Convergence S-compactifications

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ABSTRACT

Properties of continuous actions on convergence spaces are investigated. The primary focus is the characterization as to when a continuous action on a convergence space can be continuously extended to an action on a compactification of the convergence space. The largest and smallest such compactifications are studied.

1. INTRODUCTION AND PRELIMINARIES

The study of the notion of a topological transformation group or G-space dates back to the work of Gottshalk and Hedlund [4] who generalized several classical dynamical results from the theory of differential equations and other fields of mathematics. The problem of characterizing when a G-space has a compactification where the action can be continuously extended to the compactification, called a G-compactification, seems to have originated with de Vries [3]. The term "S-compactification" is used in the semigroup context. Our work is devoted to the study of S-compactifications where the underlying spaces are convergence spaces rather than topological spaces. It is known that the category of convergence spaces has nicer categorical properties. For example, quotient maps are productive in the category of convergence spaces but not in the category of topological spaces. Some results on S-spaces in the context of convergence spaces can be found in [1, 2]. A good reference on convergence spaces and categorical terminology is the book by Preuss [11].

Let X be a set, $\mathbf{P}(X)$ the power set of X, and $\mathbf{F}(X)$ the set of all filters on X. For $x \in X$, $\dot{x}$ denotes the fixed ultrafilter on X generated by $\{\{x\}\}$. Define the following partial order on $\mathbf{F}(X)$: Given $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$, $\mathcal{F} \geq \mathcal{G}$ (read "$\mathcal{F}$ is finer than $\mathcal{G}$") if and only if $\mathcal{G} \subseteq \mathcal{F}$. Given $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$, the least upper bound $\mathcal{F} \vee \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ exists provided that $F \cap G \neq \varnothing$ for each $F \in \mathcal{F}$ and $G \in \mathcal{G}$ and it is the smallest filter containing both $\mathcal{F}$ and $\mathcal{G}$.

Definition 1.1. A pair (X, q) is called a **convergence space** whenever X is a set and $q \colon \mathbf{F}(X) \to \mathbf{P}(X)$ obeys:

(CS1) $x \in q(\dot{x})$
(CS2) $\mathcal{G} \geq \mathcal{F}$ implies $q(\mathcal{F}) \subseteq q(\mathcal{G})$
(CS3) $x \in q(\mathcal{F})$ implies $x \in q(\mathcal{F} \cap \dot{x})$

A function q obeying (CS1) through (CS3) is called a **convergence structure** on X. The notation $x \in q(\mathcal{F})$ is read as "$\mathcal{F}$ q-converges to x" or as "$\mathcal{F}$ converges to x" and is usually written as $\mathcal{F} \xrightarrow{q} x$ or $\mathcal{F} \to x$. Most of the time we will not need to make explicit reference to the convergence structure so we will normally write (X, q) as X and $\mathcal{F} \xrightarrow{q} x$ as $\mathcal{F} \to x$.

A function $f \colon X \to Y$ between two convergence spaces is **continuous** provided that $f^{\to}\mathcal{F} \to x$ whenever $\mathcal{F} \to x$. Here, $f^{\to}\mathcal{F}$ denotes the filter on Y generated by $\{f(F) \colon F \in \mathcal{F}\}$. Given $\mathcal{G} \in \mathbf{F}(Y)$, we use $f^{\leftarrow}\mathcal{G}$ to denote the filter on X generated by $\{f^{-1}(G) \colon G \in \mathcal{G}\}$ whenever the latter does not contain $\varnothing$. Given two convergence structures p and q on a set X, we say that q is **finer** than p, denoted $q \geq p$, whenever the identity mapping $\mathrm{id}_X \colon (X, q) \to (X, p)$ is continuous.

Let CONV denote the category of convergence spaces and continuous maps and let X be an object in CONV. The **closure** and **interior** of a subset A of X are defined as follows:

$$\mathrm{cl}\, A = \{x \in X \colon \mathcal{F} \to x \text{ for some } \mathcal{F} \text{ and } A \in \mathcal{F}\}$$

$$\mathrm{int}\, A = \{x \in A \colon \mathcal{F} \to x \text{ implies that } A \in \mathcal{F}\}$$

The operators cl and int are in general not idempotent. The **neighborhood filter** of x is defined by

$$\mathcal{U}(x) = \{V \subseteq X \colon x \in \mathrm{int}(V)\}$$

and $A \subseteq X$ is **open** whenever $\mathrm{int}(A) = A$. The convergence structure q on X is called a **pretopology** on X if $\mathcal{U}(x) \to x$ for each $x \in X$. A pretopology on X is said to be a **topology** whenever $\mathcal{U}(x)$ has a base of open sets for each $x \in X$. We say X is **Hausdorff** provided that each filter converges to at most one point, and **regular** if $\mathrm{cl}\,\mathcal{F} \to x$ whenever $\mathcal{F} \to x$, where $\mathrm{cl}\,\mathcal{F}$ denotes the filter on X generated by $\{\mathrm{cl}\, F \colon F \in \mathcal{F}\}$. A Hausdorff regular convergence space is called $\mathbf{T_3}$. A point $x \in X$ is an **adherent point** of $\mathcal{F} \in \mathbf{F}(X)$ whenever there exists a $\mathcal{G} \geq \mathcal{F}$ such that $\mathcal{G} \to x$; $\mathrm{adh}\,\mathcal{F}$ denotes the set of all adherent points of $\mathcal{F}$. We say X is **compact** provided that $\mathrm{adh}\,\mathcal{F} \neq \varnothing$ for each $\mathcal{F} \in \mathbf{F}(X)$ or equivalently if each ultrafilter on X converges. A convergence space Y is said to be a **compactification** of X in CONV if Y is compact Hausdorff and

if there is a dense embedding of X into Y. Observe that compactifications are required to be Hausdorff. A compactification Y is called **regular** whenever Y is regular. If Y and Z are two compactifications of X in CONV, define $Y \geq Z$ to mean that there exists a continuous map $h \colon Y \to Z$ such that $h \circ f = g$, where f is the dense embedding of X into Y and g is the dense embedding of X into Z. Note that $\geq$ is a partial order on the set of compactifications of X if we agree not to distinguish between isomorphic objects in CONV.

Definition 1.2. The triple $(S, \cdot, p)$ is said to be a **convergence monoid** if it satisfies:

    (CM1) $(S, \cdot)$ is a commutative monoid with identity e.
    (CM2) (S, p) is a convergence space.
    (CM3) The binary operation $(x, y) \mapsto x \cdot y$ of S is continuous.

Let CM denote the category of convergence monoids and continuous homomorphisms. To simplify notation, we will write S for the object $(S, \cdot, p)$ in CM. Let X be a convergence space, let S be a convergence monoid and let $\lambda \colon X \times S \to X$. Consider the following conditions on λ:

    (A1) $\lambda(x, e) = x$ for all $x \in X$.
    (A2) $\lambda(\lambda(x, s), t) = \lambda(x, s \cdot t)$ for all $x \in X$ and all $s, t \in S$.
    (A3) λ is continuous.

If λ satisfies (A1) and (A2), then λ is an **action of S on X**, and if in addition it satisfies (A3), we say that λ is a **continuous action of S on X**.

Definition 1.3. Let CA be the category whose objects consist of all triples (X, S, λ), where X is a convergence space, S is a convergence monoid and λ is a continuous action of S on X, and whose morphisms are pairs (f, k) of functions of the form $(X, S, \lambda) \to (Y, T, \mu)$ such that:

    (C1) $f \colon X \to Y$ is a morphism in CONV,
    (C2) $k \colon S \to T$ is a morphism in CM and
    (C3) $\mu \circ (f \times k) = f \circ \lambda$.

Definition 1.4. A **compactification** (**regular compactification**) of an object (X, S, λ) in CA is an object (Y, S, μ) in CA such that:
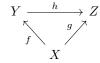
    (COM1) Y is a compact Hausdorff (compact T_3) convergence space,
    (COM2) X is densely embedded in Y, where the dense embedding f is such that
    (COM3) (f, id_S) is a morphism in CA.

Remark 1.5. Throughout the remainder of this work, S will always denote a convergence monoid, X a convergence space and λ a continuous action of S on X. We will always write p for the convergence structure on S and q for the convergence structure on X. An object in CA of the form (Y, S, μ) is called an S-**space** and for notational convenience will be denoted as (Y, μ) or Y. A morphism (f, id_S) between two S-spaces in CA will be written more simply as f. Also, any compactification of X in CA will be called an S-**compactification** of X.

