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PRODUCTS OF PAIRWISE TOTALLY PERMUTABLE GROUPS

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Abstract

In this paper finite groups factorized as product of pairwise totally permutable subgroups are studied in the framework of Fitting classes.

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1 Introduction

All groups considered in this paper are finite. Within the framework of factorized groups, products of totally permutable groups have been widely investigated (c.f. [4], [7], [5] and [9]). We recall that the subgroups H and K of a group G are totally permutable, if every subgroup of H permutes with every subgroup of K. Moreover, a group G is the totally permutable product of the subgroups H and K if $G = HK$ and H and K are totally permutable. One of the leading questions in this context asks about properties of the factors which are inherited by the whole group (and vice versa). This can be stated in the following way: Assume that $\mathcal{L}$ is a class of groups and $G = HK$ is the product of the totally permutable subgroups H and K. Then:

(1) Do $H, K \in \mathcal{L}$ imply $G \in \mathcal{L}$?

(2) Does $G \in \mathcal{L}$ imply $H, K \in \mathcal{L}$?

These questions were given positive answers for suitable formations $\mathcal{L}$ containing the formation $\mathcal{U}$ of all supersoluble groups. Even more, the corresponding natural extensions for products of finitely many pairwise totally permutable groups also hold. We refer to [1], [2] and [3] for details. For the dual type of classes, namely for Fitting classes containing $\mathcal{U}$, the questions mentioned above were considered in [9]. Although they remain open for an arbitrary Fitting class $\mathcal{L}$ containing $\mathcal{U}$, positive results were obtained for important types of such Fitting classes, among them Fischer classes. In this paper we take further this study by investigating the case of finitely many factors in the context of Fitting classes. It turns out that whenever $\mathcal{L}$ is a Fitting class containing $\mathcal{U}$ and satisfying either (1) or (2), then the respective extensions for products of finitely many pairwise totally permutable groups hold.

We refer to [8] for the notation and basic results on classes of groups.

2 Preliminaries

We recall in the next lemma a fundamental property of totally permutable groups which will be often used in the sequel:

Lemma 1. ([5], Theorem 1)
Assume that H and K are totally permutable groups. Then H centralizes $K^{\mathcal{N}}$ and K centralizes $H^{\mathcal{N}}$, where $\mathcal{N}$ denotes the class of all nilpotent groups. In particular, $H^{\mathcal{N}}$ and $K^{\mathcal{N}}$ are both normal subgroups of the product HK.

The following lemma is an extension of ([5], Corollary 2) for products of pairwise totally permutable groups.

Lemma 2. *Let $G = G_1G_2 \cdots G_r$ be a group such that G_1, G_2, …, G_r are pairwise totally permutable subgroups of G. Then $[\prod_{i\in I} G_i, \prod_{j\in J} G_j]$ is a nilpotent normal subgroup of G, for any $I, J \subseteq \{1, 2, \dots, r\}$ such that $\{I, J\}$ is a partition of $\{1, 2, \dots, r\}$.*

Proof. We denote by T_i an $\mathcal{N}$-projector of G_i, for each $i \in \{1, 2, \dots, r\}$. Then $G_i = G_i^{\mathcal{N}} T_i$, for each $i \in \{1, 2, \dots, r\}$. Since the group G_j centralizes $G_i^{\mathcal{N}}$, for each $i, j \in \{1, 2, \dots, r\}$, $i \neq j$, by Lemma 1, we have that:

$$\begin{aligned}[][\prod_{i\in I} G_i, \prod_{j\in J} G_j] &= [\prod_{i\in I} G_i^{\mathcal{N}} T_i, \prod_{j\in J} G_j^{\mathcal{N}} T_j] = \\ &= [\prod_{i\in I} T_i, \prod_{j\in J} T_j] \leq (\prod_{i\in I} T_i \prod_{j\in J} T_j)'.\end{aligned}$$

We notice that $\prod_{i=1}^{r} T_i$ is a product of pairwise totally permutable nilpotent subgroups. Then it is a supersoluble group by ([6], Theorem 1), and so the result is clear.

