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Subgroups of paratopological groups and feebly compact groups

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Abstract

It is shown that if all countable subgroups of a semitopological group G are precompact, then G is also precompact and that the closure of an arbitrary subgroup of G is again a subgroup. We present a general method of refining the topology of a given commutative paratopological group G such that the group G with the finer topology, say, σ is again a paratopological group containing a subgroup whose closure in (G,σ) is not a subgroup.
It is also proved that a feebly compact paratopological group H is perfectly κ-normal and that every G_δ-dense subspace of H is feebly compact.

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1. Introduction

Our aim is to study closures of subgroups of paratopological groups. We will say that a group G with a topology is an *SP-group* (abbreviation for Subgroup Preserving) if the closure of every subgroup of G is again a subgroup of G. Every

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topological group is clearly an SP-group. A similar statement fails in the class of paratopological groups, see [2, Example 1.4.17]. Nevertheless, there exists a wide class of paratopological groups containing the Sorgenfrey line which is closed under taking quotient groups, subgroups, and arbitrary products and which contains only SP-groups [13]. The main result of [13] was extended in [6] for the class of *almost topological groups* (the corresponding definition is given Section 6).

In Section 3 of the article we show that under certain conditions, the topology of a commutative paratopological group can be refined in such a way that the group with the new topology is again a paratopological group which fails to be an SP-group (see Propositions 3.7 and 3.8, and Corollary 3.9). In fact, we refine the topology of a given paratopological group by declaring open a 'single' subsemigroup of the group.

In Section 4 we consider compact subsets of paratopological groups and the action of internal automorphisms on open neighborhoods of the identity. It is well known that for a compact (even precompact) subset B of a topological group G, and an arbitrary neighborhood U of the identity e in G, one can find a neighborhood V of e such that $bVb^{-1} \subseteq U$, for each $b \in B$. In Example 4.1 we show that this fact is not valid in the class of Hausdorff paratopological groups.

A *feebly compact* space is a topological space in which every locally finite family of open sets is finite. In Tychonoff spaces the concepts of pseudocompactness and feeble compactness coincide. It is known that every pseudocompact (hence Tychonoff) paratopological group is a topological group [10, Theorem 2.6]. This result remains valid for regular feebly compact paratopological groups [1, Theorem 1.7]. However, a Hausdorff feebly compact paratopological group can fail to be a topological group [9, Example 3]. Furthermore, under Martin's Axiom, there exists a Hausdorff countably compact paratopological group with discontinuous inversion [9, Example 2]. Thus we focus our attention on the study of feebly compact paratopological groups in Section 5. We prove that every feebly compact paratopological group is perfectly κ-normal and that any G_δ-dense subspace of a feebly compact paratopological group is feebly compact.

In Section 6 we prove that an almost topological group G is precompact (or Baire) if and only if the underlying topological group $\overline{G}$ has the same property, and that the indices of narrowness of G and $\overline{G}$ coincide.

2. Preliminaries, notation and definitions

Let G be an abstract group with a topology. The group G is *left topological* (*right topological*) provided that the left (right) translations are continuous in G. A group G that is both left and right topological is a *semitopological group.* If multiplication on G is continuous, we say that G is a *paratopological group.* If, in addition, the inversion is continuous in G, then G is a *topological group.* A left (right) topological group G is *left* (*right*) *precompact* if for every open neighborhood U of the identity of G there is a finite subset F of G such that $FU = G$ ($UF = G$). A semitopological group G is *precompact* provided that it is both left and right precompact.

We denote by $\langle H\rangle$ the subgroup generated by a subset H of a group G. If $x \in G$, we write $\langle x\rangle$ instead of $\langle\{x\}\rangle$. A nonempty subset S of the group G is called a *subsemigroup* of G if $xy \in S$, for all $x, y \in S$. Note that S does not necessarily contain the neutral element of G.

A semitopological (paratopological) group G is called *saturated* provided that U^{-1} has nonempty interior, for every neighborhood U of the identity in G. Every precompact paratopological group is saturated [8, Proposition 3.1].

We also say that a semitopological group G is *topologically periodic* if for every neighborhood U of the neutral element e in G and every element $x \in G$, there exists a positive integer n such that $x^n \in U$. Clearly every semitopological torsion group is topologically periodic.

In this article, a *regular* space will be a topological space X such that for every $x \in X$ and every open neighborhood U of x there is an open neighborhood V of x such that $\overline{V} \subseteq U$, or equivalently, for every closed subset F of X and $x \notin F$, there are open disjoint sets U and V in X such that $x \in U$, and $F \subseteq V$. A space will be called T_3 provided that it is regular and T_1.

Let X be a topological space and $U \subseteq X$. We say that U is a *regular open* subset of X provided that $U = int\,\overline{U}$. The family of regular open sets in X is a base for a topology ρ on X weaker than the topology of X. We denote the topological space (X, ρ) by rX; in general, the space rX is called the *semiregularization* of X, for rX is a semiregular space. It turns out that for any paratopological group G, the space rG is a regular paratopological group (see [7, Proposition 1.5]). Since we are interested primarily in paratopological groups, we will refer to rG as the *regularization* of G. The results of the rest of this section are proved by Ravsky in [9].

