function. Suppose that $\{\chi_N\} \subset \mathcal{D}(K)$ is like in (48). Then, we have the following:

(a) If $WF_{\phi(u)}(K \times F) = \emptyset$, then the sequence $g_N = \chi_{Nmp} P(D)^N u$, for $p \in \mathbb{N}$ large enough independent of $N$, satisfies that there is $k \in \mathbb{N}$ such that for every $\ell \in \mathbb{N}$, there is $C_\ell > 0$ with
\[
|\tilde{g}_N(\xi)| \leq C_\ell e^{(1/k)p^*(kNm)} (1 + |\xi|)^{-\ell}, \quad \xi \in F, \ N \in \mathbb{N}. \tag{65}
\]

(b) If $WF_{\phi(u)}(K \times F) = \emptyset$, then the sequence $g_N = \chi_{Nmp} P(D)^N u$, for $p \in \mathbb{N}$ large enough independent of $N$, satisfies that for every $k, \ell \in \mathbb{N}$ there is $C_{k, \ell} > 0$ with
\[
|\tilde{g}_N(\xi)| \leq C_{k, \ell} e^{k^* p^*(Nm/k)} (1 + |\xi|)^{-\ell}, \quad \xi \in F, \ N \in \mathbb{N}. \tag{66}
\]
Proof. We make a sketch of proof of (a) only. Let $x_0 \in K, \xi_0 \in F \setminus \{0\}$ and choose $U$ and $\Gamma$, with $\Gamma$ a conic subset of $F$ and $f_N$ according to Definition 7. If the support of $X_N$ is in $U$, we have $X_N i P(D)^N u = X_N i f_N$. Now, the proof is like (ii)(b) of Proposition 9 for the set $\Gamma$ and $f_N$ instead of $P(D)^N u_{\text{norm}}$. To obtain a uniform estimate in $F$, we can proceed as in [22, Lemma 3.5] at the end of the proof of (a). See also the proof of [25, Theorem 8.4.4].

The singular support of a classical distribution $u \in \mathcal{D}'(\Omega)$ with respect to the class $\mathcal{E}_s((\Omega)$ is the complement of the biggest open set $U$, where $u|_U \in \mathcal{E}_s(U)$. As a consequence of Propositions 6 and 9 and Corollary 10, we obtain the following result.

**Corollary 11.** The projection in $\Omega$ of $WF^p(u)$ is the singular support with respect to the class $\mathcal{E}_s((\Omega)$ if $u \in \mathcal{D}'(\Omega)$.

**Proof.** Follow the lines of the proofs of [22, Theorem 3.6] and [25, Theorem 8.4.5].

**Remark 12.** We observe that from the definition it is obvious that if $P$ is hypoelliptic, then for $*= (\omega)$ or $\{\omega\}$

$$WF^p(u) = WF^p(P u) .$$

Then, by Proposition 9, the following inclusions hold:

$$WF^p(u) \subset WF^p(P u) \subset WF^p(u) \subset WF^p(u) .$$

Now, we can state an improvement of [22, Theorem 4.8] for operators with constant coefficients.

**Theorem 13.** Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, a_\alpha \in C$, be a hypoelliptic linear partial differential operator with constant coefficients and order $m$ and let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $P_m$ be the principal part of $P$ and $\Sigma = \{ (x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\} : P_m(\xi) = 0 \}$ the characteristic set of $P(D)$. Then, for any distribution $u \in \mathcal{D}'(\Omega)$

$$WF^p(u) \subset WF^p(u) \cup \Sigma .$$

**Proof.** Let $x_0, \xi_0 \notin WF^p(u)$ such that $P_m(\xi_0) \neq 0$. Then, there are a neighborhood $U$ of $x_0$, a conic neighborhood $\Gamma$ of $\xi_0$, and a sequence $\{f_N\}_{N \in \mathbb{N}} \subset \mathcal{D}'(\Omega)$ that verify (i), (ii)(a)-(ii)(b) in the Roumieu case, and (ii)(a)-(iii)(b) in the Beurling case of Definition 7. We take $F \subset \Gamma$ such that $P_m(\xi) \neq 0$ for $\xi \in F$. We take a compact neighborhood $K \subset U$ of $x_0$ and consider a sequence $\{X_N\}_{N \in \mathbb{N}} \subset \mathcal{D}(U)$ satisfying (48) such that $X_N \equiv 1$ on $K$.

We set now $u_N = X_m i u$. We want to estimate

$$\tilde{u}_N(\xi) = \langle u_N, X_m i u \rangle = \int u(x) X_m i u(x) e^{-i(\xi, x)} dx .$$

To estimate $|\tilde{u}_N(\xi)|$ in $F$, we will solve in an approximate way the following equation:

$$i P(D)^N v(x) = X_m i u(x) e^{-i(\xi, x)} .$$

As in [17], we put $v(x) = e^{-i(\xi, x)} u(\xi)/P_m(\xi)^N$. For $(x, \xi) \in K \times F$, we have

$$P(D)^N u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} a_\alpha D^\alpha (e^{-i(\xi, \xi)} P_m(\xi)^N u) .$$

Therefore, we want to give an approximate solution of

$$e^{-i(\xi, x)} (I - R)^N u = \mathcal{X}_{3m}^m (x) e^{-i(\xi, x)} .$$

A formal solution of (74) is given by the series:

$$u = (I - R)^{-N} \mathcal{X}_{3m}^m = \sum_{j=0}^N (-N)_j (-1)^j R^j \mathcal{X}_{3m}^m .$$