Lemma 3. *Let $T = \langle x\rangle\langle y\rangle$ be a product of two permutable cyclic q-groups, with q an odd prime. Assume that there exists a q'-group H acting on T*

by automorphisms such that $T = [H, T]$ and $\langle x\rangle$ and $\langle y\rangle$ are H-invariant groups. Then T is an abelian group.

Proof. According to III. Satz 11.5 of [10], T is metacyclic, that is, there exists a normal subgroup A of T such that A and T/A are cyclic. Now, we deduce that T is an M-group, that is, a group with modular subgroup lattice, by ([11], Lemma 2.3.4). Moreover, since q is odd, T does not involve Q_8, the quaternion group of order 8, and so T is an M^*-group, according to [11], page 58.

Assume that T is nonabelian. Since T is an M^*-group, by ([11], Theorem 2.3.23) there exist characteristic subgroups R and S of T such that $\Phi(T) \leq S < R$ and $[R, \mathrm{Aut}(T)] \leq S$. Since $T = [H, T]$, with $H \leq \mathrm{Aut}(T)$, it is clear that $R < T$. On the other hand, $T/\Phi(T) \cong Z_q \times Z_q$ and so $|T : R| = q$ and $S = \Phi(T)$. Consequently $[R, H] \leq \Phi(T)$. Moreover, $R = [R, H]C_R(H)$ by coprime action and we know, by ([11], Lemma 2.3.2), that any two of the subgroups of T permute. We may assume that $\langle x\rangle \not\subseteq R$. Then

$$T = R\langle x\rangle = \Phi(T)C_R(H)\langle x\rangle = C_R(H)\langle x\rangle.$$

So

$$T = [H, T] = [H, C_R(H)\langle x\rangle] = [H, \langle x\rangle] \leq \langle x\rangle,$$

a contradiction which proves the result.

3 The results

Theorem 1. *Let $\mathcal{F}$ be a Fitting class containing $\mathcal{U}$ and satisfying the following property:*

(*) *If a group $G = HK$ is the product of the totally permutable subgroups H and K such that $H \in \mathcal{F}$ and $K \in \mathcal{F}$, then $G \in \mathcal{F}$.*

Let the group $G = G_1G_2 \cdots G_r$ be a product of the pairwise totally permutable subgroups G_1, G_2, $\ldots$, G_r. If $G_i \in \mathcal{F}$, for all $i \in \{1, 2, \ldots, r\}$, then $G \in \mathcal{F}$.

Proof. Assume that the result is false and let $G = G_1G_2 \cdots G_r$ be a counterexample where G_1, G_2, $\ldots$, G_r are pairwise totally permutable $\mathcal{F}$-subgroups of G with $|G| + |G_1| + \cdots + |G_r|$ minimal. We split the proof into the following steps:

(1) We may assume that $G_2, \ldots, G_r$ are nilpotent groups and G_1 is not nilpotent. We denote $H = G_1$ and $K = G_2 \cdots G_r$. Moreover, $K \in \mathcal{U}$ and $[K, H^{\mathcal{N}}] = 1$.

If $G_i \in \mathcal{N}$ for all $i \in \{1, 2, \ldots, r\}$, then $G \in \mathcal{U} \subseteq \mathcal{F}$, by ([6], Theorem 1), which is a contradiction. Assume now that there exist $i, j \in \{1, 2, \ldots, r\}$, $i \neq j$, such that $G_i \notin \mathcal{N}$ and $G_j \notin \mathcal{N}$. From Lemma 1 it follows that $[G_k, G_t^{\mathcal{N}}] = 1$, for all $k, t \in \{1, 2, \ldots, r\}$, $k \neq t$. Then $G_t \leq C_G(G_i^{\mathcal{N}}) < G$ for every $t \neq i$, and $G_t \leq C_G(G_j^{\mathcal{N}}) < G$ for every $t \neq j$. Hence

$$C_G(G_i^{\mathcal{N}}) = \big(\prod_{\substack{t=1 \\ t \neq i}}^{r} G_t\big)\big(G_i \cap C_G(G_i^{\mathcal{N}})\big)$$

is a product of pairwise totally permutable subgroups in $\mathcal{F}$, as $C_{G_i}(G_i^{\mathcal{N}}) \trianglelefteq G_i \in \mathcal{F}$. We conclude that $C_G(G_i^{\mathcal{N}}) \in \mathcal{F}$, by the choice of G. In a similar way, $C_G(G_j^{\mathcal{N}}) \in \mathcal{F}$. Then

$$G = C_G(G_i^{\mathcal{N}}) C_G(G_j^{\mathcal{N}}) \in \mathrm{N}_0(\mathcal{F}) = \mathcal{F},$$

a contradiction.

