ON SYLOW NORMALIZERS OF FINITE GROUPS *

L. S. Kazarin
Department of Mathematics, Yaroslavl P. Demidov State University
Sovetskaya Str 14, 150000 Yaroslavl, Russia
E-mail: Kazarin@uniyar.ac.ru

A. Martínez-Pastor
Escuela Técnica Superior de Ingeniería Informática,
Instituto Universitario de Matemática Pura y Aplicada IUMPA-UPV
Universitat Politècnica de València
Camino de Vera, s/n, 46022 Valencia, Spain
E-mail: anamarti@mat.upv.es

and

M. D. Pérez-Ramos
Departament d'Àlgebra, Universitat de València
C/ Doctor Moliner 50, 46100 Burjassot (València), Spain
E-mail: Dolores.Perez@uv.es

Dedicated to the memory of Leonid A. Shemetkov (1937–2013)

Abstract

The paper considers the influence of Sylow normalizers, i.e., normalizers of Sylow subgroups, on the structure of finite groups. In the universe of finite soluble groups it is known that classes of groups with nilpotent Hall subgroups for given sets of primes are exactly the subgroup-closed saturated formations satisfying the following property: a group belong to the class if and only if its Sylow normalizers do so. The paper analyzes the extension of this research to the universe of all finite groups.

*The second and third authors have been supported by Proyecto MTM2010-19938-C03-02, Ministerio de Economía y Competitividad, Spain. The first author would like to thank the Universitat de València and the Universitat Politècnica de València for their warm hospitality during the preparation of this paper. He has been also supported by RFBR Project 13-01-00469.

Keywords: Finite group; Sylow normalizer; nilpotent Hall subgroup; saturated formation.
2010 Mathematics Subject Classification: 20D20, 20F17

1 Introduction and Preliminaries

This paper arises from the interest in the influence of the normalizers of Sylow subgroups on the structure of finite groups. We will call those subgroups Sylow normalizers to abbreviate and will consider only finite groups. This is a classical topic in the theory of groups with a good number of results in the literature. The present paper takes further a previous research in [9] and [17]. A starting point for this approach could be a classical Glauberman's result [12] which states that a group is a p-group for a prime p if and only if its Sylow subgroups are self-normalizing. Bianchi, Gillio Berta Mauri and Hauck [5] proved that a group is nilpotent if and only if its Sylow normalizers are nilpotent. Other related results and extensions can be also found in the bibliography (see for instance [4], [6], [7], [8], [11], [14, §3], [15], [18], [19], [20]). The mentioned results of Glauberman and of Bianchi et al. were partially extended in the universe of finite soluble groups by D'Aniello, De Vivo, Giordano and the third author [9], who proved that a soluble group possesses nilpotent Hall subgroups for given sets of primes if and only if its Sylow normalizers satisfy the same property. We show an easy way to construct groups with nilpotent Hall subgroups:

We denote by $\mathcal{E}$ the class of all finite groups and by $\mathcal{E}_\sigma$, for any set of primes σ, the class of all finite σ-groups.

Let π be a set of primes and consider $\{\pi_i \mid i \in I\}$ a partition of π. Set

$$\times_{i\in I}\mathcal{E}_{\pi_i} := (G \in \mathcal{E}_\pi \mid G = \times_{i\in I}G_{\pi_i},\ G_{\pi_i} \in \mathcal{E}_{\pi_i}).$$

We notice that whenever $\sigma \subseteq \pi$ satisfies the following condition:

$$\text{“if } p, q \in \sigma,\ p \neq q, \text{ with } p \in \pi_i,\ q \in \pi_j, \text{ then } i \neq j\text{”}$$

it holds that groups in the class $\times_{i\in I}\mathcal{E}_{\pi_i}$ have nilpotent Hall σ-subgroups for such set of primes σ, that is,

$$\times_{i\in I}\mathcal{E}_{\pi_i} \subseteq \mathbf{E}_\sigma^{\mathrm{n}} \cap \mathcal{E}_\pi$$

where $\mathbf{E}_\sigma^{\mathrm{n}}$ denotes the class of all groups with nilpotent Hall σ-subgroups.

We emphasize that the classes $\times_{i\in I}\mathcal{E}_{\pi_i}$ can be seen as extensions of the class of nilpotent groups, by considering direct products of Hall subgroups corresponding to pairwise disjoint sets of primes instead of Sylow subgroups. Moreover, also for such a class $\times_{i\in I}\mathcal{E}_{\pi_i}$, a group belongs to the class if and only if its Sylow normalizers do so. This is a relevant result among others of a similar nature in [17], which extend the above mentioned results to the universe of all finite groups, in particular the one for nilpotent groups in [5].

Our approach is based on a graph theoretical result in [17] which states the connectivity of the so-called Sylow graph in almost simple groups.

We introduce next a general construction involving the above mentioned classes $\times_{i\in I}\mathcal{E}_{\pi_i}$ as well as other examples in [17]. For notation and results about classes of groups and closure operations we refer to Doerk and Hawkes book [10]. In particular, we denote by $\mathrm{Syl}_p(G)$ and by $\mathrm{Hall}_\sigma(G)$, respectively, the set of Sylow p-subgroups and the set of Hall σ-subgroups of a group G, p a prime and σ a set of primes. For $i = 1, 2$, $\mathfrak{P}_i$ denotes the class of primitive groups of type $i \in \{1, 2\}$.

Let π be a set of primes. For each prime $p \in \pi$, let $\pi(p)$ be a set of primes satisfying the following conditions:

(i) $p \in \pi(p) \subseteq \pi$,

(ii) *for any $q \in \pi(p)$, then $p \in \pi(q)$.*

If $q \in \pi(p)$, we write $p \leftrightarrow q$. (This defines a reflexive and symmetric relation on π.)

Associated to such sets of primes, we construct the following class of groups:

$\mathcal{F} = LF(f)$ is the saturated formation locally defined by a formation function f given by:

$$f(p) = \mathcal{E}_{\pi(p)} \text{ if } p \in \pi, \quad f(q) = \emptyset \text{ if } q \notin \pi.$$

In particular, $\pi = \mathrm{Char}(\mathcal{F})$ is the characteristic of $\mathcal{F}$.

Such formations $\mathcal{F}$ appear in [9] where are called covering-formations.

(As mentioned in [9, Remark 1(b)] the condition $\pi(p) \subseteq \pi$ is no loss of generality in order to construct the associated saturated formation $\mathcal{F}$.)

The following description of $\mathcal{F}$ will be frequently used:

$$\mathcal{F} = \mathcal{E}_\pi \cap (\cap_{p\in\pi}\mathcal{E}_{p'}\mathcal{E}_p f(p)) = \mathcal{E}_\pi \cap (\cap_{p\in\pi}\mathcal{E}_{p'}\mathcal{E}_{\pi(p)}).$$

Let $\mathcal{S}$ denote the class of all finite soluble groups. The saturated formation $\mathcal{F} \cap \mathcal{S} = LF(g)$, being

$$g(p) = \mathcal{S}_{\pi(p)} \text{ if } p \in \pi, \quad g(p) = \emptyset \text{ if } p \notin \pi,$$

where $\mathcal{S}_{\pi(p)}$ is the class of all finite soluble $\pi(p)$-groups, is called *covering-formation of soluble groups* ([9]).

When $\pi(p) = \{p\}$ for every prime p, we notice that $\mathcal{F} = \mathcal{N}$ is the class of all finite nilpotent groups. Other relevant examples for our study can be seen in [17]. In particular, if the sets of primes $\pi(p)$ form a partition $\{\pi_i \mid i \in I\}$ of π, then the groups in $\mathcal{F}$ are characterized by being direct products of Hall subgroups corresponding to the sets of primes in the partition, that is,

$$\mathcal{F} = \times_{i\in I}\mathcal{E}_{\pi_i} := (G \in \mathcal{E}_\pi \mid G = \times_{i\in I}G_{\pi_i},\ G_{\pi_i} \in \mathcal{E}_{\pi_i}).$$

For the general case, we define in addition another set as follows:

$$\Sigma := \{\sigma \subseteq \pi \mid \ |\sigma| \geq 2,\ (p, q \in \sigma, p \neq q \ \Rightarrow\ p \not\leftrightarrow q)\},$$

and we will also consider the following classes of groups:

$$\mathcal{U} := \bigcap_{\sigma \in \Sigma} \mathbf{E}^{\mathrm{n}}_{\sigma} \cap \mathcal{E}_{\pi} \ , \qquad \mathcal{V} := \bigcap_{\sigma \in \Sigma} \mathbf{S}^{\mathrm{n}}_{\sigma} \cap \mathcal{E}_{\pi} \ ,$$

where, for any set of primes τ, $\mathbf{S}^{\mathrm{n}}_{\tau}$ denotes the class of all finite groups whose τ-subgroups are nilpotent. (In case $\Sigma = \emptyset$, it is understood $\bigcap_{\sigma \in \Sigma} \mathbf{E}^{\mathrm{n}}_{\sigma} = \bigcap_{\sigma \in \Sigma} \mathbf{S}^{\mathrm{n}}_{\sigma} = \mathcal{E}$.)

We show in Proposition 2.1 below that $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{V}$.

When restricting to the soluble universe, it was obtained in [9] that

$$\mathcal{F} \cap \mathcal{S} = \mathcal{U} \cap \mathcal{S} = \mathcal{V} \cap \mathcal{S}$$

and these classes of groups, i.e. covering formations of soluble groups, are characterized as the subgroup-closed saturated formations $\mathcal{X} \subseteq \mathcal{S}$ satisfying that a soluble group belongs to $\mathcal{X}$ if and only if its Sylow normalizers do also so.

