Fixed point theorems for generalized contractions on partial metric spaces

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Abstract

We obtain two fixed point theorems for complete partial metric space that, by one hand, clarify and improve some results that have been recently published in Topology and its Applications, and, on the other hand, generalize in several directions the celebrated Boyd and Wong fixed point theorem and Matkowski fixed point theorem, respectively.

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1 Introduction and preliminaries

In [2, Theorem 1], Altun, Sola and Simsek established the following fixed point theorem for complete partial metric spaces.

Theorem 1 ([2]). *Let* (X, p) *be a complete partial metric space and let* $f : X \to X$ *be a map such that*

$$p(fx, fy) \leq \phi\left(\max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}\left[p(x, fy) + p(y, fx)\right]\right\}\right),$$

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for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous nondecreasing function such that $\phi(t) < t$ for all $t > 0$. Then f has a unique fixed point.

In [1], Altun and Sadarangani observed that the proof of Theorem 1 was wrong (in fact, the error occurs on page 2781, line 11, as the authors noted) and then they proved the following modification of it.

Theorem 2 ([1]). *Let (X, p) be a complete partial metric space and let $f : X \to X$ be a map such that*

$$p(fx, fy) \leq \phi\left(\max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}\left[p(x, fy) + p(y, fx)\right]\right\}\right),$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that the series $\sum_{n=0}^{\infty} \phi^n(t)$ converges for all $t > 0$ (ϕ^n denotes the n-th iterate of ϕ). Then f has a unique fixed point.

In this paper we show that, regardless, Theorem 1 above is true; in fact, we prove a more general result by replacing the condition that ϕ is continuous and nondecreasing by the condition that it is upper semicontinuous from the right, obtaining, in this way, a result that generalizes in several directions the celebrated Boyd-Wong fixed point theorem [3].

Furthermore, we modify Theorem 2 by replacing the condition that the series $\sum_{n=0}^{\infty} \phi^n(t)$ converges for all $t > 0$ by simply that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$, obtaining, in this way, a result that generalizes in several directions the celebrated Matkowski fixed point theorem [6].

In the sequel the letters $\mathbb{N}$ and ω will denote the set of all positive integer numbers and the set of all nonnegative integer numbers, respectively.

Let us recall that partial metric spaces were introduced by Matthews ([5]) to the study of denotational semantics of dataflow networks. In fact, (complete) partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory (see, for instance, [4, 5, 8, 9, 10, 11]).

Following [5], a partial metric on a set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

(i) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$; (ii) $p(x, x) \leq p(x, y)$; (iii) $p(x, y) = p(y, x)$; (iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Observe that if $p(x, y) = 0$ then $x = y$.

A partial metric space is a pair (X, p) such that X is a set and p is a partial metric on X.

In the rest of this section we recall some properties of partial metric spaces which will be useful later on.

Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon + p(x, x)\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to [0, \infty)$ given by $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, is a metric on X.

Furthermore, a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X, p) converges, with respect to τ_{p^s}, to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

According to [5], a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$, and (X, p) is called complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Example 1. Let $X = [0, \infty)$ and let $p : X \times X \to [0, \infty)$ given by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. It is well known and easy to see that (X, p) is a complete partial metric space. In fact, p^s is the Euclidean metric on X.

Finally, the following crucial facts are shown in [5]:

(a) A sequence in a partial metric space (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, p^s).

(b) A partial metric space (X, p) is complete if and only if (X, p^s) is complete.

2 The results

In order to simplify the notation, given a partial metric space (X, p) and $f : X \to X$ a map, we define

$$P_f(x, y) := \max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}\left[p(x, fy) + p(y, fx)\right]\right\},$$

for all $x, y \in X$.

Lemma 1. *Let (X,p) be a partial metric space and let $f : X \to X$ be a map. Then, for each $x \in X$, we have*

$$P_f(x, fx) = \max\{p(x, fx), p(fx, f^2x)\}.$$

Proof. Let $x \in X$. Then

$$\begin{aligned}
\max\{p(x, fx), p(fx, f^2x)\} &\leq P_f(x, fx) \\
&= \max\{p(x, fx), p(fx, f^2x), \frac{1}{2}\left[p(x, f^2x) + p(fx, fx)\right]\} \\
&\leq \max\{p(x, fx), p(fx, f^2x), \frac{1}{2}\left[p(x, fx) + p(fx, f^2x)\right]\} \\
&= \max\{p(x, fx), p(fx, f^2x)\}.
\end{aligned}$$

The proof is complete. □

Lemma 2. *Let (X,p) be a partial metric space and let $f : X \to X$ be a map such that*

$$p(fx, fy) \leq \phi(P_f(x, y)),$$

for all $x, y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\phi(t) < t$ for all $t > 0$. If $x \in X$ satisfies that $f^n x \neq f^{n+1}x$ for all $n \in \omega$, then the following hold:

(a) $P_f(f^n x, f^{n+1}x) = p(f^n x, f^{n+1}x)$ *for all* $n \in \omega$.