Lemma 2.1. *Let X be a topological space. Then X is feebly compact if and only if rX is feebly compact.*

Let G be a paratopological group and $\mathcal{N}$ the family of all open neighborhoods of the neutral element e of G. Then the set $H = \bigcap_{U\in\mathcal{N}}(U \cap U^{-1})$ is an invariant subgroup of G, i.e. $xHx^{-1} = H$ for each $x \in G$. Denote by T_0G the quotient paratopological group G/H and let $\pi\colon G \to G/H$ be the quotient homomorphism. It is easy to verify that $U = \pi^{-1}\pi(U)$, for each open set $U \subseteq G$ and that the group T_0G is a T_0-space (see [9, Section 5]).

Lemma 2.2. *A paratopological group G is feebly compact if and only if T_0G is feebly compact.*

Lemma 2.3. *For every feebly compact paratopological group G, the group $T_0(rG)$ is a pseudocompact topological group.*

3. Closures of subgroups

Here we find conditions under which a semitopological group turns out to be an SP-group or when a paratopological group G admits a finer paratopological group topology, say, σ such that (G, σ) fails to be an SP-group.

Proposition 3.1. *Let G be a left (right) topological group such that every countable subgroup of G is left (right) precompact. Then G is left (right) precompact.*

Proof. Suppose, on the contrary, that G is not left precompact. Choose an open neighborhood U of the identity in G such that $FU \neq G$, for every finite subset F of G. We will define an increasing countable family $\{C_n\}_{n\in\omega}$ of countable subgroups of G as follows. Let C_0 be any countable subgroup of G. Once we have defined the subgroups $C_0, \ldots, C_n$ such that $C_0 \subseteq \cdots \subseteq C_n$, for any finite set $F \subseteq C_n$, we choose $x_F \in G \backslash FU$. We define $C_{n+1} = \langle C_n \cup \{x_F : F \subseteq C_n, |F| < \omega\}\rangle$. Let $C = \bigcup_{n\in\omega} C_n$. Since every C_n is countable, the group C is countable. By hypothesis C is left precompact, thus there is a finite set $F \subseteq C$ such that $F(U \cap C) = C$. On the other hand, there exists $k \in \omega$ such that $F \subseteq C_k$. Then $x_F \notin FU \supseteq C$, a contradiction with the definition of C. Thus G is left precompact. □

Corollary 3.2. *Let G be a semitopological group such that every countable subgroup of G is precompact. Then G is also precompact.*

It is worth mentioning that subgroups of a precompact paratopological group can fail to be precompact. One of the standard examples of this phenomenon is as follows. Let $\mathbb{T}_s$ be the circle group endowed with the Sorgenfrey topology, i.e. a local base at the neutral element 1 of $\mathbb{T}_s$ is formed by the sets $U_n = \{e^{\pi i x} : 0 \leq x < 1/n\}$, with $n \in \mathbb{N}^+$. Then $\mathbb{T}_s$ is a commutative zero-dimensional (hence Tychonoff) paratopological group. The paratopological group $\mathbb{T}_s^2$ is also precompact and it contains the closed discrete uncountable subgroup $\Delta_2 = \{(x, x^{-1}) : x \in \mathbb{T}_s\}$. Hence the subgroup Δ_2 is not precompact. Furthermore, every discrete abelian group can be embedded as a subgroup into a precompact Hausdorff paratopological group [4, Corollary 5].

For an element x of a paratopological group G, we denote by S_x the subsemigroup $\{x^n : n \in \omega\}$ of G. Note that S_x contains the neutral element of G.

Proposition 3.3. *Suppose that a semitopological group G is not an SP-group. Then there exists an element $x \in G$ of infinite order such that the subsemigroup S_x is open in the cyclic group $\langle x\rangle$.*

Proof. Let H be a subgroup of G such that $\overline{H}$ fails to be a subgroup of G. As G is a semitopological group, $\overline{H}$ is a subsemigroup of G [2, Proposition 1.4.10]. Since $\overline{H}$ is not a subgroup of G, there exists an element $y \in \overline{H}$ such that $y^{-1} \notin \overline{H}$. It is clear that the subsemigroup S_y is contained in $\overline{H}$. Since $y^{-1} \notin \overline{H}$, the element y is of infinite order. Moreover, $y^{-n} \notin \overline{H}$, for every

$n \in \mathbb{N}$. If we denote the element y^{-1} by x, then $S = \big(G \setminus \overline{H}\big) \cap \langle x \rangle$ is open in $\langle x \rangle$ and $S_x = x^{-1}S$ is open in $\langle x \rangle$ as well. □

Corollary 3.4. *Every topologically periodic semitopological group is an SP-group.*

Proof. Suppose that a semitopological group G fails to be an SP-group. Then, by Proposition 3.3, G contains an element x of infinite order such that the subsemigroup $S_x = \{x^n : n \in \omega\}$ is open in the cyclic group $\langle x \rangle$. Choose an open set U in G such that $U \cap \langle x \rangle = S_x$. Then U contains the neutral element of G and $y^n \notin U$ for each positive integer n, where $y = x^{-1}$. Hence the group G is not topologically periodic. □

The following lemma is known in the folklore. We present its proof to ease the reader's job.