Consequently there exists a unique $i \in \{1, 2, \ldots, r\}$ such that $G_i \notin \mathcal{N}$. Without loss of generality we may suppose $i = 1$. Now the conclusion is clear, by ([6], Theorem 1) and Lemma 1.

(2) $K^G \cap H \in \mathcal{N}$ and $K^G \in \mathcal{U}$.

Since $[K, H^{\mathcal{N}}] = 1$ we can deduce that $K^G \cap H^{\mathcal{N}} \leq Z(K^G \cap H)$ and so $K^G \cap H \in \mathcal{N}$. Finally, since $K^G = (K^G \cap H)K$ is a product of pairwise totally permutable nilpotent subgroups, then $K^G \in \mathcal{U}$, by ([6], Theorem 1).

(3) There exists a prime number p such that $G = H^{\mathcal{N}} H_p K^G$, with H_p a Sylow p-subgroup of H.

Since $H^{\mathcal{N}} H_q K^G$ is a normal subgroup of G, for all primes q, where H_q is a Sylow q-subgroup of H, the result follows taking into account the choice of G.

(4) For all primes $q \neq p$, $H^{\mathcal{N}} H_q [H, K]$ is a normal $\mathcal{F}$-subgroup of G, where H_q is a Sylow q-subgroup of H.

We notice first that $H^{\mathcal{N}} H_q [H, K]$ is a normal subgroup of $G = HK$ contained in $H^{\mathcal{N}} K^G$ by (3). Then the result follows from (2) and this fact.

(5) $H^{\mathcal{N}} H_p[H_p, K] \notin \mathcal{F}$

Suppose that $H^{\mathcal{N}} H_p[H_p, K] \in \mathcal{F}$. Since $H^{\mathcal{N}} H_p[H_p, K] = (H^{\mathcal{N}} H_p)^G$ is a normal subgroup of G, then

$$H^G = H[H, K] = (H^{\mathcal{N}} H_p[H_p, K])(\prod_{q \neq p} H^{\mathcal{N}} H_q[H, K]) \in \mathrm{N}_0(\mathcal{F}) = \mathcal{F},$$

by (4). Consequently $G = H^G K^G \in \mathrm{N}_0(\mathcal{F}) = \mathcal{F}$, a contradiction which proves step (5).

(6) $G = H^{\mathcal{N}} H_p K$.

If $H^{\mathcal{N}} H_p K < G$, then $H^{\mathcal{N}} H_p K \in \mathcal{F}$ by the choice of G. But this contradicts step (5), since $H^{\mathcal{N}} H_p[H_p, K]$ is a normal subgroup of $H^{\mathcal{N}} H_p K$.

(7) $H/H^{\mathcal{N}}$ is a p-group.

This follows from (6) by the choice of $(G_1, \ldots, G_r)$.

(8) $p \leq q$ for all primes q dividing $|K|$ and $G = H^{\mathcal{N}} H_p K_p K_{p'}$, where K_p is a Sylow p-subgroup of K and $K_{p'}$ is a Hall p'-subgroup of K. Moreover, $K_{p'}$ is a normal subgroup of G.

Suppose that $p \geq q$, for all primes q dividing $|K|$. Since $H_p K$ is a supersoluble group by ([6], Theorem 1), we can deduce that $H = H^{\mathcal{N}} H_p$ is a subnormal subgroup of $G = H^{\mathcal{N}} H_p K$. Hence, $G = K^G H \in \mathrm{N}_0(\mathcal{F}) = \mathcal{F}$, a contradiction. Consequently there exists a prime q dividing $|K|$ with $p < q$.