For

$$u_N := \sum_{j=0}^N (-N)_j (-1)^j R^j \mathcal{X}_{3m}^m ,$$

we can write

$$(I - R)^N u_N = \sum_{j=0}^N (-N)_j (-1)^j R^j \mathcal{X}_{3m}^m .$$

We observe that the coefficient of $R^{h+j} \mathcal{X}_{3m}^m$ is $R^h \mathcal{X}_{3m}^m$ with $h + j = k \leq mN$ is given by

$$(-1)^k \sum_{h=0}^k \binom{N}{h} \binom{N}{k-h} = 0, \quad k \geq 1,$n

by the Chu-Vandermonde identity. For $k \geq mN + 1$, the term $R^k$ does not appear anymore for $h = 0$. So, we do not have all the summands needed in the identity above and hence
the coefficients of \( R^k \) are not zero. Therefore, (we write \( \chi \) for \( \chi_{3m^2N} \) for simplicity)

\[
(I - R)^N w_N = \chi + \sum_{h=1}^N \sum_{j=mN+1-h}^{mN} \binom{N}{j} (-1)^{h+j} R^{h+j} \chi
\]

for

\[
e_N := \sum_{h=1}^N \sum_{j=mN+1-h}^{mN} \binom{N}{j} (-1)^{h+j} R^{h+j} \chi.
\]

Then,

\[
\sum_P (I - R)^N (e^{-i(\xi, \eta)} P_m^{-N} w_N) = e^{-i(\xi, \eta)} (I - R)^N w_N
\]

If we apply these identities to \( u \), we obtain

\[
\tilde{u}_N(\xi) = \int e^{-i(\xi, \eta)} \chi_{3m^2N} u(x) dx
\]

where the integrals denote action of distributions.

We suppose now that \( u \) has order \( M > 0 \) in a neighborhood of \( K \). Since \( H_1(\xi) = \langle u, e_N e^{-i(\xi, \eta)} \rangle \), we have

\[
|H_1(\xi)| \leq C \sum_{|\beta| \leq M} |D_\beta (e_N (x, \xi) e^{-i(\xi, \eta)})|
\]

We study now

\[
H_2(\xi) = \int e^{-i(\xi, \eta)} P_m^{-N} (x, \xi) P(D)^N u(x) dx
\]

where we have split \( H_2(\xi) \) in the sum of \( S_1(\xi) \) and \( S_2(\xi) \), the first when \( |\eta| \leq c|\xi| \) and the second when \( |\eta| \geq c|\xi| \), for a constant \( c > 0 \) to be chosen.
First, we estimate \( \omega_N \) defined in formula (76). Proceeding in a similar way as before with the expression of \( e_{N} \), if we take \( |\xi| > mN \) and \( |\beta| \leq 2m^2 N \) and estimate the binomials as in (85), we find a constant \( A > 0 \) such that

\[
|D_\alpha^j \omega_N| \leq \sum_{j=0}^{\infty} \left( \frac{|\xi|}{j+1} \right)^j (1 + |\xi|)^{N_M} \cdot \sum_{j=0}^{\infty} \left( \frac{|\xi|}{j+1} \right)^j j! (3m^2 N)^j
\]

for some constants \( D, B > 0 \).

For \( S_1(\xi) \) we have

\[
|S_1(\xi)| \leq |P_\alpha(\xi)|^{-N} \cdot \sup_{|\eta| \leq \xi} \left| \frac{\lambda_{\xi-N}}{\eta} \right|.
\]

As in the proof of Proposition 9, we can estimate \( S_1(\xi) \), the Roumieux case, with the use of (ii)(b) of Definition 7 in the following way: we select \( c > 0 \) for which there are \( C > 0 \) and \( k \in \mathbb{N} \) such that for \( \xi \) in some neighborhood \( \Gamma^0 \) of \( \xi_0 \), we have, as in (58),

\[
\sup_{|\eta| \leq \xi} \left| \frac{\lambda_{\xi-N}}{\eta} \right| = \frac{C^{N+1} \| e^{(1/k)^k} (Nkk) \|}{(1 + |\xi|)^{N_M}} \cdot \sum_{j=0}^{\infty} \left( \frac{|\xi|}{j+1} \right)^j j! (3m^2 N)^j.
\]

Consequently, since \( \| \lambda_{\xi-N} \| \leq A^N \), we have

\[
|S_1(\xi)| \leq D^{N+1} \| e^{(1/k)^k} (Nkk) \| \| \lambda_{\xi-N} \|^{-N} \cdot \sum_{j=0}^{\infty} \left( \frac{|\xi|}{j+1} \right)^j j! (3m^2 N)^j.
\]

Therefore, by (90), we separate Ulrich cases.

For Roumieux Case. From Definition 7(ii)(a), we have

\[
|\tilde{f}_N(\xi)| \leq C^N \left( e^{(1/N\eta)^k} + |\xi| \right)^{N_M} (1 + |\xi|)^M, \quad N \in \mathbb{N}, \quad \xi \in \mathbb{R}^n,
\]

for some constants \( C > 0 \), \( M > 0 \), and \( k \in \mathbb{N} \). Now, as \( \omega_N \in \mathcal{B}(U) \), by (90), we have, as in [22, Lemma 3.5],

\[
|\tilde{w}_N(\eta)| \leq C^{N+1} \left( e^{(1/k)^k} (Nkk) \right) \left( e^{(1/N\eta)^k} + |\eta| \right)^{N_M} (1 + |\eta|)^{-N-M}, \quad \eta \in \mathbb{R}^n.
\]

We proceed now as in the proof of (ii)(b) of Proposition 9 in order to estimate \( H_2(\xi) = S_1(\xi) + S_2(\xi) \). In \( S_2(\xi) \), we have \( |\xi - \eta| \leq (1 + c^{-1}) |\eta| \) and, by (92), we deduce

\[
|S_2(\xi)| \leq (2\pi)^{-N} |P_\alpha(\xi)|^{-N} \times \int_{|\eta| \leq \xi} |\tilde{w}_N(\eta)| \tilde{f}_N(\xi - \eta) \, d\eta
\]

\[
\leq D^{N+1} \| e^{(1/k)^k} (Nkk) \| \left( e^{(1/N\eta)^k} + |\eta| \right)^{N_M} \times \int_{|\eta| \leq \xi} (1 + |\eta|)^M \, d\eta
\]

\[
\leq B^N \left( e^{(1/k)^k} (Nkk) \right)^{|\xi|^{-N_M}},
\]