(b) $p(f^n x, f^{n+1}x) \leq \phi(p(f^{n-1}x, f^n x)) < p(f^{n-1}x, f^n x)$ *for all* $n \in \mathbb{N}$.

Proof. (a) Let $x \in X$ be such that $f^n x \neq f^{n+1}x$ for all $n \in \omega$. Then $p(f^n x, f^{n+1}x) > 0$ for all $n \in \omega$. By Lemma 1,

$$P_f(f^n x, f^{n+1}x) = \max\{p(f^n x, f^{n+1}x), p(f^{n+1}x, f^{n+2}x)\}.$$

Since

$$p(f^{n+1}x, f^{n+2}x) \leq \phi(P_f(f^n x, f^{n+1}x)) < P_f(f^n x, f^{n+1}x),$$

it follows that $P_f(f^n x, f^{n+1}x) = p(f^n x, f^{n+1}x)$ for all $n \in \omega$.

(b) Taking into account (a), we deduce that

$$p(f^n x, f^{n+1}x) \leq \phi(P_f(f^{n-1}x, f^n x)) = \phi(p(f^{n-1}x, f^n x)) < p(f^{n-1}x, f^n x),$$

for all $n \in \mathbb{N}$. $\square$

Let us recall that a function $\phi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right provided that for each $t \geq 0$ and each sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \geq t$ and $\lim_{n\to\infty} t_n = t$, it follows that $\limsup_{n\to\infty} \phi(t_n) \leq \phi(t)$.

Theorem 3. *Let (X, p) be a complete partial metric space and let $f : X \to X$ be a map such that*

$$p(fx, fy) \leq \phi(P_f(x, y)),$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a upper semicontinuous from the right function such that $\phi(t) < t$ for all $t > 0$. Then f has a unique fixed point $z \in X$. Moreover $p(z, z) = 0$.

Proof. Let $x \in X$. If there is $n \in \omega$ such that $f^n x = f^{n+1}x$, then $f^n x$ is a fixed point of f and uniqueness of $f^n x$ follows as in the last part of the proof below.

Hence, we shall assume that $f^n x \neq f^{n+1}x$ for all $n \in \omega$. Put $x_0 = x$ and construct the sequence $(x_n)_{n\in\omega}$ where $x_n = f^n x_0$ for all $n \in \omega$. Thus $x_{n+1} = fx_n$ and $p(x_n, x_{n+1}) > 0$ for all $n \in \omega$.

By Lemma 2 (b), there is $c \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} \phi(p(x_n, x_{n+1})) = c.$$

If $c > 0$, we have

$$c = \limsup_{n\to\infty} \phi(p(x_n, x_{n+1})) \leq \phi(c) < c,$$

a contradiction. So $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.

Next we show that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

This will be done by adapting a technique of Boyd and Wong [3, Theorem 1]. Indeed, assume the contrary. Then there exist $\varepsilon > 0$ and sequences $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$, with $m_k > n_k \geq k$, and such that $p(x_{n_k}, x_{m_k}) \geq \varepsilon$ for all $k \in \mathbb{N}$.

From the fact that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ we can suppose, without loss of generality, that $p(x_{n_k}, x_{m_k - 1}) < \varepsilon$.

For each $k \in \mathbb{N}$ we have

$$\varepsilon \leq p(x_{n_k}, x_{m_k}) \leq p(x_{n_k}, x_{m_k-1}) + p(x_{m_k-1}, x_{m_k}) < \varepsilon + p(x_{m_k-1}, x_{m_k}),$$

and, hence, $\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = \varepsilon$.

Now let $k_0 \in \mathbb{N}$ such that $p(x_{n_k+1}, x_{n_k}) < \varepsilon$ and $p(x_{m_k+1}, x_{m_k}) < \varepsilon$ for all $k \geq k_0$. Then

$$\begin{aligned} p(x_{n_k}, x_{m_k}) &\leq P_f(x_{n_k}, x_{m_k}) \\ &\leq p(x_{n_k}, x_{m_k}) + \frac{1}{2}(p(x_{m_k}, x_{m_k+1}) + p(x_{n_k+1}, x_{n_k})), \end{aligned}$$

for all $k \geq k_0$. So $\lim_{k\to\infty} P_f(x_{n_k}, x_{m_k}) = \varepsilon$.