To be more precise we gather the following notation and results from [9]:

For a group G and a prime p, let G_p denote a Sylow p-subgroup of G. We set $\pi(G)$ for the set of all prime numbers dividing the order of G. Then, for a class $\mathcal{X}$ of groups, the class map N is defined as follows:

$$\mathrm{N}\mathcal{X} = (G \mid N_G(G_p) \in \mathcal{X}, \text{ for every prime } p \in \pi(G)).$$

Proposition 1.1. [9, Proposition 2] *Let $\mathcal{G} = \mathcal{F} \cap \mathcal{S}$ be a covering-formation of soluble groups defined as above. Then:*

1. $\mathrm{S}\mathcal{G} := (H \leq G \mid G \in \mathcal{G}) = \mathcal{G}$ *and* $\mathrm{N}\mathcal{G} \cap \mathcal{S} = \mathcal{G}$;

2. $\mathcal{G} = \bigcap_{\sigma \in \Sigma, |\sigma|=2} \mathbf{E}^{\mathrm{n}}_{\sigma} \cap \mathcal{S} = \bigcap_{\sigma \in \Sigma} \mathbf{E}^{\mathrm{n}}_{\sigma} \cap \mathcal{S}$.

Theorem 1.1. [9, Theorem] *Let $\mathcal{H}$ be a subgroup-closed saturated formation of soluble groups. Then the following statements are equivalent:*

(i) *For any soluble group, its Sylow normalizers belong to $\mathcal{H}$ if and only if the group belongs to $\mathcal{H}$, i.e. $\mathrm{N}\mathcal{H} \cap \mathcal{S} = \mathcal{H}$;*

(ii) *$\mathcal{H}$ is a covering-formation of soluble groups.*

This nice picture in the soluble universe $\mathcal{S}$ breaks up completely when extending to the universe of all finite groups $\mathcal{E}$. We exhibit in the paper examples for any kind of failure regarding the coincidence of the classes of groups under consideration and their characterizations. Then we are interested in understanding the reasons for the failures, which allows us to provide positive results.

We keep the above introduced notation throughout the paper. Particularly, we consider prefixed sets of primes π, $\pi(p)$ for each $p \in \pi$, as given above and associated to them, $\mathcal{F}$ will always denote a covering-formation defined as *previously, and the same for the classes $\mathcal{U}$ and $\mathcal{V}$. When convenient to avoid confusions we will write $\Sigma_{\mathcal{F}} := \Sigma$ for the associated set as defined before.*

In Section 2 we analyze the classes $\mathcal{F}, \mathcal{U}$ and $\mathcal{V}$, their properties and interrelationships. The classes $\mathcal{U}$ and $\mathcal{V}$ happen to be formations, also $\mathcal{V}$ is subgroup-closed, like $\mathcal{F}$, but the key point is that they are not saturated in general as it is the case for $\mathcal{F}$. For instance, Examples 2.1 shows that $\mathbf{E}_\sigma^n$ and $\mathbf{S}_\sigma^n$ are never saturated if σ is a finite set of odd primes with $|\sigma| \geq 2$, and illustrates the kind of difficulties in this study. In the case when $\mathcal{U}$ and $\mathcal{V}$ are saturated formations, which depend on the choice of the sets of primes considered for their constructions, then it holds that $\mathcal{F} = \mathcal{U} = \mathcal{V}$.

In Section 3, for a subgroup-closed saturated formation $\mathcal{H}$ we study the interrelation between the following properties:

(C) $\mathcal{H}$ is a covering-formation;

(N) $\mathrm{N}\mathcal{H} = \mathcal{H}$

In general none of these properties implies the other, contrary to what happens in the universe of soluble groups ([9, Remark 1(c)], [8], [17, Remark, p. 270]). Theorem 3.1 provides a condition in terms of the canonical local definition of $\mathcal{H}$ in order to assure that (N) $\Rightarrow$ (C). On the other hand, in [17, Examples 1,2,3] we showed some particular constructions of covering-formations $\mathcal{F}$ satisfying that $\mathrm{N}\mathcal{F} = \mathcal{F}$ and noticed that for them $\mathcal{F} = \mathcal{U}$. We prove in Theorem 3.2 that this is not casual but the property $\mathcal{F} = \mathcal{U} = \mathcal{V}$ is a consequence of satisfying $\mathrm{N}\mathcal{F} = \mathcal{F}$. However Examples 3.1 show that the converse does not hold, as one might hope, even for covering-formations of soluble groups or assuming full characteristic. But the property $\mathcal{F} = \mathcal{U} = \mathcal{V}$ takes to an interesting relation between covering-formations and critical groups. More precisely, following [3] we say that a saturated formation $\mathcal{H}$ has the *Shemetkov property* if every $\mathcal{H}$-critical group is either a Schmidt group or a cyclic group of prime order. We recall that a group G is called *$\mathcal{H}$-critical* if G is not in $\mathcal{H}$ but all proper subgroups of G are in $\mathcal{H}$; the class of all $\mathcal{H}$-critical groups is denoted as $\mathrm{Crit_s}(\mathcal{H})$. Also the *boundary of a class* $\mathcal{H}$ is the class $b(\mathcal{H}) = (G \in \mathcal{E} \mid G \notin \mathcal{H}$ but $G/N \in \mathcal{H}$ whenever $1 \neq N \trianglelefteq G)$.

The following characterizations of the subgroup-closed saturated formations with the Shemetkov property from [2] and [3] will be of interest in the paper.

Theorem 1.2. *Let $\mathcal{H} = LF(H)$ be a subgroup-closed saturated formation with H its canonical local definition. Denote $\pi = \mathrm{Char}(\mathcal{H})$. The following statements are pairwise equivalent:*

1. *$\mathcal{H}$ has the Shemetkov property.*

2. *([2, Theorem 2]) $\mathcal{H}$ satisfies the following two conditions:*

(i) *For each prime $p \in \pi$ there exists a set of primes $\pi(p)$ with $p \in \pi(p)$ such that $\mathcal{H}$ is locally defined by a formation function h given by*

$h(p) = \mathcal{E}_{\pi(p)}$ if $p \in \pi$, and $h(q) = \emptyset$ if $q \notin \pi$; moreover this formation function h satisfies the following property:

(*) If $G \in \mathrm{Crit_S}(\mathcal{H}) \cap b(\mathcal{H})$ and G is an almost simple group such that $G \notin h(p)$ for some prime $p \in \pi(\mathrm{Soc}(G))$, then $G \notin h(q)$ for each prime $q \in \pi(\mathrm{Soc}(G))$.

(ii) *$\mathrm{Crit_S}(\mathcal{H}) \cap b(\mathcal{H})$ does not contain non-abelian simple groups.*

3. *([3, Theorem 1]) A π-group G belongs to $\mathcal{H}$ if and only if $N_G(Q)/C_G(Q)$ belongs to $\mathcal{E}_{\pi(p)}$ for each p-subgroup Q of G and each prime $p \in \pi$, where $\pi(p) := \mathrm{Char}(H(p))$.*

4. *([3, Theorem 1]) A π-group G belongs to $\mathcal{H}$ if and only if $N_G(Q)$ belongs to $\mathcal{E}_{\pi\setminus\{p\}}\mathcal{E}_{\pi(p)}$ for each p-subgroup Q of G and each prime $p \in \pi$, where $\pi(p) := \mathrm{Char}(H(p))$.*

As pointed out in [3], the equivalences (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) in the previous theorem provide a characterization of the subgroup-closed saturated formations with the Shemetkov property as the subgroup-closed saturated formations for which an extension of the well known Frobenius p-nilpotence criterion holds.

Theorem 3.3 below proves that for a covering-formation $\mathcal{F}$ it holds that $\mathcal{F} = \mathcal{U} = \mathcal{V}$ if and only if $\mathcal{F}$ has the Shemetkov property.

As a particular construction in the paper, it is obtained for instance that the class of groups whose $\{2,3\}$-subgroups are nilpotent coincides with the class of groups with nilpotent Hall $\{2,3\}$-subgroups, and this class is a covering formation which satisfies Theorem 3.3, i.e., it has also the Shemetkov property (see Examples 3.1(1) below).

2 The Classes $\mathcal{F}$, $\mathcal{U}$ and $\mathcal{V}$. When do they are saturated as Formations.

We study in this section the classes $\mathcal{F}$, $\mathcal{U}$ and $\mathcal{V}$, their properties and interrelationships.

In the universe of all finite groups we have still the following result regarding nilpotent Hall subgroups.

Proposition 2.1. *For the covering-formation $\mathcal{F}$ we have*

$$\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{V}.$$

Proof. We prove first that $\mathcal{F} \subseteq \mathcal{U}$. Assume that this is not true and let $G \in \mathcal{F}\setminus\mathcal{U}$ of minimal order. Let $\tau \in \Sigma$. We argue by induction on $|\tau|$ to prove that $\mathrm{Hall}_\tau(G) \cap \mathcal{N} \neq \emptyset$.

Since $|\tau| \geq 2$, let $\tau = \tau_1 \cup \{p\}$, $p \notin \tau_1$. By inductive hypothesis assume that there exists $G_{\tau_1} \in \mathrm{Hall}_{\tau_1}(G) \cap \mathcal{N}$. (We notice that this is true if $|\tau| = 2$.)

For any $q \in \tau_1$, $G^{f(q)} \in \mathcal{E}_{q'}$. Since $p \notin \pi(q)$, we have that $G_p \in O_{q'}(G)$. Consequently, $G_p \leq \cap_{q\in\tau_1} O_{q'}(G)$.

On the other hand, $G^{f(p)} \in \mathcal{E}_{p'}$. If $q \in \tau_1$, then $q \notin \pi(p)$, which implies again that $G_q \leq O_{p'}(G)$. Therefore, $G_{\tau_1} \leq O_{p'}(G)$.

Now $[G_p, G_{\tau_1}] \leq O_{p'}(G) \cap (\cap_{q \in \tau_1} O_{q'}(G))$. If this intersection is trivial, then $G_p G_{\tau_1} \in \mathrm{Hall}_\tau(G) \cap \mathcal{N}$ and we are done. Otherwise there exists $1 \neq N \trianglelefteq G$ such that $N \leq \cap_{r \in \tau} O_{r'}(G) = O_{\tau'}(G)$.