Assume that Y and Z are two S-compactifications of X in CA with dense embeddings $f\colon X \to Y$ and $g\colon X \to Z$. Define $Y \geq Z$ to mean that there exists a morphism $h\colon Y \to Z$ in CA such that $h \circ f = g$. We say Y and Z are **equivalent S-compactifications** of X if $Y \geq Z$ and $Z \geq Y$. Verification of the following lemma is straightforward and omitted here.

Lemma 1.6.

(i) *The relation $\geq$ between S-compactifications of X defined above is a partial order on the set of all S-compactifications of X if we agree not to distinguish between equivalent S-compactifications.*

(ii) *Suppose that Y and Z are S-compactifications of X satisfying $Y \geq Z$. Then the following diagram commutes:*

$$Y \xrightarrow{\ \ h\ \ } Z$$
$$f \diagdown \quad \diagup g$$
$$X$$

and (a) $h(Y - f(X)) = Z - g(X)$ and (b) $f(X)$ is open in Y if and only if $g(X)$ is open in Z.

Definition 1.7.

(i) We call X **adherence-restrictive** if for each $\mathcal{F} \in \mathbf{F}(X)$ and each convergent filter $\mathcal{G} \in \mathbf{F}(S)$, $\mathrm{adh}\,\mathcal{F} = \varnothing$ implies $\mathrm{adh}\,\lambda^{\to}(\mathcal{F} \times \mathcal{G}) = \varnothing$.

(ii) Let Y be an S-compactification of X, let $f\colon X \to Y$ be the dense embedding and let μ be the action of S on Y. We say that Y is **remainder-invariant** provided that $\mu((Y - f(X)) \times S) \subseteq Y - f(X)$.

2. S-COMPACTIFICATIONS

Recall from Section 1 that a compactification must be Hausdorff. Unlike the topological context, a one-point compactification in CONV is not necessarily unique up to homeomorphism. The following one-point compactification defined below is used. Pick an $\omega \notin X$, let $X_* = X \cup \{\omega\}$ and let $j\colon X \to X_*$ be the natural injection. Define convergence in X_* as follows:

$$\mathcal{H} \in \mathcal{F}(X_*) \text{ converges to } j(x) \Leftrightarrow \mathcal{H} \geq j^{\to}\mathcal{F} \text{ for some } \mathcal{F} \in \mathbf{F}(X) \text{ that converges to } x$$

$$\mathcal{H} \in \mathcal{F}(X_*) \text{ converges to } \omega \Leftrightarrow \mathcal{H} \geq j^{\to}\mathcal{F} \cap \dot{\omega} \text{ for some } \mathcal{F} \in \mathbf{F}(X) \text{ with } \mathrm{adh}\,\mathcal{F} = \varnothing$$

One can check that X_* with the above convergence is a compactification of X in CONV provided that X is non-compact and Hausdorff. Moreover, $Y \geq X_*$ in CONV for any other one-point compactification Y of X.

Let η denote the set of all ultrafilters on X which fail to converge.

Theorem 2.1. *Suppose X is not compact and let Y be an S-compactification of X. Then X is adherence-restrictive if and only if Y is remainder-invariant.*

Proof. A contrapositive argument is used in each direction. Let $f\colon X \to Y$ be the dense embedding, let μ be the action of S on Y and suppose that Y fails to be remainder-invariant. Then $\mu(y, s) = f(x)$ for some $y \in Y - f(X)$, $s \in S$ and

$x \in X$. Let $\mathcal{F} \in \eta$ such that $f^{\rightarrow}\mathcal{F} \to y$. Then $\mathrm{adh}\,\mathcal{F} = \varnothing$ and $f^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})) = (f \circ \lambda)^{\rightarrow}(\mathcal{F} \times \dot{s}) = (\mu \circ (f \times \mathrm{id}_S))^{\rightarrow}(\mathcal{F} \times \dot{s}) = \mu^{\rightarrow}(f^{\rightarrow}\mathcal{F} \times \dot{s}) \to \mu(y, s) = f(x)$. Since f is an embedding, $\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \to x$ and thus $\mathrm{adh}(\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})) \neq \varnothing$, proving that X is not adherence-restrictive.

Conversely, suppose that X is not adherence-restrictive. Then there exists $\mathcal{F} \in \mathbf{F}(X)$ with $\mathrm{adh}\,\mathcal{F} = \varnothing$ and $x \in \mathrm{adh}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}))$ for some $\mathcal{G} \xrightarrow{p} s$ and $x \in X$. Let $\mathcal{H}$ be an ultrafilter on X such that $\mathcal{H} \to x$ and $\mathcal{H} \geq \lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G})$. Let $\mathcal{K}$ be an ultrafilter such that $\mathcal{K} \geq \mathcal{F} \times \mathcal{G}$ and $\mathcal{H} = \lambda^{\rightarrow}\mathcal{K}$. Note that the projection $\mathcal{K}_X$ of $\mathcal{K}$ onto X is finer than $\mathcal{F}$. Since Y is compact and $\mathrm{adh}\,\mathcal{F} = \varnothing$, $f^{\rightarrow}\mathcal{K}_X \to y$ for some $y \in Y - f(X)$, hence $f^{\rightarrow}\mathcal{H} = f^{\rightarrow}(\lambda^{\rightarrow}\mathcal{K}) \geq (f \circ \lambda)^{\rightarrow}(\mathcal{K}_X \times \mathcal{G}) = (\mu \circ (f \times \mathrm{id}_S))^{\rightarrow}(\mathcal{K}_X \times \mathcal{G}) = \mu^{\rightarrow}(f^{\rightarrow}\mathcal{K}_X \times \mathcal{G}) \to \mu(y, s)$. However, $\mathcal{H} \to x$ implies that $f^{\rightarrow}\mathcal{H} \to f(x)$, and thus $\mu(y, s) = f(x)$. Hence Y fails to be remainder-invariant. $\square$

Theorem 2.2. *If X is non-compact and Hausdorff, then it has a one-point remainder-invariant S-compactification if and only if X is adherence-restrictive.*

Proof. Assume that X is adherence-restrictive and let X_* be the compactification of X in CONV discussed above. Define $\lambda_* \colon X_* \times S \to X_*$ by

$$\lambda_*(y, s) = \begin{cases} j(\lambda(y, s)), & \text{if } y \in j(X) \\ \omega, & \text{if } y = \omega. \end{cases}$$

Some simple algebra reveals that λ_* is an action of S on X_*. It is shown that λ_* is continuous. Suppose that $\mathcal{H} \in \mathbf{F}(X_*)$ converges to $j(x)$ and $\mathcal{G} \in \mathbf{F}(X)$ converges to s. Then there exists $\mathcal{F} \in \mathbf{F}(X)$ that converges to x such that $\mathcal{H} \geq j^{\rightarrow}\mathcal{F}$. It follows from the definition of λ_* and the continuity of λ that $\lambda_*^{\rightarrow}(j^{\rightarrow}\mathcal{F} \times \mathcal{G}) = (j \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \to j(\lambda(x, s))$. Next, suppose that $\mathcal{H} \in \mathbf{F}(X_*)$ converges to ω and $\mathcal{G} \in \mathbf{F}(X)$ converges to s. Then there exists $\mathcal{F} \in \mathbf{F}(X)$ such that $\mathrm{adh}\,\mathcal{F} = \varnothing$ and $\mathcal{H} \geq j^{\rightarrow}\mathcal{F} \cap \dot{\omega}$. Since X is adherence-restrictive, $\mathrm{adh}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G})) = \varnothing$. It follows that $\lambda_*^{\rightarrow}(\mathcal{H} \times \mathcal{G}) \geq \lambda_*^{\rightarrow}((j^{\rightarrow}\mathcal{F} \cap \dot{\omega}) \times \mathcal{G}) = \lambda_*^{\rightarrow}(j^{\rightarrow}\mathcal{F} \times \mathcal{G}) \cap \lambda_*^{\rightarrow}(\dot{\omega} \times \mathcal{G}) = j^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G})) \cap \dot{\omega} \to \omega$. This proves that λ_* is a continuous action of S on X_*. Moreover, $\lambda_* \circ (j \times \mathrm{id}_S) = j \circ \lambda$ and thus X_* (with λ_* as the action) is an S-compactification of X. The compactification is also remainder-invariant by construction. The converse follows from Theorem 2.1. $\square$

Theorem 2.3. *Assume that X is non-compact and Hausdorff. Then it has a smallest remainder-invariant S-compactification if and only if X is adherence-restrictive and the image of X is open in each of its remainder-invariant S-compactifications. Moreover, if there is a smallest remainder-invariant S-compactification, it is equivalent to the one-point S-compactification X_*.*

Proof. Suppose that X has a smallest remainder-invariant S-compactifiction. According to Theorem 2.1, X is adherence-restrictive, and thus by Theorem 2.2, X_* is a remainder-invariant S-compactification of X. Applying Lemma

1.6 (ii), it follows that X_* is equivalent to the smallest remainder-invariant S-compactification. Moreover, part (b) of Lemma 1.6 (ii) implies that the image of X is open in each of its remainder-invariant S-compactifications.

