Matrices $A$ such that $RA = A^{s+1}R$ when $R^k = I$

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Abstract
This paper examines matrices $A \in \mathbb{C}^{n \times n}$ such that $RA = A^{s+1}R$ where $R^k = I$, the identity matrix, and where $s$ and $k$ are nonnegative integers with $k \geq 2$. Spectral theory is used to characterize these matrices. The cases $s = 0$ and $s \geq 1$ are considered separately since they are analyzed by different techniques.

Keywords: Potent matrix; idempotent matrix; spectrum; Jordan form; involutory matrix.

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1 Introduction and Preliminaries

Let $R_1$ be the square matrix with ones on the cross diagonal and zeros elsewhere; note that $R_1$ is often called the centrosymmetric permutation matrix. A matrix $A_1$ that commutes with $R_1$ is called a centrosymmetric matrix [12]. Any square matrix $R_2$ satisfying $R_2^2 = I$, where $I$ is the identity matrix, is called an involution or an involutory matrix. The real eigenvalues of nonnegative matrices that commute with a real involution were studied in [13]. It is well-known that if $P$ is a permutation matrix, then $P^k = I$ for some positive integer $k$. Matrices that commute with a permutation matrix $P$ were studied in [8]. A well-known and important class of matrices that commute with a permutation matrix are the circulant matrices [3, 6], consisting of all matrices that commute with $R_3$, where $R_3$ is the irreducible permutation matrix with ones on the first superdiagonal, a one in the lower left-hand corner, and zeros elsewhere. If $A$ is an $n \times n$ circulant matrix, then $R_3A = AR_3$ can be expressed as $R_3AR_3^{n-1} = A$ since $R_3^2 = I_n$, the $n \times n$ identity matrix.

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A matrix $R \in \mathbb{C}^{n \times n}$ such that $R^k = I_n$ for some positive integer $k$ with $k \geq 2$ is called a $\{k\}$-involutory matrix [10, 11]. Throughout this paper, all matrices $R$ will be $\{k\}$-involutory. It is clear that when $k = 2$, such an $R$ is either $\pm I_n$, or else a nontrivial involution. Also, $k = n$ is the smallest positive integer for which $R_k$ is $\{k\}$-involutory, and this guarantees that there are nontrivial, non-diagonal $\{k\}$-involutory matrices for all integers $k$ and $n$ with $n \geq 2$ and $2 \leq k \leq n$. The matrix $\exp(\frac{2\pi i}{k})I_n$ is $\{k\}$-involutory for all positive integers $n$ and all integers $k \geq 2$, and, thus, it should be clear that there are $\{k\}$-involutory matrices for which $k > n$ must occur. Finally, we always assume that $R \neq I_n$, and hence, if $R^k = I_n$, then $k \geq 2$.

This paper is focused on the study of the $\{R, s + 1, k\}$-potent matrices. A matrix $A \in \mathbb{C}^{n \times n}$ is called an $\{R, s + 1, k\}$-potent matrix if $RA = A^{s+1}R$ for some nonnegative integer $s$ and some $\{k\}$-involutory matrix $R$. Note that the cases, $k = 2$ and $s \geq 1$, and $k \geq 2$ and $s = 0$, have already been analyzed in [7, 14], respectively. Spectral properties of matrices related to the $\{R, s + 1, k\}$-potent matrices are presented in [4, 9]. Other similar classes of matrices and their spectral properties have been studied in [5, 9, 10, 11].

In this paper characterizations of $\{R, s + 1, k\}$-potent matrices are given, with the cases $s \geq 1$ and $s = 0$ treated separately. In the first case, the concept of $\{t+1\}$-group involutory matrix will be used. These matrices were introduced in [2] for $t = 2$, and the definition can be extended for any integer $t > 2$ as follows: A matrix $A \in \mathbb{C}^{n \times n}$ is called a $\{t+1\}$-group involutory matrix if $A^# = A^{t-1}$, where $A^#$ denotes the group inverse of $A$. We recall that the group inverse of a square matrix $A$ is the only matrix $A^#$ (when it exists) satisfying: $AA^#A = A$, $A^#AA^# = A^#$, $AA^# = A^#A$. Moreover, $A^#$ exists if and only if $\text{rank}(A^2) = \text{rank}(A)$ [1].