(93)
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**Beurling Case.** In this setting we will proceed in a similar way. We can select \( 0 < c < 1 \) and apply now (iii)(b) of Definition 7 to obtain, for every \( k \in \mathbb{N} \), a constant \( C_k > 0 \) such that, for all \( \xi \) in some neighborhood of \( \xi_0 \),

\[
|S_1(\xi)| \leq \left| P_n(\xi) \right| \eta^{-N} \left| \left| w_N \right| \right|_{L^1} \cdot \sup_{|\eta| \leq \xi} \left| \left| \mathcal{F}_N(\xi - \eta) \right| \right| \\
\leq C_k \epsilon^{N} \eta^m (N/k) \left| \xi \right|^{M-Nm}.
\]  
(101)

In a similar way to (92), we can obtain here

\[
|\mathcal{F}_N(\eta)| \leq C_k \epsilon^{N} \eta^m (N/k) \left| \xi \right|^{-Nm}, \quad N \in \mathbb{N}, \quad \left| \xi \right| > N.
\]  
(102)

Therefore, from (101) and (103), we have \( C > 0 \) and for a fixed \( k \in \mathbb{N} \) a constant \( C_k > 0 \) such that for \( \xi \) in some conic neighborhood of \( \xi_0 \) and \( \left| \xi \right| \geq e^{(k/Nm-1)\eta^m} (Nm-1/k) \),

\[
|H_1(\xi)| \leq C_k \epsilon^{N} \eta^m (N/k) \left| \xi \right|^{M-Nm}
\]  
(104)

As in the Roumieu case, we deduce a similar estimate for \( |H_1(\xi)| \). Then, the bounds for \( H_1(\xi) \) and \( H_2(\xi) \) given a constant \( C > 0 \) and, for every \( k \in \mathbb{N} \), a constant \( C_k > 0 \) such that for \( \xi \) in some conic neighborhood of \( \xi_0 \) and \( \left| \xi \right| \geq e^{(k/Nm-1)\eta^m} (Nm-1/k) \)

\[
\left| \tilde{u}_N(\xi) \right| \leq C_k \epsilon^{N} \eta^m \left( N/k \right) \left| \xi \right|^{M-Nm}.
\]  
(105)

Finally, we also have a similar estimate when \( \left| \xi \right| \geq e^{(k/Nm-1)\eta^m} (Nm-1/k) \), which concludes the proof of the theorem.

**Remark 14.** If \( P(D) \) is elliptic, then \( \Sigma = \emptyset \) and Theorem 13 and Remark 12 imply that

\[
WF_\ast(u) = WF_{P_\ast}(u).
\]  
(106)

**Example 15.** We show that the inclusions

\[
WF_{P_\ast}(u) \subseteq WF_\ast(u),
\]  
(107)

\[
WF_{P_\ast}(u) \subseteq WF_\ast(Pu)
\]  

of Remark 12 are strict. As in [14] (see [26]), we consider a nonquasianalytic weight function \( \omega \) satisfying the following condition: there exists a constant \( H \geq 1 \) such that for all \( t \geq 0 \),

\[
2\omega(t) \leq \omega(\sqrt{t}) + H.
\]  
(108)

For example, if \( \omega \) is a Gevrey weight, then it satisfies such a property. We consider now a polynomial \( P \) with constant complex coefficients such that it is hypoelliptic but not elliptic (for instance, the heat operator). Then by [14, Theorem 4.12], there is \( u \in \mathcal{E}_P(\omega)(\Omega) \) for some open subset \( \Omega \) of \( \mathbb{R}^n \).

Then, \( WF_{P_\ast}(u) = \emptyset \) but \( WF_\ast(u) \neq \emptyset \), which implies that the inclusion

\[
WF_{P_\ast}(u) \subseteq WF_\ast(u)
\]  
(109)

is strict.

On the other hand, if we consider now a \( \{\omega\} \) -hypoelliptic polynomial \( P \) which is not elliptic (e.g., the heat operator in \( \mathbb{R}^n \) for \( \omega(t) = t^{1/2} \)), then as before there will be \( u \in \mathcal{E}_P(\omega)(\Omega) \) \( \setminus \mathcal{E}_P^Q(\omega)(\Omega) \). In particular, \( WF_{P_\ast}(u) = \emptyset \). Now, if \( WF_\ast(Pu) = \emptyset \), we will have \( Pu \in \mathcal{E}_P(\omega)(\Omega) \) and hence \( P \) is \( \{\omega\} \) -hypoelliptic, \( u \in \mathcal{E}_P(\omega)(\Omega) \), which is a contradiction. Therefore, \( WF_\ast(Pu) \neq \emptyset \) and we conclude that the inclusion

\[
WF_{P_\ast}(u) \subseteq WF_\ast(Pu)
\]  
(110)

is strict.

Let us also remark that for the heat operator \( Q(D) = \partial_t - \Delta_x \), we can explicitly write its characteristic set \( \Sigma \), so that the previous considerations give, for \( u \in \mathcal{E}_P^Q(\omega)(\Omega) \setminus \mathcal{E}_P(\omega)(\Omega) \), the following information on \( WF_\ast(u) \), because of Theorem 13:

\[
\emptyset \neq WF_\ast(u) \subseteq WF_{P_\ast}(u) \cup \Sigma
\]  
(111)

In the Beurling setting, we can proceed in a similar way. Let us finally notice that the inclusion

\[
WF_\ast(Pu) \subseteq WF_{P_\ast}(u)
\]  
(112)

of Remark 12 is strict in general.

