Document downloaded from:

http://hdl.handle.net/10251/46839

This paper must be cited as:

Gutiérrez García, J.; Romaguera Bonilla, S. (2011). Examples of non-strong fuzzy metrics. Fuzzy Sets and Systems. 162(1):91-93. doi:10.1016/j.fss.2010.09.017.

The final publication is available at

http://dx.doi.org/10.1016/j.fss.2010.09.017

Copyright Elsevier

Examples of non strong fuzzy metrics[☆]

J. Gutiérrez García[a,*], S. Romaguera[b]

[a]*Departamento de Matemáticas, Universidad del País Vasco-Euskal Herriko Unibertsitatea, Apdo. 644, 48080, Bilbao, Spain*
[b]*Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, 46071, Valencia, Spain*

Abstract

Answering a recent question posed by Gregori, Morillas and Sapena ("On a class of completable fuzzy metric spaces", Fuzzy Sets and Systems, 161 (2010), 2193–2205) we present two examples of non strong fuzzy metrics (in the sense of George and Veeramani).

Keywords: Fuzzy metric space, Strong fuzzy metric
2000 MSC: 54A40, 54E35, 54E70

The theory of fuzzy metric completion (in the sense of George and Veeramani [1]) is very different from the classical theories of metric completion and probabilistic metric completion since there are fuzzy metric spaces which are not completable. The authors of [3] became interested in strong fuzzy metrics when looking for a class of completable fuzzy metrics. In [3] they provide an example of a non-strong fuzzy metric for the minimum t-norm and state the open question to find a non-strong fuzzy metric for another t-norm different from the minimum.

In what follows we answer this question and present two examples of non-strong fuzzy metrics for the product and the Łukasiewicz t-norm, respectively. Terms and undefined concepts may be found in [3].

Definition 1. ([1]) A *fuzzy metric* on a (nonempty) set X is a pair $(M, *)$ such that M is a fuzzy set on $X \times X \times (0, +\infty)$ and $*$ is a continuous t-norm satisfying the following conditions:

(FM1) $M(x, y, t) > 0$ for all $x, y \in X$ and all $t > 0$;

(FM2) $M(x, y, t) = 1$ for $t > 0$ if and only if $x = y$;

(FM3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and all $t > 0$;

(FM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $x, y, z \in X$ and $t, s > 0$;

(FM5) $M(x, y, \cdot)$ is continuous for each $x, y \in X$.

[☆]This research was supported by the Ministry of Science and Innovation of Spain under grants MTM2009-12872-C02-01 and MTM2009-12872-C02-02. J. Gutiérrez García also acknowledges financial support from the University of the Basque Country under grant GIU07/27.

*Corresponding author
Email addresses: javier.gutierrezgarcia@ehu.es (J. Gutiérrez García), sromague@mat.upv.es (S. Romaguera)
URL: http://www.ehu.es/javiergutierrezgarcia/ (J. Gutiérrez García)

Note that condition (FM4) implies that $M(x, y, \cdot)$ is non-decreasing for each $x, y \in X$.

A special type of fuzzy metrics has been recently considered in [3] and [2]:

Definition 2. ([3]) Let $(X, M, *)$ be a fuzzy metric space. M is said to be *strong* if it satisfies the following additional axiom

(FM4') $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$ for all $x, y, z \in X$ and $t > 0$.

Clearly, if d is a metric on a set X, then the fuzzy metric $(M_d, *)$ is strong for every continuous t-norm $*$ such that $* \leq \cdot$, where M_d is defined by $M_d(x, y, t) = t/(t + d(x, y))$, for all $x, y \in X$ and $t > 0$.

In Section 3 of [3] (see also [2]) the authors observed that, however, the fuzzy metric $(M_d, \wedge)$ is strong if and only if d is an ultrametric. Then, they posed the natural question of finding a non strong fuzzy metric $(M, *)$ where $*$ is not $\wedge$.

The following examples solve this question.