Lemma 3.5. *If all cyclic subgroups of a semitopological group G are left precompact, then G is topologically periodic.*

Proof. If G is not topologically periodic, we can find an open neighborhood U of the neutral element e in G and an element $x \in G$ distinct from e such that $x^n \notin U$, for each positive integer n. In particular, the cyclic group $\langle x \rangle$ is infinite and the subsemigroup $S_y = \{y^n : n \in \omega\}$, where $y = x^{-1}$, is open in $\langle y \rangle = \langle x \rangle$. Then $FS_y \neq \langle y \rangle$ for every nonempty finite subset F of $\langle y \rangle$. Indeed, if $k = \min\{n \in \mathbb{Z} : y^n \in F\}$, then $y^{k-1} \notin FS_y$. Thus the subgroup $\langle y \rangle$ of G is not left precompact. □

Combining Corollary 3.4 and Lemma 3.5, we obtain the following fact:

Corollary 3.6. *Let G be a semitopological group such that every cyclic subgroup of G is left precompact. Then G is an SP-group.*

It is worth mentioning that our Corollary 3.4 follows from a more general result established in [14, Theorem 3.3]: The closure of every *subsemigroup* of a topologically periodic semitopological group is a subgroup. The corresponding argument in [14] is, however, quite different.

In the sequel we present several results on refinements of topologies of paratopological (SP-)groups turning them into non-SP-groups.

A non-empty subset T of an abelian group G is called *independent* if the equality $n_1x_1+\cdots+n_kx_k = 0_G$ with $n_1, \ldots, n_k$ integers and distinct $x_1, \ldots, x_k \in T$ implies $n_1x_1 = \cdots = n_kx_k = 0_G$. Thus a set $T \subseteq G$ of elements of infinite order is independent if and only if the former equality implies that $n_1 = \cdots = n_k = 0$. Given a paratopological group G with open neighborhood base $\mathcal{N}$ at the identity e, and a subsemigroup S of G containing e, we denote by G_S the group G with topology τ_S whose local base at e is the family $\mathcal{B} = \{U \cap S : U \in \mathcal{N}\} \cup \mathcal{N}$. That is, $G_S = (G, \tau_S)$. It is easy to verify that G_S is also a paratopological group. By the definition, the topology of G_S refines the topology of G. Note that if G is first countable, so is G_S.

Proposition 3.7. *Let (G,τ) be a first countable, abelian paratopological group. If G contains an infinite independent subset K of elements of infinite order which accumulates at the identity e of G, then G_S is not an SP-group for some countable subsemigroup S of G with $e \in S$.*

Proof. Choose any element $x \in K$. Let $\{U_n\}_{n\in\omega}$ be a local base at e. Since e is an accumulation point of the set K, we can choose $t_n \in (K \setminus \{x\}) \cap U_n$, for every $n \in \omega$, with $t_n \neq t_m$ if $n \neq m$. Let S be the subsemigroup of G generated by the subset $T = \{e\} \cup \{t_n : n \in \omega\}$. Consider the group G_S. Since K is an independent subset of G, the equality $\langle x\rangle \cap \langle T\rangle = \{e\}$ holds.

Let $h_n = x + t_n$, for every $n \in \omega$, and $H = \langle\{h_n : n \in \omega\}\rangle$. By the definition of H, $x \in \overline{H}$, here the closure is taken in the group G_S. For let $U_n \cap S$ be a basic open neighborhood of e in G_S, then $h_n = x+t_n \in (x + (U_n \cap S)) \cap H$. Now we prove that $-x \notin \overline{H}$. Suppose, on the contrary, that $-x \in \overline{H}$. Since S is open in G_S, $(-x + S) \cap H \neq \varnothing$. Then, since H is a subgroup, $-x + l_1 t_{n_1} + \cdots + l_i t_{n_i} = k_1 h_{m_1} + \cdots + k_j h_{m_j}$ for some positive integers $l_1, \ldots, l_i$, some nonzero integers $k_1, \ldots, k_j$, and $n_1, \ldots, n_i \in \omega$, and distinct $m_1, \ldots, m_j \in \omega$. From here it follows that $(-1 - k_1 - \cdots - k_j)\, x = k_1 t_{m_1} + \ldots k_j t_{m_j} - l_1 t_{n_1} - \cdots - l_i t_{n_i}$. Since $\langle x\rangle \cap \langle T\rangle = \{e\}$, we have on one hand that $-1 - k_1 - \cdots - k_j = 0$, hence $k_1 + \cdots + k_j = -1$. On the other hand, the independence of $T \setminus \{e\}$ implies that $i = j$ and, after reindexing the set $\{m_1, \ldots, m_i\}$, $n_r = m_r$, for every $r = 1, \ldots i$. Thus $k_r = l_r$, for each $r = 1, \ldots, i$. Then every k_r is positive, which is a contradiction. Thus $\overline{H}$ is not a subgroup of G_S. □

In the next proposition we present sufficient conditions on a subsemigroup S of a commutative, first countable paratopological group G, in order that G_S won't be an SP-group.

