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Research Article

Lattice Copies of ℓ^2 in L^1 of a Vector Measure and Strongly Orthogonal Sequences

E. Jiménez Fernández and E. A. Sánchez Pérez

Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera, s/n, 46022 Valencia, Spain

Correspondence should be addressed to E. A. Sánchez Pérez, easancpe@mat.upv.es

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Let $\mathbf{m}$ be an ℓ^2-valued (countably additive) vector measure and consider the space $L^2(\mathbf{m})$ of square integrable functions with respect to $\mathbf{m}$. The integral with respect to $\mathbf{m}$ allows to define several notions of orthogonal sequence in these spaces. In this paper, we center our attention in the existence of strongly $\mathbf{m}$-orthonormal sequences. Combining the use of the Kadec-Pelczyński dichotomy in the domain space and the Bessaga-Pelczyński principle in the range space, we construct a two-sided disjointification method that allows to prove several structure theorems for the spaces $L^1(\mathbf{m})$ and $L^2(\mathbf{m})$. Under certain requirements, our main result establishes that a normalized sequence in $L^2(\mathbf{m})$ with a weakly null sequence of integrals has a subsequence that is strongly $\mathbf{m}$-orthonormal in $L^2(\mathbf{m}^*)$, where $\mathbf{m}^*$ is another ℓ^2-valued vector measure that satisfies $L^2(\mathbf{m}) = L^2(\mathbf{m}^*)$. As an application of our technique, we give a complete characterization of when a space of integrable functions with respect to an ℓ^2-valued positive vector measure contains a lattice copy of ℓ^2.

1. Introduction

In recent years, vector measure integration has been shown to be a good framework for the analysis of the properties of Banach function spaces and the operators defined on them. In particular, it is a powerful tool for representing Banach function spaces providing an additional integration structure. For instance, every 2-convex order continuous Banach function space with weak unit can be written as a space $L^2(\mathbf{m})$ of integrable functions with respect to a suitable vector measure $\mathbf{m}$ ([1, Th. 2.4]; see also [2, Ch.3] for more information). As in the case of the Hilbert spaces of square integrable functions, sequences in $L^2(\mathbf{m})$ that satisfy some orthogonality properties with respect to the vector valued integral become useful both for studying the geometry of the space [3–5] and for applications, mainly in the context of the function approximation [3, 6, 7].

In contrast to the scalar case, several notions of $\mathbf{m}$-orthogonality are possible in the case of an ℓ^2-valued (countably additive) vector measure $\mathbf{m}$. A sequence $\{f_i\}_{i=1}^{\infty}$ in $L^2(\mathbf{m})$ is said to be *strongly* $\mathbf{m}$*-orthonormal* if the integral of the product of two different functions is 0 and the integral of each f_i^2 is e_i, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence in ℓ^2. In this paper, we center our attention in this strong version of $\mathbf{m}$-orthogonal sequence, giving a complete characterization of the spaces $L^2(\mathbf{m})$ in which such sequences exist; actually, we will show that this fact is closely connected to the existence of lattice copies of ℓ^2 in the corresponding space $L^1(\mathbf{m})$ that is preserved by the integration map. In order to do this, we develop a sort of two-sided Kadec-Pelczyński disjointification technique. Roughly speaking, this procedure allows to produce sequences of normalized functions in $L^2(\mathbf{m})$—or $L^1(\mathbf{m})$—which are almost disjoint and have integrals that are almost orthogonal in ℓ^2: after an isomorphic change of vector measure, we obtain our results both for the existence of strongly $\mathbf{m}$-orthogonal sequences in $L^2(\mathbf{m})$ and the existence of lattice copies of ℓ^2 in $L^1(\mathbf{m})$.

The paper is organized as follows. After the preliminary Section 2, we analyze in Section 3 the existence of strongly orthonormal sequences in $L^2(\mathbf{m})$, and we show that it is a genuine vector valued phenomenon, in the sense that they do not exist for scalar measures and in the case of their natural extensions, vector measures with compact integration maps. Actually, later on we prove that they do not exist for ℓ^2-valued measures with disjointly strictly singular integration maps. In the positive, we show in Theorem 3.7 that under reasonable requirements, given an $\mathbf{m}$-orthonormal sequence $\{f_i\}_{i=1}^{\infty}$ in $L^2(\mathbf{m})$, it is possible to construct another vector measure $\mathbf{m}^*$ such that

(1) $\{f_i\}_{i=1}^{\infty}$ is a *strongly* $\mathbf{m}^*$-orthonormal sequence in $L^2(\mathbf{m})$,

(2) $L^2(\mathbf{m}) = L^2(\mathbf{m}^*)$.

Combining with the Kadec-Pelczyński dichotomy, the requirement on the sequence of being $\mathbf{m}$-strongly orthogonal can be relaxed to being weakly null (Corollary 3.9), obtaining in this case a sequence of functions satisfying (1) and (2) that approximates a subsequence of the original one. Some examples and direct consequences of this result are also given. Finally, Section 4 is devoted to show some applications in the context of the structure theory of Banach function spaces, focusing our attention in Banach function lattices that are represented as spaces of square integrable functions with respect to an ℓ^2-valued vector measure and are not Hilbert spaces. For the case of positive measures, we show that the existence of strongly $\mathbf{m}$-orthonormal sequences is equivalent of the existence of lattice copies of ℓ^4 in $L^2(\mathbf{m})$ and lattice copies of ℓ^2 in $L^1(\mathbf{m})$ (Proposition 4.3 and Theorem 4.5). The translation of these results for the space $L^1(\mathbf{m})$ gives the following result on its structure that can also be written in terms of the integration map (Theorem 4.7): the space $L^1(\mathbf{m})$ contains a normalized weakly null sequence if and only if it has a reflexive sublattice if and only if it contains a lattice copy of ℓ^2.

2. Preliminaries

In this section, we introduce several definitions and comments regarding the spaces $L^2(\mathbf{m})$. We refer to [8] for definitions and basic results on vector measures. Let X be a Banach space. We will denote by B_X the unit ball of X, that is $B_X := \{x \in X : \|x\|_X \le 1\}$. X' will be the topological dual of X. Let Σ be a σ-algebra on a nonempty set Ω. Throughout the paper $\mathbf{m} : \Sigma \to X$ will be a countably additive vector measure. The *semivariation* of $\mathbf{m}$ is the nonnegative function $\|\mathbf{m}\|$ whose value on a set $A \in \Sigma$ is given by $\|\mathbf{m}\|(A) := \sup\{|\langle \mathbf{m}, x' \rangle|(A) : x' \in B_{X'}\}$. The *variation* $|\mathbf{m}|$ of $\mathbf{m}$ on a measurable set A is given by $|\mathbf{m}|(A) := \sup \sum_{B \in \Pi} \|\mathbf{m}(B)\|$ for $A \in \Sigma$, where the supremum is computed over all finite measurable partitions Π of A. The variation $|\mathbf{m}|$ is a monotone countably additive function on Σ—a positive scalar measure—, while the semivariation $\|\mathbf{m}\|$ is a monotone subadditive function on Σ, and for each $A \in \Sigma$ we have that $\|\mathbf{m}\|(A) \leq |\mathbf{m}|(A)$.

For each element $x' \in X'$, the formula $\langle \mathbf{m}, x' \rangle(A) := \langle \mathbf{m}(A), x' \rangle$, $A \in \Sigma$, defines a (countably additive) scalar measure. As usual, we say that a sequence of Σ-measurable functions converges $|\langle \mathbf{m}, x' \rangle|$-almost everywhere if it converges pointwise in a set $A \in \Sigma$ such that $|\langle \mathbf{m}, x' \rangle|(\Omega \setminus A) = 0$. A sequence converges $\mathbf{m}$-almost everywhere if it converges in a set A that satisfies that the semivariation of $\mathbf{m}$ in $\Omega \setminus A$ is 0.