Since $G/N \in \mathcal{F}$, by the choice of G we have that there exists $S/N \in \mathrm{Hall}_\tau(G/N) \cap \mathcal{N}$. But $(|N|, |S/N|) = 1$, which implies by the Schur-Zassenhaus theorem that $S = TN$ for some $T \leq S$, $T \in \mathcal{E}_\tau$ and $T \cap N = 1$. Moreover, $T \cong TN/N = S/N \in \mathcal{N}$ and $T \in \mathrm{Hall}_\tau(G)$ because $T \in \mathrm{Hall}_\tau(S)$; that is, $T \in \mathrm{Hall}_\tau(G) \cap \mathcal{N}$, which concludes the proof.

The inclusion $\mathcal{U} \subseteq \mathcal{V}$ follows by a well-known result of Wielandt ([21]; [16, Satz III.5.8]). □

Lemma 2.1. *If $N \trianglelefteq H$ and H/N is a σ-subgroup for a set of primes σ, then $H = LN$ for some σ-subgroup L of H.*

Proof. Let L be a minimal supplement of N in H; then $N \cap L \leq \Phi(L)$. We may assume that $L \neq 1$. If L is a σ-group we are done. Otherwise there exists a prime $p \in \pi(L) \setminus \sigma$. Consequently, for $L_p \in \mathrm{Syl}_p(L)$ it follows that $L_p \leq N \cap L \leq \Phi(L)$, a contradiction. □

Proposition 2.2. *Let σ be a set of primes. The classes of groups $\mathbf{E}_\sigma^{\mathrm{n}}$ and $\mathbf{S}_\sigma^{\mathrm{n}}$ are formations; $\mathbf{S}_\sigma^{\mathrm{n}}$ is also subgroup-closed but this is not in general the case for $\mathbf{E}_\sigma^{\mathrm{n}}$.*

Consequently, the classes $\mathcal{U}$ and $\mathcal{V}$ are formations and $\mathcal{V}$ is also subgroup-closed though this is not in general the case for $\mathcal{U}$.

Proof. We prove first that $\mathbf{E}_\sigma^{\mathrm{n}}$ is a formation. It is easily proved that $\mathbf{E}_\sigma^{\mathrm{n}}$ is closed under taking factor groups. Let now $N, M \trianglelefteq G$ with $G/N, G/M \in \mathbf{E}_\sigma^{\mathrm{n}}$ and $N \cap M = 1$. We argue by induction on the order of G to prove that $G \in \mathbf{E}_\sigma^{\mathrm{n}}$.

By the hypothesis G/N and G/M have nilpotent Hall σ-subgroups, say H/N and L/M, respectively. Then $HM/(NM)$ and $LN/(NM)$ are Hall σ-subgroups of $G/(NM)$. By Wielandt's result ([21]; [16, Satz III.5.8]) it follows that HM and LN are conjugate and there is no loss of generality to assume that $HM = LN$. We notice that a Hall σ-subgroup of HM would be a Hall σ-subgroup of G, so that by the inductive argument we may assume that $G = HM = LN$. Since $L/M = (H \cap L)M/M$ is a nilpotent Hall σ-subgroup of G/M we deduce that $H \cap L/H \cap M$ is a nilpotent Hall σ-subgroup of $H/H \cap M$. If $H < G$, we have by inductive hypothesis that $H \in \mathbf{E}_\sigma^{\mathrm{n}}$. But again notice that $\mathrm{Hall}_\sigma(H) \subseteq \mathrm{Hall}_\sigma(G)$ and we are done. Hence we may assume that $G = H$ and analogously that $G = L$. But this implies that G is nilpotent, which concludes the proof of this part.

For an example showing that $\mathbf{E}_\sigma^{\mathrm{n}}$ is not subgroup-closed, we consider $\sigma = \{3, 5\}$ and $\mathrm{Alt}(5) \cong \mathrm{L}_2(2^2) \leq \mathrm{L}_2(2^4)$. Then $\mathrm{L}_2(2^4) \in \mathbf{E}_\sigma^{\mathrm{n}}$ but $\mathrm{Alt}(5) \notin \mathbf{E}_\sigma^{\mathrm{n}}$.

On the other hand it is clear that $\mathbf{S}_\sigma^{\mathrm{n}}$ is subgroup-closed. We prove next that it is closed under taking factor groups. We consider $N \trianglelefteq G \in \mathbf{S}_\sigma^{\mathrm{n}}$ and a σ-subgroup H/N of G/N. By Lemma 2.1 we have that $H = LN$ for some

σ-subgroup L of H. Since $G \in \mathbf{S}^{\mathrm{n}}_{\sigma}$ we have that L is nilpotent. Then H/N is nilpotent and $G/N \in \mathbf{S}^{\mathrm{n}}_{\sigma}$. To complete the proof that $\mathbf{S}^{\mathrm{n}}_{\sigma}$ is a formation, since it is subgroup-closed, it is enough to prove that $A, B \in \mathbf{S}^{\mathrm{n}}_{\sigma}$ implies $A \times B \in \mathbf{S}^{\mathrm{n}}_{\sigma}$. Hence we consider H a σ-subgroup of $A \times B$. By the hypothesis on A and B we have that HB/B and HA/A are nilpotent, as they are isomorphic to subgroups of A and B, respectively. Then H is nilpotent and we are done.

The rest of the proof follows easily. $\square$

Examples 2.1. The classes $\mathcal{U}$ and $\mathcal{V}$ are not saturated in general.

For instance, if σ is a finite set of odd primes, $|\sigma| \geq 2$, we may consider $m \geq 1$ such that $\prod_{p\in\sigma} p \mid 2^m - 1$. Then $G = \mathrm{L}_2(2^m) \in \mathbf{E}^n_{\sigma} \subseteq \mathbf{S}^n_{\sigma}$. Let us consider a prime $r \in \sigma$ and $E_r(G)$ the universal Frattini r-elementary G-extension, and let $A_r(G)$ be the r-Frattini module of G. So $E_r(G)/A_r(G) \cong G$ and $A_r(G) \leq \Phi(E_r(G))$. Moreover we have that

$$\mathrm{Ker}(G \text{ on } A_r(G)) \leq \mathrm{Ker}(G \text{ on } \mathrm{Soc}(A_r(G))) = O_{r'r}(G) = 1$$

by a Griess-Schmid result ([13]; see [10, Appendix β]). Whence $E_r(G) \notin \mathbf{E}^n_{\sigma}$ and also $E_r(G) \notin \mathbf{S}^n_{\sigma}$, which proves that $\mathbf{E}^n_{\sigma}$ and $\mathbf{S}^n_{\sigma}$ are not saturated. (Here $\mathcal{U} = \mathbf{E}^n_{\sigma}$ and $\mathcal{V} = \mathbf{S}^n_{\sigma}$ by defining $\pi = \mathbb{P}$ the set of all prime numbers and $\pi(p) = (\mathbb{P} \setminus \sigma) \cup \{p\}$ if $p \in \sigma$, $\pi(q) = \mathbb{P}$ if $q \notin \sigma$.)

We will see next that for any of the classes $\mathcal{U}$ and $\mathcal{V}$, it is saturated if and only if it coincides with $\mathcal{F}$.

Lemma 2.2 (Zsigmondy)**.** *Let q and n be integers, $q, n \geq 2$. A prime number r is called* primitive prime divisor with respect to the pair (q, n) *if r divides $q^n - 1$ but r does not divide $q^i - 1$ for $i < n$. Then:*

1. *There exists a primitive prime divisor with respect to the pair (q, n) unless $n = 2$ and q is a Mersenne prime or $(q, n) = (2, 6)$.*

2. *If r is a primitive prime divisor with respect to the pair (q, n), then $r - 1 \equiv 0$ (mod n).*

Examples 2.2.

1. In general $\mathcal{F} \neq \mathcal{U} \neq \mathcal{V}$ in Proposition 2.1.

Take for instance $\pi = \{2, 3, 5, 17\}$ and

$$f(2) = f(17) = \mathcal{E}_{\pi},\ f(3) = \mathcal{E}_{\{2,3,17\}},\ f(5) = \mathcal{E}_{\{2,5,17\}},\ f(r) = \emptyset\ \forall r \notin \pi.$$

In this case, $\mathcal{U} = \mathbf{E}^{\mathrm{n}}_{\{3,5\}} \cap \mathcal{E}_{\pi}$ and $\mathcal{V} = \mathbf{S}^{\mathrm{n}}_{\{3,5\}} \cap \mathcal{E}_{\pi}$.

For $G = \mathrm{L}_2(2^4)$, $|G| = 2^4 \cdot 3 \cdot 5 \cdot 17$ and $G \in \mathcal{U}$ but $G \notin \mathcal{F}$.

On the other hand, $\mathrm{Alt}(5) \in \mathcal{V} \setminus \mathcal{U}$.

2. The following example shows that the case $\mathcal{F} = \mathcal{U} \neq \mathcal{V}$ is possible. Moreover, in this example $\mathcal{F} = \mathcal{U} \subseteq \mathcal{S}$.

Take $\pi = \{2,3,5\}$ and

$$f(2) = \mathcal{E}_\pi,\ f(3) = \mathcal{E}_{\{2,3\}},\ f(5) = \mathcal{E}_{\{2,5\}},\ f(r) = \emptyset\ \forall r \notin \pi.$$

As in the previous example we have that $\mathrm{Alt}(5) \in \mathcal{V} \setminus \mathcal{U}$.

But we claim that here $\mathcal{F} = \mathcal{U}$. (We notice that $\mathcal{F} \subseteq \mathcal{E}_\pi \cap (\mathcal{E}_{3'}\mathcal{E}_3 f(3)) = \mathcal{E}_{\{2,5\}}\mathcal{E}_{\{2,3\}} \subseteq \mathcal{S}$.) Since $\mathcal{F} \subseteq \mathcal{U}$, by Proposition 1.1 we will be done if we prove that $\mathcal{U} \subseteq \mathcal{S}$. But if $G \in \mathcal{U}$, then $\pi(G) \subseteq \{2,3,5\}$ and G has a nilpotent Hall $\{3,5\}$-subgroup, say $G_{\{3,5\}}$. Consequently, if G_2 is a Sylow 2-subgroup of G, then $G = G_{\{3,5\}}G_2$ is the product of two nilpotent groups, which implies that G is soluble by the well-known Kegel-Wielandt theorem.