If T is an independent set of elements of infinite order in a commutative group G with identity 0, every non-zero element $t \in \langle T\rangle$ can be written in a unique way as $t = k_1 t_1 + \cdots + k_n t_n$, with $k_1, \ldots, k_n$ non-zero integers and distinct $t_1, \ldots, t_n \in T$. We put $Exp_T(t) = k_1 + \cdots + k_n$.

Proposition 3.8. *Let (G,τ) be a commutative, first countable paratopological group with identity e and $S = S \cup \{e\}$ a subsemigroup of G. Suppose that*

1. *there is a countable infinite independent set $K \subseteq S$ of elements of infinite order such that e is an accumulation point of K;*
2. *there exists $x^* \in G$ of infinite order such that $\langle x^*\rangle \cap \langle S\rangle = \{e\}$;*
3. *if $T = \langle K\rangle$, and $t \in T \cap S$, then $Exp_T(t) \geq 0$.*

Then the first countable paratopological group $G_S = (G, \tau_S)$ is not an SP-group.

Proof. Let $\{V_n\}_{n\in\omega}$ be a local base at e in (G,τ). For every $n \in \omega$, choose an element $v_n \in V_n \cap K$ in such a way that $v_n \neq v_m$ if $n \neq m$. Consider the subgroup H of G generated by the set $\{x^* + v_n : n \in \omega\}$. Since the family $\{V_n\}_{n\in\omega}$ is a local base at e, we have that $x^* \in \overline{H}$, here the closure is taken in G_S. We claim that $-x^* \notin \overline{H}$. Suppose, on the contrary, that $-x^* \in \overline{H}$. Since S is an open neighborhood of e in G_S, we have that $(-x^* + S) \cap H \neq \varnothing$.

Then $-x^* + s = k_1 (x^* + v_{n_1}) + \cdots + k_i (x^* + v_{n_i})$, for some $s \in S$, $k_1, \ldots, k_i$ integers, and distinct $n_1, \ldots, n_i \in \omega$. From here, $(k_1 + \cdots + k_i + 1) x^* = s - k_1 v_{n_1} - \cdots - k_i v_{n_i} \in \langle x^* \rangle \cap \langle S \rangle$, which implies by (2) that $k_1 + \cdots + k_i + 1 = 0$. But then $s = k_1 v_{n_1} + \cdots + k_i v_{n_i} \in S \cap T$, and $Exp_T(s) < 0$, which contradicts (3). Hence, $-x^* \notin \overline{H}$, and $\overline{H}$ is not a subgroup of G_S. □

Corollary 3.9. *There exists a first countable topology containing the usual topology of* $\mathbb{R}$ *that makes the additive group* $\mathbb{R}$ *a paratopological group but not an* SP*-group.*

Proof. The additive group $\mathbb{R}$ with its usual topology is first countable. We choose a countable neighborhood base $\{U_n : n \in \omega\}$ at 0 in $\mathbb{R}$ such that $U_{n+1} \subset U_n$, for every $n \in \omega$. We can define by induction an infinite set $K = \{x_n : n \in \omega\} \subset \mathbb{R}$ as follows. Let $x_0 \in (\mathbb{R} \setminus \mathbb{Q}) \cap U_0$. Once we have defined independent elements $x_0, \ldots, x_n$, with $x_i \in U_i$ for every $i = 1, \ldots, n$, we put $A_n = \mathbb{Q}x_0 + \cdots + \mathbb{Q}x_n$. Since A_n is countable, we can choose an irrational number $x_{n+1} \in (\mathbb{R} \setminus A_n) \cap U_{n+1}$. Let us verify that the set K is independent.

Suppose not, then we can find a linear combination $k_1 x_{i_1} + \cdots + k_n x_{i_n} = 0$ of elements of K, with non-zero integers $k_1, \ldots, k_n$ and $i_1 < \cdots < i_n$, where $n > 1$. Then $x_{i_n} = -\frac{k_1}{k_n} x_{i_1} - \cdots - \frac{k_{n-1}}{k_n} x_{i_{n-1}} \in \mathbb{Q}x_{i_1} + \cdots + \mathbb{Q}x_{i_{n-1}} \subseteq A_{i_{n-1}}$, a contradiction. Thus K is an independent set of elements of infinite order. Clearly K accumulates at 0. Let S be the subsemigroup of $\mathbb{R}$ generated by $K \setminus \{x_0\}$ and $x^* = x_0$. By Proposition 3.8, the paratopological group $\mathbb{R}_S$ is first countable and fails to be an SP-group. By its definition, the topology of $\mathbb{R}_S$ is first countable and refines the usual topology of $\mathbb{R}$. □

The character of the group (G, τ) in Propositions 3.7 and 3.8 is countable. With some adjustments, these results can be extended to groups (G, τ) of an arbitrary character $\lambda \geq \omega$. We just need to require that $|K \cap U| \geq \lambda$ for each $U \in \mathcal{N}(e)$, where $\mathcal{N}(e)$ is a local base at the identity e for (G, τ).