Conversely, suppose that X is adherence-restrictive and that the image of X is open in each of its remainder-invariant S-compactifications. According to Theorem 2.2, X_* is a one-point S-compactification of X and $j(X)$ is open in X_*. Let Y be any remainder-invariant S-compactication of X having dense embedding f and action μ. Define $h\colon Y \to X_*$ by $h(f(x)) = j(x)$ and $h(y) = \omega$ for each $x \in X$ and $y \in Y - f(X)$. We show that h is continuous: If a filter $\mathcal{H}$ on Y converges to $f(x)$, then $f(X) \in \mathcal{H}$ since $f(X)$ is open in Y, and thus $\mathcal{F} := f^{\leftarrow}\mathcal{H} \to x$ and $h^{\to}\mathcal{H} = (h \circ f)^{\to}\mathcal{F} = j^{\to}\mathcal{F} \to j(x) = h(f(x))$. Next, assume that $\mathcal{H} \to y \in Y - f(X)$. If $Y - f(X) \in \mathcal{H}$, then $h^{\to}\mathcal{H} = \dot{\omega} \to \omega = h(y)$. Otherwise, $\mathcal{F} = f^{\leftarrow}\mathcal{H}$ exists, $\operatorname{adh}\mathcal{F} = \varnothing$ and $h^{\to}\mathcal{H} \geq j^{\to}\mathcal{F} \cap \dot{\omega} \to \omega$.

Next, we show $\lambda_* \circ (h \times \operatorname{id}_S) = h \circ \mu$: If $x \in X$ and $s \in S$, then $(\lambda_* \circ (h \times \operatorname{id}_S))(f(x), s) = \lambda_*(h(f(x)), s) = j(\lambda(x,s)) = (h \circ f)(\lambda(x,s)) = (h \circ (f \circ \lambda))(x,s) = h(\mu \circ (f \times \operatorname{id}_S))(x,s) = (h \circ \mu)(f(x), s)$, and thus $(\lambda_* \circ (h \times \operatorname{id}_S))(f(x), s) = (h \circ \mu)(f(x), s)$. Finally, if $y \in Y - f(X)$, then $(\lambda_*(h \circ \operatorname{id}_S))(y, s) = \lambda_*(h(y), s) = \lambda_*(\omega, s) = \omega = (h \circ \mu)(y, s)$. The last equality is valid since Y is remainder-invariant. Therefore, $\lambda_* \circ (h \times \operatorname{id}_S) = h \circ \mu$, and thus h is a morphism in CA. This concludes the proof that X_* is the smallest remainder-invariant S-compactification of X. $\qquad\square$

We say that S is a **convergence group** whenever S is a group and the function that maps elements of S to their inverses is continuous. In what follows, we will write s^{-1} for the inverse of $s \in S$. Also, given a filter $\mathcal{G}$ on S, we will write $\mathcal{G}^{-1}$ for the filter on S generated by $\{G^{-1}\colon G \in \mathcal{G}\}$, where G^{-1} denotes the set of inverses of the elements of G. The next result should be contrasted with Theorem 2.1.

Theorem 2.4. *If S is a convergence group, then X is adherence-restrictive.*

Proof. Suppose X fails to be adherence-restrictive. Then there exists $\mathcal{F} \in \mathbf{F}(X)$, $\mathcal{G} \in \mathbf{F}(S)$, $s \in S$ and $x \in X$ such that $\operatorname{adh}\mathcal{F} = \varnothing$ and $\mathcal{G} \to s$ and $x \in \operatorname{adh}(\lambda^{\to}(\mathcal{F} \times \mathcal{G}))$. Let $\mathcal{H}$ be an ultrafilter such that $\mathcal{H} \geq \lambda^{\to}(\mathcal{F} \times \mathcal{G})$ and $\mathcal{H} \to x$. Choose an ultrafilter $\mathcal{L} \geq \mathcal{F} \times \mathcal{G}$ for which $\mathcal{H} = \lambda^{\to}\mathcal{L}$. Since S is a convergence group, $\mathcal{G}^{-1} \to s^{-1}$, and thus $\lambda^{\to}(\mathcal{H} \times \mathcal{G}^{-1}) \to \lambda(x, s^{-1})$. It is shown that $\lambda^{\to}(\lambda^{\to}\mathcal{L} \times \mathcal{G}^{-1}) \vee \mathcal{L}_X$ exists, $\mathcal{L}_X$ being the projection of $\mathcal{L}$ on X. Let $L \in \mathcal{L}$, let L_X denote the projection of L on X and let $G \in \mathcal{G}$. Choose any $F \in \mathcal{F}$. Since $\mathcal{L} \geq \mathcal{F} \times \mathcal{G}$, $L \cap (F \times G) \neq \varnothing$. Let $(y, t) \in L \cap (F \times G)$. Then $(\lambda(y,t), t^{-1}) \in \lambda(L) \times G^{-1}$ and thus $y \in \lambda(\lambda(L) \times G^{-1}) \cap L_X$, which means $\lambda^{\to}(\lambda^{\to}\mathcal{L} \times \mathcal{G}^{-1}) \vee \mathcal{L}_X$ exists. Since $\lambda^{\to}(\mathcal{H} \times \mathcal{G}^{-1}) \to \lambda(x, s^{-1})$, we have that $\mathcal{F} \leq \mathcal{L}_X \leq \lambda^{\to}(\lambda^{\to}\mathcal{L} \times \mathcal{G}^{-1}) = \lambda^{\to}(\mathcal{H}^{\to} \times \mathcal{G}^{-1}) \vee \mathcal{L}_X \to \lambda(x, s^{-1})$, which contradicts $\operatorname{adh}\mathcal{F} = \varnothing$. $\qquad\square$

The next result appears as Theorem 3.2 [9].

Theorem 2.5 ([9])**.** *If X is non-compact and Hausdorff, then the following are equivalent:*

   (i) *X has a smallest compactification in* CONV*.*

   (ii) *The image of X is open in each of its compactifications.*

   (iii) *The set η of non-convergent ultrafilters on X is finite.*

   (iv) *X has a largest compactification in* CONV*.*

Example 2.6. Suppose $S = \{e\}$ is the trivial group, X Hausdorff and η is infinite. Define $\lambda\colon X \times S \to X$ by $\lambda(x, s) = x$. Then λ is a continuous action and thus X is an S-space. Let Y be any compactification of X with dense embedding f. Define $\mu\colon Y \times S \to S$ by $\mu(y, s) = y$. Then μ is a continuous action of S on Y and $\mu \circ (f \times \mathrm{id}_S) = f \circ \lambda$. It follows that Y is an S-compactification of X. In this case, there is a bijection between the compactifications of X and the S-compactifications of X. Since η is infinite, it follows from Theorem 2.5 that there fails to exists either a smallest or a largest S-compactification of X.

Example 2.7. Let $X := [0, 1)$ and $S := (0, 1]$ be equipped with the usual topologies and let the operation on S be multiplication. Define $\lambda\colon X \times S \to X$ by $\lambda(x, s) = xs$. Then λ is a continuous action and thus X is an S-space. Let $X_* = [0, 1]$ have the usual topology (note that $\omega = 1$ in this case). Define $\mu\colon X_* \times S \to X_*$ by $\mu(y, s) = ys$. Then μ is a continuous action of S on X_* and the S-space (X_*, μ) is an S-compactification of X. Since $\mu(\omega, s) = s \neq \omega$ whenever $s \neq 1$, (X_*, μ) is not remainder-invariant. Since the action λ_* of S on X_* fails to be continuous at each (ω, s), $s \neq 1$, the S-space (X_*, λ_*) is not an S-compactification of X. Ergo, X is not adherence-restrictive.

3. Regular S-compactifications

Recall that $\mathcal{U}(x)$ denotes the neighborhood filter of $x \in X$, and that the convergence structure q on X is a pretopology provided that $\mathcal{U}(x) \xrightarrow{q} x$ for each $x \in X$. If q is not a pretopology, define $\mathcal{F} \xrightarrow{\pi q} x$ if and only if $\mathcal{F} \geq \mathcal{U}(x)$. Then πq is a pretopology on X and $\pi X := (X, \pi q)$ is called the **pretopological modification** of X.