3. The previous argument may be easily extended to prove that $\mathcal{F} = \mathcal{U} \subseteq \mathcal{S}$ in the following more general situation:

Let π and σ be sets of primes such that $\pi = \sigma \cup \{q\}, q \notin \sigma$, and define $\pi(q) = \pi$ and $\pi(p) = \{p,q\}\ \forall p \in \sigma$. In this case, $\mathcal{U} = \mathbf{E}^n_\sigma \cap \mathcal{E}_\pi$ and $\mathcal{F} = \mathcal{U} \subseteq \mathcal{S}$.

4. We construct next an example showing that also the case $\mathcal{F} \neq \mathcal{U} = \mathcal{V}$ is possible.

The construction is based on properties of the Suzuki group $\mathrm{Sz}(2^7)$. It might be helpful to notice the following:

$$|\,\mathrm{Sz}(2^7)| = 2^{14}(2^7-1)(2^{14}+1) = 2^{14} \cdot 127 \cdot (5 \cdot 29) \cdot 113.$$

Let $\pi = \{2,5,29,127,113\}$ and consider the covering-formation $\mathcal{F}$ of characteristic π defined by the following sets of primes:

$$\begin{aligned} &\pi(2) = \pi(127) = \pi(113) = \pi, \\ &\pi(5) = \{2,5,127,113\}, \quad \pi(29) = \{2,29,127,113\}, \\ &\pi(p) = \emptyset, \text{ if } p \notin \pi. \end{aligned}$$

In this case, we have:

$$\Sigma = \{\sigma \subseteq \pi \mid\ |\sigma| \geq 2\,,\ (p,q \in \sigma,\, p \neq q\ \Rightarrow\ p \not\leftrightarrow q)\} = \{\sigma\} \text{ with } \sigma = \{5,29\}$$

$$\mathcal{U} = \mathbf{E}^{\mathrm{n}}_{\{5,29\}} \cap \mathcal{E}_\pi\ ; \quad \mathcal{V} = \mathbf{S}^{\mathrm{n}}_{\{5,29\}} \cap \mathcal{E}_\pi$$

We prove next that $\mathcal{F} \neq \mathcal{U} = \mathcal{V}$ by the following steps:

Step 1. If X is a non-abelian simple group and $\pi(X) \subseteq \pi$, then $X \cong \mathrm{Sz}(2^7)$.

Since $3 \notin \pi(X)$, then $X \cong \mathrm{Sz}(q)$, $q = 2^{2m+1}, m \geq 1$, and $|\,\mathrm{Sz}(q)| = q^2(q-1)(q^2+1)$. We apply now Lemma 2.2. Since $(2^{4(2m+1)}-1) = q^4 - 1 = (q^2-1)(q^2+1) = (2^{2(2m+1)}-1)(2^{2(2m+1)}+1)$, if r is a primitive prime divisor with respect to the pair $(2, 4(2m+1))$, then r divides $2^{2(2m+1)}+1$; in particular, $r \in \{2,5,29,127,113\}$. Moreover, $r = 1 + k4(2m+1)$ with $k \geq 1$.

It is easily checked that the case $r = 1 + 4(2m+1)$, for $k = 1$, is not possible unless $m = 3$ and $r = 29$; in this case $X \cong \mathrm{Sz}(2^7)$. Assume that $r = 1 + k4(2m+1)$ with $k \geq 2$. Then $1 + 8(2m+1) \leq r \leq 127$, which implies

$m \leq 7$. But again only the case $m = 3$ satisfies that $\pi(q-1) = \pi(2^{2m+1}-1) \subseteq \{2, 5, 29, 127, 113\}$. Consequently, $X \cong \mathrm{Sz}(2^7)$ and we are done.

Step 2. $\mathrm{Sz}(2^7) \in \mathcal{U}$. *Consequently,* $\mathrm{Sz}(2^7)$ *satisfies the* $D_{\{5,29\}}$*-property, that is,* $\mathrm{Sz}(2^7)$ *has a unique conjugacy class of Hall* $\{5, 29\}$*-subgroups and any* $\{5, 29\}$*-subgroup is contained in some Hall* $\{5, 29\}$*-subgroup.*

It follows by known properties of Suzuki groups and a well-known Wielandt's result.

Step 3. $\mathcal{U} = \mathcal{V}$.

Assume that this is not true and let $G \in \mathcal{V} \setminus \mathcal{U}$ of minimal order. Notice that $G \in \mathcal{V} \subseteq \mathcal{E}_\pi$ with $\pi = \{2, 5, 29, 127, 113\}$. Since $\mathcal{V}$ and $\mathcal{U}$ are formations, G has a unique minimal normal subgroup N, and $G/N \in \mathcal{U}$. Then, by Lemma 2.1, there exists a $\{5, 29\}$-subgroup H of G such that $HN/N \in \mathrm{Hall}_{\{5,29\}}(G/N) \cap \mathcal{N}$.

If N were a p-group, for some prime p, it would follow that either H or HN would be a nilpotent Hall $\{5, 29\}$-subgroup of G, depending on the cases that either $p \notin \{5, 29\}$ or $p \in \{5, 29\}$, and so $G \in \mathcal{U}$, a contradiction. Consequently, and using Steps 1 and 2, we deduce that N is a direct product of copies of $\mathrm{Sz}(2^7)$ and $N < G$. In particular, it follows that N satisfies the $\mathrm{D}_{\{5,29\}}$-property, whence $G = NN_G(N_{\{5,29\}})$, for $N_{\{5,29\}} \in \mathrm{Hall}_{\{5,29\}}(N)$, by the Frattini Argument. Moreover, $N_G(N_{\{5,29\}}) < G \in \mathcal{V}$. Since $\mathcal{V}$ is subgroup-closed, the choice of G implies that $N_G(N_{\{5,29\}}) \in \mathcal{U}$. Now we notice that $\mathrm{Hall}_{\{5,29\}}(N_G(N_{\{5,29\}})) \subseteq \mathrm{Hall}_{\{5,29\}}(G)$. Hence $G \in \mathcal{U}$, a contradiction which proves that $\mathcal{U} = \mathcal{V}$.

Step 4. $\mathrm{Sz}(2^7) \notin \mathcal{F}$. *Hence* $\mathcal{F} \neq \mathcal{U} = \mathcal{V}$.

This is clear by the definition of $\mathcal{F}$ and Steps 2 and 3.

Remark 2.1. *Constructions in Examples 2.2(2),(4) show that it is not enough that* $\mathcal{U}$ *is subgroup-closed in order to guarantee that* $\mathcal{F} = \mathcal{U} = \mathcal{V}$. *(Notice that both* $\mathcal{F}$ *and* $\mathcal{V}$ *are subgroup-closed.)*

We characterize next when $\mathcal{U}$ and $\mathcal{V}$ are saturated.

Proposition 2.3. Let $\mathcal{X} \in \{\mathcal{U}, \mathcal{V}\}$. Structure of $G \in \mathcal{X} \setminus \mathcal{F}$ of minimal order: *If* G *is such a group then either*

(i) G *is non-abelian simple; or*

(ii) $G = [N]\langle x \rangle \in \mathfrak{P}_2$ *where* N *is a non-abelian simple group,* $N \in \mathcal{F}$ *and* $\langle x \rangle \cong C_q$ *for some prime* q *such that* $q \notin \pi(N)$.

Proof. Since $\mathcal{X}$ is closed under taking factor groups and $\mathcal{F}$ is a saturated formation, G has a unique minimal normal subgroup, say N, $G/N \in \mathcal{F}$ and $G \in \mathfrak{P}_1 \cup \mathfrak{P}_2$.

Assume first that $G \in \mathfrak{P}_1$. In this case, $N = C_G(N)$ is a p-group for some prime p. If $G/N \in f(p) = \mathcal{E}_{\pi(p)}$, then $G \in \mathcal{F}$, a contradiction. This means that there exists $q \in \pi(G/N)$ such that $q \notin \pi(p)$. Let $G_q \in \mathrm{Syl}_q(G)$. Since $G \in \mathcal{X}$, $NG_q \in \mathcal{N}$. Consequently $G_q \leq C_G(N) = N$ and $G_q = 1$, a contradiction.

Consider the case $G \in \mathfrak{P}_2$. If $G = N$, we are in the case (i). Otherwise we may assume that $N = A \times \overset{n)}{\cdots} \times A$, A a non-abelian simple group and $N < G \leq Aut(A) \wr_{nat} \mathrm{Sym}(n) = [Aut(A) \times \overset{n)}{\cdots} \times Aut(A)]\,\mathrm{Sym}(n)$, $n \geq 1$. Let us denote $B := Aut(A) \times \overset{n)}{\cdots} \times Aut(A)$. Since $\mathcal{X}$ is closed under taking normal subgroups, the choice of G implies that $A \in \mathcal{F}$. We notice that this is equivalent to the fact that $t \leftrightarrow s$ for all $t, s \in \pi(A)$.

Since $G/N \in \mathcal{F}$, if we suppose that $G/C_G(N) \in \mathcal{E}_{\pi(p)}$ for all $p \in \pi(N)$, then $G \in \mathcal{F}$, which is not the case. Consequently there exist $p \in \pi(N)$ and $q \in \pi(G) \setminus \pi(N)$ such that $p \not\leftrightarrow q$.

By the Frattini argument $G = NN_G(N_p)$ for $N_p \in \mathrm{Syl}_p(N)$ and so there exists $G_q \in \mathrm{Syl}_q(N_G(N_p)) \subseteq \mathrm{Syl}_q(G)$.