4. Precompact subsets and internal automorphisms

It is known that, given a precompact subset B of a topological group G and any neighborhood U of the identity e of G, there is a neighborhood V of e such that $b^{-1}Vb \subseteq U$, for every $b \in B$. In particular, if B is compact and U is open, then the set $\bigcap_{b \in B} bUb^{-1}$ is again an open neighborhood of e. This property does not hold in the class of paratopological groups, as Example 4.1 below shows.

Following [5], we say that a paratopological group G is $\flat$-*separated* provided that G admits a continuous one-to-one homomorphism onto a Hausdorff topological group or, equivalently, if G admits a weaker Hausdorff topological group topology.

Example 4.1. There exist a Hausdorff saturated paratopological group H, a compact subset B of H, and a neighborhood U of the identity e of H such that no neighborhood V of e satisfies $b^{-1}Vb \subseteq U$, for every $b \in B$.

Proof. Let $G = F^{(\omega)}$ be the direct sum of ω copies of the free group F on two generators, x and y, and S be the minimal subsemigroup of F containing the set $\{1, x, y\}$, where 1 is the identity of F.

We continue as in [5, Proposition 6]. For every $n \in \omega$, let U_n be the set of elements z of G such that $z(k) = 1$, for all $k \leq n$, and $z(k) \in S$, for every $k > n$. The family $\mathcal{N} = \{U_n : n \in \omega\}$ satisfies the Pontryagin conditions for a neighborhood base at the identity e_G of G of a paratopological group topology τ on G (see [11, Proposition 2.1]). It is easy to verify that the topology τ on G is Hausdorff, for let $z \neq e_G$ be an element of G and let $k = \min\{n \in \omega : z(n) \neq 1\}$. The sets zU_k and U_k are disjoint open neighborhoods of z and e_G, respectively. Thus G is a Hausdorff and [5, Proposition 6] implies that G is a $\flat$-separated paratopological group.

For every $n \in \omega$, let b_n the element of G defined by $b_n(n) = x$, and $b_n(k) = 1$ if $k \neq n$. Let $B' = \{b_n : n \in \omega\}$, and $B = B' \cup \{e_G\}$. Every element of $\mathcal{N}$ contains all but a finite number of elements of B, thus B is a compact subset of G.

Let $n \in \omega$, and consider the element $z \in U_n$ defined as follows: $z(n+1) = y$ and $z(k) = 1$ for every $k \neq n+1$. We have that $t = b_{n+1}^{-1} z b_{n+1} \notin U_0$, for $t(n+1) = x^{-1}yx \notin S$. The element $n \in \omega$ is arbitrary, so there is no neighborhood V of e_G in G such that $b^{-1}Vb \subseteq U_0$ for each $b \in B$.

Since G is a $\flat$-separated paratopological group, [4, Corollary 3] implies that the group G can be embedded as a subgroup into a Hausdorff saturated paratopological group H. Suppose that the conclusion of the example does not hold. Choose an open neighborhood U of the identity e in H such that $U \cap G = U_0$. Then for some open neighborhood V of e, $b^{-1}Vb \subseteq U$, for each $b \in B$. Let $V_0 = V \cap G$. Clearly B and V_0 are subsets of G and it follows that $b^{-1}V_0 b \subseteq U \cap G = U_0$, for each $b \in B$, a contradiction. This proves that there is no open neighborhood V of e in H such that $b^{-1}Vb \subseteq U$ for each $b \in B$. □

It is not clear whether one can refine Example 4.1 by choosing the group H precompact. Notice that every precompact paratopological group is saturated [8].

5. Feebly compact paratopological groups

Here we prove that a feebly compact paratopological group H is perfectly κ-normal and that every G_δ-dense subspace of H is feebly compact.

We start with a simple lemma in which rX denotes the semiregularization of a given space X (see Section 2).

Lemma 5.1. *Let X be a space and $i_X \colon X \to rX$ the identity mapping of X onto the semiregularization of X. If D is a dense subspace of a space X, then rD is naturally homeomorphic to the subspace $i_X(D)$ of rX.*

Proof. It follows from the definition of the operation of semiregularization that the families of regular open sets in X and rX coincide. Since D is dense in X, we obtain the required conclusion. □

Corollary 5.2. *A dense subset D of a space X is feebly compact as a subspace of X if and only if D is feebly compact as a subspace of rX.*

Lemma 5.3. *Let $f\colon X \to Y$ be an open continuous mapping of the space X onto a feebly compact space Y, and suppose that $f^{-1}f(U) = U$ for every open set U in X. Then X is feebly compact.*

Proof. Let $\mathcal{U}$ be a locally finite family of open sets in X. We claim that $\mathcal{U}' = \{f(U) : U \in \mathcal{U}\}$ is a locally finite family of open sets in Y. Since f is open, $\mathcal{U}'$ is a family of open subsets of Y. Let $y \in Y$. Choose $x \in f^{-1}(y)$ and an open neighborhood V of x in X meeting only a finite number of elements of $\mathcal{U}$. Suppose that $f(V)$ meets $f(U)$ for some $U \in \mathcal{U}$. Then $V \cap U = f^{-1}f(V) \cap f^{-1}f(U) \neq \varnothing$. Thus the open neighborhood $f(V)$ of y meets only a finite number of elements of $\mathcal{U}'$. Hence $\mathcal{U}'$ is locally finite, and since Y is feebly compact, $\mathcal{U}'$ is finite. From here, using that for every open set U of X the equality $f^{-1}f(U) = U$ holds, we conclude that $\mathcal{U}$ is finite. Thus X is feebly compact. □