Let μ be a positive scalar measure. The measure $\mathbf{m}$ is *absolutely continuous* with respect to μ if $\lim_{\mu(A) \to 0} \mathbf{m}(A) = 0$; in this case we write $\mathbf{m} \ll \mu$ and we say that μ is a control measure for $\mathbf{m}$. Countably additive vector measures always have control measures. It is known that there exists always an element $x' \in X'$ such that $\mathbf{m} \ll |\langle \mathbf{m}, x' \rangle|$. We call such a scalar measure a *Rybakov measure* for $\mathbf{m}$ (see [8, Ch.IX,2]). If $|\langle \mathbf{m}, x' \rangle|$ is a Rybakov measure for $\mathbf{m}$, a sequence of Σ-measurable functions converges $\mathbf{m}$-almost everywhere if and only if it converges $|\langle \mathbf{m}, x' \rangle|$-almost everywhere.

A Σ-measurable function f is integrable with respect to $\mathbf{m}$ if it is integrable with respect to each scalar measure $\langle \mathbf{m}, x' \rangle$, and for every $A \in \Sigma$ there is an element $\int_A f d\mathbf{m} \in X$ such that $\langle \int_A f d\mathbf{m}, x' \rangle = \int_A f d\langle \mathbf{m}, x' \rangle$ for every $x' \in X'$. The set of all the (classes of $\mathbf{m}$—a.e. equal) $\mathbf{m}$—integrable functions $L^1(\mathbf{m})$ defines an order continuous Banach function space with weak unit χ_Ω—in the sense of [9, p.28]—over *any* Rybakov measure for $\mathbf{m}$ that is endowed with the norm

$$\|f\|_{L^1(\mathbf{m})} := \sup_{x' \in B_{X'}} \int_\Omega |f| d |\langle \mathbf{m}, x' \rangle|, \quad f \in L^1(\mathbf{m}). \tag{2.1}$$

The reader can find the definitions and fundamental results concerning the space $L^1(\mathbf{m})$ in [2, 10–12].

The spaces $L^p(\mathbf{m})$ are defined extending the definition above in a natural manner [1, 2, 13]. They are p-convex order continuous Banach function spaces with weak unit χ_Ω over any Rybakov measure, with the norm

$$\|f\|_{L^p(\mathbf{m})} := \left\| |f|^p \right\|_{L^1(\mathbf{m})}^{1/p}, \qquad f \in L^p(\mathbf{m}). \tag{2.2}$$

It is also known that if $1/p + 1/q = 1$, $f_1 \in L^p(\mathbf{m})$ and $f_2 \in L^q(\mathbf{m})$, then the pointwise product $f_1 \cdot f_2$ belongs to $L^1(\mathbf{m})$ (see for instance [2, Ch.3]). We will consider the integration operator $I_\mathbf{m} : L^1(\mathbf{m}) \to X$ associated to the vector measure $\mathbf{m}$, that is defined by $I_\mathbf{m}(f) := \int_\Omega f d\mathbf{m}$, $f \in L^1(\mathbf{m})$. The properties of the integration map have been largely studied in several recent papers (see [2, 14–17] and the references therein). If i, j are indexes of a set I, we write $\delta_{i,j}$ for the Kronecker delta as usual. A sequence $\{f_i\}_{i=1}^\infty$ in $L^2(\mathbf{m})$ is called $\mathbf{m}$-*orthogonal* if $\| \int_\Omega f_i f_j \, d\mathbf{m}\| = \delta_{i,j} k_i$ for positive constants k_i. If $\|f_i\|_{L^2(\mathbf{m})} = 1$ for all $i \in \mathbb{N}$, it is called $\mathbf{m}$-*orthonormal*. The properties of these sequences have been recently analyzed in a series of papers, and some applications have been already developed (see [3–7, 18]). In this paper, we deal with the following more restrictive version of orthogonality for ℓ^2-valued measures.

Definition 2.1. Let $\mathbf{m} : \Sigma \to \ell^2$ be a vector measure. We say that $\{f_i\}_{i=1}^\infty \subset L^2(\mathbf{m})$ is a *strongly* $\mathbf{m}$*-orthogonal sequence* if $\int_\Omega f_i f_j d\mathbf{m} = \delta_{i,j} e_i k_i$ for an orthonormal sequence $\{e_i\}_{i=1}^\infty$ in ℓ^2 and for $k_i > 0$. If $k_i = 1$ for every $i \in \mathbb{N}$, we say that it is a *strongly* $\mathbf{m}$*-orthonormal sequence*.

We need some elements on Banach-lattice-valued vector measures; in particular, on ℓ^2-valued measures when the order in ℓ^2 is considered. If X is a Banach lattice, we say that a vector measure $\mathbf{m} : \Sigma \to X$ is positive if $\mathbf{m}(A) \geq 0$ for all $A \in \Sigma$. Note that if $\mathbf{m}$ is positive and x' is a positive element of the Banach lattice X', then the measure $\langle \mathbf{m}, x' \rangle$ coincides with its variation. We refer to [2, 9, 19] for general questions concerning Banach lattices and Banach function spaces. An operator between Banach lattices is called strictly singular if no restriction to an infinite dimensional subspace give an isomorphism, and ℓ^2-singular if this happens for subspaces isomorphic to ℓ^2. It is called disjointly strictly singular if no restriction to the closed linear span of a disjoint sequence is an isomorphism.

We use standard Banach spaces notation. A sequence $\{x_n\}_{n=1}^\infty$ in a Banach space X is called a *Schauder basis* of X (or simply a *basis*) if for every $x \in X$ there exists a unique sequence of scalars $\{\alpha_n\}_{n=1}^\infty$ such that $x = \lim_{n\to\infty} \sum_{k=1}^n \alpha_k x_k$. A sequence $\{x_n\}_{n=1}^\infty$ which is a Schauder basis of its closed span is called a *basic sequence*. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two basis for the Banach spaces X and Y, respectively. Then $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are equivalent if and only if there is an isomorphism between X and Y that carries each x_n to y_n.

Let $\{x_i\}_{i=1}^\infty$ be a basic sequence of a Banach space X and take two sequences of positive integers $\{p_i\}_{i=1}^\infty$ and $\{q_i\}_{i=1}^\infty$ satisfying that $p_i < q_i < p_{i+1}$ for every $i \in \mathbb{N}$. A *block basic sequence* $\{y_i\}_{i=1}^\infty$ associated to $\{x_i\}_{i=1}^\infty$ is a sequence of vectors of X defined as finite linear combinations as $y_i = \sum_{k=p_i}^{q_i} \alpha_{i,k} x_k$, where $\alpha_{i,k}$ are real numbers. We refer to [20, Ch.V] for the definition of block basic sequence and to [9, 20] for general questions concerning Schauder basis.

3. Strongly m-Orthogonal Sequences in L^2(m)

This section is devoted to show how to construct strongly $\mathbf{m}$-orthonormal sequences in $L^2(\mathbf{m})$. Let us start with an example of the kind of sequences that we are interested in.