2 Main results

Clearly, $I_n$ and $n \times n$ zero matrix $O$ are always $\{R, s + 1, k\}$-potent matrices. For any given positive integers $n$, $s$, and $k$ (with $k \geq 2$), and for any given $n \times n$ $\{k\}$-involutory matrix $R$, there exists a nontrivial $\{R, s + 1, k\}$-potent matrix. Consider $A = \omega I_n$, where $\omega$ is a primitive $s^{th}$ root of unity. Note that when $s = 0$, $A = R$ is an $\{R, 1, k\}$-potent matrix that is nontrivial when $R$ is nontrivial.

The question arising in this paper follows from the observation that if $A \in \mathbb{C}^{n \times n}$ is an $\{R, s + 1, k\}$-potent matrix, then $A^{(s+1)k} = A$. To see this, note that from $RA = A^{s+1}R$, it follows that $R^2A = R(AA^sR) = A^{s+1}RA^sR = A^{s+1}A^{s+1}R = \cdots = A^{(s+1)(s+1)}R^2$, and similarly, $R^{k}A = A^{(s+1)k}R^k$. (The equality in the observation is uninformative when $s = 0$; the $s = 0$ case will be addressed in Subsection 2.2.) The necessity of $A^{(s+1)k} = A$ is clear, but is this condition sufficient to guarantee that a matrix $A$ is an $\{R, s + 1, k\}$-potent matrix for an arbitrary $\{k\}$-involution $R$? Not surprisingly, since $R$ does not appear in the equality, the condition is not sufficient as the following example
demonstrates:

\[ A = \exp \left( \frac{2\pi i}{3} \right) I_2, \quad R = \text{diag}(i, -1), \quad s = 1, \quad k = 4. \]

Consequently, we seek a complementary condition that in conjunction with \( A^{(s+1)k} = A \) implies \( A \) is an \( \{R, s+1, k\} \)-potent matrix.

2.1 The case \( s \geq 1 \)

Assume that \( A \) is an \( \{R, s+1, k\} \)-potent matrix. Let \( n_s = (s+1)^k - 1 \). Since \( A^{(s+1)k} = A \), the polynomial \( t^{(s+1)k} - t \), whose roots all have multiplicity 1, is divisible by the minimal polynomial of \( A \). Thus, \( A \) is diagonalizable with spectrum \( \sigma(A) \subseteq \{0\} \cup \{\omega^1, \omega^2, \ldots, \omega^{n_s-1}, \omega^{n_s} = 1\} \) where \( \omega := \exp \left( \frac{2\pi i}{n_s} \right) \).

Hence, the spectral theorem [1] assures that there exist disjoint projectors

\[ P_0, P_1, P_2, \ldots, P_{n_s-1}, P_{n_s} \]

such that

\[ A = \sum_{j=1}^{n_s} \omega^j P_j \quad \text{and} \quad \sum_{j=0}^{n_s} P_j = I_n, \quad (1) \]

where \( P_{j_0} = O \) if there exists \( j_0 \in \{1, 2, \ldots, n_s\} \) such that \( \omega^{j_0} \notin \sigma(A) \) and moreover that \( P_0 = O \) when \( 0 \notin \sigma(A) \).

Pre-multiplying the previous expressions given in (1) by the matrix \( R \) and post-multiplying by \( R^{-1} \) gives

\[ RAR^{-1} = \sum_{j=1}^{n_s} \omega^j RP_j R^{-1} \]

and

\[ \sum_{j=0}^{n_s} RP_j R^{-1} = I_n. \]

(2)

It is clear that the nonzero \( RP_j R^{-1} \) are disjoint projectors for each \( j = 0, 1, \ldots, n_s \).

From (1),

\[ A^{s+1} = \sum_{j=1}^{n_s} \omega^{j(s+1)} P_j \]

because the nonzero \( P_j \) are disjoint projectors.

