

Graphs and Classes of Finite Groups

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Abstract. There are different ways to associate to a finite group a certain graph. An interesting question is to analyse the relations between the structure of the group, given in group-theoretical terms, and the structure of the graph, given in the language of graph theory. This survey paper presents some contributions to this research line.

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All groups in this paper are finite. We will consider only graphs which are undirected, simple (that is, with no parallel edges), and without loops. These graphs will be characterised by the set of vertices and the adjacency relation between the vertices. Only basic concepts about graphs will be needed for this paper. They can be found in any book about graph theory or discrete mathematics, for example [12].

Given a group G , there are many ways to associate a graph with G by taking families of elements or subgroups as vertices and letting two vertices be joined by an edge if and only if they satisfy a certain relation. We may ask about characterising group structural properties by means of the properties of the associated graph. In recent years, there has been considerable interest in this line of research (see [1, 2, 3, 5, 7, 10]).

If \mathfrak{X} is a class of groups, Delizia, Moravec, and Nicotera [10] associate a graph $\Gamma_{\mathfrak{X}}(G)$ with a group G by taking the nontrivial elements of G as vertices and letting $a, b \in G \setminus \{1\}$ be joined by an edge if $\langle a, b \rangle \in \mathfrak{X}$. If we choose $\mathfrak{X} = \mathfrak{A}$, where \mathfrak{A} is the class of all abelian groups, then the graph is just the *commuting graph* of G and so we may think of this graph as a generalisation of

the commuting graph. This graph has been used to study simple groups since the paper of Stellmacher [18].

For an arbitrary class, the graph considered by Delizia, Moravec, and Nicotera is too general to give much information and so we restrict the classes we consider. We suppose that our class \mathfrak{X} is *subgroup closed and contains \mathfrak{A}* .

Delizia, Moravec, and Nicotera restrict their attention to groups which are \mathfrak{X} -transitive. A group G is said to be \mathfrak{X} -transitive, or an $\mathfrak{X}T$ -group, if whenever $\langle a, b \rangle \in \mathfrak{X}$ and $\langle b, c \rangle \in \mathfrak{X}$, then $\langle a, c \rangle \in \mathfrak{X}$ ($a, b, c \in G$). Note that the \mathfrak{A} -transitive groups are just the CT -groups, or groups in which the centraliser of any non-identity element is an abelian subgroup. CT -groups are of historical importance as an early example of the type of classification that would be used in the FeitThompson theorem and the classification of simple groups and were classified by several mathematicians (Weisner [20], Suzuki [19], Wu [23]). Characterisations of $\mathfrak{X}T$ -groups are known for several classes of groups (see [11] for a survey).

Let G be an $\mathfrak{X}T$ -group. If \mathcal{C} is a connected component of $\Gamma_{\mathfrak{X}}(G)$, then \mathcal{C} is a complete graph, $\mathcal{C} \cup \{1\}$ is a subgroup of G and $\{\mathcal{C} \cup \{1\} : \mathcal{C} \text{ a connected component of } \Gamma_{\mathfrak{X}}(G)\}$ is a partition of G ([17] p.145). As a consequence of the classification of groups with a partition (see for example [17, Theorems 3.5.10 and 3.5.1]) we have the following result.

Theorem 1. Let G be an $\mathfrak{X}T$ -group.

- (1) The connected components of $\Gamma_{\mathfrak{X}}(G)$ form a normal partition of $G \setminus \{1\}$, that is, conjugates of connected components are again connected components.
- (2) Either $\Gamma_{\mathfrak{X}}(G)$ is connected or G is one of the following groups: a Frobenius group, $\text{PSL}(2, 2^h)$, $\text{Sz}(2^h)$ (a Suzuki group, $h = 2k + 1 > 1$).

Note that a group in the partition has the property that every 2-generator subgroup is an \mathfrak{X} -group. In general this will not ensure that the group is an \mathfrak{X} -group. For instance, \mathfrak{N}_2 , the class of (finite) nilpotent groups of class at most 2, is an example, since the free group of exponent 3 on 3 generators has class 3, but every 2-generator subgroup has class at most 2 (see [16]). A class \mathfrak{X} of groups has been called 2-recognisable (2-erkennbar) by Brandl [9] if a group G is in \mathfrak{X} if and only if every 2-generator subgroup of G is in \mathfrak{X} .

Observe that G is an $\mathfrak{X}T$ -group and $\Gamma_{\mathfrak{X}}(G)$ is connected, then every 2-generator subgroup of G is an \mathfrak{X} -group and so if \mathfrak{X} is 2-recognisable then $G \in \mathfrak{X}$.

Many classes of groups are known to be 2-recognisable; abelian, nilpotent, supersoluble and soluble among them. Some of the classes of supersoluble groups that have been extensively investigated in recent years are 2-recognisable.