Example 3.1. Let $([0,\infty), \Sigma, \mu)$ be Lebesgue measure space (Figure 1). Let $r_k(x) := \operatorname{sign}\{\sin(2^{k-1}x)\}$ be the *Rademacher* function of period 2π defined at the interval $E_k = [2(k-1)\pi, 2k\pi]$, $k \in \mathbb{N}$. Consider the vector measure $\mathbf{m} : \Sigma \to \ell^2$ given by $\mathbf{m}(A) := \sum_{k=1}^\infty (-1/2^k)(\int_{A\cap E_k} r_k d\mu) e_k \in \ell^2$, $A \in \Sigma$.

Note that if $f \in L^2(\mathbf{m})$ then $\int_{[0,\infty)} f d\mathbf{m} = ((-1/2^k)\int_{E_k} f r_k d\mu)_k \in \ell^2$. Consider the sequence of functions

$$\begin{aligned}
f_1(x) &= \sin(x) \cdot \chi_{[\pi,2\pi]}(x) \\
f_2(x) &= \sin(2x) \cdot \left(\chi_{[0,2\pi]}(x) + \chi_{[(7/2)\pi,4\pi]}(x)\right) \\
f_3(x) &= \sin(4x) \cdot \left(\chi_{[0,4\pi]}(x) + \chi_{[(23/4)\pi,6\pi]}(x)\right) \\
&\ \ \vdots \\
f_n(x) &= \sin\left(2^{n-1}x\right) \cdot \left(\chi_{[0,2(n-1)\pi]}(x) + \chi_{[(2n-2/2^n)\pi,2n\pi]}(x)\right), \quad n \geq 2.
\end{aligned} \tag{3.1}$$

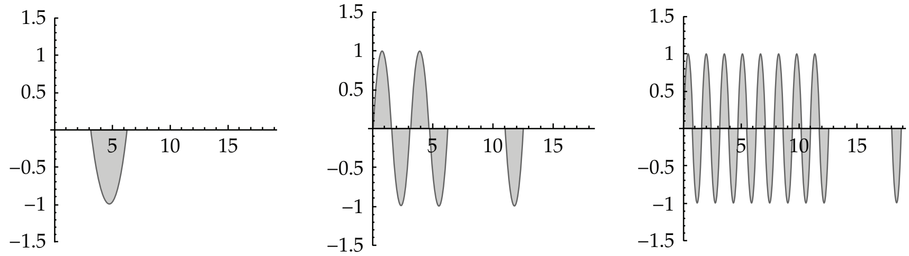

Figure 1: Functions $f_1(x)$, $f_2(x)$, and $f_3(x)$ in Example 3.1.

This sequence can be used to define a strongly $\mathbf{m}$-orthogonal sequence, since

$$\begin{aligned}
\left\langle \int_{[0,\infty)} f_n^2 d\mathbf{m}, e_n \right\rangle &= -\frac{1}{2^n}\int_{E_n} f_n^2 r_n d\mu = \frac{\pi}{2^{2n}}, \\
\left\langle \int_{[0,\infty)} f_n^2 d\mathbf{m}, e_k \right\rangle &= -\frac{1}{2^k}\int_{E_k} f_n^2 r_k d\mu = 0, \quad \forall k \neq n, \\
\left\langle \int_{[0,\infty)} f_n f_m d\mathbf{m}, e_k \right\rangle &= -\frac{1}{2^k}\int_{E_k} f_n f_m r_k d\mu = 0, \quad \text{for } n \neq m \text{ and } \forall k.
\end{aligned} \tag{3.2}$$

If we define the functions of the sequence $\{F_n\}_{n=1}^{\infty}$ by $F_n(x) := (2^n/\sqrt{\pi}) f_n(x)$, we get

$$\int_{[0,\infty)} F_n^2 d\mathbf{m} = e_n, \quad \forall n \in \mathbb{N} \qquad \int_{[0,\infty)} F_n F_k d\mathbf{m} = 0, \quad \forall n, k \in \mathbb{N}, n \neq k. \tag{3.3}$$

The starting point of our analysis is the Bessaga-Pelczyński selection principle. It establishes that if $\{x_k\}_{k=1}^{\infty}$ is a basis of the Banach space X and $\{x_k^*\}_{k=1}^{\infty}$ is the sequence of coefficient functionals, if we take a normalized sequence $\{y_n\}_{n=1}^{\infty}$ such that $\lim_n \langle y_n, x_k^* \rangle = 0$, then $\{y_n\}_{n=1}^{\infty}$ admits a basic subsequence that is equivalent to a block basic sequence of $\{x_n\}_{n=1}^{\infty}$ (see for instance Theorem 3 in [20, 21], Ch.V). We adapt this result for sequences of square integrable functions $\{f_n\}_{n=1}^{\infty}$ in order to identify when the sequence of integrals $\{\int_{\Omega} f_n^2 d\mathbf{m}\}_{n=1}^{\infty} \subset \ell^2$ is a basic sequence in ℓ^2. The following result is a direct consequence of the principle mentioned above. Notice that the first requirement in Proposition 3.2 is obviously satisfied in the case of $\mathbf{m}$-orthonormal sequences. The second condition constitutes the key of the problem.

Proposition 3.2. *Let $\mathbf{m} : \Sigma \to \ell^2$ be a vector measure, and consider the canonical basis $\{e_k\}_{k=1}^{\infty}$ of ℓ^2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^2(\mathbf{m})$. If there is an $\varepsilon > 0$ such that the sequence $\{\int_{\Omega} f_n^2 d\mathbf{m}\}_{n=1}^{\infty}$ satisfies*

(1) $\inf_n \| \int_{\Omega} f_n^2 d\mathbf{m} \|_{\ell^2} = \varepsilon > 0$,

(2) $\lim_n \langle \int_{\Omega} f_n^2 d\mathbf{m}, e_k \rangle = 0, \forall k \in \mathbb{N}$,

then $\{\int_{\Omega} f_n^2 d\mathbf{m}\}_{n=1}^{\infty}$ has a subsequence which is a basic sequence. Moreover, it is equivalent to a block basic sequence of $\{e_k\}_{k=1}^{\infty}$.

Let us highlight with an example the geometrical meaning of the requirements above. This is, in a sense, the canonical situation involving disjointness.

Example 3.3 ([5, Ex.10]). Let (Ω, Σ, μ) be a probability measure space. Let us consider the following vector measure $\mathbf{m} : \Sigma \to \ell^2$,

$$\mathbf{m}(A) := \sum_{i=1}^{\infty} \mu(A \cap A_i) e_i \in \ell^2, \quad A \in \Sigma, \tag{3.4}$$

where $\{A_i\}_{i=1}^{\infty}$ is a disjoint measurable partition of Ω, with $\mu(A_i) \neq 0$ for all $i \in \mathbb{N}$. Notice that $\int_\Omega f^2 d\mathbf{m} = \sum_{i=1}^{\infty} (\int_{A_i} f^2 d\mu) e_i \in \ell^2$ for all $f \in L^2(\mathbf{m})$. Consider a sequence of norm one functions $\{f_n\}_{n=1}^{\infty}$ in $L^2(\mathbf{m})$ such that $f_n := f_n \chi_{A_n}$ for all n. For every $k \in \mathbb{N}$, the following equalities hold:

$$\begin{gathered} \lim_n \left\langle \int_\Omega f_n^2 d\mathbf{m}, e_k \right\rangle = \lim_n \left\langle \sum_{i=1}^{\infty} \left(\int_{A_i} f_n^2 d\mu \right) e_i, e_k \right\rangle = \lim_n \int_{A_k} f_n^2 d\mu = 0, \\ \left\| \int_\Omega f_n^2 d\mathbf{m} \right\|_{\ell^2} = 1 \quad \forall n \in \mathbb{N}. \end{gathered} \tag{3.5}$$

Therefore, condition (2.2) of Proposition 3.2 is fulfilled in this example: the role of disjointness is clear.