Let $\pi(K) \cup \{p\} = \{p_1, p_2, \ldots, p_t = p, p_{t+1}, \ldots, p_n\}$, with $p_1 < p_2 < \cdots < p_t = p < p_{t+1} < \cdots < p_n$. We denote $\pi = \{p, p_{t+1}, \ldots, p_n\}$ and $\pi' = (\pi(K) \cup \{p\}) \backslash \pi$. Since $H_p K$ is a supersoluble group, $K_{\pi'}$ normalizes $H_p K_\pi$, where $K_{\pi'}$ and K_π are a Hall π'-subgroup and the Hall π-subgroup of K, respectively. Hence $H^{\mathcal{N}} H_p K_\pi$ is a normal subgroup of G. Assume that $H^{\mathcal{N}} H_p K_\pi < G$. We notice that $K_\pi = O_\pi(G_2) \cdots O_\pi(G_r)$ is a product of pairwise totally permutable nilpotent subgroups each of which is totally permutable with H. Then it follows that $H^{\mathcal{N}} H_p K_\pi \in \mathcal{F}$ by the choice of G. Therefore $G = (H^{\mathcal{N}} H_p K_\pi) K^G \in \mathrm{N}_0(\mathcal{F}) = \mathcal{F}$, a contradiction which implies that $G = H^{\mathcal{N}} H_p K_\pi$. Now, by the choice of $(G_1, \ldots, G_r)$, it follows that $K = K_\pi$ and $p \leq q$ for all primes $q \in \pi(K)$. Since $H_p K \in \mathcal{U}$, it is clear that $K_{p'}$ is a normal subgroup of $G = H^{\mathcal{N}} H_p K_p K_{p'}$ and we are done.

(9) K is a normal p'-subgroup of G.

We notice that $K_{p'} = O_{p'}(G_2)\cdots O_{p'}(G_r)$. If $HK_{p'} < G$, then $HK_{p'} \in \mathcal{F}$ by the choice of G. Now, since $H^{\mathcal{N}}K_{p'}$ is a normal subgroup of G and $G/H^{\mathcal{N}}K_{p'}$ is a p-group by (8), it follows that $HK_{p'}$ is a subnormal subgroup of G. This means that $G = (HK_{p'})K^G \in \mathrm{N}_0(\mathcal{F}) = \mathcal{F}$, a contradiction. Hence $G = HK_{p'}$ and $K = K_{p'}$ by the choice of $(G_1, \ldots, G_r)$. By (8), K is normal in G.

(10) For all $j \in \{2, \ldots, r\}$, $G_j = [G_j, H]$. Moreover, $\prod_{k=1}^{t} G_{j_k} = [\prod_{k=1}^{t} G_{j_k}, H]$ for each set of indices $\{j_1, \ldots, j_t\} \subseteq \{2, \ldots, r\}$. In particular, $H^G = G$ and $K = [H, K]$ is a nilpotent group.

First, we remark that for every $j \in \{2, \ldots, r\}$, $[G_j, H] = [G_j, H_p] \leq G_j$ because $H = H^{\mathcal{N}}H_p$, $[H^{\mathcal{N}}, K] = 1$ and H_pK is a supersoluble group, with p the smallest prime dividing its order. Now, since G_j is a p'-group, by coprime action we know that $G_j = [G_j, H_p]C_{G_j}(H_p)$, for all $j \in \{2, \ldots, r\}$. Then we have that $H^G = \langle H^{G_2\cdots G_r}\rangle \leq\leq H(\prod_{j=2}^{r}[G_j, H_p]) \leq H^G$. Since $H^G = H(\prod_{j=2}^{r}[G_j, H_p])$ is a product of pairwise totally permutable subgroups in $\mathcal{F}$, if we assume $H^G < G$, then by the choice of G we deduce that $H^G \in \mathcal{F}$ and $G = KH^G \in \mathrm{N}_0(\mathcal{F}) = \mathcal{F}$, a contradiction. Hence $G = H^G = H(\prod_{j=2}^{r}[G_j, H_p])$. By the choice of $(G_1, \ldots, G_r)$ we conclude that $G_j = [G_j, H_p] = [G_j, H]$, for all $j \in \{2, \ldots r\}$.

