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Strong mixing measures and invariant sets in linear dynamics

TESI DOCTORAL REALITZADA PER:

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CERTIFICA:

que la presente memoria ‘‘Strong mixing measures and invariant sets in linear dynamics’’ ha sido realizada bajo mi dirección por Marina Murillo Arcila y constituye su tesis para optar al grado de Doctor en Ciencias Matemáticas, con mención ‘‘Doctor Internacional’’.

Y para que así conste en cumplimiento de la legislación vigente presentamos y apadrinamos ante la Escuela de Doctorado de la Universitat Politècnica de València la referida tesis firmando el presente certificado.

Valencia, Diciembre de 2014

El director:

Alfred Peris Manguillot

A mis padres

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Resumen

La presente memoria "Medidas fuertemente mezclantes y subconjuntos invariantes en dinámica lineal" se estructura en tres partes. En el Capítulo 0 se introducen la notación, definiciones y resultados básicos que se necesitarán a lo largo de la tesis. La primera parte consta de dos capítulos, los Capítulos 1 y 2, donde estudiamos la relación entre el Criterio de Hiperciclicidad Frecuente y la existencia de medidas de probabilidad borelianas fuertemente mezclantes. La segunda parte la compone el Capítulo 3, donde centramos nuestra atención en el estudio de la hiperciclicidad frecuente de los C_0-semigrupos de traslación. En la última parte, consistente en los Capítulos 4 y 5, estudiamos propiedades dinámicas que satisfacen los sistemas dinámicos lineales autónomos y no autónomos sobre ciertos subconjuntos invariantes. A continuación proporcionamos una breve descripción de cada capítulo:

En el Capítulo 1, construimos medidas de probabilidad borelianas fuertemente mezclantes y T-invariantes con soporte total para operadores definidos en F- espacios que satisfacen el Criterio de Hiperciclicidad Frecuente. Además, proporcionamos ejemplos de operadores que verifican este criterio y mejoramos el resultado en el caso de operadores backward shifts unilaterales y caóticos con la obtención de medidas exactas. Los contenidos de este capítulo han sido publicados en [88] y [12].

En el Capítulo 2, demostramos que el Criterio de Hiperciclicidad Frecuente para C_0-semigrupos, obtenido por Mangino y Peris en [82], asegura la existencia de medidas invariantes fuertemente mezclantes con soporte total. Proporcionaremos diversos ejemplos que ilustran este resultado y que varían desde el modelo de nacimiento y muerte hasta la ecuación de Black-Scholes. Todos los resultados de este capítulo han sido publicados en [86].

En el Capítulo 3, centramos nuestra atención en uno de los C_0-semigrupos más importantes, el semigrupo traslación. Inspirados en el trabajo de Bayart y Ruzsa [22] que caracteriza la hiperciclicidad frecuente de los operadores backward shifts con pesos, caracterizamos los semigrupos traslación frecuentemente hipercíclicos en los espacios $C_0^\rho(\mathbb{R})$ y $L_p^\rho(\mathbb{R})$. En primer lugar, repasamos los resultados ya existentes sobre la dinámica de los semigrupos traslación. A continuación, obtenemos una caracterización de la hiperciclicidad frecuente para operadores pseudo-shifts con pesos en función de los mismos, que se usará más tarde para caracterizar el C_0-semigrupo traslación en $C_0^\rho(\mathbb{R})$. Finalmente, estudiamos el caso de $L_p^\rho(\mathbb{R})$. También estableceremos una analogía entre el estudio de la hiperciclicidad frecuente para el semigrupo traslación en $L_p^\rho(\mathbb{R})$ y el operador backward shift en espacios de sucesiones con pesos. Los contenidos de este capítulo han sido incluidos en [81].

En el Capítulo 4 hemos estudiado la hiperciclicidad, el caos de Devaney y las propiedades de tipo mezclante tanto en el sentido topológico como en el medible para operadores definidos en espacios vectoriales topológicos que presentan subconjuntos invariantes. Hemos establecido relaciones entre el hecho de que un operador satisfaga las propiedades dinámicas sobre ciertos subconjuntos invariantes y que las satisfaga sobre la envoltura lineal cerrada del propio subconjunto o sobre la unión de subconjuntos invariantes. Además, proporcionamos condiciones sobre el operador (o C_0-semigrupo), que aseguren que al restringirlo sobre el subconjunto invariante, satisface ciertas propiedades dinámicas. En particular, centraremos nuestra atención en el caso de operadores positivos y semigrupos positivos definidos en retículos, y el cono positivo. Los contenidos de este capítulo han sido publicados en [85].