In what follows, we show that *if the integration operator is compact then there are no strongly* $\mathbf{m}$-orthonormal *sequences*. In particular, this shows that the existence of such sequences is a pure vector measure phenomenon, since the integration map is obviously compact when the measure is scalar. Compactness of the integration map is nowadays well characterized (see [2, Ch.3] and the references therein); it is a strong property, in the sense that it implies that the space $L^1(\mathbf{m})$ is lattice isomorphic to the L^1 space of the variation of $\mathbf{m}$, that is a scalar measure (see [2, Prop.3.48]). We need the next formal requirement for the elements of the sequence $\{f_n\}_{n=1}^{\infty}$. We say that a function $f \in L^2(\mathbf{m})$ is *normed by the integral* if $\|f\|_{L^2(\mathbf{m})} = \| \int_\Omega f^2 \, d\mathbf{m}\|^{1/2}$. This happens for instance when the vector measure $\mathbf{m}$ is positive (see [22] or [2, Lemma 3.13]), since in this case the norm can be computed using the formula $\|f\|_{L^1(\mathbf{m})} = \| \int_\Omega |f| d\mathbf{m}\|$ for all $f \in L^1(\mathbf{m})$. We impose this requirement for the aim of clarity; some of the results could be adapted using a convenient renorming process in order to avoid it.

Remark 3.4. Let $\mathbf{m} : \Sigma \to \ell^2$ be a countably additive vector measure. If there exists a strongly $\mathbf{m}$-orthonormal sequence in $L^2(\mathbf{m})$ which elements are normed by the integrals, then the integration operator $I_\mathbf{m} : L^1(\mathbf{m}) \to \ell^2$ is not compact. To see this, let $\{f_i\}_{i=1}^{\infty} \subset L^2(\mathbf{m})$ be a strongly $\mathbf{m}$-orthonormal sequence in $L^2(\mathbf{m})$ and consider a orthonormal sequence $\{e_i\}_{i=1}^{\infty}$. Then $\int_\Omega f_i f_j \, d\mathbf{m} = \delta_{i,j} e_i$, an thus

$$\int_\Omega f_i^2 d\mathbf{m} = e_i = I_\mathbf{m}\left(f_i^2\right). \tag{3.6}$$

Therefore $\{f_i^2\}_{i=1}^{\infty} \subset B_{L^1(\mathbf{m})}$, and so the sequence $\{I_\mathbf{m}(f_i^2)\}_{i=1}^{\infty}$ that satisfies that $\{I_\mathbf{m}(f_i^2)\}_{i=1}^{\infty} \subset I_\mathbf{m}(B_{L^1(\mathbf{m})}) \subset \overline{I_\mathbf{m}(B_{L^1(\mathbf{m})})}$ does not admit any convergent subsequence. It follows that $\overline{I_\mathbf{m}(B_{L^1(\mathbf{m})})}$

is not compact and so, $I_{\mathbf{m}}(B_{L^1(\mathbf{m})})$ is not relatively compact. This allows to conclude that $I_{\mathbf{m}}$ is not compact.

Theorem 3.7 below gives a necessary condition—and, in a sense, also a sufficient condition—for the existence of strongly orthonormal sequences in a space of functions $L^2(\mathbf{m})$ starting from a given $\mathbf{m}$-orthonormal sequence. The existence of such $\mathbf{m}$-orthonormal sequences is always assured: just consider a sequence of normalized disjoint functions in $L^2(\mathbf{m})$. The following result is an application of the Kadec-Pelczyński disjointification procedure for order continuous Banach function spaces—also called the Kadec-Pelczyński dichotomy, see Theorem 4.1 in [23, 24]—, in the following version, that can be found in [25] (see the comments after Proposition 1.1). *Let $X(\mu)$ be an order continuous Banach function space over a finite measure μ with a weak unit (this implies $X(\mu) \hookrightarrow L^1(\mu)$). Consider a normalized sequence $\{x_n\}_{n=1}^{\infty}$ in $X(\mu)$. Then*

(1) *either $\{\|x_n\|_{L^1(\mu)}\}_{n=1}^{\infty}$ is bounded away from zero,*

(2) *or there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and a disjoint sequence $\{z_k\}_{k=1}^{\infty}$ in $X(\mu)$ such that $\|z_k - x_{n_k}\| \to_k 0$.*

Recall that the space $L^2(\mathbf{m})$ is an order continuous Banach function space over any Rybakov (finite) measure $\mu = |\langle \mathbf{m}, x_0' \rangle|$ for $\mathbf{m}$.

Proposition 3.5. *Let $\{g_n\}_{n=1}^{\infty}$ be a normalized sequence in $L^2(\mathbf{m})$. Suppose that there exists a Rybakov measure $\mu = |\langle \mathbf{m}, x_0' \rangle|$ for $\mathbf{m}$ such that $\{\|g_n\|_{L^1(\mu)}\}_{n=1}^{\infty}$ is not bounded away from zero. Then there are a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ and an $\mathbf{m}$-orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ such that $\|g_{n_k} - f_k\|_{L^2(\mathbf{m})} \to_k 0$.*

Proof. By the criterion given above, there is a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ and a disjoint sequence $\{f'_k\}_{k=1}^{\infty}$ such that $\|g_{n_k} - f'_k\|_{L^2(\mathbf{m})} \to_k 0$. Consider the sequence given by the functions $f_k := f'_k / \|f'_k\|$. Then $\|g_{n_k} - f_k\|_{L^2(\mathbf{m})} \to_k 0$. Since $\int_\Omega f_k f_j d\mathbf{m} = 0$ for every $k \neq j$ due to the fact that they are disjoint, we obtain the result. □

Although the existence of a strongly $\mathbf{m}$-orthonormal subsequence of an $\mathbf{m}$-orthogonal sequence cannot be assured in general, we show in what follows that under the adequate requirements it is possible to find a vector measure $\mathbf{m}^*$ satisfying that $L^2(\mathbf{m}) = L^2(\mathbf{m}^*)$ and with respect to which there is a subsequence that is strongly $\mathbf{m}^*$-orthonormal. We use the following lemma, which proof is elementary (see Lemma 3.27 in [2]).

Lemma 3.6. *Let $\mathbf{m} : \Sigma \to \ell^2$ be a vector measure. Let $\varphi : \ell^2 \to H$ be an isomorphism, where H is a separable Hilbert space, and consider the vector measure $\mathbf{m}^* = \varphi \circ \mathbf{m}$. Then the spaces $L^2(\mathbf{m})$ and $L^2(\mathbf{m}^*)$ are isomorphic, and for every $f \in L^2(\mathbf{m})$, $\int_\Omega f^2 d\mathbf{m}^* = \varphi(\int_\Omega f^2 d\mathbf{m})$.*

Theorem 3.7. *Let us consider a vector measure $\mathbf{m} : \Sigma \to \ell^2$ and an $\mathbf{m}$-orthonormal sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $L^2(\mathbf{m})$ that are normed by the integrals. Let $\{e_n\}_{n=1}^{\infty}$ be the canonical basis of ℓ^2. If $\lim_n \langle \int_\Omega f_n^2 d\mathbf{m}, e_k \rangle = 0$ for every $k \in \mathbb{N}$, then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ and a vector measure $\mathbf{m}^* : \Sigma \to \ell^2$ such that $\{f_{n_k}\}_{k=1}^{\infty}$ is strongly $\mathbf{m}^*$-orthonormal.*

Moreover, $\mathbf{m}^*$ can be chosen to be as $\mathbf{m}^* = \phi \circ \mathbf{m}$ for some Banach space isomorphism ϕ from ℓ^2 onto ℓ^2, and so $L^2(\mathbf{m}) = L^2(\mathbf{m}^*)$.