Since $P_f(x_{n_k}, x_{m_k}) \geq \varepsilon$ for all $k \in \mathbb{N}$, and ϕ is upper semicontinuous from the right, we deduce that

$$\limsup_{k\to\infty} \phi(P_f(x_{n_k}, x_{m_k})) \leq \phi(\varepsilon).$$

On the other hand, for each $k \in \mathbb{N}$ we have

$$\begin{aligned} \varepsilon &\leq p(x_{n_k}, x_{m_k}) \leq p(x_{n_k}, x_{n_k+1}) + p(x_{n_k+1}, x_{m_k+1}) + p(x_{m_k+1}, x_{m_k}) \\ &\leq p(x_{n_k}, x_{n_k+1}) + \phi(P_f(x_{n_k}, x_{m_k})) + p(x_{m_k+1}, x_{m_k}), \end{aligned}$$

so

$$\varepsilon \leq \limsup_{k\to\infty} \phi(P_f(x_{n_k}, x_{m_k})) \leq \phi(\varepsilon),$$

a contradiction because $\phi(\varepsilon) < \varepsilon$.

Consequently $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, and, thus, $(x_n)_{n\in\omega}$ is a Cauchy sequence in the complete partial metric space (X, p). Hence, there is $z \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(z, x_n) = p(z, z) = 0.$$

We show that z is a fixed point of f.

To this end we first note that $\lim_{n\to\infty} P_f(z, x_n) \leq p(z, fz)$.

Moreover, since $p(z, fz) \leq P_f(z, x_n)$ for all $n \in \omega$, we deduce that

$$p(z, fz) = \lim_{n\to\infty} P_f(z, x_n),$$

so

$$\limsup_{n\to\infty} \phi(P_f(z, x_n)) \leq \phi(p(z, fz)).$$

On the other hand, since for each $n \in \omega$,

$$p(z, fz) \leq p(z, x_n) + p(x_n, fz),$$

it follows that

$$\begin{aligned} p(z, fz) &\leq \limsup_{n\to\infty} (p(z, x_n) + p(x_n, fz)) = \limsup_{n\to\infty} p(x_n, fz) \\ &\leq \limsup_{n\to\infty} \phi(P_f(x_{n-1}, z)) \leq \phi(p(z, fz)). \end{aligned}$$

Therefore $p(z, fz) = 0$ and thus $z = fz$.
Finally, let $u \in X$ such that $fu = u$. Then,

$$p(u, z) = p(fu, fz) \leq \phi(P_f(u, z)) = \phi((p(u, z)).$$

Hence $p(u, z) = 0$, i.e., $u = z$. This concludes the proof. □

Corollary 1. *Let (X, p) be a complete partial metric space and let $f : X \to X$ be a map such that*

$$p(fx, fy) \leq \phi(p(x, y)),$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a upper semicontinuous from the right function such that $\phi(t) < t$ for all $t > 0$. Then f has a unique fixed point $z \in X$. Moreover $p(z, z) = 0$.

Corollary 2 (Boyd and Wong [3]). *Let (X, d) be a complete metric space and let $f : X \to X$ be a map such that*

$$d(fx, fy) \leq \phi(d(x, y)),$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a upper semicontinuous from the right function such that $\phi(t) < t$ for all $t > 0$. Then f has a unique fixed point.

The following is a typical instance where Theorem 1 (and also Corollary 1) can be applied but Theorem 2 not.

Example 2. Let (X, p) be the complete partial metric space of Example 1, and let $f : X \to X$ given by $fx = x/2$ for all $x \in X$.

Now let $\phi : [0, \infty) \to [0, \infty)$ defined by

$$\begin{aligned}
\phi(0) &= 0, \\
\phi(t) &= \frac{nt}{n+2} + \frac{1}{(n+1)(n+2)} \quad \text{if } t \in [\frac{1}{n+1}, \frac{1}{n}), \; n \in \mathbb{N}, \text{ and} \\
\phi(t) &= \frac{t}{2} \quad \text{if } t \geq 1.
\end{aligned}$$

It is routine to check that ϕ is continuous on $[0, \infty)$ with $t/2 < \phi(t) < t$ for all $t > 0$. Hence ϕ satisfies the conditions of Theorem 1 and thus of Corollary 1. Note that, in fact, the graph of the restriction of ϕ to $[1/(n+1), 1/n]$, $n \in \mathbb{N}$, is the straight line segment with origin at $(1/(n+1), 1/(n+2))$ and end at $(1/n, 1/(n+1))$.

Nevertheless, since $\phi(1/n) = 1/(n+1)$ for all $n \in \mathbb{N}$, and $\phi(t) = t/2$ for all $t > 1$, it follows that $\sum_{n=0}^{\infty} \phi^n(t) = \infty$ for all $t > 0$. So ϕ does not satisfy the conditions of Theorem 2.

Finally, we have $p(fx, fy) = \max\{x/2, y/2\} \leq \phi(\max\{x, y\}) = \phi(p(x, y))$, for all $x, y \in X$, and thus, all conditions of Theorem 1 (and also of Corollary 1) are satisfied.

