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Radicals in the class of compact right topological rings

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Abstract

We construct in this article three radicals in the class of compact right topological rings. We also prove that a simple left Noetherian compact right topological ring of prime characteristic is finite.

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1. Introduction

The notion of a compact left (right) topological ring was introduced in [6] and later in [3].

A fundamental question occuring at the study of any class of rings is: What are simple rings which are in this class?

It is known, for instance, that a compact ring is simple if and only if it is a matrix ring over a finite division ring [4].

We study in this paper simple compact right topological rings. We prove that every left Noetherian compact right topological simple ring with identity of prime characteristic is finite. Answering a question posed by Professor Richard Wiegandt, we construct some radicals in the class of compact right topological rings.

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2. Notation and conventions

All rings are assumed associative not necessarily with identity. Topological groups are assumed Hausdorff. The closure of a subset A of a topological space is denoted by $\overline{A}$. A J-semismiple ring is a ring whose Jacobson radical is zero.

3. A few elementary results

Differences between elementary properties of topological rings and right topological rings are essential. We will give an example of a right topological ring for which the right annihilator is not closed. We construct also a compact right topological ring with nonclosed center.

Let R be a compact ring with identity and $\alpha : R \to R$ a group endomorphism. Define on the group $A = R \times R$ with the product topology the multiplication $(a, x)(b, y) = (0, a\alpha(b))$. Then $A^3 = 0$ and A is a compact right topological ring.

Let p be a prime number and $R = \mathbb{F}_p^{\aleph_0}$ with the product topology or $R = \mathbb{F}_p[[t]]$ be the ring of formal power series over $\mathbb{F}_p$. Then the additive group of R contains a dense subgroup B of index p. Let $\alpha : R \to R$ be a group homomorphism for which $\ker \alpha = B$. Then $\mathrm{Ann}_r A \supset R \times B$, hence $\mathrm{Ann}_r A$ is dense. Since $\mathrm{Ann}_r A \neq A$, it is not closed.

Theorem 3.1. *Let $(A, \mathcal{T})$ be a compact abelian group. Then the ring* $\mathrm{End}A$ *of all not necessarily continuous endomorphisms of A with the topology of pointwise convergence is a compact right topological ring.*

Proposition 3.2. *For any right topological ring R the closure of its abstract center $Z = Z(R)$ is a subring.*

Proof. Indeed, $\overline{Z} \cdot \overline{Z} \subset \overline{Z \cdot \overline{Z}} = \overline{\overline{Z} \cdot Z} \subset \overline{\overline{Z}} = \overline{Z}$. □

Example 3.3. A compact topological ring with nonclosed center.

The group $\mathbb{R}$ admits a compact group topology (see [2]). According to Theorem 3.1 $\mathrm{End}\mathbb{R}$ admits a compact right ring topology. Since the center of $\mathrm{End}\mathbb{R}$ is $\mathbb{Q}$, the center is not closed. The subring $\overline{\mathbb{Q}}$ of $\mathrm{End}\mathbb{R}$ is connected.

4. Simple compact right topological rings

As usual, a ring R is called *simple* if $R^2 \neq 0$ and R has no ideals different from 0 and R. Recall that an ideal M of a ring R is called *maximal* if the factor ring R/M is a simple ring.

Lemma 4.1. *There are no infinite compact right topological simple Artinian rings of prime characteristic.*

Proof. Assume on the contrary that there exists an infinite compact right topological simple Artinian ring R of prime characteristic.

There exists an infinite cardinal α such that $R(+) \cong_{\text{top}} \mathbb{Z}(p)^{\alpha}$. We identify the additive group $R(+)$ of R with $\mathbb{Z}(p)^{\alpha}$. Consider the ring $S = End_c(R(+))$

of continuous endomorphisms of $R\,(+)$ with the compact-open topology. We note that $End_c\,(R\,(+))$ has a fundamental system of neighborhoods of 0 consisting of right ideals. Indeed, denote for each neighborhood V of 0_R by $T\,(V)$ the right ideal $\{\rho \in S \mid \rho\,(R) \subset V\ \}$. The family $\{T\,(V)\}$, where V runs all neighborhoods of R, forms a fundamental system of neighborhoods of 0_S consisting of right ideals. We shall estimate now the weight of S. Fix $\beta < \alpha$. Put

$$H_\beta = \left\{ \rho \in S \mid \rho\,(R) \subset \mathbb{Z}\,(p)_\beta \times \prod_{\gamma \neq \beta} \{0\}_\gamma \right\}.$$

Fact 1. H_β is a discrete right ideal of S.

Indeed, $H_\beta \cap T(\{0\} \times \prod_{\gamma\neq\beta} Z(p)_\gamma\}) = 0$.