The fact $G \in \mathcal{X}$ implies now that there exists $N_pG_q \in \mathcal{N}$. In particular, $G_q \leq C_G(N_p)$, $N_p = A_p^{(1)} \times \ldots \times A_p^{(n)} \in \mathrm{Syl}_p(N)$, $A_p^{(i)} \in \mathrm{Syl}_p(A)$ for every $i = 1, \ldots, n$.

We claim that $G_q \leq G \cap B$. Let $x = a\sigma \in G_q$, where $a \in B$ and $\sigma \in \mathrm{Sym}(n)$. We have that x centralizes N_p. On the other hand, the element $a \in B$ normalizes each component A of $N = A \times \overset{n)}{\cdots} \times A$ and the element $\sigma \in \mathrm{Sym}(n)$ permutes the components of $N = A \times \overset{n)}{\cdots} \times A$. It follows that $\sigma = 1$ and $x = a \in B$.

Take $x \in G_q$ such that $o(x) = q$ and consider $L := N\langle x \rangle$. We claim that $L \in \mathcal{X}$.

In case $\mathcal{X} = \mathcal{V}$, the claim is clear because $\mathcal{V}$ is subgroup-closed.

Assume that $\mathcal{X} = \mathcal{U}$. Let $r \in \pi(N)$ and $N_r \in \mathrm{Syl}_r(N)$. Arguing as before $L = NN_L(N_r)$ and $\mathrm{Syl}_q(N_L(N_r)) \subseteq \mathrm{Syl}_q(L)$. Consequently there exists $L_{r,q} \in \mathrm{Hall}_{\{r,q\}}(L)$. If $q \not\leftrightarrow r$, then $G \in \mathcal{U} \subseteq \mathbf{E}^n_{\{q,r\}} \subseteq \mathbf{S}^n_{\{q,r\}}$. Whence $L_{r,q} \in \mathcal{N}$, which proves that $L \in \mathcal{U}$ as claimed.

If $L < G$, the choice of G implies that $L \in \mathcal{F} \subseteq \mathcal{E}_{p'}\mathcal{E}_p\mathcal{E}_{\pi(p)}$. Consequently $L \in \mathcal{E}_{\pi(p)}$ and $q \in \pi(p)$, a contradiction.

Therefore $G = L = N\langle x \rangle$. Since $x \in B$, it follows that $n = 1$ and $N = A$ is a non-abelian simple group. Now G has the structure described in (ii) and we are done. $\square$

Remark 2.2. *1. Cases in Proposition 2.3(i) may happen.*

For instance, consider Examples 2.2(1), where $G = L_2(2^4) \in \mathcal{U} \subseteq \mathcal{V}$ *but* $G \notin \mathcal{F}$*; also* $\mathrm{Alt}(5) \in \mathcal{V} \setminus \mathcal{F}$.

2. Cases in Proposition 2.3(ii) may happen.

Let us consider for instance $G = L_2(2^5)$*;* $|G| = 2^5 \cdot 31 \cdot 11 \cdot 3$*. Let* σ *be a field automorphism of* G *of order* 5 *and take* $H = [G]\langle\sigma\rangle$ *the natural semidirect product of* G *with* $\langle\sigma\rangle$*. We notice that* $H \in \mathbf{E}^{\mathrm{n}}_{\{3,5\}}$.

Set $\pi = \{2, 31, 11, 3, 5\} = \pi(2) = \pi(11) = \pi(31)$*,* $\pi(3) = \{2, 31, 11, 3\}$*,* $\pi(5) = \{2, 31, 11, 5\}$.

Let $\mathcal{F} = LF(f)$ *be locally defined by:*

$$f(p) = \mathcal{E}_{\pi(p)} \text{ if } p \in \pi, \quad f(r) = \emptyset \text{ if } r \notin \pi.$$

In this case, $\mathcal{U} = \mathbf{E}^{\mathrm{n}}_{\{3,5\}} \cap \mathcal{E}_{\pi}$.

We notice that $H \in \mathcal{U} \subseteq \mathcal{V}$, $H \notin \mathcal{F}$, $G \in \mathcal{F}$ is a non-abelian simple group and $5 \notin \pi(G)$.

Theorem 2.1. *The following statements are pairwise equivalent:*

(i) $\mathcal{V} = \mathrm{E}_\Phi \mathcal{V} := (G \mid \exists N \trianglelefteq G$ *with* $N \leq \Phi(G)$ *and* $G/N \in \mathcal{V})$;

(ii) $\mathcal{V} \cap \mathcal{M} = \emptyset$ *being*

$$\begin{aligned} \mathcal{M} := (X \mid\ & \textit{either } X \textit{ is a non-abelian simple group} \\ & \textit{or } X = [E]C_q \in \mathfrak{P}_2,\ E \textit{ a non-abelian simple group, } q \notin \pi(E); \\ & \exists r, s \in \pi(X) \subseteq \pi \textit{ such that } r \not\leftrightarrow s); \end{aligned}$$

(iii) $\mathcal{V} = \mathcal{F}$;

(iv) $\mathcal{V}$ *is a saturated formation.*

Proof. We notice first that if either $X = E$ or $X = [E]C_q \in \mathfrak{P}_2$, being E a non-abelian simple group, then $X \in \mathcal{F}$ if and only if $r \leftrightarrow s$ for all $r, s \in \pi(X) \subseteq \pi$.

(i)$\Rightarrow$(iii) Assume that $\mathcal{V} = \mathrm{E}_\Phi \mathcal{V}$ but $\mathcal{V} \neq \mathcal{F}$ and let $X \in \mathcal{V} \setminus \mathcal{F}$ of minimal order. Then either $X = E$ or $X = [E]C_q \in \mathfrak{P}_2$, $q \notin \pi(E)$, E a non-abelian simple group, by Proposition 2.3.

Since $X \notin \mathcal{F}$, there exists $r, s \in \pi(X)$ such that $r \not\leftrightarrow s$. We may assume $r \in \pi(E)$ and $s \in \pi(E) \cup \{q\}$. Let us consider $E_r(X)$ the universal Frattini r-elementary X-extension and let $A_r(X)$ be the r-Frattini module of X. So $E_r(X)/A_r(X) \cong X$ and $A_r(X) \leq \Phi(E_r(X))$. Moreover, as mentioned before, we have that

$$\mathrm{Ker}(X \text{ on } A_r(X)) \leq \mathrm{Ker}(X \text{ on } \mathrm{Soc}(A_r(X))) = O_{r'r}(X) = 1$$

by a Griess-Schmid result ([13]; see [10, Appendix β]).

Since $X \in \mathcal{V} = \mathrm{E}_\Phi \mathcal{V}$ by hypothesis, it follows that $E_r(X) \in \mathcal{V}$. Since $r \not\leftrightarrow s$ and $A_r(X)T_s$ is a $\{r, s\}$-subgroup of $E_r(X)$, for $1 \neq T_s \in \mathrm{Syl}_s(E_r(X))$, we deduce that $A_r(X)T_s$ is nilpotent and then $[T_s, A_r(X)] = 1$. This implies that the Sylow s-subgroups of X are contained in $\mathrm{Ker}(X \text{ on } A_r(X)) = 1$, a contradiction which proves that $\mathcal{V} = \mathcal{F}$.

It is clear the (iii)$\Rightarrow$(iv)$\Rightarrow$(i).

(ii)$\Rightarrow$(iii) Assume that $\mathcal{V} \cap \mathcal{M} = \emptyset$ but $\mathcal{V} \neq \mathcal{F}$ and let $G \in \mathcal{V} \setminus \mathcal{F}$ of minimal order. By Proposition 2.3 and the hypothesis we deduce that $p \leftrightarrow q$ for all $p, q \in \pi(G)$ which implies $G \in \mathcal{F}$, a contradiction which proves $\mathcal{V} = \mathcal{F}$.

(iii)$\Rightarrow$(ii) We notice that $X \in \mathcal{M}$ implies $X \notin \mathcal{F}$. Hence if $\mathcal{V} = \mathcal{F}$ then $\mathcal{V} \cap \mathcal{M} = \emptyset$. □

Theorem 2.2. *The following statements are pairwise equivalent:*

(i) $\mathcal{U} = \mathrm{E}_\Phi \mathcal{U} := (G \mid \exists N \trianglelefteq G$ *with* $N \leq \Phi(G)$ *and* $G/N \in \mathcal{U})$;

(ii) *$\mathcal{U} \cap \mathcal{M} = \emptyset$ being*

$$\begin{aligned}\mathcal{M} := (X \ :\ & \text{either } X \text{ is a non-abelian simple group} \\ & \text{or } X = [E]C_q \in \mathfrak{P}_2,\ E \text{ a non-abelian simple group, } q \notin \pi(E); \\ & \exists r, s \in \pi(X) \subseteq \pi \text{ such that } r \not\leftrightarrow s);\end{aligned}$$

(iii) *$\mathcal{U} = \mathcal{F}$;*

(iv) *$\mathcal{U}$ is a saturated formation.*

Proof. (i)⇒(iii) Assume that $\mathcal{U} = \mathrm{E}_\Phi \mathcal{U}$ but $\mathcal{U} \neq \mathcal{F}$ and let $X \in \mathcal{U} \backslash \mathcal{F}$ of minimal order. Then either $X = E$ or $X = [E]C_q \in \mathfrak{P}_2$, $q \notin \pi(E)$, E a non-abelian simple group, by Proposition 2.3.

Since $X \notin \mathcal{F}$, there exists $r, s \in \pi(X)$ such that $r \not\leftrightarrow s$. We may assume $r \in \pi(E)$ and $s \in \pi(E) \cup \{q\}$. As in Theorem 2.1 we consider $E_r(X)$ the universal Frattini r-elementary X-extension and note that the hypothesis implies that $E_r(X) \in \mathcal{U}$. But we know that $\mathcal{U} \subseteq \mathcal{V}$. Then we can argue as in the proof of Theorem 2.1 to deduce that the Sylow s-subgroups of X are trivial, a contradiction which proves that $\mathcal{U} = \mathcal{F}$.