**4. Distributions with Prescribed Wave Front Set**

The proof of the following lemma is straightforward.

**Lemma 16.** Let \( \omega \) be a weight function. Then, for every \( a > 0 \) and \( m \in \mathbb{N} \)

\[
(i) \ t^m e^{-\omega(t)} \leq \epsilon^{\omega^m(m/a)} \forall t \geq 1;
\]

\[
(ii) \ inf_{t \in [0,1]} t^m e^{\omega^m(jm/a)} \leq t^m e^{-\omega(t)} \forall t \geq 1.
\]

Now, we show that the product of a Gevrey function with a function in \( \mathcal{E}_P^Q(\omega)(\Omega) \) belongs to the last space.

**Proposition 17.** Let \( \omega \) be a nonquasianalytic weight function such that \( \omega(t^\gamma) = o(\sigma(t)) \) as \( t \to \infty \), where \( \gamma > 0 \) is the constant in (28) and \( \sigma(t) = t^{1/3} \) is a Gevrey weight, with \( s > 1 \). If \( \chi \in \mathcal{E}_P^Q(\omega)(\Omega) \) and \( u \in \mathcal{E}_P^Q(\omega)(\Omega) \), where \( \ast = \{\omega\} \) or \( \omega \), then the multiplication \( \chi \ u \in \mathcal{E}_P^Q(\omega)(\Omega) \).
Proof. We will analyse the $L^2$-norms of $P(D)^j(\chi u)$ on a compact set $K$ in $\Omega$. First, we observe that, by the generalized Leibniz rule over $P(D)$ applied $j$ times,

$$P(D)^j(\chi u) = P(D) \left[ \prod_{|\alpha| \leq m} \frac{1}{\alpha!} \partial^\alpha \chi \cdot \prod_{|\alpha| \leq m} p^{(\alpha)}(D) \left( p^{(\alpha_1)}(D) \cdots p^{(\alpha_j)}(D) u \right) \right]$$

(113)

We fix now a compact set $K$ in $\Omega$ such that $\text{dist}(K, \partial \Omega) \geq r > 0$. We apply $L^2$-norms in the compact set $K$

$$\left\| P(D)^j(\chi u) \right\|_{2,K} \leq \sum_{|\alpha| \leq m} \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \cdot \prod_{|\alpha| \leq m} p^{(\alpha)}(D) \left( p^{(\alpha_1)}(D) \cdots p^{(\alpha_j)}(D) u \right) \right\|_{2,K}.$$  

(114)

Since $\chi \in \mathcal{D}(\alpha)(\Omega)$, there is a constant $A > 0$ such that, for each $\alpha \in \mathbb{N}_0^d$ and $x \in K$ we have

$$|D^\alpha \chi(x)| \leq A^{|\alpha|} |\alpha|^{|\alpha|}.$$  

(115)

Consequently,

$$\sup_{x \in K} |D^{\alpha_1} \cdots D^{\alpha_j} \chi(x)| \leq A^{k_{\alpha_1} + \cdots + k_{\alpha_j}} |\alpha_1 + \cdots + \alpha_j|^{|\alpha_1| + \cdots + |\alpha_j|} \leq A^{\frac{j}{j-1}}.$$  

(116)

Therefore,

$$\left\| P(D)^j(\chi u) \right\|_{2,K} \leq \sum_{|\alpha| \leq m} \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \cdot \prod_{|\alpha| \leq m} p^{(\alpha)}(D) \left( p^{(\alpha_1)}(D) \cdots p^{(\alpha_j)}(D) u \right) \right\|_{2,K}$$

(117)

$$\leq \sum_{|\alpha| \leq m} A^\frac{j}{j-1} \prod_{|\alpha| \leq m} p^{(\alpha)}(D) \left( p^{(\alpha_1)}(D) \cdots p^{(\alpha_j)}(D) u \right) \right\|_{2,K}.$$  

Now, we apply (28) $j$ times to the factor $\| p^{(\alpha_1)} \cdots p^{(\alpha_j)} u \|_{2,K}$. We will use the notation $K(\varepsilon) = K + B(0, \varepsilon)$, for $\varepsilon > 0$. In the first step,

$$\left\| p^{(\alpha_1)} \cdots p^{(\alpha_j)} u \right\|_{2,K} \leq C \left( \varepsilon_{\alpha_1} + \varepsilon_{\alpha_2} \right) \left( \sum_{|\alpha| \leq m} \varepsilon_{\alpha_1} \cdots \varepsilon_{\alpha_j} \right) \left\| p^{(\alpha_1)} \cdots p^{(\alpha_j)} u \right\|_{2,K(\varepsilon_1)}$$

(118)

In the second step, $K(\varepsilon_1)$ is replaced by $K(\varepsilon_1 + \varepsilon_2)$ and so on in the next steps. Therefore, to avoid that, after $j$ steps, the set $K(\varepsilon_1 + \cdots + \varepsilon_j)$ leaves $\Omega$ and to keep it bounded for all $j$, we may take $\varepsilon_k$ depending on $k$ for all $1 \leq k \leq j$. We take

$$\varepsilon_k = B k^{-\gamma}$$

(119)

for all $j$. It is obvious that $\varepsilon_k^{-\gamma} \leq \varepsilon_{k+1}^{-\gamma}$ for all $1 \leq k \leq j - 1$. Moreover, we can assume that $\varepsilon_k < 1$ for all $1 \leq k \leq j$.