Proof. Consider an $\mathbf{m}$-orthonormal sequence $\{f_n\}_{n=1}^{\infty}$ in $L^2(\mathbf{m})$ and the sequence of integrals $\{\int_\Omega f_n^2 d\mathbf{m}\}_{n=1}^{\infty}$. As an application of Proposition 3.2, we get a subsequence $\{\int_\Omega f_{n_k}^2 d\mathbf{m}\}_{k=1}^{\infty}$

that is equivalent to a block basic sequence $\{e'_{n_k}\}_{k=1}^{\infty}$ of the canonical basis of ℓ^2. Recall that according to the notation given in Section 2, $\alpha_{i,j}$ are the constants that appear in the definition of the block basic sequence. Associated to this sequence, there is an isomorphism φ

$$A := \overline{\operatorname{span}(e'_{n_k})}^{\ell^2} \xrightarrow{\varphi} B := \overline{\operatorname{span}\left(\int_{\Omega} f_{n_k}^2 d\mathbf{m}\right)}^{\ell^2} \tag{3.7}$$

such that $\varphi(e'_{n_k}) := \int_{\Omega} f_{n_k}^2 d\mathbf{m}$, $k \in \mathbb{N}$.

We can suppose without loss of generality that the elements of the sequence $\{e'_{n_k}\}_{k=1}^{\infty}$ have norm one. To see this, it is enough to consider the following inequalities. First note that there are positive constants Q and K such that for every $n \in \mathbb{N}$, $Q = Q\| \int_{\Omega} f_{n_k}^2 d\mathbf{m}\| \le \|e'_{n_k}\| \le K\| \int_{\Omega} f_{n_k}^2 d\mathbf{m}\| = K$ as a consequence of the existence of the isomorphism φ. Let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of real numbers. Then

$$\left\| \sum_{i=1}^{\infty} \lambda_i \frac{e'_i}{\|e'_i\|} \right\|_2^2 = \sum_{i=1}^{\infty} \frac{|\lambda_i|^2 \left(\sum_{j=p_i}^{q_i} |\alpha_{i,j}|^2 \right)}{\|e'_i\|^2} = \sum_{i=1}^{\infty} \sum_{j=p_i}^{q_i} \frac{|\lambda_i|^2 |\alpha_{i,j}|^2}{\|e'_i\|^2}. \tag{3.8}$$

The existence of an upper and a lower bound for the real numbers $\|e'_i\|$ given above provides the equivalence between this quantity and $\| \sum_{i=1}^{\infty} \lambda_i e'_i \|_2^2$ for each sequence of real numbers $\{\lambda_i\}_{i=1}^{\infty}$.

Since each closed subspace of a Hilbert space is complemented, there is a subspace B^c such that $\ell^2 = B \oplus_2 B^c$ isometrically, where this direct sum space is considered as a Hilbert space (with the adequate Hilbert space norm). We write P_B and P_{B^c} for the corresponding projections. Let us consider the linear map $\phi := \varphi^{-1} \oplus \mathrm{Id} : B \oplus_2 B^c \xrightarrow{\phi} A \oplus_2 B^c$, where $\mathrm{Id} : B^c \to B^c$ is the identity map.

Note that $H := A \oplus_2 B^c$ is a Hilbert space with the scalar product

$$\langle x + y, x' + y' \rangle_H = \langle x, x' \rangle_H + \langle y, y' \rangle_H, \qquad x + y, x' + y' \in A \oplus_2 B^c, \tag{3.9}$$

that can be identified with ℓ^2. Obviously, ϕ is an isomorphism. Let us consider now the vector measure $\mathbf{m}^* := \phi \circ \mathbf{m} : \Sigma \xrightarrow{\mathbf{m}} \ell^2 \xrightarrow{\phi} A \oplus_2 B^c$. By Lemma 3.6, $L^2(\mathbf{m}) = L^2(\phi \circ \mathbf{m}) = L^2(\mathbf{m}^*)$. Let us show that $\{f_{n_k}\}_{k=1}^{\infty}$ is a strongly $\mathbf{m}^*$-orthonormal sequence. We consider the orthonormal sequence $\{(e'_{n_k}, 0)\}_{k=1}^{\infty}$ in H. The first condition in the definition of strongly orthonormal sequence is fulfilled, since

$$\begin{aligned} \int_{\Omega} f_{n_k}^2 d\mathbf{m}^* &= \int_{\Omega} f_{n_k}^2 d(\phi \circ \mathbf{m}) = \phi\left(P_B\left(\int_{\Omega} f_{n_k}^2 d\mathbf{m} \right), P_{B^c}\left(\int_{\Omega} f_{n_k}^2 d\mathbf{m} \right) \right) \\ &= \left(\varphi^{-1}\left(\int_{\Omega} f_{n_k}^2 d\mathbf{m} \right), 0 \right) = (e'_{n_k}, 0), \end{aligned} \tag{3.10}$$

for every $k \in \mathbb{N}$. The second one is given by the following calculations. For $k \neq l$,

$$\left\| \int_{\Omega} f_{n_k} f_{n_l} d\mathbf{m}^* \right\| = \left\| \int_{\Omega} f_{n_k} f_{n_l} d(\phi \circ \mathbf{m}) \right\| = \left\| \phi\left(\int_{\Omega} f_{n_k} f_{n_l} d\mathbf{m} \right) \right\| = \|\phi(0)\| = 0, \tag{3.11}$$

since ϕ is continuous and $\{f_{n_k}\}_{k=1}^{\infty}$ is an $\mathbf{m}$-orthonormal sequence. Thus we get $\int_\Omega f_{n_k} f_{n_l} d\mathbf{m}^* = 0$. This proves the theorem. □

Remark 3.8. In a certain sense, the converse of Theorem 3.7 also holds. Take as the vector measure $\mathbf{m}^*$ the measure $\mathbf{m}$ itself with values in ℓ^2 and consider the canonical basis $\{e_n\}_{n=1}^{\infty}$. Clearly, every strongly $\mathbf{m}$-orthonormal sequence is $\mathbf{m}$-orthonormal and satisfies the condition $\lim_n \langle \int_\Omega f_n^2 d\mathbf{m}^*, e_k\rangle = 0$, since

$$\left\langle \int_\Omega f_n^2 d\mathbf{m}^*, e_k \right\rangle = \langle e_n, e_k\rangle = 0, \quad k \neq n. \tag{3.12}$$

Corollary 3.9. *Let $\mathbf{m} : \Sigma \to \ell^2$ be a countably additive vector measure. Let $\{g_n\}_{n=1}^{\infty}$ be a normalized sequence of functions in $L^2(\mathbf{m})$ that are normed by the integrals. Suppose that there exists a Rybakov measure $\mu = |\langle \mathbf{m}, x_0'\rangle|$ for $\mathbf{m}$ such that $\{\|g_n\|_{L^1(\mu)}\}_{n=1}^{\infty}$ is not bounded away from zero. If $\lim_n \langle \int_\Omega g_n^2 d\mathbf{m}, e_k\rangle = 0$ for every $k \in \mathbb{N}$, then there is a (disjoint) sequence $\{f_k\}_{k=1}^{\infty}$ such that*

(1) *$\lim_k \|g_{n_k} - f_k\|_{L^2(\mathbf{m})} = 0$ for a given subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$, and*

(2) *it is strongly $\mathbf{m}^*$-orthonormal for a certain Hilbert space valued vector measure $\mathbf{m}^*$ defined as in Theorem 3.7 that satisfies that $L^2(\mathbf{m}) = L^2(\mathbf{m}^*)$.*

This is a direct consequence of Proposition 3.5 and Theorem 3.7. For the proof, just take into account the continuity of the integration map and the fact that the elements of the sequence $\{g_n\}_{n=1}^{\infty}$ are normed by the integrals.