Now, if we take $\{j_1, \ldots, j_t\} \subseteq \{2, \ldots r\}$, then we have that $\prod_{k=1}^{t} G_{j_k} = \prod_{k=1}^{t}[G_{j_k}, H] \leq [\prod_{k=1}^{t} G_{j_k}, H] \leq \prod_{k=1}^{t} G_{j_k}$. In particular, $K = [H, K]$ is a nilpotent group, by Lemma 2.

(11) For all $j \in \{2, \ldots, r\}$, G_j is an abelian group and, moreover, H normalizes each subgroup of G_j.

Choose any $j \in \{2, \ldots, r\}$. We claim that H does not centralize any non-trivial Sylow subgroup of G_j. Assume not and let $(G_j)_q \neq 1$, a Sylow q-subgroup of G_j, for some prime q, such that $[(G_j)_q, H] = 1$. Then $H^G \leq H(\prod_{1\neq i\neq j} G_i)(G_j)_{q'}$ and, by (10), we deduce that $G = H(\prod_{1\neq i\neq j} G_i)(G_j)_{q'}$. Now, by the choice of $(G_1, \ldots, G_r)$, we obtain that $G_j = (G_j)_{q'}$ and $(G_j)_q = 1$, a contradiction.

Since $H_p(G_j)_q$ is a product of totally permutable subgroups and it is a supersoluble group, it follows that H_p normalizes each subgroup of $(G_j)_q$ but does not centralize $(G_j)_q$, for all primes $q \in \pi(G_j)$. By ([5], Lemma 1), $(G_j)_q$ is an abelian group, for all primes $q \in \pi(G_j)$, and hence G_j is an abelian group.

Again, since H_pG_j is a supersoluble group which is a product of two totally permutable subgroups, and p is the smallest prime dividing its order, we deduce that H_p normalizes each subgroup of G_j. Now, since $[H^{\mathcal{N}}, G_j] = 1$, the result follows by (7).

(12) G_j is a cyclic p_j-group, for some prime p_j, for all $j \in \{2, \ldots, r\}$.

Choose any $j \in \{2, \ldots, r\}$. Since G_j is abelian by (11), it is a direct product of cyclic subgroups of prime power orders. Let $G_j = \times_i T_{j_i}$, $T_{j_i} \cong Z_{p_i^{\alpha_i}}$ for some primes $p_i > 2$ and some integers $\alpha_i \geq 0$ for each i. Then $G = H(\prod_{1 \neq k \neq j} G_k)(\times_i T_{j_i})$ is a product of pairwise totally permutable subgroups in $\mathcal{F}$. Now, $|H| + \sum_{1 \neq k \neq j} |G_k| + \sum_i |T_{j_i}| < |G_1| + \cdots + |G_r|$, since $\sum_i |T_{j_i}| < \prod_i |T_{j_i}|$ unless $G_j = T_{j_i}$ for some index j_i. It follows that each G_j is a cyclic p_j-group, for some prime p_j, by the choice of $(G_1, \ldots, G_r)$.

(13) $K = G_2 \cdots G_r$ is an abelian group.

Since K is a nilpotent group by (10), it suffices to show that any Sylow q-subgroup of K, for any prime q, is abelian. Take any pair of indices $i, j \in \{2, \ldots r\}$ such that G_i and G_j are q-groups and denote $T_{ij} = G_iG_j$. By (10) $T_{ij} = [T_{ij}, H] = [T_{ij}, H_p]$. And, moreover, by (12), T_{ij} is the product of two permutable cyclic H-invariant q-subgroups, where q is an odd prime (we recall that $p < q$). Then T_{ij} is an abelian group by Lemma 3. This means that $[G_i, G_j] = 1$ for every pair of q-groups G_i and G_j. Consequently, we deduce that any Sylow q-subgroup of K is abelian, by (12), and the result follows.

(14) The final contradiction.

Since $K = G_2 \cdots G_r$ is an abelian group and H normalizes each subgroup of G_r, it is clear that $HG_2 \cdots G_{r-1}$ is totally permutable with G_r. If $HG_2 \cdots G_{r-1}$ is a proper subgroup of G, then it is an $\mathcal{F}$-group by the choice of G. Consequently $G = (HG_2 \cdots G_{r-1})G_r$ is a product of two totally permutable subgroups in $\mathcal{F}$. From our assumption we obtain that $G \in \mathcal{F}$, a contradiction. This implies that $G = HG_2 \cdots G_{r-1}$. By the choice of $(G_1, \ldots, G_r)$ we have that $G \in \mathcal{F}$, the final contradiction.