After $j$ steps we get

$$\| p^{(\alpha_1)} \cdots p^{(\alpha_j)} u \|_{2,K} \leq C \left( \sum_{|\alpha| \leq m} \varepsilon_{\alpha_1} \cdots \varepsilon_{\alpha_j} \right) \left\| p^{(\alpha_1)} \cdots p^{(\alpha_j)} u \right\|_{2,K(\varepsilon_1)}$$

(120)

With our selection of $\varepsilon_k$ for $1 \leq k \leq j$, we have

$$\varepsilon_{\alpha_1} \cdots \varepsilon_{\alpha_j} \leq \frac{B (\alpha_1 + \cdots + \alpha_j)}{2 (\alpha_1 + \cdots + \alpha_j)}$$

(121)

for all $\alpha_1, \ldots, \alpha_j$. Moreover, for all $j$, $K(\varepsilon_1 + \cdots + \varepsilon_j) \subset K(r/2)$, which is compact and a subset of $\Omega$. Consequently, since $j^j \leq \varepsilon_{\alpha_1} \cdots \varepsilon_{\alpha_j}$ for all $j = 1, 2, \ldots$, we have (we can assume that the constant $B < 1$ and then $B (\alpha_1 + \cdots + \alpha_j) < 1$ for all $1 \leq k \leq j$)

$$\varepsilon_{\alpha_1} \cdots \varepsilon_{\alpha_j} \leq \frac{(j^j)^j}{B (j^j)^j} \leq \frac{e^{mj}}{(j^j)^m}.$$  

(122)

Summing up, we obtain

$$\left\| p^{(\alpha_1)} \cdots p^{(\alpha_j)} u \right\|_{2,K} \leq \left( \sum_{|\alpha| \leq m} \frac{(j^j)^j}{B (j^j)^j} \right) \frac{\alpha_1 \cdots \alpha_j}{(j^j)^m}.$$  

(123)
If we use the multinomial theorem,
\[
\sum_{|\alpha|, |\beta| \leq m} \frac{m!}{|\alpha|! |\beta|!} (2C/B') ^{m-|\alpha|} A^m_j \leq \sum_{|\alpha|, |\beta| \leq m} \frac{m!}{|\alpha|!} \leq e^{mn},
\]
where \( n \) is the dimension of the multi-index \(|\alpha|\) or \(|\beta|\). Then, it is clear that
\[
\sum_{|\alpha|, |\beta| \leq m} \frac{(2C/B') ^{m-|\alpha|} A^m_j}{|\alpha|! |\beta|!} \leq E^j
\]
for some constant \( E > 0 \) that depends on \( P(D) \), \( \chi \), and the compact set \( K(r/2) \).

Now, we control the sequence \( (j(j-1) \cdots (j-k+1))^\nu \) for \( k = 1, \ldots, j \), which is the factor of \( \|P^{j-k}u\|_{2,K(r/2)} \) and less than or equal to
\[
\binom{j}{k}^\nu k!^\nu \leq 2^{j\nu} k!^\nu.
\]

For \( * = \{0\} \), since \( \omega(t^J) = o(t^{1/j}) \) as \( t \to +\infty \), there is a constant \( F > 0 \) such that
\[
(k!)^\nu \leq Fe^{\nu (k)}, \quad k \in \mathbb{N}.
\]

Since \( \varphi^* (x) / x \to \infty \) as \( t \to \infty \), for any constant \( h \in \mathbb{N} \),
\[
(k!)^\nu \leq Fe^{(1/h)^{\nu}} (kh) \leq Fe^{(1/h)^{\nu}} (knh).
\]

On the other hand, since \( u \in \mathcal{E}^P_{\{\nu\}}(\Omega) \), there are constants \( G > 0 \) and \( h \in \mathbb{N} \) that depend on \( K(r/2) \) such that
\[
\|P^{j-k}u\|_{2,K(r/2)} \leq Ge^{(1/h)^{\nu} ((j-k)nh)}, \quad k = 0, 1, \ldots, j \in \mathbb{N}.
\]

Then, from the convexity of \( \varphi^* \),
\[
\|P(D) / \chi u\|_{2,K} \leq F^{j\nu} \left( \|P^{j-k}u\|_{2,K(r/2)} + Fe^{\nu ((j-k)nh)} \|P^{j-1}u\|_{2,K(r/2)} + Fe^{\nu (2nh)} \|P^{j-2}u\|_{2,K(r/2)} + \cdots + Fe^{\nu (jnh)} \|P^{1}u\|_{2,K(r/2)} \right)
\]
\[
\leq (j + 1) 2^{j\nu} F^j Ge^{\nu (jnh)}.
\]

If \( * = \{\omega\} \), since \( \omega(t^J) = o(t^{1/j}) \) as \( t \to +\infty \) for every \( \ell \in \mathbb{N} \), there is \( D_\ell > 0 \) such that
\[
(k!)^\nu \leq D_\ell e^{\varphi^*(k/\ell)}, \quad k \in \mathbb{N}.
\]

Moreover, if \( u \in \mathcal{E}^P_{\{\nu\}}(\Omega) \) for each \( \ell \in \mathbb{N} \), there is \( C_\ell > 0 \) such that
\[
\|P^{j-k}u\|_{2,K(r/2)} \leq C_\ell e^{\varphi^* ((j-k)/\ell)}, \quad k = 0, 1, \ldots, j \in \mathbb{N}.
\]

Now, we can proceed as in the Roumieu case to obtain
\[
\|P(D)/\chi u\|_{2,K(r/2)} \leq (j + 1) 2^{j\nu} F^j Ge^{\varphi^* (j/\ell)}, \quad j \in \mathbb{N},
\]
which concludes the proof.

Let us recall that, by Proposition 9 and Theorem 13 if \( \omega \) is a nonquasianalytic weight and \( P(D) \) is elliptic, then
\[
WF^P_{\{\nu\}} u = WF^P_{\{\nu\}} \forall u \in \mathcal{D}',
\]
for \( * \) being equal to \( \{\omega\} \) or \( \{\omega\} \). Let us then assume \( P(D) \) is not elliptic and prove the following result, which generalizes Theorems 8.1.4 and 8.1.14 of [25].