Fact 2. $|H_\beta| \leq \alpha$.

Indeed, fix $\beta_0, \ldots, \beta_n < \alpha, n \in \omega$.

Put $S_{\beta_0 \ldots \beta_n} = \{\chi \in H_\beta \mid \chi(\{0\}_{\beta_0} \times \cdots \times \{0\}_{\beta_n} \times \prod_{\gamma \notin \{\beta_0,\ldots,\beta_n\}} Z(p)_\gamma) = 0\}$. Since the values of every $\pi \in S_{\beta_0\ldots\beta_n}$ are determined by the set $Z(p)_{\beta_0} \times \cdots \times Z(p)_{\beta_n} \times \prod_{\gamma \notin \{\beta_0,\ldots,\beta_n\}} \{0\}_\gamma$, $S_{\beta_0\ldots\beta_n}$ is finite. This implies that $|H_\beta| \leq \alpha$.

Fact 3. $|\sum_{\beta<\alpha} H_\beta| \leq \alpha$.

Since $|H_\beta| \leq \alpha$ for every $\beta < \alpha$, the cardinality of $\sum_{\beta<\alpha} H_\beta$ is $\leq \alpha \times \alpha = \alpha$.

Fact 4. $\sum_{\beta<\alpha} H_\beta$ is a dense subgroup of S.

Denote for each $\beta < \alpha$ the endomorphism q_β of $R(+)$: $q_\beta : R \to R, (x_\gamma) \mapsto x_\beta \times \prod_{\delta\neq\beta} 0_\delta$ and by pr_β the projection of R on $Z(p)_\beta = Z(p)$. Evidently, $q_\beta \in H$. We have that $q_{\beta_0}\varphi + \cdots + q_{\beta_n}\varphi \in H$ for each $\beta_0, \ldots, \beta_n < \alpha, n \in \omega, \varphi \in S$.

Set $V = \{0\}_{\beta_0} \times \cdots \times \{0\}_{\beta_n} \times \prod_{\beta\neq\beta_0,\ldots,\beta_n} Z(p)_\beta$. Then we have:

$pr_{\beta_i}[(\varphi - \Sigma_{j=0}^n q_{\beta_j}\varphi)(x)] = pr_{\beta_i}[\varphi(x) - \Sigma_{j=0}^n q_{\beta_j}\varphi(x)]$
$= pr_{\beta_i}[\varphi(x)] - \Sigma_{j=0}^n pr_{\beta_i}[q_{\beta_j}\varphi(x)]$
$= pr_{\beta_i}[\varphi(x)] - pr_{\beta_i}[\varphi(x)]$
$= 0$,

for every $i \in \{0, n\}$ and every $x \in R$.

We obtain $(\varphi - \Sigma_{j=0}^n q_{\beta_j}\varphi)(R) \subset V$, i.e., $\varphi - \Sigma_{j=0}^n q_{\beta_j}\varphi \in T(V)$, hence $\varphi \in H + T(V)$.

We have proved that $S = H + T(V)$. The topological group S has a fundamental system of neighborhoods of 0 of cardinality $\leq \alpha$. It follows that $w(S) \leq \alpha$, where $w(S)$ is the weight of S. Define $\chi : R \to S, x \mapsto R_x$. Then χ is a ring anti-isomorphism of R in S. Since S has a fundamental system of neighborhoods of 0 consisting of right ideals, $|\chi(R)| \leq \alpha$, a contradiction, since $|R| = 2^\alpha > \alpha$. □

Theorem 4.2. *Every simple left Noetherian compact right topological ring R with identity of prime characteristic is finite.*

Proof. Since R is a compact right topological ring, every principal left ideal of R is closed. By compactness, every finitely generated left ideal is closed. Since R is left Noetherian, every left ideal of R is closed. If $\mathcal{F}$ is a filter basis consisting of cosets with respect to left ideals, then by compactness of R its intersection is non-empty. Thus, R is a discrete left linearly compact simple ring. It follows that R is a left linearly compact discrete simple ring. Then, (see [7], [5]), R is an Artinian simple ring. By Lemma 4.1, R is finite. $\square$

5. Radicals in the class of compact right topological rings

Lemma 5.1. *If R is a compact right topological ring and I an ideal with identity e, then I is a topological direct summand.*

Proof. It is well-known that e will be a central idempotent. We have that Re and $R(1-e) = \{x \in R \mid xe = 0\}$ are closed ideals of R. Since $R = Re \oplus R(1-e)$ is a direct sum, by compactness of R, this sum is topological. $\square$

Let $\mathcal{L}$ be a class of Hausdorff right topological rings. We will say that the operator ρ is a *radical* in the class $\mathcal{L}$ if:

(i) $\rho(R)$ is a closed ideal of R for each $R \in \mathcal{L}$;

(ii) $\rho(\rho(R)) = \rho(R)$;

(iii) $f(\rho(R)) \subset \rho(f(R))$ for any continuous homomorphism $f : R \to S$, where $R, S \in \mathcal{L}$;

(iv) $\rho(R/\rho(R)) = 0$.