Theorem 3.1 ([6])**.** *The convergence space X has a regular compactification in* CONV *if and only if*

   (i) *X and πX agree on convergence of ultrafilters, and*

   (ii) *πX is a completely regular topological space.*

We say X is **completely regular** provided that it is T_3 and agrees on ultrafilter convergence with a completely regular topological space. With this definition, we can restate Theorem 3.1 above as: X has a regular compactification in CONV if and only if X is a completely regular convergence space.

Recall that an object Y in CA is a **regular S-compactification** of X if Y is a regular compactification of X and if $\mu \circ (f \times \mathrm{id}_S) = f \circ \lambda$, where $f\colon X \to Y$ is

the dense embedding. A convergence space is **locally compact** provided that each convergent filter contains a compact subset. According to Proposition 2.3 [8], a convergence space is locally compact if and only if each convergent ultrafilter contains a compact subset.

Theorem 3.2. *If X is adherence-restrictive and completely regular, then the statements* (i) – (iii) *are equivalent and* (iii) *implies* (iv), *where*

(i) X *has a one-point regular remainder-invariant S-compactification.*
(ii) X *is locally compact.*
(iii) X *has a regular remainder-invariant S-compactification and the image of X is open in each such compactification.*
(iv) X *has a smallest regular remainder-invariant S-compactification.*

Proof. (i) implies (ii): Let Y be a one-point regular S-compactification of X and let $f\colon X \to Y$ be the dense embedding. According to Theorem 3.1, $f\colon \pi X \to \pi Y$ is a dense embedding in the category TOP of topological spaces and thus πY is a T$_3$-compactification of πX in TOP, which means πX is locally compact. Again, by Theorem 3.1, X and πX agree on ultrafilter convergence so X is also locally compact.

(ii) implies (i): Let X_* be the one-point S-compactification defined in Section 2 with dense embedding $j\colon X \to X_*$. It must be shown that X_* is regular. Assume that $\mathcal{H}$ is a filter on X_* that converges to $j(x)$. Then $\mathcal{H} \geq j^{\to}\mathcal{F}$ for some $\mathcal{F} \in \mathbf{F}(X)$ that converges to x. Since X is completely regular and locally compact, $\operatorname{cl}\mathcal{F} \to x$ and $\mathcal{F}$ contains a compact subset A of X. Observe that $\omega \notin \operatorname{cl} j(A)$: otherwise, if there is an ultrafilter $\mathcal{K}$ on X_* that contains $j(A)$ and converges to ω, then $j^{\leftarrow}\mathcal{K}$ is a filter on X that contains A and has empty adherence, contradicting that A is compact. It follows that $X_* - j(A) \in \mathcal{K}$ for each ultrafilter $\mathcal{K}$ on X_* that converges to ω. Hence $\operatorname{cl} j(A) = j(\operatorname{cl} A) = j(A)$ and thus $\operatorname{cl}\mathcal{H} \geq \operatorname{cl} j^{\to}\mathcal{F} = j^{\to}(\operatorname{cl}\mathcal{F}) \to j(x)$. Moreover, if $\mathcal{H} \to \omega$, then $\mathcal{H} \geq j^{\to}\mathcal{F} \cap \dot{\omega}$ for some $\mathcal{F} \in \mathbf{F}(X)$ such that $\operatorname{adh}(\mathcal{F}) = \varnothing$. Since X is completely regular, it has the same ultrafilter convergence as πX, hence $\operatorname{adh}(\operatorname{cl}\mathcal{F}) = \varnothing$ and thus $\operatorname{cl}\mathcal{H} \geq \operatorname{cl}(j^{\to}\mathcal{F} \cap \dot{\omega}) = j^{\to}(\operatorname{cl}\mathcal{F}) \cap \dot{\omega} \to \omega$. This proves that X_* is regular and thus X_* is a one-point regular remainder-invariant S-compactification of X.

(ii) implies (iii): Since (ii) implies (i), X has a regular remainder-invariant S-compactification. Let Y be any regular remainder-invariant S-compactification with dense embedding $f\colon X \to Y$. We now show that $f(X)$ is open in Y. Since $f\colon \pi X \to \pi Y$ is a dense embedding in TOP, πY is a Hausdorff compactification of πX in TOP. Since X and πX agree on ultrafilter convergence, πX and X are locally compact. Thus, $f(X)$ is open in πY and hence open in Y.

(iii) implies (ii): Let Y be a regular S-compactification of X with dense embedding $f\colon X \to Y$ such that $f(X)$ is open in Y. Since πY is a compactification of πX in TOP, πX is locally compact and thus X is locally compact.

(iii) implies (iv): Since (iii) implies (i), X_* is a regular remainder-invariant S-compactification. Suppose that Y is any regular remainder-invariant S-compactification of X with dense embedding $f\colon X \to Y$. By hypothesis, $f(X)$ is

open in Y, so the proof of Theorem 2.3 shows that $Y \geq X_*$, hence X_* is the smallest regular remainder-invariant S-compactification of X. $\qquad\square$

Theorem 3.3. *If X has a regular S-compactification, then it has a largest regular S-compactification, which is remainder invariant whenever X is adherence-restrictive.*

Proof. Let (Y_α) be the family of all regular S-compactifications of X. For each α, let $f_\alpha \colon X \to Y_\alpha$ be the dense embedding and let μ_α be the action of S on Y_α. Let $Y = \prod Y_\alpha$ be the product space in CONV and define $\mu \colon Y \times S \to Y$ so that $\mu(y,s) = (\mu_\alpha(y_\alpha), s)$, where $y = (y_\alpha)$. It is straightforward to check that μ is an action of S on Y. Let $\pi_\alpha \colon Y \to Y_\alpha$ denote the projection onto Y_α. Since the diagram below is commutative,
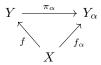
$$
\begin{array}{ccc}
Y \times S & \xrightarrow{\ \mu\ } & Y \\
{\scriptstyle \pi_\alpha \times \mathrm{id}_S} \big\downarrow & & \big\downarrow {\scriptstyle \pi_\alpha} \\
Y_\alpha \times S & \xrightarrow{\ \mu_\alpha\ } & Y_\alpha
\end{array}
$$

it follows that μ is continuous. This proves that Y is an S-space as well as a compact T_3 object in CONV.

Define $f \colon X \to Y$ by $f(x) = (f_\alpha(x))$. Since $f_\alpha = \pi_\alpha \circ f$ for each α, f is continuous. Moreover, if $\mathcal{F} \in \mathbf{F}(X)$ and $f^\to \mathcal{F} \to f(x)$, then $f_\alpha^\to \mathcal{F} = \pi_\alpha^\to(f^\to \mathcal{F}) \to f_\alpha(x)$. Since f_α is an embedding, $\mathcal{F} \to x$ and consequently f is an embedding. Since $\mu \circ (f \times \mathrm{id}_S) = f \circ \lambda$, f is an embedding in CA. Note that the following diagram is commutative in CA:

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \pi_\alpha\ } & Y_\alpha \\
{\scriptstyle f} \nwarrow & & \nearrow {\scriptstyle f_\alpha} \\
& X &
\end{array}
$$

Finally, let $Z = \operatorname{cl} f(X)$ and let δ denote the restriction of μ on $Z \times S$. Observe that Z is an S-space since $\delta(Z \times S) = \delta(\operatorname{cl} f(X) \times S) \subseteq \operatorname{cl} \delta(f(X) \times S) = \operatorname{cl} \mu(f(X) \times S) = \operatorname{cl}(\mu \circ (f \times \mathrm{id}_S))(X \times S) = \operatorname{cl}(f \circ \lambda)(X \times S) = \operatorname{cl} f(\lambda(X \times S)) \subseteq \operatorname{cl} f(X) = Z$. It follows that Z is the largest regular S-compactification of X. In the event that Z is adherence-restrictive, it follows from Theorem 2.1 that Z is remainder-invariant. $\qquad\square$

Consideration is now given to the question as to when an S-space has a regular S-compactification. For this discussion, it is convenient to introduce the notion of a Cauchy space.

Definition 3.4. The pair $(X, \mathscr{C})$ is called a **Cauchy space** and $\mathscr{C}$ a Cauchy structure whenever $\mathscr{C} \subseteq \mathbf{F}(X)$ obeys:

(CA1) $\dot{x} \in \mathscr{C}$ for each $x \in X$.
(CA2) $\mathcal{F} \geq \mathcal{G} \in \mathscr{C}$ implies $\mathcal{F} \in \mathscr{C}$.
(CA3) $\mathcal{F} \cap \mathcal{G} \in \mathscr{C}$ whenever $\mathcal{F}, \mathcal{G} \in \mathscr{C}$ and $\mathcal{F} \vee \mathcal{G}$ exists.