We consider here the following classes: the class \mathfrak{T} of soluble T -groups (soluble groups in which normality is transitive), the class \mathfrak{PT} of soluble PT -groups (soluble groups in which permutability is transitive) and the class \mathfrak{PST} of soluble PST -groups (soluble groups in which Sylow permutability is transitive). That each of these classes is subgroup closed, it is contained in the class of all supersoluble groups and contains all abelian groups is well known (a description of these classes and their properties can be found in [6]).

Lemma 2. The classes \mathfrak{T} , \mathfrak{PT} , and \mathfrak{PST} are all 2-recognisable.

We now define \mathfrak{D} to be the class of groups G with all Sylow subgroups cyclic (these are just the groups with a cyclic normal subgroup whose quotient is cyclic and whose order and index are coprime, see [24, Theorem V.3.11]). We then define $\mathfrak{F}_{\mathfrak{X}}$ to be the class of Frobenius groups with the property that the kernel K is an \mathfrak{X} -group, $G/K \in \mathfrak{D}$ and for each prime $p \mid |G/K|$ and each prime $q \mid |K|$, p does not divide $q - 1$.

Theorem 3. Let \mathfrak{X} be a 2-recognisable class of soluble groups containing $\mathfrak{A} \cup \mathfrak{D}$. Then the class of all \mathfrak{XT} -groups is contained in $\mathfrak{X} \cup \mathfrak{F}_{\mathfrak{X}}$.

Theorem 4. Let \mathfrak{X} be a 2-recognisable class of soluble groups containing $\mathfrak{A} \cup \mathfrak{D}$ and G be an \mathfrak{XT} -group.

- (1) $G \in \mathfrak{X}$ if and only if $\Gamma_{\mathfrak{X}}(G)$ is connected.
- (2) $G \notin \mathfrak{X}$ if and only if G is a Frobenius group and $K \setminus \{1\}$ is a connected component of $\Gamma_{\mathfrak{X}}(G)$ (where K is the kernel of G).

Bearing in mind the above results, it is natural to ask for the smallest 2-recognisable class of groups containing $\mathfrak{A} \cup \mathfrak{D}$.

Proposition 5. Let \mathfrak{T}_0 be the class of \mathfrak{T} -groups G with G/G' cyclic or G abelian. Then \mathfrak{T}_0 is the smallest 2-recognisable class of groups containing $\mathfrak{A} \cup \mathfrak{D}$.

The following variation of the commuting graph gives a characterisation for the groups in which all subgroups are permutable. We will call it the *graph of permutability of cyclic subgroups* (see [4]). Given a group G , consider the graph in which the vertices are the cyclic subgroups of G and in which every two vertices are adjacent when they permute. A group has all subgroups permutable if and only if the graph of permutability of cyclic subgroups is complete. A related graph whose vertices are the non-normal subgroups was studied by Bianchi, Gillio, and Verardi (see [7, 8, 13]).

The *prime graph* of a group has also attracted the attention of many researchers. The vertices of this graph are the prime numbers dividing the order of the group G and two different vertices p and q are connected if and only if G has an element of order pq . For instance, in the cyclic group of order 6,

this graph is complete, but in the symmetric group of degree 3, this graph consists of two isolated vertices. The first references known to the authors of this graph correspond to Gruenberg and Kegel, in an unpublished manuscript, and to Williams, who studied the number of connected components of the prime graph of a finite group (see [14, 21, 22]). Abe and Iiyori studied in [2] a generalisation of the prime graph in the following way: given a group G , they construct the graph Γ_G whose vertices are the prime numbers dividing the order of G and in which given two different vertices p and q , they are adjacent if and only if G possesses a soluble group of order divisible by pq . Abe and Iiyori [2] proved:

Theorem 6. If G is a non-abelian simple group, then Γ_G is connected, but not complete.

Herzog, Longobardi, and Maj [15] have considered the graph whose vertices are the non-trivial conjugacy classes of a group G and in which two non-trivial conjugacy classes C and D of G are adjacent if and only if there exists $c \in C$ and $d \in D$ such that $cd = dc$. They show that if G is a soluble group, then this graph has at most two connected components, each of diameter at most 15. They also study the structure of the groups for which there exist no edges between non-central conjugacy classes and the relation between this graph and the prime graph.

In [5] (see also [4]), the authors define for a group G a graph $\Gamma(G)$ whose vertices are the conjugacy classes of cyclic subgroups of G and in which two vertices $\text{Cl}_G(\langle x \rangle)$ and $\text{Cl}_G(\langle y \rangle)$ are adjacent if and only if we can find an element $g \in G$ such that $\langle x \rangle$ permutes with $\langle y^g \rangle$. The main result of [5] is:

Theorem 7. A group G is a soluble PT-group if and only if the graph $\Gamma(G)$ is complete.

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