Remark 5.2. Our definition differs from the definition of a radical in classes of topological rings introduced in [1]. In [1] it is assumed that the class of topological rings for which is defined a radical is closed under taking of ideals. Evidently, this condition is not filled for compact right topological rings.

We will construct three radicals in the class of compact right topological rings. First, we note that the connected component of zero is a radical in the class of compact right topological rings. Indeed, if R is a compact right topological ring, then its component R_0 of zero is equal to $\cap_{n\in\mathbb{N}} nR$, hence R_0 is a closed ideal of R.

Theorem 5.3. *Let $\mathcal{L}$ be the class of all compact right topological rings and $\rho(R) = \cap M$, where M runs all open maximal ideals of R. Then ρ is a radical.*

Proof. Since every open ideal is closed, $\rho(R)$ is a closed ideal of R. Then $\rho(R/\rho(R) = 0$ and by Kaplansky's Theorem $R/\rho(R)$ has identity. The factor ring $R/\rho(R)$ is a compact J-semisimple ring.

Denote by H the two-sided ideal generated by $\rho(\rho(R))$. Then $\rho(\rho(R)) \subset H \subset \rho(R)$. By Andrunakievich's Lemma, $H^3 \subset \rho(\rho(R))$. Since $\rho(R)/\rho(\rho(R))$ is semiprime, $H \subset \rho(\rho(R))$, hence $H = \rho(\rho(R))$.

We claim that if $f : R \to S$ is a continuous homomorphism, then $f(\rho(R)) \subset \rho(f(R))$.

Indeed, consider f as a surjective homomorphism of R on $L = f(R)$. Then $\rho(L) = \cap M'$ where M' runs all maximal ideals of L. It follows that $f^{-1}(\rho(L)) = \cap f^{-1}(M')$ where M' runs all open maximal ideals of L. If M' is a maximal open ideal of L, then $f^{-1}(M')$ will be an open maximal ideal of R. It follows that $f^{-1}(\rho(L)) \supset \rho(R)$, hence $f(\rho(R)) \subset \rho(f(L))$.

Now we will prove that $\rho^2 = \rho$. Indeed, consider the factor ring $R/\rho(\rho(R))$. The ideal $\rho(R)/\rho(\rho(R))$ has identity, therefore is a topological direct summand. We obtain that $R/\rho(\rho(R))$ is a topological direct sum of $R/\rho(R)$ and $\rho(\rho(R))$. It follows that $\rho(R/\rho(\rho(R))) = 0$, hence $\rho(R) \subset \rho(\rho(R))$ and so $\rho(\rho(R)) = \rho(R)$. $\square$

We will obtain another radical ϕ different from ρ if we will put $\phi(R)$ the intersection of all open ideals V of a compact right topological ring R for which R/V is a field.

Theorem 5.4. *For any compact right topological ring R and for any its closed ideal I holds $\rho(I) = I \cap \rho(R)$.*

Proof. If $\phi : R \to R/\rho(R)$ is the natural homomorphism, then the restriction $\phi\,|_I\colon I \to I + \rho(R)/\rho(R)$ is surjective and $I + \rho(R)/\rho(R)$ is a closed ideal of $R/\rho(R)$. Since $R/\rho(R)$ is a product of finite simple rings ([4]), $I + \rho(R)/\rho(R)$ is ρ-semisimple. Thus $\rho(I) \subset \rho(R) \cap I$.

Denote by H the two-sided ideal of R generated by $\rho(I)$. Then $\rho(R) \subset H \subset I$. By Andrunakievich's Lemma, $H^3 \subset \rho(R)$. Since $I/\rho(I)$ is semiprime, $H \subset \rho(I)$, hence $H = \rho(I)$.

Since $I/\rho(I)$ is a compact semisimple ring, it has an identity [4]. By Lemma 5.1 there exists a closed ideal S_1 of $R/\rho(R)$ such that $R = S_1 \oplus (I/\rho(I))$, a topological direct sum. By a standard result, $S_1 = S/\rho(R)$, where S is a closed ideal of R. We have that $R = S + I$ and $S \cap I = \rho(I)$.

The factor ring R/S is topologically isomorphic to $I/I \cap S = I/\rho(I)$, hence is ρ-semisimple. It follows that $\rho(R) \subset S$, hence $\rho(R) \cap I \subset S \cap I = \rho(I)$. $\square$

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