The elements of a Cauchy structure are called **Cauchy filters**. A function f between Cauchy spaces is called **Cauchy continuous** if $f^{\rightarrow}\mathcal{F}$ is a Cauchy filter whenever $\mathcal{F}$ is a Cauchy filter. We use CHY to denote the category of Cauchy spaces and Cauchy continuous functions.

The first study of completions resembling a Cauchy space defined above seems to be due to Keller [5]. The interested reader is referred to Lowen [10], Preuss [11] and Reed [12] for a thorough treatment of Cauchy spaces.

If for all $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$ and all $x \in X$, $\mathcal{F} \to x$ and $\mathcal{G} \to x$ imply $\mathcal{F} \cap \mathcal{G} \to x$, then X is called a **limit space**. We use LIM to denote the full subcategory of CONV whose objects are limit spaces. If $\mathscr{C} \subseteq \mathbf{F}(X)$ is a Cauchy structure, then $q_{\mathscr{C}} \colon \mathbf{F}(X) \to \mathbf{P}(X)$ defined by $\mathcal{F} \overset{q_{\mathscr{C}}}{\to} x$ if and only if $\mathcal{F} \cap \dot{x} \in \mathscr{C}$ is a convergence structure making $(X, q_{\mathscr{C}})$ a limit space. If $f \colon (X, \mathscr{C}) \to (Y, \mathscr{D})$ is Cauchy continuous, then $f \colon (X, q_{\mathscr{C}}) \to (Y, q_{\mathscr{D}})$ is continuous, but the converse is in general false. If X is a Hausdorff limit space, then $\mathscr{C} = \{\mathcal{F} \in \mathbf{F}(X) \colon \mathcal{F}$ converges$\}$ is a Cauchy structure and $q_{\mathscr{C}} = q$.

A compactification Y of X in CONV with dense embedding f is said to be **strict** provided that whenever $\mathcal{H} \in \mathbf{F}(Y)$ converges to some $y \in Y$, there exists an $\mathcal{F} \in \mathbf{F}(X)$ such that $f^{\rightarrow}\mathcal{F} \to y$ and $\mathcal{H} \geq \mathrm{cl}\, f^{\rightarrow}\mathcal{F}$. All our previous definitions (S-compactifications, etc.) apply to the objects in LIM.

Theorem 3.5. *Suppose both X and S are Hausdorff limit spaces. Let Y be any strict regular compactification of X in* LIM *with dense embedding f, let $\mathscr{C} = \{\mathcal{F} \in \mathbf{F}(X) \colon f^{\rightarrow}\mathcal{F}$ converges$\}$ and let $\mathscr{S} = \{\mathcal{G} \in \mathbf{F}(S) \colon \mathcal{G}$ converges$\}$. Then there exists a continuous action μ of S on Y making Y into a regular S-compactification of X if and only if $\lambda \colon (X, \mathscr{C}) \times (S, \mathscr{S}) \to (X, \mathscr{C})$ is Cauchy continuous.*

Proof. Suppose there exists a continuous action μ of S on Y making Y into a regular S-compactification of X. Then $\mathscr{C}$ is a Cauchy structure. We now show that $\lambda \colon (X, \mathscr{C}) \times (S, \mathscr{S}) \to (X, \mathscr{C})$ is Cauchy continuous. Let $\mathcal{F} \in \mathscr{C}$ and $\mathcal{G} \in \mathscr{S}$. Then $f^{\rightarrow}\mathcal{F} \to y$ and $\mathcal{G} \to s$ for some $y \in Y$ and $s \in S$. It follows that $f^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G})) = (\mu \circ (f \times \mathrm{id}_S))^{\rightarrow}(\mathcal{F} \times \mathcal{G}) = \mu^{\rightarrow}(f^{\rightarrow}\mathcal{F} \times \mathcal{G}) \to \mu(y, s)$, which means $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \in \mathscr{C}$, which means λ is Cauchy continuous.

Conversely, suppose that λ is Cauchy continuous. Define $\mu \colon Y \times S \to Y$ by

$$\mu(y, s) = \begin{cases} f(\lambda(x, s)), & \text{if } y = f(x) \text{ for some } x \in X, \\ \lim (f \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{G}), & \text{where } f^{\rightarrow}\mathcal{F} \to y \in Y - f(X) \text{ and } \mathcal{G} \to s. \end{cases}$$

Note that the above is well-defined. Indeed, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on X such that $f^{\rightarrow}\mathcal{F}_1$ and $f^{\rightarrow}\mathcal{F}_2$ converge to y and $\mathcal{G}_1$ and $\mathcal{G}_2$ are filters on S that converge to s, then $f^{\rightarrow}(\mathcal{F}_1 \cap \mathcal{F}_2) \to y$ and $\mathcal{G}_1 \cap \mathcal{G}_2 \to s$ since Y and S are limit spaces, hence $\mathcal{F}_1 \cap \mathcal{F}_2 \in \mathscr{C}$. Moreover, since λ is Cauchy continuous, $\lambda^{\rightarrow}(\mathcal{F}_1 \times \mathcal{G}_1) \cap \lambda^{\rightarrow}(\mathcal{F}_2 \times \mathcal{G}_2) \geq \lambda^{\rightarrow}((\mathcal{F}_1 \cap \mathcal{F}_2) \times (\mathcal{G}_1 \cap \mathcal{G}_2)) \in \mathscr{C}$. Thus, $f^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F}_1 \times \mathcal{G}_1) \cap \lambda^{\rightarrow}(\mathcal{F}_2 \times \mathcal{G}_2))$ converges and $\lim (f \circ \lambda)^{\rightarrow}(\mathcal{F}_1 \times \mathcal{G}_1) = \lim (f \circ \lambda)^{\rightarrow}(\mathcal{F}_2 \times \mathcal{G}_2)$ in Y.

We now show that μ is an action. First, we have that $\mu(f(x), e) = f(\lambda(x, e)) = f(x)$ and $\mu(y, e) = \lim (f \circ \lambda)^{\rightarrow}(\mathcal{F} \times \dot{e}) = \lim f^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \dot{e})) = y$, where $f^{\rightarrow}\mathcal{F} \to y \in Y - f(X)$. Second, if $x \in X$ and $s, t \in S$, then $\mu(\mu(f(x), s), t) =$

$\mu(f(\lambda(x,s)),t) = f(\lambda(\lambda(x,s),t)) = f(\lambda(x,s \cdot t)) = \mu(f(x),s \cdot t)$. Third, suppose $y \in Y - f(X)$ and $f^{\rightarrow}\mathcal{F} \to y$ and let $s,t \in S$. If $\mu(y,s) = f(x)$ for some $x \in X$, then since f is an embedding and $f(x) = \mu(y,s) = \lim(f \circ \lambda)^{\rightarrow}(\mathcal{F} \times \dot{s}) = \lim f^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}))$, we have that $\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \to x$ and so $\mu(y,s \cdot t) = \lim(f \circ \lambda)^{\rightarrow}(\mathcal{F} \times s \dot{\cdot} t) = \lim f^{\rightarrow}(\lambda^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \times \dot{t})) = f(\lambda(x,t)) = \mu(f(x),t) = \mu(\mu(y,s),t)$. Otherwise, if $\mu(y,s) \in Y - f(X)$, then since $f^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})) \to \mu(y,s)$ we have that $\mu(\mu(y,s),t) = \lim(f \circ \lambda)^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \times \dot{t}) = \lim(f \circ \lambda)^{\rightarrow}(\mathcal{F} \times s \dot{\cdot} t)$.