Theorem 2. *Let $\mathcal{F}$ be a Fitting class containing $\mathcal{U}$ and satisfying the following property:*

(*) *If a group $G = HK$ is the product of the totally permutable subgroups H and K such that $G \in \mathcal{F}$, then $H \in \mathcal{F}$ and $K \in \mathcal{F}$.*

Let the group $G = G_1 G_2 \cdots G_r$ be a product of the pairwise totally permutable subgroups G_1, G_2, $\ldots$, G_r. If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i \in \{1, 2, \ldots, r\}$.

Proof. Assume the result is false and let $G = G_1 G_2 \cdots G_r \in \mathcal{F}$ be a counterexample where G_1, G_2, $\ldots$, G_r are pairwise totally permutable subgroups of G, not all of them in $\mathcal{F}$, with $|G| + |G_1| + \cdots + |G_r|$ minimal. We split the proof into the following steps:

(1) We may assume that G_2, $\ldots$, G_r are nilpotent groups and G_1 is not nilpotent. We denote $H = G_1$ and $K = G_2 \cdots G_r$. Moreover, $K \in \mathcal{U}$ and $[K, H^{\mathcal{N}}] = 1$.

Obviously not all $G_1, G_2, \ldots, G_r$ are nilpotent. If we assume that there exists $i, j \in \{1, 2, \ldots, r\}$, $i \neq j$, such that $G_i \notin \mathcal{N}$ and $G_j \notin \mathcal{N}$, we can deduce, as in Theorem 1, step (1), that $C_G(G_i^{\mathcal{N}}) = (\prod_{t=1,\, t\neq i}^{r} G_t)$ $(G_i \cap C_G(G_i^{\mathcal{N}}))$ is a proper normal subgroup of G. By the choice of G, we obtain that $G_t \in \mathcal{F}$, for all $t \neq i$. In a similar way, arguing with G_j, we deduce that $G_l \in \mathcal{F}$, for all $l \neq j$. Then $G_k \in \mathcal{F}$ for all $k \in \{1, 2, \ldots, r\}$, a contradiction.

Consequently there exists a unique $i \in \{1, 2, \ldots, r\}$ such that $G_i \notin \mathcal{N}$. Without loss of generality we may assume that $i = 1$. Now, the conclusion is clear by ([6], Theorem 1) and Lemma 1.

(2) $K^G \cap H \in \mathcal{N}$ and $K^G \in \mathcal{U}$.

We can argue as in Theorem 1, step (2).

(3) There exists a prime number p such that $G = H^{\mathcal{N}} H_p K^G$, with H_p a Sylow p-subgroup of H.

Assume that $H^{\mathcal{N}} H_q K^G < G$, for all primes q, where H_q is a Sylow q-subgroup of H. For every prime q, since $H^{\mathcal{N}} H_q K^G$ is a normal subgroup of G, we have that $H^{\mathcal{N}} H_q K^G \in \mathrm{S}_n(\mathcal{F}) = \mathcal{F}$. But $H^{\mathcal{N}} H_q K^G = H^{\mathcal{N}} H_q (K^G \cap H) K$ is a product of pairwise totally permutable subgroups. By the choice of G we deduce that $H^{\mathcal{N}} H_q (K^G \cap H) \in \mathcal{F}$. In particular, $H^{\mathcal{N}} H_q \in \mathrm{S}_n(\mathcal{F}) = \mathcal{F}$, for all primes q, and so $H \in \mathrm{N}_0(\mathcal{F}) = \mathcal{F}$, a contradiction.

(4) For all primes $q \neq p$, $H^{\mathcal{N}} H_q \in \mathcal{F}$, where H_q is a Sylow q-subgroup of H. Moreover, $H^{\mathcal{N}} H_p \notin \mathcal{F}$.