The rest of the proof follows as in the proof of Theorem 2.1. □

3 Sylow Normalizers

In the universe of finite soluble groups, for a subgroup-closed saturated formation $\mathcal{H} \subseteq \mathcal{S}$, the following equivalence holds (Theorem 1.1):

$$\mathrm{N}\mathcal{H} \cap \mathcal{S} = \mathcal{H} \iff \mathcal{H} \text{ is a covering-formation of soluble groups.}$$

In contrast, none of the implications in this equivalence remains valid when extending to the universe of all finite groups. An example in [9, Remark 1(c)] shows that a covering-formation does not need to be N-closed in general. Also, there is an example in [8] of a subgroup-closed saturated formation $\mathcal{X}$ with the property $\mathrm{N}\mathcal{X} = \mathcal{X}$ which is not a covering-formation. (See also [17, Remark, p. 270] for some additional information.)

We analyze in this section possible approaches for positive results in the finite universe. First, Theorem 3.1 below provides additional conditions in terms of the canonical local definition of a subgroup-closed saturated formation $\mathcal{H}$ to guarantee that $\mathrm{N}\mathcal{H} = \mathcal{H}$ implies that $\mathcal{H}$ is a covering-formation. We will study afterwards covering-formations $\mathcal{F}$ which satisfy $\mathrm{N}\mathcal{F} = \mathcal{F}$.

Proposition 3.1. *Let $\mathcal{H}$ be a subgroup-closed saturated formation and let H denote its canonical local definition. Set $\pi := \mathrm{Char}(\mathcal{H})$ and $\pi(p) := \mathrm{Char}(H(p))$ for each $p \in \pi$. Assume that $\mathrm{N}\mathcal{H} = \mathcal{H}$. Then:*

1. *If $p \in \pi$, then $\mathrm{N}H(p) = H(p)$ if and only if $\mathcal{E}_{\pi(p)} \cap \mathcal{H} = H(p)$.*

2. *If $2 \in \pi$, then $\mathcal{E}_{\pi(2)} \cap \mathcal{H} = H(2)$. In particular, $\mathrm{N}H(2) = H(2)$ and $H(2)$ is a subgroup-closed saturated formation.*

Proof. Since $\mathcal{H}$ is subgroup closed, it follows that $H(p)$ is also subgroup-closed by [10, IV.3.16] and, consequently, $H(p) \subseteq \mathrm{N}H(p)$ and $H(p) \subseteq \mathcal{E}_{\pi(p)}$ for all $p \in \pi$.

We prove first that for $p \in \pi$, if $\mathcal{E}_{\pi(p)} \cap \mathcal{H} = H(p)$, then $\mathrm{N}H(p) = H(p)$.

Let $p \in \pi$, assume that $\mathcal{E}_{\pi(p)} \cap \mathcal{H} = H(p)$ and consider $G \in \mathrm{N}H(p)$. We need to prove that $G \in H(p)$. But $G \in \mathrm{N}H(p) \subseteq \mathrm{N}\mathcal{H} = \mathcal{H}$ because $H(p) \subseteq \mathcal{H}$. Moreover $\pi(G) \subseteq \pi(p)$ because $H(p) \subseteq \mathcal{E}_{\pi(p)}$. Whence $G \in \mathcal{H} \cap \mathcal{E}_{\pi(p)} = H(p)$ and we are done.

Now we consider $p \in \pi$. If $p \neq 2$, we assume in addition that $\mathrm{N}H(p) = H(p)$. We prove next that $\mathcal{E}_{\pi(p)} \cap \mathcal{H} = H(p)$, which will conclude the proof.

Assume that this is not true; then $H(p)$ is properly contained in $\mathcal{E}_{\pi(p)} \cap \mathcal{H}$. We mimic the proof of [9, Theorem]. Then let $\mathcal{X} = (G \in (\mathcal{E}_{\pi(p)} \cap \mathcal{H}) \setminus H(p) \mid |(\pi(G) \cup \{p\}) \setminus \{p\}|$ is minimal). Let us consider $G \in \mathcal{X}$ such that $|G|$ is minimal. By the choice of G, there exists a unique minimal normal subgroup N of G and $G/N \in H(p)$. If N were a p-group, then $G \in \mathcal{S}_p H(p) = H(p)$, a contradiction. Consequently, N is not a p-group.

Step 1. $p \notin \pi(G)$. We point out that in case $p = 2$, G is soluble by the Feit-Thompson theorem.

Argue as in [9, Proof of Theorem, Step 1] with obvious changes.

Step 2. N is abelian; in particular, N is a q-group for some prime $q \neq p$. Moreover, $\Phi(G) < F(G) = O_q(G)$.

If $p = 2$, then G is soluble by step 1 and the result is clear.

Assume now that $p \neq 2$. Then we have in addition that $\mathrm{N}H(p) = H(p)$.

If N were not abelian, then $N_G(G_r) < G$ for all $r \in \pi(G)$ and $G_r \in \mathrm{Syl}_r(G)$.

If N were a q-group but $\Phi(G) = F(G) = O_q(G)$, then $q \in \pi(G/O_q(G))$, which implies that $N_G(G_q) < G$ for $G_q \in \mathrm{Syl}_q(G)$. If $r \in \pi(G)$, $r \neq q$, then $N_G(G_r) < G$ for $G_r \in \mathrm{Syl}_r(G)$.

Hence, in both considered cases, the choice of G would imply that $N_G(G_r) \in H(p)$ for all $r \in \pi(G)$. Since $\mathrm{N}H(p) = H(p)$, it would follow that $G \in H(p)$, a contradiction which proves step 2.

The rest of the proof follows arguing as in [9, Proof of Theorem, Step 2 – Step 5] with obvious changes. □

Proposition 3.2. *Let $\mathcal{H}$ be a subgroup-closed saturated formation and let H denote its canonical local definition. Set $\pi := \mathrm{Char}(\mathcal{H})$ and $\pi(p) := \mathrm{Char}(H(p))$ for each $p \in \pi$. Assume that $\mathrm{N}\mathcal{H} = \mathcal{H}$. Then the following conditions are equivalent:*

1. $\mathcal{E}_{\pi(p)} \cap \mathcal{H} = H(p)$ *for all* $p \in \pi$,

2. *$\mathcal{H}$ is a covering-formation (defined by the sets of primes $\pi(p)$ for each prime $p \in \pi$).*

Proof. It is clear that condition 2 implies condition 1.

Assume now that condition 1 holds. We notice first that $\mathrm{N}\mathcal{H} = \mathcal{H}$ implies that $(p \in \pi(q) \Leftrightarrow q \in \pi(p))$ for all $p, q \in \pi$, by [9, Proposition 1]. Then we need to prove that $\mathcal{H} = \mathcal{F}$ for $\mathcal{F}$ the covering-formation of characteristic π and defined by the sets of primes $\pi(p)$ for each $p \in \pi$.

Since $\mathcal{H}$ is subgroup-closed, it is known that for any $p \in \pi$, $H(p)$ is also subgroup-closed, which implies that $H(p) \subseteq \mathcal{E}_{\pi(p)}$. It follows that $\mathcal{H} \subseteq \mathcal{F}$. Assume that $\mathcal{H} \neq \mathcal{F}$ and let $G \in \mathcal{F} \setminus \mathcal{H}$ of minimal order. Then G has a unique minimal normal subgroup N and $G/N \in \mathcal{H}$.

If $N_G(G_p) < G$ for all $p \in \pi(G)$ and $G_p \in \mathrm{Syl}_p(G)$, then the choice of G implies that $G \in \mathrm{N}\mathcal{H} = \mathcal{H}$, a contradiction. Consequently, there exists $p \in \pi(G)$ such that $N_G(G_p) = G$ with $G_p \in \mathrm{Syl}_p(G)$. Then N is a p-group, $G \in \mathcal{E}_{\pi(p)}$ as $G \in \mathcal{F}$, and so $G/N \in \mathcal{E}_{\pi(p)} \cap \mathcal{H} = H(p)$. Hence $G \in \mathcal{S}_p H(p) = H(p) \subseteq \mathcal{H}$, the final contradiction. □

Remark 3.1. *Let $\mathcal{H}$ be a (subgroup-closed) saturated formation, let H denote its canonical local definition and set $\pi := \mathrm{Char}(\mathcal{H})$ and $\pi(p) := \mathrm{Char}(H(p))$ for each $p \in \pi$. In contrast to the behaviour of saturated formations of soluble groups (see [10, IV.3.8(b)]), the fact that $\mathcal{E}_{\pi(p)} \cap \mathcal{H} = H(p)$ for all $p \in \pi$ does not imply in general that the saturated formation $\mathcal{H}$ is locally defined by the formation function h given by $h(p) = \mathcal{E}_{\pi(p)}$ if $p \in \pi$ and $h(q) = \emptyset$ if $q \notin \pi$.*

To see this it is enough to consider π any non-empty set of primes and $\mathcal{H} = \mathcal{S}_\pi$.

This shows that in Proposition 3.2 the hypothesis $\mathrm{N}\mathcal{H} = \mathcal{H}$ can not be omited.

Theorem 3.1. *Let $\mathcal{H}$ be a subgroup-closed saturated formation and let H denote its canonical local definition. Set $\pi := \mathrm{Char}(\mathcal{H})$ and $\pi(p) := \mathrm{Char}(H(p))$ for each $p \in \pi$. Assume that $\mathrm{N}\mathcal{H} = \mathcal{H}$. Then:*

$\mathrm{N}H(p) = H(p)$ *for all* $p \in \pi$, $p \neq 2 \iff \mathcal{H}$ *is a covering formation.*

Proof. It follows by Propositions 3.1 and 3.2. □

Remark 3.2. *From example in [8] giving a subgroup-closed saturated formation which is N-closed but not a covering-formation and Theorem 3.1 we may deduce that for a subgroup-closed saturated formation $\mathcal{H}$, the fact that $\mathrm{N}\mathcal{H} = \mathcal{H}$ does not imply in general that $\mathrm{N}H(p) = H(p)$ for its canonical local definition H and $p \in \mathrm{Char}\,\mathcal{H}$ (unless $p = 2$; see Proposition 3.1).*

In [17, Examples 1,2,3] we showed some particular constructions of covering-formations $\mathcal{F}$ satisfying that $\mathrm{N}\mathcal{F} = \mathcal{F}$ and noticed that for them $\mathcal{F} = \mathcal{U}$. We prove next that this is not casual but the property $\mathcal{F} = \mathcal{U} = \mathcal{V}$ is a consequence of satisfying $\mathrm{N}\mathcal{F} = \mathcal{F}$. However we will show afterwards that the converse does not hold.