We now show that μ is continuous. First, we prove that if $A \subseteq X$ and $G \subseteq S$, then $\mu(\mathrm{cl}\, f(A) \times G) \subseteq \mathrm{cl}(f \circ \lambda)(A \times G)$: Let $(y,s) \in \mathrm{cl}\, f(A) \times G$. If $y = f(x)$, then $x \in \mathrm{cl}\, A$, and since $(f \circ \lambda)(\mathrm{cl}\, A \times G) \subseteq \mathrm{cl}(f \circ \lambda)(A \times G)$, $\mu(f(x),s) = f(\lambda(x,s)) \in \mathrm{cl}(f \circ \lambda)(A \times G)$. Otherwise, if $y \in Y - f(X)$ and $f^{\rightarrow}\mathcal{F} \to y$ with $A \in \mathcal{F}$, then $A \times G \in \mathcal{F} \times \dot{s}$, $(f \circ \lambda)^{\rightarrow}(\mathcal{F} \times \dot{s}) \to \mu(y,s)$, hence $\mu(y,s) \in \mathrm{cl}(f \circ \lambda)(A \times G)$, hence $\mu(\mathrm{cl}\, f(A) \times G) \subseteq \mathrm{cl}(f \circ \lambda(A \times G))$ as claimed. Now suppose $\mathcal{H}$ is a filter on Y and that $\mathcal{G}$ is a filter on S converging to s. If $\mathcal{H} \to f(x)$, then since Y is a strict regular compactification of X in LIM, there exists an $\mathcal{F} \in \mathbf{F}(X)$ such that $f^{\rightarrow}\mathcal{F} \to y$ and $\mathcal{H} \geq \mathrm{cl}\, f^{\rightarrow}\mathcal{F}$, hence $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \to \lambda(x,s)$ and $\mu^{\rightarrow}(\mathcal{H} \times \mathcal{G}) \geq \mu^{\rightarrow}(\mathrm{cl}\, f^{\rightarrow}\mathcal{F} \times \mathcal{G}) \geq \mathrm{cl}(f \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \to f(\lambda(x,s)) = \mu(f(x),s)$. If $\mathcal{H} \to y \in Y - f(X)$, then since Y is strict, there is an $\mathcal{F} \in \mathbf{F}(X)$ that converges to x such that $\mathcal{H} \geq \mathrm{cl}\, f^{\rightarrow}\mathcal{F}$, hence $\mu^{\rightarrow}(\mathcal{H} \times \mathcal{G}) \geq \mu^{\rightarrow}(\mathrm{cl}\, f^{\rightarrow}\mathcal{F} \times \mathcal{G}) \geq \mathrm{cl}(f \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{G})$, hence by the regularity of Y and the fact that $(f \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \to \mu(y,s)$ we have that $\mu^{\rightarrow}(\mathcal{H} \times \mathcal{G}) \to \mu(y,s)$. This proves that μ is continuous, and since $\mu \circ (f \times \mathrm{id}_S) = f \circ \lambda$, Y is a regular S-compactification of X. $\qquad\square$

Recall from Theorem 3.1 that a T_3 convergence space has a regular compactification in CONV if and only if it is compeletely regular. This result is also valid in the subcategory LIM. The following additional result is proved in Theorem 2 [6].

Theorem 3.6. *If X is a completely regular convergence (limit) space, then it has a largest regular compactification in CONV (respectively, LIM). The largest regular compactification is strict.*

The largest regular compactification guaranteed by Theorem 3.6 is called the **Stone-Čech compactification** for convergence (limit) spaces. Combining the two previous theorems yields the next result.

Theorem 3.7. *Suppose X is a completely regular limit space with Stone-Čech compactification βX in LIM and dense embedding $f\colon X \to \beta X$ and suppose S is a Hausdorff limit space. Let $\mathscr{S} = \{\mathcal{G} \in \mathbf{F}(S)\colon \mathcal{G} \text{ converges}\}$ and $\mathscr{C} = \{\mathcal{F} \in \mathbf{F}(X)\colon f^{\rightarrow}\mathcal{F} \text{ converges}\}$. Then λ can be extended to a continuous action of S on βX making it into a regular S-compactification of X if and only if λ is Cauchy continuous. Moreover, βX is the largest regular S-compactification of X whenever λ is Cauchy continuous.*

Proof. Let μ be the action of S on βX. According to Theorem 3.6, βX is strict, so by Theorem 3.5, βX is a regular S-compactification of X if and only

if λ is Cauchy continuous. Suppose λ is Cauchy continuous. Then f is a dense embedding in CA and $\mu \circ (f \times \mathrm{id}_S) = f \circ \lambda$. It remains to show that βX is the largest regular S-compactification of X. Let Y be any regular S-compactification of X with dense embedding g and action δ. Then $\delta \circ (g \times \mathrm{id}_S) = g \circ \lambda$. Since βX is the Stone-Čech compactification of X in LIM, there exists a continuous function $h \colon \beta X \to Y$ such that $h \circ f = g$. We show that $\delta \circ (h \times \mathrm{id}_S) = h \circ \mu$ on $f(X)$: If $x \in X$ and $s \in S$, then $(\delta \circ (h \times \mathrm{id}_S))(f(x), s) = \delta((h \circ f)(x), s) = \delta(g(x), s) = (\delta \circ (g \times \mathrm{id}_S))(x, s) = (g \circ \lambda)(x, s) = (h \circ f \circ \lambda)(x, s) = (h \circ (f \circ \lambda))(x, s) = (h \circ (\mu \circ (f \times \mathrm{id}_S)))(x, s) = (h \circ \mu)(f(x), s)$. Since $f(X)$ is dense in βX, $\delta \circ (h \times \mathrm{id}_S) = h \circ \mu$ on βX and thus h is a morphism in CA. This means $\beta X \geq Y$, hence βX is the largest regular S-compactification of X. $\qquad\square$

4. Generalized Quotient Spaces

Let GQ denote the full subcategory of CA whose objects (Y, T, μ) satisfy:

(i) T is a commutative.
(ii) $\mu(\cdot, t)$ is injective for each fixed $t \in T$.

Suppose X is an object in GQ. Define a relation $\backsim$ on $X \times S$ so that $(x, s) \backsim (y, t)$ if and only if $\lambda(x, t) = (y, s)$. One can readily check that $\backsim$ is an equivalence relation. We use $\langle x, s \rangle$ for the equivalence class containing (x, s) and $\theta \colon X \times S \to (X \times S)/\backsim$ for the quotient map. We call $(X \times S)/\backsim$ the **generalized quotient** of X and we denote it by $\mathbf{B}(X)$.

Define $\Lambda \colon (X \times S) \times S \to (X \times S)$ by $\Lambda((x, s), t) = (\lambda(x, t), s)$ and define $\lambda_{\mathbf{B}} \colon \mathbf{B}(X) \times S \to \mathbf{B}(X)$ by $\lambda_{\mathbf{B}}(\langle x, s \rangle, t) = \langle \lambda(x, t), s \rangle$. It is shown in Theorem 2.4 (b) of [2] that Λ and $\lambda_{\mathbf{B}}$ are continuous actions, which means that $X \times S$ and $\mathbf{B}(X)$ are S-spaces. In fact, the following diagram is commutative:

$$\begin{array}{ccc} (X \times S) \times S & \xrightarrow{\Lambda} & X \times S \\ {\scriptstyle (\theta, \mathrm{id}_S)}\big\downarrow & & \big\downarrow{\scriptstyle \theta} \\ \mathbf{B}(X) \times S & \xrightarrow{\lambda_{\mathbf{B}}} & \mathbf{B}(X) \end{array}$$

Theorem 4.1. *Suppose X is an object in GQ. If S is compact and X is not compact and Hausdorff, then $\mathbf{B}(X_*)$ is a one-point S-compactification of $\mathbf{B}(X)$, where X_* is the one-point S-compactification of X given in Thereom 2.2.*

Proof. First we show that $\lambda_{\mathbf{B}}^* \colon \mathbf{B}(X) \times S \to \mathbf{B}(X)$ defined by $\lambda_{\mathbf{B}}^*(\langle j(x), s \rangle, t) = \langle (j \circ \lambda)(x, t), s \rangle$ and $\lambda_{\mathbf{B}}^*(\langle \omega, s \rangle, t) = \langle \lambda_*(\omega, t), s \rangle = \langle \omega, s \rangle$ is an action. Note that $\lambda_{\mathbf{B}}^*(\langle j(x), s \rangle, e) = \langle j \circ \lambda(x, e), s \rangle = \langle j(x), s \rangle$ and $\lambda_{\mathbf{B}}^*(\langle \omega, s \rangle, e) = \langle \omega, s \rangle$. Also,

$$\begin{aligned} \lambda_{\mathbf{B}}^*(\lambda_{\mathbf{B}}^*(\langle j(x), t \rangle, s), u) &= \lambda_{\mathbf{B}}^*(\langle (j \circ \lambda)(x, t), s \rangle, u) \\ &= \langle (j \circ \lambda)(\lambda(x, t), u), s \rangle \\ &= \langle (j \circ \lambda)(x, t \cdot u), s \rangle \\ &= \lambda_{\mathbf{B}}^*(\langle j(x), s \rangle, tu) \end{aligned}$$

and

$$\lambda_{\mathbf{B}}^*(\lambda_{\mathbf{B}}^*(\langle \omega, s \rangle, t), u) = \lambda_{\mathbf{B}}^*(\langle \lambda_*(\omega, t), s \rangle, u)$$
$$= \lambda_{\mathbf{B}}^*(\langle \omega, s \rangle, u)$$
$$= \langle \lambda_*(\omega, u), s \rangle$$
$$= \langle \omega, s \rangle = \lambda_{\mathbf{B}}^*(\langle \omega, s \rangle, t \cdot u)$$

This proves that $\lambda_{\mathbf{B}}$ is an action.