Example 1. Let $X = \{x, y, z\}$, $* = \cdot$ and $M : X \times X \times (0, +\infty) \to [0, 1]$ defined for each $t > 0$ as

$$M(x, x, t) = M(y, y, t) = M(z, z, t) = 1,$$
$$M(x, z, t) = M(z, x, t) = M(y, z, t) = M(z, y, t) = \frac{t}{t+1},$$
$$M(x, y, t) = M(y, x, t) = \frac{t^2}{(t+2)^2}.$$

It is easy to check that M satisfies (FM1), (FM2), (FM3) and (FM5). With respect to (FM4), we have that

$$M(x, y, t+s) - M(x, z, t) * M(z, y, s) = \frac{(t+s)^2}{(t+s+2)^2} - \frac{t}{t+1} \cdot \frac{s}{s+1} = \frac{(t-s)^2(s+t+1)}{(t+s+2)^2(t+1)(s+1)} \geq 0$$

Hence $M(x, y, t+s) \geq M(x, z, t) * M(z, y, s)$. Also

$$M(x, z, t+s) = \frac{t+s}{t+s+1} \geq \frac{s}{s+1} = M(z, y, s) \geq M(x, z, t) * M(z, y, s)$$
$$M(y, z, t+s) = \frac{t+s}{t+s+1} \geq \frac{t}{t+1} = M(y, x, t) \geq M(y, x, t) * M(x, z, s).$$

Consequently, $(M, \cdot)$ is a fuzzy metric.

However, for each $t > 0$ we have that

$$M(x, y, t) = \frac{t^2}{(t+2)^2} < \frac{t^2}{(t+1)^2} = M(x, z, t) * M(z, y, t).$$

It follows that $(M, \cdot)$ is not strong.

Example 2. Let $X = \{x, y, z\}$, $* = T_L$ (the Łukasiewicz t-norm), and $M : X \times X \times (0, +\infty) \to [0, 1]$ defined for each $t > 0$ as

$$M(x, x, t) = M(y, y, t) = M(z, z, t) = 1,$$
$$M(x, z, t) = M(z, x, t) = M(y, z, t) = M(z, y, t) = \frac{2t+1}{2t+2},$$
$$M(x, y, t) = M(y, x, t) = \frac{t}{t+2}.$$

It is easy to check that M satisfies (FM1), (FM2), (FM3) and (FM5). With respect to (FM4), we have that

$$\begin{aligned} M(x,y,t+s) - M(x,z,t) * M(z,y,s) &= \frac{t+s}{t+s+2} - \max\left\{\frac{2t+1}{2t+2} + \frac{2s+1}{2s+2} - 1, 0\right\} \\ &= \frac{2(t-s)^2}{(t+s+2)(2t+2)(2s+2)} \geq 0. \end{aligned}$$

Hence $M(x,y,t+s) \geq M(x,z,t) * M(z,y,s)$. Also

$$M(x,z,t+s) = \frac{2(t+s)+1}{2(t+s)+2} \geq \frac{2s+1}{2s+2} = M(z,y,s) \geq M(x,z,t) * M(z,y,s)$$

$$M(y,z,t+s) = \frac{2(t+s)+1}{2(t+s)+2} \geq \frac{2t+1}{2t+2} = M(y,x,t) \geq M(y,x,t) * M(x,z,s).$$

Consequently, (M, T_L) is a fuzzy metric.

However, for each $t > 0$ we have that

$$M(x,y,t) = \frac{t}{t+2} < \frac{t}{t+1} = \frac{2t+1}{2t+2} + \frac{2t+1}{2t+2} - 1 = M(x,z,t) * M(z,y,t).$$

It follows that (M, T_L) is not strong.

Conclusion and perspectives

In this short note we answer a question posed in [3] and present two examples of non strong fuzzy metrics (in the sense of [1]). Note that in our first example we have used the product t-norm. In this direction it would be interesting to find further examples for continuous t-norms that are greater than the product but different from minimum.

References

[1] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994), 395–399.

[2] A. Sapena, S. Morillas, On strong fuzzy metrics, Proceedings of the Workshop in Applied Topology WiAT09: "Applied Topology: Recent progress for Computer Science, Fuzzy Mathematics and Economics", 135–141.

[3] V. Gregori, S. Morillas, A. Sapena, On a class of completable fuzzy metric spaces, Fuzzy Sets and Systems 161 (2010), 2193–2205.