4. Applications: Copies of ℓ^2 in $L^1(\mathbf{m})$ That Are Preserved by the Integration Map

One of the consequences of the results of the previous section is that the existence of strongly $\mathbf{m}$-orthonormal sequences in $L^2(\mathbf{m})$ is closely related to the existence of lattice copies of ℓ^2 in $L^1(\mathbf{m})$. In this section, we show how to apply our arguments for finding some information on the structure of the spaces $L^1(\mathbf{m})$ and the properties of the associated integration map.

Our motivation has its roots in the general problem of finding subspaces of Banach function spaces that are isomorphic to ℓ^2. It is well known that in general these copies are related to weakly null normalized sequences; the arguments that prove this relation go back to the Kadec-Pelczyński dichotomy and have been applied largely in the study of strictly singular embeddings between Banach function spaces [25, 26]. In some relevant classes of Banach function spaces—L_p-spaces, Lorentz spaces, Orlicz spaces, and general rearrangement invariant (r.i.) spaces—these copies are related to subspaces generated by Rademacher-type sequences (see [27–30] and the references therein). For instance, Corollary 2 in [27] states that for a r.i. Banach function space E on $[0,1]$, if the norms on E and L^1 are equivalent on some infinite dimensional subspace of E, then the Rademacher functions span a copy of ℓ^2 in E. However, our construction generates copies of ℓ^2 that are essentially different. Actually, they are defined by positive or even disjoint functions, and so the copies of ℓ^2 that our results produce allow to conclude that if there is a normalized sequence of positive functions with a weakly null sequence of integrals, the integration map is neither disjointly strictly singular nor ℓ^2-singular.

On the other hand, it is well known that strongly orthonormal sequences—that are called λ-orthonormal systems in Definition 2 of [5]—define isometric copies of ℓ^4 in spaces

$L^2(\mathbf{m})$ of a positive vector measure $\mathbf{m}$ (see Propositions 8 and 11 in [5]). In particular, this makes clear that the existence of these sequences imply that $L^2(\mathbf{m})$ is not a Hilbert space, and so $L^1(\mathbf{m})$ is not an L^1-space. However, there is a big class of Banach function spaces that can be represented as $L^1(\mathbf{m})$ of an ℓ^2-valued positive vector measure $\mathbf{m}$ (see for instance Example 10 in [5] or Example 8 in [4]). The $L^1(\mathbf{m})$-spaces associated to such vector measures are sometimes called ℓ-sums of L^1-spaces. In Section 4 of [31], a first attempt to study 2-convex subspaces—the natural extension of ℓ^2-copies in this setting—of ℓ-sums of L^1-spaces was made. Also, a first analysis of the question of when $L^1(\mathbf{m})$ is a Hilbert space—based on the behavior of specific sequences too—was made in [11, Section 4]. In what follows, we provide more information on the existence of copies of ℓ^2 in spaces $L^1(\mathbf{m})$ of a positive vector measure, and the closely related problem of the existence of ℓ^4 in $L^2(\mathbf{m})$. After that, some contributions to the analysis to the study of strictly singular integration maps are given. Recently, a new considerable effort has been made in order to find the links between the belonging of the integration map $I_{\mathbf{m}}$ to a particular class of operators and the structure properties of the space $L^1(\mathbf{m})$. For integration maps belonging to relevant operator ideals, this has been done in [16, 17, 32] (see also [2, Ch.5] and the references therein). For geometric and order properties of the integration map—mainly concavity and positive p-summing type properties—, we refer to [33, 34] and [2, Ch.6].

For the aim of clarity, in this section we deal with positive vector measures, that—as we said in the previous section—satisfy that all the elements of the spaces $L^2(\mathbf{m})$ are normed by the integrals. In this case, it can be shown that there is an easy characterization of strongly $\mathbf{m}$-orthonormal sequences, which simplifies the arguments.

Remark 4.1. Suppose that a vector measure satisfies that the set $P := \{x' \in \ell^2 : \langle \mathbf{m}, x' \rangle \geq 0\}$ separates the points of ℓ^2 and assume that for a given sequence $\{f_n\}_{n=1}^\infty$, $\langle \int_\Omega f_n^2 d\mathbf{m}, x' \rangle \cdot \langle \int_\Omega f_k^2 d\mathbf{m}, x' \rangle = 0$ for every $n, k \in \mathbb{N}$ such that $n \neq k$ and $x' \in P$. Then $\int_\Omega f_n f_k d\mathbf{m} = 0$ for every $n \neq k$. This is a direct consequence of Hölder's inequality and the integrability with respect to $\mathbf{m}$ of all the functions involved. For the particular case of *positive* vector measures, the standard basis $\{e_n\}_{n=1}^\infty$ of ℓ^2 plays the role of P; this means that the requirement $\int_\Omega f_n^2 d\mathbf{m} = e_n$ for all n automatically implies that $\{f_n\}_{n=1}^\infty$ is a strongly $\mathbf{m}$-orthonormal sequence.

Lemma 4.2. *Let $\mathbf{m} : \Sigma \to \ell^2$ be a positive vector measure, and suppose that the bounded sequence $\{g_n\}_{n=1}^\infty$ in $L^2(\mathbf{m})$ satisfies that $\lim_n \langle \int_\Omega g_n^2 d\mathbf{m}, e_k \rangle = 0$ for all $k \in \mathbb{N}$. Then there is a Rybakov measure μ for $\mathbf{m}$ such that $\lim_n \|g_n\|_{L^1(\mu)} = 0$.*

Proof. Take for instance the sequence $x_0' = \{(1/2)^{n/2}\}_{n=1}^\infty \in \ell^2$. Since $\mathbf{m}$ is positive, the measure $\mu := \langle \mathbf{m}, x_0' \rangle$ is positive and defines a Rybakov measure for $\mathbf{m}$. Since $\langle \int_\Omega g_n^2 d\mathbf{m}, x_0' \rangle = \|g_n\|_{L^2(\mu)}^2$ for all $n \in \mathbb{N}$ and the requirement on $\{\int_\Omega g_n^2 d\mathbf{m}\}_{n=1}^\infty$ imply that it is weakly null, we obtain by Hölder's inequality that $\lim_n \|g_n\|_{L^1(\mu)} \leq \lim_n \|g_n\|_{L^2(\mu)} \|\mathbf{m}\|^{1/2} = 0$. □

Proposition 4.3. *Let $\mathbf{m} : \Sigma \to \ell^2$ be a positive (countably additive) vector measure. Let $\{g_n\}_{n=1}^\infty$ be a normalized sequence in $L^2(\mathbf{m})$ such that for every $k \in \mathbb{N}$, $\lim_n \langle \int_\Omega g_n^2 d\mathbf{m}, e_k \rangle = 0$. Then $L^2(\mathbf{m})$ contains a lattice copy of ℓ^4. In particular, there is a subsequence $\{g_{n_k}\}_{k=1}^\infty$ of $\{g_n\}_{n=1}^\infty$ that is equivalent to the unit vector basis of ℓ^4.*