Let \( S = \{1, 2, \ldots, n_s - 1\} \). Now consider \( \varphi : S \cup \{0\} \to S \cup \{0\} \) as the function defined by \( \varphi(j) = b_j \), where \( b_j \) is the smallest nonnegative integer such that \( b_j \equiv j(s+1) \mod n_s \). Then \( \varphi \) is a bijection [7]. It follows that

\[ A^{s+1} = \sum_{j=1}^{n_s-1} \omega^{\varphi(j)} P_j + P_{n_s} \]
and since $A$ is an $\{R, s + 1, k\}$-potent matrix, 

$$A^{s+1} = RAR^{-1}.$$ 

Hence, 

$$\sum_{i=1}^{n_s-1} \omega^i RP_i R^{-1} + RP_{n_s} R^{-1} = \sum_{j=1}^{n_s-1} \omega^{\varphi(j)} P_j + P_{n_s}.$$ 

Since $\varphi$ is a bijection, for each $i \in S$, there exists a unique $j \in S$ such that $i = \varphi(j)$. From the uniqueness of the spectral decomposition, it follows that for every $i \in S$, there exists a unique $j \in S$ such that 

$$RP_i R^{-1} = RP_{\varphi(j)} R^{-1} = P_j.$$ 

(3) 

It is clear that uniqueness also implies that 

$$RP_{n_s} R^{-1} = P_{n_s}.$$ 

(4) 

Finally, from (1) 

$$P_0 = I_n - \sum_{j=1}^{n_s} P_j.$$ 

Taking into account (2) and the definition of the bijection $\varphi$, 

$$RP_0 R^{-1} = P_0$$ 

(5) 

because of the uniqueness of the spectral decomposition. Observe that in the case where there exists $j_0 \in S$ such that $\omega^{j_0} \notin \sigma(A)$, it has been indicated that $P_{j_0} = O$. In this situation, $P_{\varphi(j_0)} = RP_{j_0} R^{-1} = O$ is also true.

Conversely, assuming $A^{(s+1)k} = A$ and that the relationships on the projectors obtained in (3), (4), and (5) hold, we can consider 

$$A = \sum_{j=1}^{n_s} \omega^j P_j$$ 

(6) 

It is now easy to check that $A^{s+1} = RAR^{-1}$.

The matrices $P_j$’s satisfying relations (3), (4), and (5) where 

$$P_0, P_1, \ldots, P_{n_s}$$ 

are the projectors appearing in the spectral decomposition of $A$ associated to the eigenvalues 

$0, \omega^1, \ldots, \omega^{n_s-1}, 1,$

are said to satisfy condition $(\mathcal{P})$. Then, the complementary condition we were looking for is condition $(\mathcal{P})$.

These results are summarized in what follows. Before that, note 

$$\text{rank}(A) = \text{rank}(A^{(s+1)k}) \leq \text{rank}(A^2) \leq \text{rank}(A)$$
when \(A^{(s+1)^k} = A\). Then, in this case, the group inverse of \(A\) exists, and it is easy to check that \(A^\# = A^{(s+1)^{k-2}}\), that is, \(A\) is a \(((s+1)^k)\)-group involutory matrix.

The main result of this subsection is now stated.

**Theorem 1** Let \(R \in \mathbb{C}^{n \times n}\) be a \(\{k\}\)-involutory matrix, \(s \in \{1, 2, 3, \ldots\}\), \(n_s = (s+1)^k - 1\), and, \(A \in \mathbb{C}^{n \times n}\). Then the following conditions are equivalent:

1. \(A\) is \(\{R, s + 1, k\}\)-potent.
2. \(A^{(s+1)^k} = A\) and there exist \(P_0, P_1, P_2, \ldots, P_{n_s}\) satisfying condition \((P)\).
3. \(A\) is diagonalizable,
   \[
   \sigma(A) \subseteq \{0\} \cup \{\omega^1, \omega^2, \ldots, \omega^{n_s} = 1\},
   \]
   with \(\omega = \exp\left(\frac{2\pi i}{n_s}\right)\), and there exist \(P_0, P_1, P_2, \ldots, P_{n_s}\) satisfying condition \((P)\).
4. \(A\) is an \(((s+1)^k)\)-group involutory matrix and there exist \(P_0, P_1, P_2, \ldots, P_{n_s}\) satisfying condition \((P)\).