**Theorem 18.** Let \( \omega \) be a nonquasianalytic weight function such that \( \omega(t^J) = o(\sigma(t)) \) ast tends to infinity, where \( \sigma(t) = t^{1/j} \) is a Gevrey weight function, with \( s > 1 \) and \( b = \max (\nu, 3/2) \), with the constant in (28). Let \( P(D) \) be a linear partial differential operator with constant coefficients which is hypoelliptic but not elliptic. Given an open subset \( \Omega \) of \( \mathbb{R}^n \) and a closed conic subset \( S \) of \( \Omega \times (\mathbb{R}^n \setminus \{0\}) \), then there is a distribution \( u \in \mathcal{D}'(\Omega) \) with \( \theta \neq WF^P_{\{\nu\}} u \in S. \) In particular, if \( S = \{x_0, t\xi_0\}, t > 0 \) for some \( x_0 \in \Omega \) and \( \xi_0 \in \mathbb{R}^n \) with \( |\xi_0| = 1 \), we have \( WF^P_{\{\nu\}} u = S. \)

**Proof.** Let us first remark that it is sufficient to prove the statement when \( \Omega = \mathbb{R}^n \).

Moreover, since \( P \) is hypoelliptic but not elliptic, we can find \( \delta > 0 \) and \( 0 < d < m \) such that
\[
|P(\xi)| \geq \delta |\xi|^d,
\]
for \( \xi \) big enough. Choose a sequence \( (x_k, \theta_k) \in S \) with \( |\theta_k| = 1 \) so that every \( (x, \theta) \in S \) with \( |\theta| = 1 \) is the limit of a subsequence.

Let us now set \( \sigma(t) = \omega(t^{3/2}) \) and separate Beurling and Roumieu cases.

**Roumieu Case.** Take \( \phi \in \mathcal{D}^P_{\{\nu\}}(\mathbb{R}^n) \) with \( \phi(0) = 1 \). Then, there exist \( c > 0 \) and \( h \in \mathbb{N} \) such that
\[
|\phi(\xi)| \leq ce^{-1/(h+d|\xi|)} \quad \forall \xi \in \mathbb{R}^n.
\]

Since \( \log t = o(\sigma(t)) \) as \( t \to +\infty \), by definition of weight function, by Lemma 1.7 of [15], there exists a weight function \( \alpha \) such that \( \log t = o(\sigma(t)) \) and \( \alpha(t) = o(\sigma(t)) \) for \( t \to +\infty \). Note that for every \( \ell \in \mathbb{N} \), there is \( k_\ell \in \mathbb{N} \) such that
\[
\exp \left\{-\frac{\sigma(k_\ell/m)}{\alpha(k_\ell/m)} \log k \right\} < k^{-\ell} \quad \forall k \geq k_\ell
\]
and define then
\[
u(x) = \sum_{k=1}^{+\infty} e^{-\sigma(k_\ell/m)\alpha(k_\ell/m)} \log k (k^{-\ell} \delta(t)).
\]
This is a continuous function in \( \mathbb{R}^n \) and we will prove that \( \emptyset \neq WF^P_{\{\nu\}} u \subset S. \)
To prove first that $\WF_P^{\varphi} u \subset S$, we take $(x_0, \xi_0) \not\in S$ and prove that $(x_0, \xi_0) \not\in \WF_P^{\varphi} u$. To this aim, we choose an open neighborhood $U$ of $x_0$ and an open conic neighborhood $\Gamma$ of $\xi_0$ such that

$$(U \times \Gamma) \cap S = \emptyset. \quad (139)$$

Write $u = u_1 + u_2$, where $u_1$ is the sum of terms in $(138)$ with $\xi_k \not\in \Gamma$ and $u_2$ is the sum of terms with $\xi_k \in \Gamma$.

Therefore, there is a neighborhood $U_1$ of $x_0$, with $\overline{U_1} \subset U$ such that $u_1$ is in $\mathcal{E}_1(U_1)$ since all but a finite number of terms vanish in $U_1$. Moreover, every weight function $\omega$ is increasing by definition, so that $\omega \leq \sigma$, $\mathcal{E}_1(\omega) \subset \mathcal{E}_1(\sigma)$ and hence $u_1 \in \mathcal{E}_1(\omega)(U_1)$.

Consider then

$$f_N = P(D)^{N} u_2 (x) = \sum_{x_k \in U} e^{-\sigma(k^{(\imath)}/\alpha(k^{(\imath)})) \log k} P(D)^{N} \times \left[ \phi \left( k(x-x_k) \right) e^{i k^{(\imath)}(x, \beta_k)} \right].$$

Note that it is a totally convergent series since

$$\sup_{x \in \mathbb{R}^n} \left| P(D)^{N} \left[ \phi \left( k(x-x_k) \right) e^{i k^{(\imath)}(x, \beta_k)} \right] \right| \leq C_N k^{3mN} \quad (141)$$

for some $C_N > 0$ and because of $(137)$ with $\ell \geq 3mN + 2$.

Let us then compute the Fourier transform

$$\widehat{f}_N (\xi) = \sum_{x_k \in U} e^{-\sigma(k^{(\imath)}/\alpha(k^{(\imath)})) \log k} P(\xi)^{N} \times \mathcal{F} \left( \phi \left( k(x-x_k) \right) e^{i k^{(\imath)}(x, \beta_k)} \right)$$

$$\quad = \sum_{x_k \in U} e^{-\sigma(k^{(\imath)}/\alpha(k^{(\imath)})) \log k} \mathcal{F} \left( e^{i k^{(\imath)}(x, \beta_k)} \right)$$

with $\theta_k \not\in \Gamma$ because of $(139)$.