Theorem 3.2. *For the covering-formation $\mathcal{F}$, if $\mathrm{N}\mathcal{F} = \mathcal{F}$, then $\mathcal{F} = \mathcal{U} = \mathcal{V}$.*

Proof. We know by Proposition 2.1 that for the covering-formation $\mathcal{F}$, $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{V}$. We prove next that $\mathcal{V} \subseteq \mathcal{F}$ if $\mathrm{N}\mathcal{F} = \mathcal{F}$.

Assume that this is not true and let $G \in \mathcal{V} \setminus \mathcal{F}$ of minimal order. Since $\mathcal{V}$ is subgroup-closed, the choice of G implies that any proper subgroup of G

belong to $\mathcal{F}$. If $N_G(G_p) < G$ for all $p \in \pi(G)$, we deduce that $G \in \mathrm{N}\mathcal{F} = \mathcal{F}$ by hypothesis, a contradiction. Hence $G_p \trianglelefteq G$ for some $p \in \pi(G)$.

Since, in addition, $\mathcal{V}$ is closed under taking factor groups and $\mathcal{F}$ is a saturated formation, we deduce that G is a primitive group of type 1, with a unique minimal normal subgroup $N = G_p$, $C_G(N) = N$, a maximal subgroup M such that $G = NM$ with $N \cap M = 1$ and $G/N \in \mathcal{F}$.

We claim that $\pi(M) \subseteq \pi(p)$. Let $q \in \pi(M)$. If $q \notin \pi(p)$ then NM_q is nilpotent because $G \in \mathcal{V}$ and $M \leq C_G(N) = N$, a contradition which proves the claim.

Hence $G/N \in \mathcal{E}_{\pi(p)} = f(p)$ and $G/N \in \mathcal{F}$, which implies that $G \in \mathcal{F}$, a contradiction which concludes the proof. □

Examples 3.1.

1. For a covering-formation $\mathcal{F}$, the fact that $\mathcal{F} = \mathcal{U} = \mathcal{V}$ is not enough to guarantee that $\mathrm{N}\mathcal{F} = \mathcal{F}$.

We construct the covering-formation $\mathcal{F}$ defined by the following sets of primes:

$$\pi(2) = \mathbb{P} \setminus \{3\}, \quad \pi(3) = \mathbb{P} \setminus \{2\}, \quad \pi(p) = \mathbb{P} \text{ if } p \neq 2, 3,$$

$\mathbb{P}$ the set of all prime numbers. We notice that $\mathcal{F}$ has full characteristic, i.e., $\mathrm{Char}(\mathcal{F}) = \mathbb{P}$. Moreover, for this covering-formation $\mathcal{F}$, we have:

$$\mathcal{F} = \mathcal{E}_{2'}\mathcal{E}_{\pi(2)} \cap \mathcal{E}_{3'}\mathcal{E}_{\pi(3)} = \mathcal{E}_{2'}\mathcal{E}_{3'} \cap \mathcal{E}_{3'}\mathcal{E}_{2'}$$

$$\Sigma_{\mathcal{F}} = \{\sigma \subseteq \mathbb{P} \mid \ |\sigma| \geq 2\,, (p, q \in \sigma,\, p \neq q \;\Rightarrow\; p \not\leftrightarrow q)\} = \{\sigma\} \text{ with } \sigma = \{2, 3\}$$

$$\mathcal{U} = \mathbf{E}^{\mathrm{n}}_{\{2,3\}} \ ; \quad \mathcal{V} = \mathbf{S}^{\mathrm{n}}_{\{2,3\}}$$

We know that $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{V}$. We claim that $\mathcal{F} = \mathcal{U} = \mathcal{V}$.

We prove first that $\mathcal{F} = \mathcal{U}$. Assume that this is not true and let $G \in \mathcal{U} \setminus \mathcal{F}$ of minimal order. By Proposition 2.3, G is either non-abelian simple, or $G = [N]\langle x \rangle \in \mathfrak{P}_2$ where N is a non-abelian simple group, $N \in \mathcal{F}$ and $\langle x \rangle \cong C_p$ for some prime p such that $p \notin \pi(N)$.

Assume that G is simple. It is known that no simple group E possesses nilpotent Hall ν-subgroups with ν a set of primes such that $|\nu \cap \pi(E)| > 1$ and $2 \in \nu$ (see [18, Corollary] and [1, Proposition 1]). Since $G \in \mathcal{U}$ it follows that $3 \notin \pi(G)$; i.e., $G \in \mathcal{E}_{3'} \subseteq \mathcal{F}$, a contradiction.

Consider now the case $G = [N]\langle x \rangle$ as above. Since $G \notin \mathcal{F}$ and $N \in \mathcal{U}$, we deduce that $3 \in \pi(G)$ but $3 \notin \pi(N)$, again by the same above-mentioned result. Consequently, $p = 3$ and $N = \mathrm{Sz}(q)$. From $G \in \mathcal{U}$ it follows that x centralizes a Sylow 2-subgroup of $\mathrm{Sz}(q)$, but this is a contradiction because no outer-automorphism of $\mathrm{Sz}(q)$ satisfies these facts. This proves that $\mathcal{F} = \mathcal{U}$.

We notice that Suzuki groups $\mathrm{Sz}(q)$ satisfy $\mathrm{Sz}(q) \in \mathcal{E}_{3'} \subseteq \mathcal{F}$; consequently, $\mathcal{F} = \mathcal{U} \not\subseteq \mathcal{S}$. (In fact, Suzuki groups $\mathrm{Sz}(q)$ are the only non-abelian simple groups in $\mathcal{F}$.)

We prove next that $\mathcal{F} = \mathcal{U} = \mathcal{V}$. We know that $\mathcal{F} = \mathcal{U} \subseteq \mathcal{V}$. Assume that the result is not true and let $G \in \mathcal{V} \setminus \mathcal{F} = \mathcal{V} \setminus \mathcal{U}$ of minimal order. As above, by Proposition 2.3, either G is a non-abelian simple, or $G = [N]\langle x \rangle \in \mathfrak{P}_2$ where N

is a non-abelian simple group, $N \in \mathcal{F}$ and $\langle x \rangle \cong C_p$ for some prime p such that $p \notin \pi(N)$. Since $G \notin \mathcal{F}$, arguing as above, we deduce that $3 \in \pi(G)$.

Assume that G is simple. Then it follows that $|G| \equiv 0 \pmod{12}$ and G contains non-nilpotent $\{2,3\}$-subgroups, whence $G \notin \mathcal{V}$, a contradiction.

Consider now the case $G = [N]\langle x \rangle$ as above. Since $N \in \mathcal{F} = \mathcal{U}$ we deduce again that N is a $3'$-group, as N is simple, and $p = 3$. Moreover, by the Frattini Argument, $G = NN_G(N_2)$ with $N_2 \in \mathrm{Syl}_2(N)$. It is clear that $N_G(N_2)$ is a proper subgroup of G, whence $1 \neq N_G(N_2) \in \mathcal{U}$. Consequently, $\emptyset \neq \mathrm{Hall}_{\{2,3\}}(N_G(N_2)) \cap \mathcal{N} \subseteq \mathrm{Hall}_{\{2,3\}}(G) \cap \mathcal{N}$, which implies that $G \in \mathcal{U}$, a contradiction.

We have now proved that $\mathcal{F} = \mathcal{U} = \mathcal{V} \nsubseteq \mathcal{S}$.

Finally we show that $\mathrm{N}\mathcal{F} \neq \mathcal{F}$. Consider $G = \mathrm{L}_2(3^n)$ with n odd. It is easy to check that $G \notin \mathcal{F}$. We claim that $G \in \mathrm{N}\mathcal{F}$.

Let $p \in \pi(G)$, $p \neq 2, 3$. For any Sylow p-subgroup G_p of G we have that $N_G(G_p) \in \mathcal{S}_{3'} \subseteq \mathcal{F}$.

We notice that $|G| = \frac{1}{2}3^n(3^n+1)(3^n-1)$ and $\frac{3^n-1}{2}$ is odd, as n is odd. Whence $|G|_2 = (3^n+1)_2$ and $N_G(G_2) = G_2 \in \mathcal{F}$ for $G_2 \in \mathrm{Syl}_2(G)$.

Moreover, for $G_3 \in \mathrm{Syl}_3(G)$, it follows that $N_G(G_3) \in \mathcal{E}_{2'} \subseteq \mathcal{F}$, which proves the claim and shows finally that $G \in \mathrm{N}\mathcal{F} \setminus \mathcal{F}$.

2. We modify now the covering-formation $\mathcal{F}$ constructed above, in part 1, by considering $\mathcal{G} := \mathcal{F} \cap \mathcal{E}_{5'}$. Then $\mathcal{G}$ is again a covering-formation defined by the sets of primes:

$$\pi(2) = \mathbb{P} \setminus \{3,5\},\ \pi(3) = \mathbb{P} \setminus \{2,5\},\ \pi(p) = \mathbb{P} \setminus \{5\} \text{ if } p \neq 2,3,5,\ \pi(5) = \emptyset,$$

and $\mathrm{Char}(\mathcal{G}) = \mathbb{P} \setminus \{5\}$. Corresponding to $\mathcal{G}$, set

$$\mathcal{U}_{\mathcal{G}} := \bigcap_{\sigma \in \Sigma_{\mathcal{G}}} \mathbf{E}^{\mathrm{n}}_{\sigma} \cap \mathcal{E}_{5'} = \mathbf{E}^{\mathrm{n}}_{\{2,3\}} \cap \mathcal{E}_{5'}$$

and

$$\mathcal{V}_{\mathcal{G}} := \bigcap_{\sigma \in \Sigma_{\mathcal{G}}} \mathbf{S}^{\mathrm{n}}_{\sigma} \cap \mathcal{E}_{5'} = \mathbf{S}^{\mathrm{n}}_{\{2,3\}} \cap \mathcal{E}_{5'}.$$

We show next that $\mathcal{G} = \mathcal{U}_{\mathcal{G}} = \mathcal{V}_{\mathcal{G}} \subseteq \mathcal{S}$ and $\mathrm{N}\mathcal{G} \nsubseteq \mathcal{G}$, in contrast to known results in the soluble universe (see Proposition 1.1).