We notice that $H^{\mathcal{N}} H_q$ is contained in $H^{\mathcal{N}}(K^G \cap H)$ by (3). But $H^{\mathcal{N}}$ $(K^G \cap H) \in \mathcal{F}$ because it is the product of two normal $\mathcal{F}$-subgroups of H. Then $H^{\mathcal{N}} H_q \in \mathrm{S}_n(\mathcal{F}) = \mathcal{F}$. Finally, if $H^{\mathcal{N}} H_p \in \mathcal{F}$, then $H \in \mathrm{N}_0(\mathcal{F}) = \mathcal{F}$, a contradiction which proves (4).

(5) $H/H^{\mathcal{N}}$ is a p-group, $p \leq q$ for all primes q dividing $|K|$ and $G = H^{\mathcal{N}} H_p K_p K_{p'}$, where K_p is a Sylow p-subgroup of K and $K_{p'}$ is a Hall p'-subgroup of K. Moreover, $K_{p'}$ is a normal subgroup of G.

Since $H^{\mathcal{N}} H_p [H_p, K] = (H^{\mathcal{N}} H_p)^G$ is a normal subgroup of G, we have that $H^{\mathcal{N}} H_p [H_p, K] \in \mathcal{F}$.

We claim first that there exists a prime q dividing $|K|$ with $p < q$. Otherwise, we can obtain, as in Theorem 1, step (8), that $H^{\mathcal{N}} H_p$ is a subnormal subgroup of $H^{\mathcal{N}} H_p [H_p, K]$. Then $H^{\mathcal{N}} H_p \in \mathrm{S}_n(\mathcal{F}) = \mathcal{F}$, which contradicts step (4).

Let $\pi(K) \cup \{p\} = \{p_1, p_2, \ldots, p_t = p, p_{t+1}, \ldots, p_n\}$, with $p_1 < p_2 < \cdots < p_t = p < p_{t+1} < \cdots < p_n$. We denote $\pi = \{p, p_{t+1}, \ldots, p_n\}$ and $\pi' = (\pi(K) \cup \{p\}) \backslash \pi$. We recall that $K = G_2 \cdots G_r = K_\pi K_{\pi'}$ is a supersoluble group, where $K_{\pi'} \in \mathrm{Hall}_{\pi'}(K)$ and $K_\pi \in \mathrm{Hall}_\pi(K)$. We may assume that $K_{\pi'} = O_{\pi'}(G_2) \cdots O_{\pi'}(G_r)$ is a product of pairwise totally permutable nilpotent subgroups each of which is totally permutable with H. Then $H_p K_{\pi'}$ is a supersoluble group by ([6], Theorem 1). Consequently $K_{\pi'}$ normalizes both K_π and H_p and so we have that

$$[H_p, K] = [H_p, K_{\pi'}][H_p, K_\pi] \leq H_p [H_p, K_\pi].$$

Therefore $H^{\mathcal{N}} H_p [H_p, K_\pi] = H^{\mathcal{N}} H_p [H_p, K]$ is a normal $\mathcal{F}$-subgroup of G. On the other hand, arguing as above, we have that $H_p K_\pi$ is also a supersoluble group. This implies that K_π is a subnormal subgroup of $H_p K_\pi$. Hence $H^{\mathcal{N}} H_p K_\pi = H^{\mathcal{N}} H_p [H_p, K_\pi] K_\pi \in \mathrm{N}_0(\mathcal{F}) = \mathcal{F}$.

If $H^{\mathcal{N}} H_p K_\pi < G$, then $H^{\mathcal{N}} H_p \in \mathcal{F}$, by the choice of G, which contradicts step (4). So we may assume that $G = H^{\mathcal{N}} H_p K_\pi$. Now, by the choice of $(G_1, \ldots, G_r)$, we can deduce that $H = H^{\mathcal{N}} H_p$, $K = K_\pi$ and $p \leq q$ for all primes $q \in \pi(K)$. Moreover, since $H_p K \in \mathcal{U}$, it is clear that $K_{p'}$ is a normal subgroup of G, and the result follows.

(6) K is a normal p'-subgroup of G.