Lemma 5.4. *Let $f\colon X \to Y$ be an open continuous mapping of a space X onto a feebly compact space Y. Suppose that $f^{-1}f(U) = U$ for every open set U of X, and that D is a dense subspace of X. Then the mapping $g = f\restriction_D\colon D \to f(D)$ is open and $g^{-1}g(V) = V$ for every open set V in D.*

Proof. Clearly g is continuous. Let $V = U \cap D$ be an open set in D, with U open in X. Then $g(V) = f(U \cap D) \subseteq f(U) \cap f(D)$. Let $y \in f(U) \cap f(D)$. Choose $x \in f^{-1}(y) \cap D$. Since $f^{-1}f(U) = U$, we have that $f^{-1}(y) \subseteq U$. Thus $x \in U \cap D$, and $y \in f(U \cap D) = g(V)$. We conclude that $g(V) = f(U) \cap f(D)$ is open in $f(D)$.

Now we prove that $g^{-1}g(V) = V$ for every open set V in D. Let U be an open set in X such that $V = U \cap D$. Then $g^{-1}g(V) = f^{-1}f(V) \cap D \subseteq f^{-1}f(U) \cap D = U \cap D = V$. Since $V \subseteq g^{-1}g(V)$, we conclude that $g^{-1}g(V) = V$. □

Corollary 5.5. *A dense subset D of a paratopological group G is feebly compact provided that the subspace $\pi(D)$ of T_0G is feebly compact, where $\pi\colon G \to T_0G$ is the quotient homomorphism.*

Proof. The homomorphism π is open and $U = \pi^{-1}\pi(U)$, for every open set U in G. Hence the required conclusion follows from Lemmas 5.3 and 5.4. □

A subset D of a space X is G_δ-*dense* in X if it meets every nonempty G_δ-set in X.

Proposition 5.6. *Let G be a feebly compact paratopological group and D a G_δ-dense subset of G. Then D is feebly compact.*

Proof. We know that D is feebly compact if and only if rD is feebly compact. By Lemma 5.1, rD is homeomorphic to D considered as a subspace of rG. Then, by Corollary 5.5, D is feebly compact if $\pi(D)$ is a feebly compact subspace of $T_0(rG)$. By Lemma 2.3, $T_0(rG)$ is a pseudocompact topological group, and clearly $\pi(D)$ is a G_δ-dense subset of $T_0(rG)$. By Corollary 6.6.3 in [2], $\pi(D)$ is feebly compact. We conclude that D is feebly compact. □

Corollary 5.7. *Every G_δ-dense subgroup of a feebly compact paratopological group is feebly compact.*

A set $A \subseteq X$ is a *zero-set* in a space X if there exists a continuous function $f\colon X \to \mathbb{R}$ such that $A = f^{-1}(0)$. A subset C of a space X is *regular closed* in X provided that $C = \overline{int\, C}$.

Definition 5.8. A space X is *perfectly κ-normal* if every regular closed subset of X is a zero-set in X.

A subspace A of a space X is *z-embedded* in X if every zero-set in A is the intersection with A of a zero-set in X, i.e. B is a zero-set of A if and only if $B = A \cap C$, with C a zero-set in X. In [3], Blair proves that a space X is perfectly κ-normal if and only if every dense subset of X is z-embedded in X.

The next lemma follows the definition of semiregularization.

Lemma 5.9. *Every regular closed subset of a space X is regular closed in rX.*

Lemma 5.10. *Let C be a regular closed set in a paratopological group G. Then $\pi(C)$ is regular closed in T_0G, where $\pi\colon G \to T_0G$ is the quotient homomorphism.*

Proof. Let C be a regular closed set in G. We prove first that $C = \pi^{-1}\pi(C)$. It suffices to show that $CH \subseteq C$, where H is the kernel of π. Let $x = ch \in CH$, with $c \in C$ and $h \in H$, and $U \in \mathcal{N}(e)$. Then $xU \in \mathcal{N}(x)$. We have that $xU = chU \in \mathcal{N}(c)$. Since $C = \overline{int\, C}$, we have that $xU \cap int\, C = chU \cap int\, C \neq \varnothing$. Thus $x \in \overline{int\, C} = C$. Therefore $C = \pi^{-1}\pi(C)$.