Proof. By Lemma 4.2, we can use Corollary 3.9 to produce a disjoint sequence $\{f_k\}_{k=1}^\infty$ in $L^2(\mathbf{m})$ that approximates a subsequence $\{g_{n_k}\}_{k=1}^\infty$ of $\{g_n\}_{n=1}^\infty$ and is strongly $\mathbf{m}^*$-orthogonal. The same computations that can be found in the proof of Proposition 8 in [5] show that for finite sums $\sum_{k=1}^N \alpha_k f_k$, the norm in $L^2(\mathbf{m})$ is equivalent to the norm of $\{\alpha_k\}_{k=1}^N$ in ℓ^4. Consequently, the closure of these finite sums in $L^2(\mathbf{m})$ provides a copy of ℓ^4. The disjointness of $\{f_k\}_{k=1}^{\infty}$ implies that in fact it is a lattice copy. Note also that $\{f_k\}_{k=1}^{\infty}$ is equivalent to $\{g_{n_k}\}_{k=1}^{\infty}$ and so to the unit vector basis of ℓ^4. □

As a direct consequence, we obtain that for a positive vector measure $\mathbf{m}$, the existence of a normalized sequence of functions such that the sequence of square integrals is weakly null implies that $L^2(\mathbf{m})$ cannot be a Hilbert space. On the other hand, if the integration map is compact, then $L^1(\mathbf{m}) = L^1(|\mathbf{m}|)$ isomorphically (see Proposition 3.48 in [2]), and thus $L^2(\mathbf{m})$ is (isomorphic to) a Hilbert space. Notice first the following obvious consequence of this fact: *$L^1(\mathbf{m})$ is isomorphic to an $L^1(\mu)$-space of a finite measure μ if and only if there is a positive ℓ^2-valued vector vector measure $\mathbf{m}_0$ such that $L^1(\mathbf{m}) = L^1(\mathbf{m}_0)$ such that the integration map is compact*; the converse statement is proved by considering the vector measure $n(A) := \mu(A)e_1 \in \ell^2$, $A \in \Sigma$. However, as the next example shows, there are spaces $L^2(\mathbf{m})$ for positive ℓ^2-valued vector measures with noncompact integration map that are Hilbert spaces. We will find in Corollary 4.6 that this conclusion—$L^1(\mathbf{m})$ not being an L^1-space, and so $L^2(\mathbf{m})$ not to be a Hilbert space—can be extended to the case of strictly singular integration maps.

Example 4.4. (1) *An ℓ^2-valued measure such that $L^1(\mathbf{m})$ is a Hilbert space and the integration map is not compact.* Consider the Hilbert space $L^2[0,1]$ and a orthonormal basis S for it. Consider the associated isomorphism $\phi_S : L^2[0,1] \to \ell^2$ that carries each function to the 2-summable sequence of its fourier coefficients. Take the vector measure $m_S : \Sigma \to \ell^2$ given by $m_S(A) := \phi_S(\chi_A)$ for each Lebesgue measurable set $A \in \Sigma$. Then $L^1(m_S) = L^2[0,1]$, although I_{m_S} is in fact an isomorphism.

(2) *A positive ℓ^2-valued measure with noncompact integration map such that $L^1(\mathbf{m})$ is a Hilbert space.* Consider a vector measure $\mathbf{m}$ as in Example 3.3 and define the positive measure $n : \Sigma \to \ell^2$ by $n(A) := \mu(A)e_1 + \mathbf{m}(A)$, $A \in \Sigma$. A direct computation shows that the norm in $L^1(n)$ is equivalent to the one in $L^1(\mu)$. Then $L^2(n) = L^2(\mu)$ isomorphically, and $I_n : \Sigma \to \ell^2$ is clearly noncompact.

Next result shows the consequences on the structure of $L^1(\mathbf{m})$ of our arguments about the existence of strongly orthonormal sequences in $L^2(\mathbf{m})$.

Theorem 4.5. *Let $\mathbf{m} : \Sigma \to \ell^2$ be a positive (countably additive) vector measure. Let $\{g_n\}_{n=1}^{\infty}$ be a normalized sequence in $L^2(\mathbf{m})$ such that for every $k \in \mathbb{N}$, $\lim_n \langle \int_\Omega g_n^2 d\mathbf{m}, e_k \rangle = 0$ for all $k \in \mathbb{N}$. Then there is a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ such that $\{g_{n_k}^2\}_{k=1}^{\infty}$ generates an isomorphic copy of ℓ^2 in $L^1(\mathbf{m})$ that is preserved by the integration map. Moreover, there is a normalized disjoint sequence $\{f_k\}_{k=1}^{\infty}$ that is equivalent to the previous one and $\{f_k^2\}_{k=1}^{\infty}$ gives a lattice copy of ℓ^2 in $L^1(\mathbf{m})$ that is preserved by $I_{\mathbf{m}^*}$.*

Proof. By Corollary 3.9 and Lemma 4.2, there is a (normalized) disjoint sequence $\{f_k\}_{k=1}^{\infty}$ in $L^2(\mathbf{m})$ that is equivalent to a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$. Let us prove directly that $\{g_{n_k}^2\}_{k=1}^{\infty}$ generates an isomorphic copy of ℓ^2 in $L^1(\mathbf{m})$. Let $\mathbf{m}^* = \phi \circ \mathbf{m}$ the vector measure given in Corollary 3.9 and let K be the norm of ϕ^{-1}. Since $\lim_k \|g_{n_k} - f_k\|_{L^2(\mathbf{m})} = 0$ for every $\varepsilon > 0$ there is a subsequence of the one above (that we denote as the previous one) that satisfies that

$$\left(\sum_{k=1}^{n} \left\| \int_\Omega |g_{n_k} - f_k|^2 d\mathbf{m} \right\| \right)^{1/2} < \frac{\varepsilon}{2}. \tag{4.1}$$

Fix an $\varepsilon > 0$. We have that, by Hölder inequality,

$$\begin{aligned}\left(\sum_{k=1}^{n}\left\|\int_{\Omega}\left|g_{n_k}^2 - f_k^2\right|d\mathbf{m}\right\|_{\ell^2}^2\right)^{1/2} &= \left(\sum_{k=1}^{n}\left\|\int_{\Omega}\left|g_{n_k} - f_k\right|\cdot\left|g_{n_k} + f_k\right|d\mathbf{m}\right\|_{\ell^2}^2\right)^{1/2} \\ &\leq \left(\sum_{k=1}^{n}\left\|\int_{\Omega}\left|g_{n_k} - f_k\right|^2 d\mathbf{m}\right\|\cdot\left\|\int_{\Omega}\left|g_{n_k} + f_k\right|^2 d\mathbf{m}\right\|\right)^{1/2} \\ &\leq 2\left(\sum_{k=1}^{n}\left\|\int_{\Omega}\left|g_{n_k} - f_k\right|^2 d\mathbf{m}\right\|\right)^{1/2} < \varepsilon.\end{aligned} \tag{4.2}$$

This means that

$$\begin{aligned}\left\|\int_{\Omega}\left|\sum_{k=1}^{n}\alpha_k g_{n_k}^2\right|d\mathbf{m}\right\|_{\ell^2} &\leq \left\|\int_{\Omega}\left|\sum_{k=1}^{n}\alpha_k\left(g_{n_k}^2 - f_k^2\right)\right|d\mathbf{m}\right\|_{\ell^2} + \left\|\int_{\Omega}\sum_{k=1}^{n}|\alpha_k| f_k^2 d\mathbf{m}\right\|_{\ell^2} \\ &\leq \left(\sum_{k=1}^{n}\alpha_k^2\right)^{1/2}\varepsilon + K\left\|\int_{\Omega}\sum_{k=1}^{n}|\alpha_k| f_k^2 d\mathbf{m}^*\right\|_{\ell^2} \leq (\varepsilon + K)\left(\sum_{k=1}^{n}\alpha_k^2\right)^{1/2}.\end{aligned} \tag{4.3}$$