In order to state our next theorem we shall need the following well-known and easy, but useful, observation.

Lemma 3. ([6, 7]). *Let $\phi : [0, \infty) \to [0, \infty)$ be nondecreasing and let $t > 0$. If $\lim_{n\to\infty} \phi^n(t) = 0$, then $\phi(t) < t$.*

Theorem 4. *Let (X, p) be a complete partial metric space and let $f : X \to X$ be a map such that*

$$p(fx, fy) \leq \phi(M_f(x, y)),$$

where $M_f(x, y) = \max\{p(x, y), p(x, fx), p(y, fy)\}$ for all $x, y \in X$, and $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$. Then f has a unique fixed point $z \in X$. Moreover $p(z, z) = 0$.

Proof. Let $x \in X$. If there is $n \in \omega$ such that $f^n x = f^{n+1}x$, then $f^n x$ is a fixed point of f and uniqueness of $f^n x$ follows as in the last part of the proof below.

Hence, we shall assume that $f^n x \neq f^{n+1} x$ for all $n \in \omega$. Put $x_0 = x$ and construct the sequence $(x_n)_{n\in\omega}$ where $x_n = f^n x_0$ for all $n \in \omega$. Thus $x_{n+1} = fx_n$ and $p(x_n, x_{n+1}) > 0$ for all $n \in \omega$. By Lemma 2 (b),

$$p(x_n, x_{n+1}) \leq \phi(p(x_{n-1}, x_n)),$$

for all $n \in \omega$. Then, since ϕ is nondecreasing, we deduce that

$$p(x_n, x_{n+1}) \leq \phi^n(p(x_0, x_1)),$$

for all $n \in \omega$. Hence

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Now choose an arbitrary $\varepsilon > 0$. Since $\lim_{n\to\infty} \phi^n(\varepsilon) = 0$ it follows from Lemma 3 that $\phi(\varepsilon) < \varepsilon$, so there is $n_\varepsilon \in \mathbb{N}$ such that

$$p(x_n, x_{n+1}) < \varepsilon - \phi(\varepsilon),$$

for all $n \geq n_\varepsilon$. Therefore

$$\begin{aligned} p(x_n, x_{n+2}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) \\ &< \varepsilon - \phi(\varepsilon) + \phi(p(x_n, x_{n+1})) \\ &\leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon, \end{aligned}$$

for all $n \geq n_\varepsilon$. So

$$\begin{aligned} p(x_n, x_{n+3}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+3}) \\ &< \varepsilon - \phi(\varepsilon) + \phi(M_f(x_n, x_{n+2})) \\ &\leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon, \end{aligned}$$

and following this process

$$p(x_n, x_{n+k}) < \varepsilon,$$

for all $n \geq n_\varepsilon$ and $k \in \mathbb{N}$. Consequently

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0,$$

and thus $(x_n)_{n\in\omega}$ is a Cauchy sequence in the complete partial metric space (X, p). Hence there is $z \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(z, x_n) = p(z, z) = 0.$$

We show that z is a fixed point of f.

Assume the contrary. Then $p(z, fz) > 0$. For each $n \in \mathbb{N}$ we have

$$p(z, fz) \leq p(z, x_n) + p(x_n, fz) \leq p(z, x_n) + \phi(M_f(z, x_{n-1})).$$

From our assumption that $p(z, fz) > 0$, it easily follows that there is $n_0 \in \mathbb{N}$ such that $M_f(z, x_{n-1}) = p(z, fz)$ for all $n \geq n_0$ (observe that, in particular, $p(x_{n-1}, fz) \leq p(x_{n-1}, z) + p(z, fz)$ and that $\lim_{n\to\infty} p(x_n, z) = 0$).

So

$$p(z, fz) \leq p(z, x_n) + \phi(p(z, fz)),$$

for all $n \geq n_0$.

Taking limits as $n \to \infty$, we obtain that $p(z, fz) \leq \phi(p(z, fz)) < p(z, fz)$, a contradiction. Consequently $z = fz$.

Finally, uniqueness of z follows as in Theorem 3. $\square$

Corollary 3. *Let (X, p) be a complete partial metric space and let $f : X \to X$ be a map such that*

$$p(fx, fy) \leq \phi(p(x, y)),$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$. Then f has a unique fixed point $z \in X$. Moreover $p(z, z) = 0$.

Corollary 4 (Matkowski [6])**.** *Let (X, d) be a complete metric space and let $f : X \to X$ be a map such that*

$$d(fx, fy) \leq \phi(d(x, y)),$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$. Then f has a unique fixed point $z \in X$.

Remark. Note that Theorem 4 can be also applied to Example 2, because in this example the function ϕ is nondecreasing and $\lim_{n\to\infty} \phi^n(t) = 0$, for all $t > 0$

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