From the definition of an \(\{R, s + 1, k\}\)-potent matrix, if \(A\) is \(\{R, s + 1, k\}\)-potent, then \(A\) is similar to \(A^{s+1}\). Hence, the uniqueness of the spectral decomposition of \(A\) allows us to state the correspondence between the distinct eigenvalues of \(A\) as well as between their corresponding projectors. Specifically:

**Corollary 2** Let \(R \in \mathbb{C}^{n \times n}\) be a \(\{k\}\)-involutory matrix, \(s \in \{1, 2, 3, \ldots\}\), and \(A \in \mathbb{C}^{n \times n}\) with spectrum

\[
\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}, \quad \text{with } m \geq 1
\]

where the \(\lambda_n\) are the distinct eigenvalues of \(A\). Then \(A\) is \(\{R, s + 1, k\}\)-potent if and only if \(A\) is diagonalizable and for each \(i \in \{1, 2, \ldots, m\}\) there is a unique \(j \in \{1, 2, \ldots, m\}\) such that \(\lambda_i = \lambda_j^{s+1}\) and \(P_i R = R P_j\) where \(P_1, P_2, \ldots, P_m\) are the projectors satisfying condition \((P)\).

Note that from condition (c) in Theorem 1 we know if \(\sigma(A) \not\subseteq \{0\} \cup \{\omega^0, \omega^1, \ldots, \omega^{(s+1)^k-2}\}\) then \(A\) is not \(\{R, s + 1, k\}\)-potent. Even more, Corollary 2 gives us another simple sufficient condition for \(A\) to not be \(\{R, s + 1, k\}\)-potent. The following example illustrates this situation. Let

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & -i & 1 \\
0 & 0 & 1
\end{bmatrix}, \quad \text{and} \quad R = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}.
\]

It is obvious that the eigenvalues of \(A\) are its diagonal elements. Then, we can conclude that \(A\) is not \(\{R, 3, 2\}\)-potent because cubing the eigenvalue \(-i\) of \(A\) gives the value \(i\) which is not an eigenvalue.

The general situation is given in the following result.
Corollary 3 Let $R \in \mathbb{C}^{n \times n}$ be a $\{k\}$-involutory matrix and $s \in \{1, 2, 3, \ldots\}$. If the matrix $A \in \mathbb{C}^{n \times n}$ has an eigenvalue $\lambda$ such that one of the following conditions holds:

1. $\lambda^{s+1} \notin \sigma(A)$
2. $\lambda^{(s+1)^k} \neq \lambda$

then $A$ is not $\{R, s + 1, k\}$-potent.

Up to now we have considered $s \in \{1, 2, 3, \ldots\}$ where the diagonalizability of $A$ is a consequence of the fact that $A$ is $\{R, s + 1, k\}$-potent. The case $s = 0$ is now examined.

2.2 The case $s = 0$

This situation corresponds to those matrices $A \in \mathbb{C}^{n \times n}$ such that $RA = AR$ and $R^k = I_n$. Such matrices are called $\{R, k\}$-generalized centrosymmetric matrices or, for consistency, $\{R, 1, k\}$-potent matrices. These matrices are in general not diagonalizable, as is shown by the following example:

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}, \quad R = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix} \quad \text{and} \quad k = 2.
\]

When the diagonalizability is assumed, the uniqueness of the spectral decomposition (see [1], pp. 62) gives the following result.

Theorem 4 Let $A \in \mathbb{C}^{n \times n}$ be a diagonalizable matrix with $m$ distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_m$, and spectral decomposition $A = \sum_{i=1}^{m} \lambda_i P_i$. Suppose that $R \in \mathbb{C}^{n \times n}$ is $\{k\}$-involutory for some integer $k \geq 2$. Then $A$ is an $\{R, k\}$-generalized centrosymmetric matrix if and only if $RP_i = P_i R$ for all $i \in \{1, 2, \ldots, m\}$.

Note that all of the cases $k < m$, $k = m$, and $k > m$ can occur as the following examples show:

1. If $A = \text{diag}(1, 2, 3, 2, 1)$ and $R$ is the $5 \times 5$ centrosymmetric permutation matrix then $AR = RA$ and $k = 2 < 3 = h$.
2. If $A = \text{diag}(1, 2)$ and $R = \text{diag}(1, -1)$ then $AR = RA$ and $k = 2 = h$.
3. If $A = I_2$ and $R = \exp\left(\frac{2\pi i}{25}\right) I_2$ then $AR = RA$ and $k = 25 > 1 = h$.