Since $H^{\mathcal{N}} K_{p'}$ is a normal subgroup of G and $G/H^{\mathcal{N}} K_{p'}$ is a p-group by (5), then $HK_{p'}$ is a subnormal subgroup of G. In particular, $HK_{p'} \in$

$\mathrm{s}_n(\mathcal{F}) = \mathcal{F}$. If $HK_{p'} < G$, then by the choice of G we can deduce that $H \in \mathcal{F}$, a contradiction. Hence $G = HK_{p'}$ and $K = K_{p'}$ by the choice of $(G_1, \ldots, G_r)$.

(7) For all $j \in \{2, \ldots, r\}$, $G_j = [G_j, H]$. Moreover, $\prod_{k=1}^{t} G_{j_k} = [\prod_{k=1}^{t} G_{j_k}, H]$ for each set of indices $\{j_1, \ldots, j_t\} \subseteq \{2, \ldots, r\}$. In particular, $H^G = G$ and $K = [H, K]$ is a nilpotent group.

From (1), (5) and (6), we can argue as in Theorem 1, step (10), to obtain that $[G_j, H] = [G_j, H_p] \leq G_j$, for every $j \in \{2, \ldots, r\}$, and that $H^G = H(\prod_{j=2}^{r}[G_j, H_p])$. In particular, H^G is a normal subgroup of $G \in \mathcal{F}$, which is a product of pairwise totally permutable subgroups. If $H^G < G$, then $H \in \mathcal{F}$ by the choice of G, a contradiction. Consequently $G = H^G = H(\prod_{j=2}^{r}[G_j, H_p])$ and, by the choice of $(G_1, \ldots, G_r)$, we conclude that $G_j = [G_j, H_p] = [G_j, H]$, for all $j \in \{2, \ldots, r\}$.

The remainder follows easily as in Theorem 1, step (10).

(8) For all $j \in \{2, \ldots, r\}$, G_j is an abelian group and, moreover, H normalizes each subgroup of G_j.

It follows by arguing as in Theorem 1, step (11).

(9) G_j is a cyclic p_j-group, for some prime p_j, for all $j \in \{2, \ldots, r\}$.

Arguing as in Theorem 1, step (12), and with the same notation, we obtain that, for any $j \in \{2, \ldots, r\}$, $G = H(\prod_{1 \neq k \neq j} G_k)(\times_i T_{j_i})$ is a product of pairwise totally permutable subgroups with $|H| + \sum_{1 \neq k \neq j} |G_k| + \sum_i |T_{j_i}| < |G_1| + \cdots + |G_r|$, unless G_j is a cyclic p_j-subgroup, for some prime p_j. Since $H \notin \mathcal{F}$, the result follows analogously by the choice of $(G_1, \ldots, G_r)$.

(10) $K = G_2 \cdots G_r$ is an abelian group.

It follows from Lemma 3, (7) and (9), arguing as in Theorem 1, step (13).

(11) The final contradiction.

By (8) and (10), it is clear that $HG_2 \cdots G_{r-1}$ is totally permutable with G_r. Then we can apply our assumption on the group $G = (HG_2 \cdots G_{r-1})G_r \in \mathcal{F}$ to obtain that $HG_2 \cdots G_{r-1} \in \mathcal{F}$. By the choice of $(G, G_1, \ldots, G_r)$ we can deduce that $H \in \mathcal{F}$, which provides the final contradiction.

Final remarks.

(1) If $\mathcal{F}$ is either a Fischer class containing $\mathcal{U}$ or the Fitting class product $\mathcal{N}\diamond\mathcal{H}$, $\mathcal{H}$ being a Fitting class containing $\mathcal{N}$, then $\mathcal{F}$ satisfies properties (*) in Theorems 1 and 2 (see [9], Theorem 2 and Theorem 5).

(2) If $\mathcal{F}$ is a Fitting class containing $\mathcal{U}$ and satisfying the property that $G/N \in \mathcal{F}$, whenever $G \in \mathcal{F}$ and $N \leq Z_{\mathcal{U}}(G)$ (in particular, if $\mathcal{F}$ is a Q-closed Fitting class), then $\mathcal{F}$ satisfies the property (*) in Theorem 1 (see [9], Theorem 3).

(3) If $\mathcal{F}$ is an R_0–closed Fitting class containing $\mathcal{U}$, then $\mathcal{F}$ satisfies the property (*) in Theorem 2 (see [9], Theorem 4).

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