Define $\Lambda^* \colon (X_* \times S) \times S \to X_* \times S$ so that $\Lambda^*((j(x), s), t) = (\lambda_*(j(x), t), s) = ((j \circ \lambda)(x, t), s)$ and $\Lambda^*((\omega, s), t) = (\lambda_*(\omega, t), s) = (\omega, s)$. Note that Λ^* is an action of S on X_* and it is continuous since it is the composition:

$$((z, s), t) \mapsto ((z, t), s) \mapsto (\lambda_*(z, t), s)$$

It follows that $X_* \times S$ is an object in GQ. Consider the following commutative diagram:

$$
\begin{array}{ccc}
(X_* \times S) \times S & \xrightarrow{\Lambda^*} & X_* \times S \\
{\scriptstyle (\theta^*, \mathrm{id}_S)} \downarrow & & \downarrow {\scriptstyle \theta} \\
\mathbf{B}(X_*) \times S & \xrightarrow{\lambda_{\mathbf{B}}^*} & \mathbf{B}(X_*)
\end{array}
$$

where $\theta^* \colon X_* \times S \to \mathbf{B}(X_*)$ is the quotient map in CONV. Since $(\theta^*, \mathrm{id}_S)$ is a quotient map in CONV, the diagram above shows that $\lambda_{\mathbf{B}}^*$ is continuous, which means that $\mathbf{B}(X_*)$ is an S-space.

Since $\langle x, s \rangle \in \mathbf{B}(X)$ if and only if $\langle j(x), s \rangle \in \mathbf{B}(X_*)$ and since $\langle \omega, s \rangle = \{(\omega, t) \colon t \in S\}$, it follows that $\gamma \colon \mathbf{B}(X) \to \mathbf{B}(X_*)$ defined by $\gamma(\langle x, s \rangle) = \langle j(x), s \rangle$ is an injection and $\mathbf{B}(X_*) - \gamma(\mathbf{B}(X))$ is a singleton set containing $\langle \omega, s \rangle$. We now proceed to show that γ is a dense embedding.

First, observe that the diagram below is commutative:

$$
\begin{array}{ccc}
X \times S & \xrightarrow{\theta} & \mathbf{B}(X) \\
{\scriptstyle (j, \mathrm{id}_S)} \downarrow & & \downarrow {\scriptstyle \gamma} \\
X_* \times S & \xrightarrow{\theta^*} & \mathbf{B}(X_*)
\end{array}
$$

Moreover, since θ is a quotient map and $\gamma \circ \theta = \theta^* \circ (j, \mathrm{id}_S)$ is continuous in CONV, it follows that γ is continuous. Next, assume that $\mathcal{H} \in \mathbf{F}(\mathbf{B}(X))$ such that $\gamma^{\to}\mathcal{H} \to \langle j(x), s \rangle$. Then there exists $(j(x'), s') \backsim (j(x), s)$ and $\mathcal{K} \to (j(x'), s')$ such that $\theta^{* \to}\mathcal{K} = \gamma^{\to}\mathcal{H}$. The product convergence structure on $X_* \times S$ implies that for some $\mathcal{F} \in \mathbf{F}(X)$ and some $\mathcal{G} \in \mathbf{F}(S)$, $\mathcal{F} \to x'$, $\mathcal{G} \to s'$ and $\mathcal{K} \geq j^{\to}\mathcal{F} \times \mathcal{G}$, hence $(\gamma \circ \theta)^{\to}(\mathcal{F} \times \mathcal{G}) = (\theta^* \circ (j \times \mathrm{id}_S))^{\to}(\mathcal{F} \times \mathcal{G}) = \theta^{* \to}(j^{\to}\mathcal{F} \times \mathcal{G}) \leq \theta^{* \to}\mathcal{K} = \gamma^{\to}\mathcal{H}$, hence $\mathcal{L} := \theta^{\to}(\mathcal{F} \times \mathcal{G}) \to \langle x', s' \rangle$ and $\gamma^{\to}\mathcal{L} \leq \gamma^{\to}\mathcal{H}$, hence $\mathcal{L} \leq \mathcal{H}$ since γ is an injection. It follows that $\mathcal{H} \to \langle x', s' \rangle = \langle x, s \rangle$, proving that γ is an embedding.

For any fixed $s \in S$, $\mathbf{B}(X_*) - \gamma(\mathbf{B}(X)) = \{\langle \omega, s \rangle\}$. Since X is not compact, there exists an ultrafilter $\mathcal{F} \in \mathbf{F}(X)$ that fails to converge, hence $j^{\to}\mathcal{F} \to \omega$, $j^{\to}\mathcal{F} \times \dot{s} \to (\omega, s)$ and $\theta^{* \to}(j^{\to}\mathcal{F} \times \dot{s}) \to \langle \omega, s \rangle$. It follows that $\gamma^{\to}(\theta^{\to}(\mathcal{F} \times \mathcal{G})) =$

$(\theta^* \circ (j, \mathrm{id}_S))^{\rightarrow}(\mathcal{F} \times \mathcal{G}) = \theta^{*\rightarrow}(j^{\rightarrow}\mathcal{F} \times \mathcal{G}) \to \langle \omega, s \rangle$, and since $\theta^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \in \mathbf{F}(\mathbf{B}(X))$, we conclude that γ is a dense embedding. Moreover, since X_* and S are compact, $\mathbf{B}(X_*)$ is a one-point S-compactification of $\mathbf{B}(X)$ in CONV.

We now verify that γ is a morphism in CA: Since $(\lambda_{\mathbf{B}}^* \circ (\gamma \times \mathrm{id}_S))(\langle x, s \rangle, t) = \lambda_{\mathbf{B}}^*(\langle j(x), s \rangle, t) = \langle (j \circ \lambda)(x, t), s \rangle = \gamma(\langle \lambda(x, t), s \rangle) = (\gamma \circ \lambda_{\mathbf{B}})(\langle x, s \rangle, t)$, we have that $\lambda_{\mathbf{B}}^* \circ (\gamma \times \mathrm{id}_S) = \gamma \circ \lambda_{\mathbf{B}}$. In conclusion, $\mathbf{B}(X_*)$ is a one-point S-compactification of $\mathbf{B}(X)$. $\square$

A regular S-compactification of a generalized quotient S-space is given below. The following definition is needed: A continuous surjection $f \colon X \to Y$ between convergence spaces is said to be **proper** if for each ultrafilter $\mathcal{F}$ on X, $f^{\rightarrow}\mathcal{F} \to y$ implies $\mathcal{F} \to x$ for some $x \in f^{-1}(\{y\})$. It is shown in Proposition 3.2 in [7] that proper maps preserve closures. Observe that if $f \colon X \to Y$ is a continuous surjection and X is compact and Y is Hausdorff, then f is a proper map.

Theorem 4.2. *Suppose X is an adherence-restrictive object in GQ and let Y be a strict regular S-compactification of X with dense embedding f and action μ. Assume that S is compact and regular and define $h \colon \mathbf{B}(X) \to \mathbf{B}(Y)$ by $h(\langle x, s \rangle) = \langle f(x), s \rangle$. Then $\mathbf{B}(Y)$ is a regular S-compactification of $\mathbf{B}(X)$ with dense embedding h.*

Proof. Consider the following commutative diagram:

$$
\begin{array}{ccc}
X \times S & \xrightarrow{\theta} & \mathbf{B}(X) \\
{\scriptstyle (f, \mathrm{id}_S)}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
Y \times S & \xrightarrow{\theta_Y} & \mathbf{B}(Y)
\end{array}
$$

Note that h is continuous since θ is a quotient map in CONV and $h \circ \theta = \theta_Y \circ (f \times \mathrm{id}_S)$ is continuous. We now prove that h is an injection: Suppose that $\langle x, s \rangle \neq \langle z, t \rangle$. Then $\lambda(x, t) \neq \lambda(z, s)$, $h(\langle x, s \rangle) = \langle f(x), s \rangle$ and $h(\langle z, t \rangle) = \langle f(z), t \rangle$. However, f is injective, so $\mu(f(x), t) = f(\lambda(x, t)) \neq f(\lambda(z, s)) = \mu(f(z), s)$. It follows that $h(\langle x, s \rangle) = \langle f(x), s \rangle \neq \langle f(z), t \rangle = h(\langle z, t \rangle)$, proving that h is an injection as claimed.