Suppose $R \in \mathbb{C}^{n \times n}$, $R^k = I$, and $R$ has $n$ distinct eigenvalues. Then $k \geq n$, and $R$ is diagonalizable. Further, $AR = RA$ exactly when $R$ and $A$ are simultaneously diagonalizable. Consequently, if $A$ is an $\{R, k\}$-potent matrix then $A$ is diagonalizable. Further, when $k = n$, the spectrum of $R$ is the complete set of $n^{th}$ roots of unity, so $R$ is similar to the $n \times n$ circulant permutation matrix
that there is a nonsingular matrix $Q$ such that $QRQ^{-1} = R_3$. Further, $AR = RA$ exactly when $QAQ^{-1}$ is a circulant matrix (see for example Theorem 3.1.1 in [3]). Next, we investigate the cases where $R$ does not have $n$ distinct eigenvalues.

First, we present a classic result, and we include its proof for the sake of completeness.

**Lemma 5** For each $k$-involutory matrix $R \in \mathbb{C}^{n \times n}$, there exists an integer $t$ with $1 \leq t \leq n$ and a nonsingular matrix $Q \in \mathbb{C}^{n \times n}$ such that the Jordan form of $R$, $J_R = QRQ^{-1}$ is the diagonal matrix $J_R = \text{diag}(\omega_1 I_{n_1}, \omega_2 I_{n_2}, \ldots, \omega_t I_{n_t})$, where the $\omega_i$ are distinct $k^{th}$ roots of unity and $n_1 + n_2 + \cdots + n_t = n$.

**Proof.** Assume that $k > 1$. Let $\omega = \exp \left( \frac{2\pi i}{k} \right)$. Since $R^k = I_n$, the minimum polynomial $m_R(\lambda)$ of $R$ divides $\lambda^k - 1 = \prod_{j=1}^k (\lambda - \omega^j)$, and consequently, every factor of $m_R(\lambda)$ must be a distinct linear factor. It follows that $R$ is diagonalizable, and hence, that $J_R$ has the specified form where the $\omega_j$ are distinct elements from $\{\omega^1, \omega^2, \ldots, \omega^k\}$ whose sum of multiplicities is $n$.

**Theorem 6** Suppose that $R \in \mathbb{C}^{n \times n}$ is a $k$-involutory matrix with nonsingular matrix $Q$ and Jordan form $J_R$ as given in the preceding lemma. Then $AR = RA$ for $A \in \mathbb{C}^{n \times n}$ if and only if the blocks of $Y = Q^{-1}AQ$ satisfy $Y_{ij} = 0$ when $i \neq j$, and $Y_{ii} \in \mathbb{C}^{n_i \times n_i}$ is arbitrary for $1 \leq i, j \leq t$. The matrices $Y$ contain exactly

$$d = \sum_{j=1}^t n_j^2$$

arbitrary parameters, so $C(R) = \{A \in \mathbb{C}^{n \times n} : RA = AR\}$ is a vector space of dimension $d$. Further,

$$C(R) \simeq \bigoplus_{i=1}^t \mathbb{C}^{n_i \times n_i}$$

where $\mathbb{C}^{n_i \times n_i}$ is the full matrix algebra of $n_i \times n_i$ matrices over the complex field and where the isomorphism sends $A$ to $Q^{-1}AQ$.

**Proof.** $AR = RA$ if and only if $Y = Q^{-1}AQ$ satisfies $YJ_R = J_RY$. For $1 \leq i, j \leq t$, $Y_{ij} (\omega_j I_{n_j}) = (\omega_i I_{n_i}) Y_{ij}$. Since $\omega_i \neq \omega_j$ when $i \neq j$, $Y_{ij} = 0$. When $i = j$, $Y_{ij}$ is an arbitrary $n_i \times n_i$ matrix. Thus, $Y$ is a direct sum of arbitrary submatrices containing $\sum_{j=1}^t n_j^2$ arbitrary entries.

**References**