Using the above property of π as well as the assumption that π is open and continuous, we conclude that $\pi(C)$ is a regular closed set in T_0G. □

Lemma 5.11. *Every pseudocompact topological group is a perfectly κ-normal space.*

Proof. It follows from [2, Corollary 5.3.29] that every compact topological group is a perfectly κ-normal space. The Raĭkov completion of a pseudocompact topological group G, say, ϱG is a compact topological group which is thus perfectly κ-normal. Hence so is G as a dense subspace of ϱG. □

Theorem 5.12. *Every feebly compact paratopological group is perfectly κ-normal.*

Proof. Let G be a feebly compact paratopological group and C a regular closed set in G. By Lemmas 2.3, 5.9, 5.10, and 5.11, $\pi(C)$ is a regular closed set in the perfectly κ-normal topological group T_0G, where $\pi\colon G \to T_0G$ is the quotient mapping. Thus $\pi(C)$ is a zero-set in T_0G. It follows that $C = \pi^{-1}\pi(C)$ is a zero-set in G. We conclude that G is perfectly κ-normal. □

Proposition 5.13. *Suppose that every cyclic subgroup of a Hausdorff feebly compact paratopological group G is precompact. Then G is a topological group.*

Proof. Let us show that G is topologically periodic. Suppose that $x \in G$ and U is an open neighborhood of the identity e in G. If x has finite order, there is nothing to verify. We can assume therefore that the cyclic subgroup $H = \langle x \rangle$ of G is infinite. Since H is precompact, there is a finite subset F of H such that $H = F \cdot V$, where $V = U \cap H$. Since F is finite, there exists a finite set $C \subset \mathbb{Z}$ such that $F = \{x^n : n \in C\}$. Take $k \in \mathbb{N}^+$ such that $k > n$ for each $n \in C$. By the equality $H = F \cdot V$, there exist $n \in C$ and $m \in \mathbb{Z}$ such that $x^k = x^n x^m$. Then $k = n + m$ and from $k > n$ it follows that $m = k - n > 0$. Hence $x^m \in V \subseteq U$ and the group G is topologically periodic.

Finally, by Proposition 5 in [9], every feebly compact topologically periodic paratopological group is a topological group. $\square$

6. Precompact almost topological groups

The following notion was introduced by the first listed author in [6].

Definition 6.1. An *almost topological group* is a paratopological group (G, τ) that satisfies the following conditions:

(a) The group G admits topological group topology σ weaker than τ.

(b) There exists a local base $\mathcal{B}$ at the identity e of the paratopological group (G, τ) such that the set $\widetilde{U} = U \setminus \{e\}$ is open in $\overline{G} = (G, \sigma)$, for every $U \in \mathcal{B}$.

If G and $\overline{G}$ are as in the above definition, we say that $\overline{G}$ is the underlying topological group of G.

Proposition 6.2. *Let G be an almost topological group with underlying topological group $\overline{G}$. Then G is precompact if and only if $\overline{G}$ is precompact.*

Proof. Suppose that G is precompact. Since the topology of G is finer than the topology of $\overline{G}$, the group $\overline{G}$ is precompact. Suppose that $\overline{G}$ is precompact. We assume that G is non-discrete, otherwise there is nothing to prove. Let $\mathcal{B}$ be a local base at the identity e of G as in part (b) of Definition 6.1 and take $U \in \mathcal{B}$. Choose an arbitrary element $x \in \widetilde{U} = U \setminus \{e\}$ and put $V = x^{-1}\widetilde{U}$. The set V is an open neighborhood of e in the precompact group $\overline{G}$. Let K be a finite subset of $\overline{G}$ such that $KV = G$ and $VK = G$. For the finite set $F = K \cup Kx^{-1}$, we have that $G = xG = xVK = \widetilde{U}K \subseteq UF$, and $G = KV = Kx^{-1}xV = Kx^{-1}\widetilde{U} \subseteq FU$. Thus $G = UF$ and $G = FU$. We conclude that G is precompact. $\square$

It is worth mentioning that 'precompact' cannot be replaced with 'pseudocompact' or 'feebly compact' in Proposition 6.2. Indeed, let $\mathbb{T}_s$ be the circle group endowed with the Sorgenfrey topology (see the comment after Corollary 3.2). It is clear that the underlying topological group of $\mathbb{T}_s$ is the circle group $\mathbb{T}$ with the usual compact topology. The group $\mathbb{T}_s$ is Hausdorff and zero-dimensional, hence Tychonoff. However, it is not pseudocompact.

Let us say that the *left index of narrowness* of a paratopological group G is less than or equal to an infinite cardinal τ or, in symbols, $In_l(G) \leq \tau$, if G can be covered by at most τ translates of any neighborhood of the identity in G. The least cardinal $\tau \geq \omega$ such that the group G satisfies $In_l(G) \leq \tau$ is called the left index of narrowness of G. One defines the *right index of narrowness* of a paratopological group in a similar way. The left and right indices of narrowness of G are denoted respectively by $In_l(G)$ and $In_r(G)$. Then the *index of narrowness* of G is defined as $In(G) = In_l(G) \cdot In_r(G)$.

In general, $In_l(G)$ and $In_r(G)$ can be different. Modifying slightly the argument in proof of Proposition 6.2, we obtain the following:

Proposition 6.3. *Every almost topological group G satisfies $In_l(G) = In_l(\overline{G})$ and $In_r(G) = In_r(\overline{G})$, where $\overline{G}$ is the underlying topological group of G.*

It is easy to see that $In_l(H) = In_r(H)$ for every topological group H, since inversion in topological groups is continuous. Hence the next fact is immediate from Proposition 6.3.