Next, suppose that $\mathcal{H}$ is a filter on $\mathbf{B}(X)$ such that $h^{\rightarrow}\mathcal{H} \to h(\langle x, s \rangle) = \langle f(x), s \rangle$. We show that $\mathcal{H} \to \langle x, s \rangle$: The quotient convergence structure on $\mathbf{B}(Y)$ implies that there is a filter $\mathcal{K}$ on Y that converges to y and a filter $\mathcal{G}$ on S that converges to t such that $h^{\rightarrow}\mathcal{H} \geq \theta_Y^{\rightarrow}(\mathcal{K} \times \mathcal{G})$ and $\langle y, t \rangle = \langle f(x), s \rangle$. Since Y is a strict regular compactification of X in CONV, there exists an $\mathcal{F} \in \mathbf{F}(X)$ such that $f^{\rightarrow}\mathcal{F} \to y$ and $\mathcal{K} \geq \mathrm{cl}\, f^{\rightarrow}\mathcal{F}$. It follows that $h^{\rightarrow}\mathcal{H} \geq \theta_Y^{\rightarrow}(\mathcal{K} \times \mathcal{G}) \geq \theta_Y^{\rightarrow}(\mathrm{cl}\, f^{\rightarrow}\mathcal{F} \times \mathcal{G})$, hence $\mathcal{H} \geq h^{\leftarrow}(\theta_Y^{\rightarrow}(\mathrm{cl}\, f^{\rightarrow}\mathcal{F} \times \mathcal{G}))$. We now show that $h^{\leftarrow}(\theta_Y^{\rightarrow}(\mathrm{cl}\, f^{\rightarrow}\mathcal{F} \times \mathcal{G})) \geq \theta^{\rightarrow}(\mathrm{cl}\, \mathcal{F} \times \mathcal{G})$ by verifying that $h^{-1}(\theta_Y(\mathrm{cl}\, f(F) \times G)) \subseteq \theta(\mathrm{cl}\, F \times G)$ for arbitrary $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Let $\langle v, a \rangle \in h^{-1}(\theta_Y(\mathrm{cl}\, f(F) \times G))$. Then $h(\langle v, a \rangle) = \langle f(v), a \rangle \in \theta_Y(\mathrm{cl}\, f(F) \times G)$ implies that $(w, b) \in \mathrm{cl}\, f(F) \times G$ for some $\langle w, b \rangle = \langle f(v), a \rangle$, hence $\mu(w, a) = \mu(f(v), b) = f(\lambda(v, b)) \in f(X)$. However, by Theorem 2.1, Y is remainder-invariant and thus $w = f(z) \in$

cl $f(F)$ for some $z \in X$. Since f is an embedding, it follows that $z \in \text{cl}\, F$. Moreover, $\mu(w, a) = \mu(f(z), a) = f(\lambda(z, a)) = f(\lambda(v, b))$ and so $\lambda(z, a) = \lambda(v, b)$ because f is an injection. Hence $\langle v, a \rangle = \langle z, b \rangle \in \theta(z, b) \in \theta(\text{cl}\, F \times G)$ shows that $h^{-1}(\theta_Y(\text{cl}\, f(F) \times G)) \subseteq \theta(\text{cl}\, F \times G)$, and thus $h^{\leftarrow}(\theta_Y^{\rightarrow}(\text{cl}\, f^{\rightarrow}\mathcal{F} \times \mathcal{G})) \geq \theta^{\rightarrow}(\text{cl}\, \mathcal{F} \times \mathcal{G})$ as claimed.

Recall $\mathcal{K}$ is a filter on Y that converges to y and that $\mathcal{G}$ is a filter on S that converges to t where $\langle y, t \rangle = \langle f(x), s \rangle$. Again, since Y is remainder-invariant, $\mu(y, s) = \mu(f(x), t) = f(\lambda(x, t)) \in f(X)$ and so $y = f(x')$ for some $x' \in X$. Thus, $\mu(y, s) = \mu(f(x'), s) = f(\lambda(x', s)) = f(\lambda(x, t))$ and so $\lambda(x, t) = \lambda(x', s)$, which means $\langle x, s \rangle = \langle x', t \rangle$. Since $f^{\rightarrow}\mathcal{F} \to y = f(x')$, $\mathcal{F} \to x'$ and X is regular, we have that $\text{cl}\, \mathcal{F} \to x'$, hence $\mathcal{H} \geq \theta^{\rightarrow}(\text{cl}\, \mathcal{F} \times \mathcal{G}) \to \theta(x', t) = \langle x', t \rangle = \langle x, s \rangle$, hence h is an embedding. Since S is compact, it follows that $Y \times S$ is compact, proving that $\mathbf{B}(Y)$ is compact.

Let $\langle y, s \rangle \in \mathbf{B}(Y)$. Then there exists $\mathcal{F} \in \mathbf{F}(X)$ such that $f^{\rightarrow}\mathcal{F} \to y$, hence $(h \circ \theta)^{\rightarrow}(\mathcal{F} \times \dot{s}) = (\theta_Y \circ (f \times \text{id}_S))^{\rightarrow}(\mathcal{F} \times \dot{s}) = \theta_Y^{\rightarrow}(f^{\rightarrow}\mathcal{F} \times \dot{s}) \to \theta_Y(y, s) = \langle (y, s) \rangle$, which proves that h is a dense embedding. Since Y is Hausdorff, it follows from Theorem 4.1 in [1] that $\mathbf{B}(Y)$ is also Hausdorff. In conclusion, $\mathbf{B}(Y)$ is a compactification of $\mathbf{B}(X)$ in CONV, and by employing Theorem 2.4 (b) in [2], $\mathbf{B}(Y)$ is an S-compactification of $\mathbf{B}(X)$.

We now show that $\mathbf{B}(Y)$ is regular: Suppose that $\mathcal{H}$ is a filter on $\mathbf{B}(Y)$ that converges to $\langle y, s \rangle$. Then there exists a filter $\mathcal{K}$ on Y that converges to some y' and a filter $\mathcal{G}$ on S that converges to s' such that $\mathcal{H} \geq \theta_Y^{\rightarrow}(\mathcal{K} \times \mathcal{G})$ and $\langle y, s \rangle = \langle y', s' \rangle$. Since Y and S are compact and $\mathbf{B}(Y)$ is Hausdorff, it follows from Proposition 3.2 in [7] that θ_Y is a proper map and thus closure preserving. This means that $\text{cl}\, \mathcal{H} \geq \text{cl}\, \theta_Y^{\rightarrow}(\mathcal{K} \times \mathcal{G}) = \theta_Y^{\rightarrow}(\text{cl}\, \mathcal{K} \times \text{cl}\, \mathcal{G}) \to \theta_Y(y', s') = \langle y, s \rangle$ since Y and S are regular. Therefore, $\text{cl}\, \mathcal{H} \to \langle y, s \rangle$, proving that $\mathbf{B}(Y)$ is a regular. $\qquad\square$

Employing Theorems 3.2, 4.1, and 4.2 gives the following result.

Corollary 4.3. *Suppose X is an object in GQ that is adherence-restrictive, locally compact and completely regular and suppose S is compact and completely regular. Then $\mathbf{B}(X_*)$ is a one-point regular S-compactification of $\mathbf{B}(X)$, where X_* is the one-point regular S-compactification of X.*

Let us now consider the topological case. Let τq be the finest topology on X coarser than the convergence structure q of X and let $\tau X := (X, \tau q)$ denote the topological modification of X. Recall that $\theta \colon X \times S \to \mathbf{B}(X)$ is a quotient map in CONV. According to Theorem 4.2 in [8], if either X or S is a locally compact Hausdorff topological space, then $\tau(X \times S) = \tau X \times \tau S$ and thus θ is a quotient map in TOP. In this case, $\tau \mathbf{B}(X)$ is the generalized quotient of τX (acted upon by τS) in TOP. The final result pertains to generalized quotient spaces in TOP and the proof follows from the preceeding remarks along with Theorem 4.2.

Corollary 4.4. *Suppose X is an adherence-restrictive object in* GQ *and that S is a compact Hausdorff topological space. Let Y be a topological S-compactification of X. Then $\tau\mathbf{B}(X)$ and $\tau\mathbf{B}(Y)$ are generalized quotients in* TOP *and $\tau\mathbf{B}(Y)$ is a topological S-compactification of $\tau\mathbf{B}(X)$.*

5. Conclusion

In general, an S-space has neither a smallest nor a largest S-compactification. Conditions are given for the existence of a smallest (regular) S-compactification in Theorem 2.3 (respectively, Theorem 3.2). Whether or not (iv) of Theorem 3.2 is equivalent to the other three statements listed in the theorem is unknown to the authors. Each S-space having a regular S-compactification has a largest such. Theorem 3.5 and Theorem 3.7 address the question as to when a regular compactification in LIM can be used to form a regular S-compactification in CA. It is shown in Theorem 4.2 that the process of forming a regular S-compactification remains invariant under the operation of taking "generalized quotients."

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