It is clear from part 1 that $\mathcal{G} = \mathcal{U}_{\mathcal{G}} = \mathcal{V}_{\mathcal{G}}$. We notice in addition that for $G = \mathrm{L}_2(3^3)$, it holds that $5 \notin \pi(G)$. Then, again by part 1, we have that $G \in \mathrm{N}\mathcal{G} \setminus \mathcal{G}$. Moreover, if G were a non-abelian simple group in $\mathcal{G} = \mathcal{U}_{\mathcal{G}} = \mathcal{V}_{\mathcal{G}}$, then $3, 5 \notin \pi(G)$, which is not possible. Whence $\mathcal{G} = \mathcal{U}_{\mathcal{G}} = \mathcal{V}_{\mathcal{G}} \subseteq \mathcal{S}$.

3. There exists also a covering-formation satisfying the properties as in part 2, and which has in addition full characteristic.

Consider the covering-formation $\mathcal{G}$ defined in part 2 and construct

$$\mathcal{H} := \mathcal{G} \times \mathcal{S}_5 := (G = A \times B \mid A \in \mathcal{G},\ B \in \mathcal{S}_5).$$

Then $\mathcal{H}$ is a covering-formation defined by the sets of primes:

$$\pi(2) = \mathbb{P} \setminus \{3,5\},\ \pi(3) = \mathbb{P} \setminus \{2,5\},\ \pi(p) = \mathbb{P} \setminus \{5\} \text{ if } p \neq 2,3,5,\ \pi(5) = \{5\},$$

and $\mathrm{Char}(\mathcal{H}) = \mathbb{P}$. Corresponding to $\mathcal{H}$, set

$$\mathcal{U}_{\mathcal{H}} := \bigcap_{\sigma \in \Sigma_{\mathcal{H}}} \mathbf{E}^{\mathrm{n}}_{\sigma} = \mathbf{E}^{\mathrm{n}}_{\{2,3\}} \cap \mathbf{E}^{\mathrm{n}}_{\{2,3,5\}} \cap (\bigcap_{p \in \mathbb{P} \setminus \{5\}} \mathbf{E}^{\mathrm{n}}_{\{5,p\}})$$

and

$$\mathcal{V}_{\mathcal{H}} := \bigcap_{\sigma \in \Sigma_{\mathcal{H}}} \mathbf{S}^{\mathrm{n}}_{\sigma} = \mathbf{S}^{\mathrm{n}}_{\{2,3\}} \cap \mathbf{S}^{\mathrm{n}}_{\{2,3,5\}} \cap (\bigcap_{p \in \mathbb{P} \setminus \{5\}} \mathbf{S}^{\mathrm{n}}_{\{5,p\}}).$$

We claim that $\mathcal{H} = \mathcal{U}_{\mathcal{H}} = \mathcal{V}_{\mathcal{H}} \subseteq \mathcal{S}$, $\mathrm{Char}(\mathcal{H}) = \mathbb{P}$ and $\mathrm{N}\mathcal{H} \nsubseteq \mathcal{H}$.

It is clear that $\mathcal{H} \subseteq \mathcal{S}$ and it is known that $\mathcal{H} \subseteq \mathcal{U}_{\mathcal{H}} \subseteq \mathcal{V}_{\mathcal{H}}$. We prove first that $\mathcal{H} = \mathcal{V}_{\mathcal{H}}$, which will imply $\mathcal{H} = \mathcal{U}_{\mathcal{H}} = \mathcal{V}_{\mathcal{H}}$. Assume that the result is not true and let $G \in \mathcal{V}_{\mathcal{H}} \setminus \mathcal{H}$ of minimal order. We know that G is an almost simple group by Proposition 2.3. We notice that $5 \in \pi(G)$ since, from part 1, $G \in \mathcal{V}_{\mathcal{H}} \subseteq \mathbf{S}^{\mathrm{n}}_{\{2,3\}} = \mathcal{F}$ and the only non-abelian simple groups in $\mathcal{F}$ are Suzuki groups, which are $3'$-groups. Moreover, for each prime $p \in \pi(G)$, $p \neq 5$, if G_p is a Sylow p-subgroup of G, then $5 \notin \pi(N_G(G_p)/C_G(G_p))$, and also $N_G(G_5)/C_G(G_5) \in \mathcal{S}_5$. It follows that the Sylow graph of G defined in [17] is not connected, which is a contradiction by [17, Main Theorem].

For $G = \mathrm{L}_2(3^3)$ we know from part 2 that $G \notin \mathcal{G}$ and then $G \notin \mathcal{H}$. Moreover, $G \in \mathrm{N}\mathcal{G} \subseteq \mathrm{N}\mathcal{H}$ as $\mathcal{G} \subseteq \mathcal{H}$, which proves finally the claim.

Theorem 3.3. *For the covering-formation $\mathcal{F}$ the following statements are equivalent:*

(i) *$\mathcal{F}$ has the Shemetkov property;*

(ii) $\mathcal{F} = \mathcal{U} = \mathcal{V}$.

Proof. We assume first that $\mathcal{F}$ verifies condition (ii) and we prove that $\mathcal{F}$ satisfies condition 3 in Theorem 1.2, and then, equivalently, $\mathcal{F}$ has the Shemetkov property.

We need then to prove that a π-group G belongs to $\mathcal{F}$ if and only if $N_G(P)/C_G(P)$ belongs to $\mathcal{E}_{\pi(p)}$ for each p-subgroup P of G and each prime $p \in \pi$.

Let $G \in \mathcal{F}$ and P be a p-subgroup of G for some $p \in \pi$. Let $q \in \pi(N_G(P))$ and $Q \in \mathrm{Syl}_q(N_G(P))$. Since $\mathcal{F}$ is subgroup-closed, $PQ \in \mathcal{F}$. Moreover $\mathcal{F} = \mathcal{V}$. Hence if $q \notin \pi(p)$, it follows that PQ is nilpotent and $Q \leq C_G(P)$. Consequently $N_G(P)/C_G(P) \in \mathcal{E}_{\pi(p)}$.

Conversely assume now that $N_G(P)/C_G(P) \in \mathcal{E}_{\pi(p)}$ for each p-subgroup P of G and each prime $p \in \pi$. We prove that $G \in \mathcal{F} = \mathcal{V}$.

Let $\tau \in \Sigma$ and let H be a τ-subgroup of G. We claim that H is p-nilpotent for every $p \in \tau$. Hence H will be nilpotent and we will be done.

Let $p \in \tau$ and P be a p-subgroup of H. Since $N_H(P) \leq N_G(P)$ it follows that $N_H(P)/C_H(P) \in \mathcal{E}_{\pi(p)} \cap \mathcal{E}_{\tau} = \mathcal{E}_p$. It follows that H is p-nilpotent, by the Frobenious p-nilpotence criterion, as claimed.

Now assume that $\mathcal{F}$ has the Shemetkov property but condition (ii) does not hold. Let $X \in \mathcal{V} \setminus \mathcal{F}$ of minimal order. Then, by Proposition 2.3, either $X = E$ or $X = [E]C_q \in \mathfrak{P}_2$, being E a non-abelian simple group and $q \notin \pi(E)$. In the second case, $E \in \mathcal{F}$.

Assume $X = E$. Since $\mathcal{V}$ is subgroup-closed, $X \in \mathrm{Crit}_{\mathrm{S}}(\mathcal{F}) \cap b(\mathcal{F})$ but this contradicts that $\mathcal{F}$ has the Shemetkov property by Theorem 1.2.

Assume $X = [E]C_q$. Since $\mathcal{V}$ is subgroup-closed, $X \in \mathrm{Crit}_{\mathrm{S}}(\mathcal{F}) \cap b(\mathcal{F})$ and X is an almost simple group.

Since $X \notin \mathcal{F}$ but $E \in \mathcal{F}$, there exists $p \in \pi(E)$, $E = \mathrm{Soc}(X)$, such that $q \notin \pi(p)$. In particular, $X \notin f(p) = \mathcal{E}_{\pi(p)}$. By Theorem 1.2 we have now that $X \notin f(r) = \mathcal{E}_{\pi(r)}$ for all $r \in \pi(E)$. Since $E \in \mathcal{F}$, this means that $q \notin \pi(r)$ for all $r \in \pi(E)$.

For each $r \in \pi(E)$, there exists $E_r \in \mathrm{Syl}_r(E)$ such that C_q normalizes E_r. Since $r \not\leftrightarrow q$ and $X \in \mathcal{V}$, we deduce that E_rC_q is nilpotent. Consequently, $C_q \leq C_X(E) = 1$, a contradiction. □

Remark 3.3. *In Theorem 3.3 the hypothesis that $\mathcal{F}$ is a covering-formation can not be avoid in order to prove that condition (i) implies condition (ii).*

To see this consider the class $\mathcal{H} = \mathcal{E}_{p'}\mathcal{E}_p$ of all p-nilpotent groups for a prime p. Then $\mathcal{H}$ is a subgroup-closed saturated formation with the Shemetkov property by a result of Ito (c.f. [16, IV.5.4]; see also [2, Example 4]). But $\mathcal{H}$ is not contained in any class of groups characterized by having nilpotent Hall subgroups for some set of primes.

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