If $\Gamma_1$ is a conic neighborhood of $\xi_0$ with $\overline{\Gamma_1} \subset \Gamma \cup \{0\}$, then $|\xi - \eta| \geq c_0 (|\xi| + |\eta|)$ when $\xi \in \Gamma_1$ and $\eta \not\in \Gamma$, for some $c_0 > 0$, since this is true when $|\xi| + |\eta| = 1$. Thus,

$$|\xi - k^{\theta_k} | \geq c_0 (|\xi| + k^{3})$$

$$\geq c_0 \frac{1}{3} (|\xi| + |\xi| + k^{3})$$

$$\geq c_0 \sqrt{|\xi| \cdot |\xi| + k^{3}}$$

$$= c_0 |\xi|^{2/3} k, \quad \xi \in \Gamma_1. \quad (143)$$

It follows from $(136)$ that

$$\left| \hat{\varphi} \left( k \left( \frac{\xi - k^{\theta_k}}{k} \right) \right) \right| \leq c \exp \left\{ \frac{1}{\sigma} \left( \frac{\xi - k^{\theta_k}}{k} \right) \right\}$$

$$\leq c e^{-1/(\sigma(\alpha k^{3}))}$$

$$\leq c' e^{-1/(\sigma(\alpha k^{3}))} \quad (144)$$

for some $c' > 0$, since $\omega(2t) \leq L(\omega(t) + 1)$ for some $L > 0$ by definition of weight function. Therefore, by $(142)$ and Lemma $16(\ell)$, if we fix $\ell \in \mathbb{N}$, for $\xi \in \Gamma_1$, $|\xi| \geq 1$,

$$(1 + |\xi|)^\ell \left| \hat{f}_N (\xi) \right| \leq (1 + |\xi|)^\ell \sum_{x_k \in U} e^{-\sigma(k^{(\imath)}/\alpha(k^{(\imath)})) \log k} k^{-m}$$

$$\times \left| P(\xi)^{N} \right| c \epsilon^{-1/(\sigma(\alpha k^{3}))}$$

$$\leq c_1^\ell |\xi|^{\ell N} \epsilon^{-1/(\sigma(\alpha k^{3}))}$$

$$\leq c_1^\ell \epsilon^{1/(\sigma(\alpha k^{3}))} (mN + 1), \quad (145)$$

for some $c'' > 0$. Now, from the convexity of $\varphi^*$, it follows easily that condition (ii)(b) of Definition 7 is satisfied. But also condition (ii)(a) of Definition 7 is satisfied

$$\left| \hat{f}_N (\xi) \right| \leq \sum_{x_k \in U} e^{-\sigma(k^{(\imath)}/\alpha(k^{(\imath)})) \log k} \mathcal{F} \left( e^{i k^{(\imath)}(x, \beta_k)} \right)$$

$$\leq c'' |\xi|^{\ell N} e^{-1/(\sigma(\alpha k^{3}))} \quad (146)$$

for some $c'' > 0$. This, together with $u_1 \in \mathcal{E}_1(\omega)(U_1)$, proves that $(x_0, \xi_0) \not\in \WF_P^{\varphi} u_1$.

Let us now prove that $\WF_P^{\varphi} u \not= \emptyset$.

Choose $\chi \in \mathcal{D}_\sigma(\mathbb{R}^n)$ equal to 1 near $x_k \in \Omega$, where $\sigma$ is the Gevrey weight of the hypotheses. To prove that $\WF_P^{\varphi} u \not= \emptyset$, we proceed by contradiction and assume that the wave front set is empty. Then, $u \in \mathcal{E}_1(\omega)(\Omega)$.

Set

$$\phi_k \left( k(x-x_k) \right) := \chi(x) \phi \left( k(x-x_k) \right). \quad (147)$$

By hypothesis $\sigma = o(\sigma)$ which implies in particular that $\mathcal{D}_\sigma(\mathbb{R}^n) \subset \mathcal{E}_1(\omega)(\mathbb{R}^n)$. Then, the sequence $\phi_k(y) = \chi(y/k + x_k) \phi(y)$ is a bounded set in $\mathcal{D}_\sigma(\mathbb{R}^n)$ and, in fact, the supports $\text{supp} \phi_k \subset \text{supp} \phi$ for all $k$. We can use [15, Proposition 3.4] to obtain constants $c, h > 0$ such that

$$\left| \hat{\phi}_j (\xi) \right| \leq c e^{-1/(\sigma(\alpha k^{3}))} \quad (148)$$

for all $j \in \mathbb{N}$ and all $\xi \in \mathbb{R}^n$. 

Abstract and Applied Analysis
The Fourier transform of $P(D)^N(\chi u)$ is a sum of the form (142) with $\phi$ replaced by $\phi_k$. We observe that

$$|k^3 \theta_k - j^3 \theta_j| \geq |k^3 - j^3| \geq k^2 + kj + j^2 \geq kj, \quad \text{if } k \neq j.$$  

(149)

Moreover, for $x_k$ close to $x_0$ and $k$ large enough, the equality $\phi_k = \phi$ is satisfied. Consequently, from (135), we have, for some $\varepsilon' > 0$,

$$\left| \mathcal{F} \left[ P(D)^N(\chi u) \right] (k^3 \theta_k) \right|$$

$$= \left| e^{-\sigma(k^{d/m})\log k} P(k^3 \theta_k)^N \right|$$

$$+ \sum_{j \neq k} \left| e^{-\sigma(j^{d/m})\log j} P(k^3 \theta_k) \right|$$

$$\geq \left| P(k^3 \theta_k) \right| \left( e^{-\sigma(k^{d/m})\log k} - e^{-\sigma(j^{d/m})\log j} \right)$$

(150)

where $D > 0$ is a constant that depends on the Lebesgue measure of supp $\chi$. Consequently, from (150), we have

$$\frac{\delta^N}{2} k^{3Nd - n} e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k}$$

$$\leq \left| \mathcal{F} \left[ P(D)^N(\chi u) \right] (k^3 \theta_k) \right|$$

(153)

$$\leq C e^{(1/h')\psi^*(Nmh')}$$

for every $N \in \mathbb{N}$ and $k$.