Similar computations give the converse inequality. The construction of $\mathbf{m}^*$ and the disjointness of the functions of the sequence $\{f_k^2\}_{k=1}^{\infty}$ give the last statement. □

Corollary 4.6. *Let $\mathbf{m} : \Sigma \to \ell^2$ be a positive (countably additive) vector measure. The following assertions are equivalent.*

(1) *There is a normalized sequence in $L^2(\mathbf{m})$ satisfying that $\lim_n\langle\int_{\Omega} g_n^2 d\mathbf{m}, e_k\rangle = 0$ for all the elements of the canonical basis $\{e_k\}_{k=1}^{\infty}$ of ℓ^2.*

(2) *There is an ℓ^2-valued vector measure $\mathbf{m}^* = \phi\circ\mathbf{m}$—$\phi$ an isomorphism—such that $L^2(\mathbf{m}) = L^2(\mathbf{m}^*)$ and there is a disjoint sequence in $L^2(\mathbf{m})$ that is strongly $\mathbf{m}^*$-orthonormal.*

(3) *There is a subspace $S \subseteq L^1(\mathbf{m})$ that is fixed by the integration map $I_{\mathbf{m}}$ which satisfies that there are positive functions $h_n \in S$ such that $\{\int_{\Omega} h_n d\mathbf{m}\}_{n=1}^{\infty}$ is an orthonormal basis for $I_{\mathbf{m}}(S)$.*

(4) *There is an ℓ^2-valued vector measure $\mathbf{m}^*$ defined as $\mathbf{m}^* = \phi\circ\mathbf{m}$—$\phi$ an isomorphism—such that $L^1(\mathbf{m}) = L^1(\mathbf{m}^*)$ and a subspace S of $L^1(\mathbf{m})$ such that the restriction of $I_{\mathbf{m}^*}$ to S is a lattice isomorphism in ℓ^2.*

Proof. (1) ⇒ (2) is a direct consequence of Theorem 4.5. For (2) ⇒ (3), just notice that the strong $\mathbf{m}^*$-orthogonality of a disjoint sequence $\{g_n\}_{n=1}^{\infty}$ implies that $\{g_n^2\}_{n=1}^{\infty}$ gives a lattice copy of ℓ^2 preserved by the integration map $I_{\mathbf{m}^*}$. Since $\phi^{-1}\circ\mathbf{m}^* = \mathbf{m}$, we obtain that $\{\phi^{-1}(\int_{\Omega} g_n^2 d\mathbf{m}^*)\}_{n=1}^{\infty} = \{\int_{\Omega} g_n^2 d\mathbf{m}\}_{n=1}^{\infty}$ is a basis for ℓ^2.

(3) ⇒ (1). There is a bounded sequence $\{h_n\}_{=1}^{\infty}$ in $L^1(\mathbf{m})$ such that $\int_{\Omega} h_n d\mathbf{m} = a_n$, where a_n is a orthonormal basis of closure of the subspace $A := \operatorname{span}\{\int_{\Omega} h_n d\mathbf{m} : n \in \mathbb{N}\}$, and an isomorphism ϕ from A to ℓ^2 such that $\phi(\int_{\Omega} h_n d\mathbf{m}) = e_n$. By composing with ϕ the integration map, the copy of ℓ^2 that is fixed by $I_{\mathbf{m}}$ can be considered in such a way that $\phi(\int_{\Omega} g_n^2 d\mathbf{m}) = e_n$. Consequently, $\lim_n\langle\phi(\int_{\Omega} h_n d\mathbf{m}), e_k\rangle = 0$ for all $k \in \mathbb{N}$, and so $\lim_n\langle e_k, \int_{\Omega} h_n d\mathbf{m}\rangle = 0$ for all $k \in \mathbb{N}$. It is enough to take $g_n = h_n^{1/2}$.

(3) ⇒ (4) is obvious.

(4) $\Rightarrow$ (1). Take the normalized sequence of positive functions $\{h_n\}_{n=1}^{\infty}$ in S such that $\{\int_{\Omega} h_n d\phi \circ \mathbf{m}\}_{n=1}^{\infty}$ is equivalent to the standard basis of ℓ^2, and define $g_n := h_n^{1/2}$. The weak to weak continuity of ϕ gives the result. □

We have shown that the existence of lattice copies of ℓ^4 in $L^2(\mathbf{m})$ is directly connected with the existence of lattice copies of ℓ^2 in $L^1(\mathbf{m})$. Thus, and summarizing the results in this section, we finish the paper with a complete characterization of this property for $L^1(\mathbf{m})$ of a positive ℓ^2-valued vector measure $\mathbf{m}$.

Theorem 4.7. *The following assertions for a positive vector measure $\mathbf{m} : \Sigma \to \ell^2$ are equivalent.*

(1) $L^1(\mathbf{m})$ *contains a lattice copy of ℓ^2.*

(2) $L^1(\mathbf{m})$ *has a reflexive infinite dimensional sublattice.*

(3) $L^1(\mathbf{m})$ *has a relatively weakly compact normalized sequence of disjoint functions.*

(4) $L^1(\mathbf{m})$ *contains a weakly null normalized sequence.*

(5) *There is a vector measure $\mathbf{m}^*$ defined by $\mathbf{m}^* = \phi \circ \mathbf{m}$ such that integration map $I_{\mathbf{m}^*}$ fixes a copy of ℓ^2.*

(6) *There is a vector measure $\mathbf{m}^*$ defined as $\mathbf{m}^* = \phi \circ \mathbf{m}$ that is not disjointly strictly singular.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. For (3) $\Rightarrow$ (4), just take into account that disjoint normalized sequences in weakly compact sets of Banach lattices are weakly null (see for instance the proof of Proposition 3.6.7 in [19]).

(4) $\Rightarrow$ (5). Take a weakly null normalized sequence $\{g_j\}_{j=1}^{n}$ in $L^1(\mathbf{m})$. By (the arguments used in) Lemma 4.2 we can find a Rybakov measure μ for $\mathbf{m}$ such that $\|g_j\|_{L^1(\mu)} \to 0$. Now we use the same arguments that lead to Theorem 3.7 and Corollary 3.9; by the Kadec-Pelczyński dichotomy, there exists a subsequence $\{g_{j_l}\}_{l=1}^{\infty}$ of $\{g_j\}_{j=1}^{\infty}$ and a disjoint sequence $\{z_l\}_{l=1}^{\infty}$ in the unit sphere of $L^1(\mathbf{m})$ such that $\lim_l \|g_{j_l} - z_l\| = 0$. Notice that $\{z_l\}_{l=1}^{\infty}$ also converges weakly to 0, so by taking a subsequence and after restricting the supports of the functions z_l and renorming if necessary, we obtain a normalized weakly null positive disjoint sequence $\{v_k\}_{k=1}^{\infty}$. This gives the copy of ℓ^2 that is fixed by the integration map associated to a vector measure $\mathbf{m}^* = \phi \circ \mathbf{m}$ satisfying $L^1(\mathbf{m}) = L^1(\mathbf{m}^*)$ by $I_{\mathbf{m}}(v_k) = e_k$ for the canonical basis $\{e_k\}_{k=1}^{\infty}$ of ℓ^2. Finally, (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1) are evident. □

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