Corollary 6.4. *Every almost topological group G satisfies $In_l(G) = In(G) = In_r(G)$.*

It turns out that the Baire property behaves similarly to precompactness in almost topological groups:

Proposition 6.5. *An almost paratopological group G is Baire iff the underlying topological group $\overline{G}$ is Baire.*

Proof. We can assume without loss of generality that G is not discrete. The basic fact we are going to use is that the groups G and $\overline{G}$ have the same nowhere dense subsets. Indeed, take a nowhere dense set A in G and an arbitrary nonempty open set U in $\overline{G}$. Then U is open in G, so there exists a nonempty open set V in G such that $V \subseteq U \setminus A$. Pick an element $x \in V$. As the group G is almost topological, we can find an open neighborhood W of the identity e in G such that $\widetilde{W} = W \setminus \{e\}$ is open in $\overline{G}$ and $xW \subseteq V$. Hence $O = x\widetilde{W}$ is a nonempty open set in $\overline{G}$ satisfying $O \subseteq U \setminus A$. Hence A is nowhere dense in $\overline{G}$.

Conversely, suppose that B is a nowhere dense subset of $\overline{G}$ and consider a nonempty open set U in G. Arguing as above, we take an element $x \in U$ and an open neighborhood V of e in G such that $xV \subseteq U$ and the set $\widetilde{V} = V \setminus \{e\}$ is open in $\overline{G}$. Then $x\widetilde{V}$ is a nonempty open set in $\overline{G}$, so there exists a nonempty open set W in $\overline{G}$ such that $W \subseteq x\widetilde{V} \setminus B$. Then W is open in G and $W \subseteq U \setminus B$, whence it follows that B is nowhere dense in G.

Finally, since the families of nowhere dense sets in G and $\overline{G}$ coincide, we conclude that G is Baire iff so is $\overline{G}$. □

Proposition 6.5 can be given an alternative proof as follows. It is known that every almost topological group is saturated [6, Proposition 2.6]. Given a saturated paratopological group (G, τ), denote by σ the finest topological group topology on G weaker than τ. According to [5, Theorem 5], the family $\sigma \setminus \{\varnothing\}$

is a π-base for (G, τ). It follows from [6, Proposition 2.5] that the groups (G, σ) and $\overline{G}$ are topologically isomorphic. Hence the family of nonempty open sets in $\overline{G}$ forms a π-base for G as well. We conclude that the families of nowhere dense sets in G and $\overline{G}$ coincide and G is Baire iff so is $\overline{G}$.

The fact that the nonempty open sets in $\overline{G}$ form a π-base for G can be applied to deduce the coincidence of several cardinal characteristics of an almost topological group G and its "twin" $\overline{G}$. For example, one easily verifies that $c(G) = c(\overline{G})$, $d(G) = d(\overline{G})$, and $\pi w(G) = \pi w(\overline{G})$, where the symbols '$c$', '$d$', and '$\pi w$' stand for the cellularity, density, and π-weight, respectively.

7. Some Questions

It is an interesting task to find out the permanence properties of the class of paratopological (or semitopological) SP-groups. For example, the quotient group of an SP-group is again an SP-group (see [13, Proposition 3.3]). It is also clear that an arbitrary subgroup of an SP-group is an SP-group. We do not know, however, if the class of SP-groups is finitely productive:

Question 7.1. *Is the product of two paratopological (semitopological) SP-groups an SP-group?*

If the answer to Question 7.1 were in the affirmative, then the class of SP-groups would be productive, i.e. arbitrary products of SP-groups would again be SP-groups. This fact can be easily verified using the argument from [13, Theorem 3.2].

The relations between the properties of an almost topological group G and the underlying topological group $\overline{G}$ are not completely clear. We present here only one question in this respect (for the concept of $\mathbb{R}$-factorizability, see [2, Chapter 8] and [12, 15]).

Question 7.2. *Suppose that H is an $\mathbb{R}$-factorizable almost topological group. Is the underlying topological group $\overline{H}$ of H $\mathbb{R}$-factorizable?*

It is easy to see that the inverse implication is false. Indeed, the additive topological group $\mathbb{R}$ with its usual topology is $\mathbb{R}$-factorizable, while the Sorgenfrey line $\mathbb{S}$ is not. Clearly the underlying topological group of $\mathbb{S}$ is $\mathbb{R}$.

Question 7.3. *Suppose that H is a Hausdorff precompact paratopological group. Is every regular closed subset of H a G_δ-set?*

A topological space X is an *Efimov space* if for every family γ of G_δ-sets in X, the closure of $\bigcup \gamma$ is again a G_δ-set in X [2, Section 1.6, p. 52]. We also recall that a paratopological group G is 2-*pseudocompact* if the set $\bigcap_{n\in\omega} \overline{U_n^{-1}}$ is not empty, for every decreasing sequence $\{U_n\}$ of nonempty open sets in G [9].

Question 7.4. *Is every 2-pseudocompact (or feebly compact) paratopological group an Efimov space?*

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