Now, (153) implies, by Lemma 16(ii),

$$e^{-(\omega(k^{d/m})/\alpha(k^{d/m}))\log k} = e^{\sigma(k^{d/m})/\alpha(k^{d/m})\log k}$$

$$\leq 2CK' \inf_{N \in \mathbb{N}} \left\{ \left( \frac{1}{m} k^{d/m} \right)^{-Nn} e^{(1/h')\psi^*(Nmh')} \right\}$$

(154)

$$\leq 2C \delta k^{mNd} e^{-(1/h')\omega(k^{d/m})k^{d/m}}.$$ But for every fixed $h'$, there is $k$ large enough so that

$$\frac{\omega(k^{d/m})}{\alpha(k^{d/m})} \log k$$

$$< \frac{1}{h'} \omega(k^{d/m}) - (n + 3d) \log k - (2C\delta),$$

since we can argue as in (151), which is a contradiction. Therefore, $WF^p_{(\omega)} u \neq \emptyset$.

**Beurling Case.** Take $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^n)$ with $\widehat{\phi}(0) = 1$.

For every $h \in \mathbb{N}$, there exists then a constant $c_h > 0$ such that

$$\left| \mathcal{F} (\xi) \right| \leq c_h e^{-\sigma(\xi)} \quad \forall \xi \in \mathbb{R}^n.$$  

(156)

Note that for every fixed $\ell \in \mathbb{N}$,

$$\exp \left\{ -\sigma \left( k^{d/m} \right) \right\} = \exp \left\{ -\frac{\sigma \left( k^{d/m} \right)}{\log \left( k^{d/m} \right)} \cdot m \cdot \log k \right\} < k^{-\ell},$$

(157)

for $k$ large enough since $\log k = o(\sigma(k))$ as $k \to \infty$. Define then

$$u(x) = \sum_{k=1}^{\infty} e^{-\sigma(k^{d/m})} \phi(k(x - x_k)) e^{k^\ell(x\partial^\ell_k)}.$$  

(158)

This is a continuous function in $\mathbb{R}^n$ and we will prove that $0 \neq WF^p_{(\omega)} u \subset S$.

The proof of the inclusion $WF^p_{(\omega)} u \subset S$ is similar to that in the Roumieu case. We take $(x_{0}, \xi_{0}) \neq \emptyset$, choose an open neighborhood $U$ of $x_{0}$ and an open conic neighborhood $\Gamma$ of $\xi_{0}$ such that $(U \times \Gamma) \cap S \neq \emptyset$, and write $u = u_{1} + u_{2}$, where $u_{1}$ is the sum of terms in (158) with $x_k \notin U$ and $u_{2}$ is the sum of terms with $x_k \in U$.

We choose a neighborhood $U_{1}$ of $x_{0}$ with $\overline{U_{1}} \subset U$ such that $u_{2}$ is in $\mathcal{E}_{(\omega)}(U_{1}) \subset \mathcal{E}_{(\omega)}(U_{1})$ since all but a finite number of terms vanish in $U_{1}$. 


Then, we consider the totally convergent series (because of (157) with $\ell$ large enough)

$$f_N = P(D)^N u_2(x)$$

$$= \sum_{x_k \in U} e^{-\sigma(k^{(m)})} P(D)^N \left( \phi_k(x - x_k) \right) e^{\xi_k(x - x_k)}$$

and compute its Fourier transform

$$\widehat{f}_N(\xi) = \sum_{x_k \in U} e^{-\sigma(k^{(m)})} k^{-\eta} P(\xi)^N \phi_k \left( \frac{\xi - k^3 \theta_k}{k} \right) e^{(x_k, k^3 \theta_k - \xi)},$$

with $\theta_k \not\in \Gamma$.

For a conic neighborhood $\Gamma_1$ of $\xi_0$ with $\Gamma_1 \subset \Gamma \cup \{0\}$, we have that (143) is satisfied and hence, from (156),

$$\phi \left( \frac{\xi - k^3 \theta_k}{k} \right) \leq \varphi_n \exp \left\{ -h \sigma \left( \frac{\xi - k^3 \theta_k}{k} \right) \right\}$$

$$\leq \varphi_n e^{-h \sigma(\xi - k \theta_k)} \leq \varphi_n e^{-h \sigma(\xi)}, \quad \xi \in \Gamma_1,$n
for some $\varphi_n > 0$. From the convexity of $\varphi^*$, we conclude that condition (iii)(b) of Definition 7 is satisfied. But also condition (iii)(a) of Definition 7 is satisfied

$$\left| \widehat{f}_N(\xi) \right| \leq \sum_{x_k \in U} e^{-\sigma(k^{(m)})} k^{-\eta} |P(\xi)|^N \varphi_n e^{-h \sigma(\xi)}$$

$$\leq \varphi_n e^{-h \sigma(\xi)}, \quad \xi \in \Gamma_1,$n

for some $\varphi_n > 0$. This, together with $u_1 \in \mathcal{B}(\omega_1(U_1))$, proves that $(x_0, \xi_0) \not\in W^{P, H}(\omega_1)$ and hence $W^P(\omega_1) \subset S$.

Let us prove now that $W^P(\omega_1) \neq \emptyset$.

Choose $\chi \in \mathcal{D}(\mathbb{R}^n)$ equal to 1 near $x_0$. We proceed by contradiction and assume that $W^P(\omega_1) = \emptyset$. Then, $u \in \mathcal{D}(\omega_1(\mathbb{R}^n))$ must hold for every $h > 0$ and $k$ large enough.

However, since $\omega(2t) \leq L(t) + 1$ for some $L > 0$, there exists a constant $c_1 > 0$ such that

$$\omega \left( k^{(m)} \right) \leq c_1 \left( \omega \left( k^{(m)} \right) + 1 \right),$$

contradicting (167) for $k$ large enough. Then $W^P(\omega_1) \neq \emptyset$. $\square$

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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