En el último capítulo, motivados por el trabajo de Balibrea y Oprocha [4], donde obtuvieron resultados sobre propiedades mezclantes y caos para sistemas discretos no autónomos sobre compactos, estudiamos propiedades mezclantes para sistemas dinámicos lineales no autónomos sobre ciertos subconjuntos invariantes. Todos los resultados de este capítulo han sido publicados en [87].

Resum

La tesi “Mesures fortament mesclants i subconjunts invariants en dinàmica lineal” s’estructura en tres parts. En el Capítol 0 s’introdueix la notació, definicions i resultats bàsics que es necessitaran al llarg de la tesi. La primera part de la tesi consta de dos capítols, els Capítols 1 i 2, on estudiem la relació entre el Criteri d’Hiperciclicitat Freqüent i l’existència de mesures de probabilitat borelianes fortament mesclants. La segona part la compon el Capítol 3, on centrem la nostra atenció en l’estudi de la hiperciclicitat freqüent dels C_0-semigrups de translació i l’última part consistent en els Capítols 4 i 5, on estudiem propietats dinàmiques que satisfan els sistemes dinàmics lineals autònoms i no autònoms sobre certs subconjunts invariants. A continuació proporcionem una breu descripció de cada capítol:

En el Capítol 1, construïm mesures de probabilitat borelianes fortament mesclants i T-invariants amb suport total per a operadors definits en F-espais que satisfan el Criteri d’Hiperciclicitat Freqüent. A més, proporcionem exemples d’operadors que verifiquen aquest criteri i millorem aquest resultat en el cas d’operadors backward shifts unilaterals i caòtics. Els continguts d’aquest capítol han sigut publicats en [88] i [12].

En el Capítol 2, mostrem que el Criteri d’Hiperciclicitat Freqüent per a C_0-semigrups, obtingut per Mangino i Peris en [82], assegura l’existència de mesures invariants fortament mesclants amb suport total. Proporcionarem diversos exemples que il·lustren aquest resultat i que varien des del model de naixement i mort fins a l’equació de Black-Scholes. Tots els resultats d’aquest capítol han sigut publicats en [86] .

En el Capítol 3, centrem la nostra atenció en un dels C_0-semigrups més importants, el semigrup translació. Inspirats en el treball de Bayart i Ruzsa [22] que caracteritza la hiperciclicitat freqüent dels operadors back-

ward shifts amb pesos, caracteritzem els semigrups translació freqüentment hipercíclics en els espais $C_0^\rho(\mathbb{R})$ i $L_p^\rho(\mathbb{R})$. En primer lloc, repassem els resultats ja existents sobre la dinàmica dels semigrups translació. A continuació, obtenim una caracterització de la hiperciclicitat freqüent per a operadors pseudo-shifts amb pesos en funció dels mateixos que s'usarà més tard per a caracteritzar el C_0-semigrup translació en $C_0^\rho(\mathbb{R})$. Finalment, estudiem el cas de $L_p^\rho(\mathbb{R})$. També establirem una analogia entre l'estudi de la hiperciclicitat freqüent per al semigrup translació en $L_p^\rho(\mathbb{R})$ i l'operador backward shift en espais de successions amb pesos. Els continguts d'aquest capítol han sigut inclosos en [81].

En el Capítol 4 hem estudiat la hiperciclicitat, el caos de Devaney i les propietats de tipus mesclant tant en el sentit topològic com en el mesurable per a operadors definits en espais vectorials topològics que presenten subconjunts invariants. Hem establit relacions entre el fet que un operador satisfaça les propietats dinàmiques sobre certs conjunts invariants i que les satisfaà sobre l'embolcall lineal tancat del propi subconjunt o sobre la unió de subconjunts invariants. A més, donem condicions a l'operador (o C_0-semigrup), que asseguren que en restringir-ho sobre el subconjunt invariant, satisfaà certes propietats dinàmiques. En particular, centrarem la nostra atenció en el cas d'operadors positius i semigrups positius definits en reticles i el con positiu. Els continguts d'aquest capítol han sigut publicats en [85].

En l'últim capítol, motivats pel treball de Balibrea i Oprocha [4], on van obtenir resultats sobre propietats mesclants i caos per a sistemes discrets no autònoms en conjunts compactes, estudiem propietats mesclantes per a sistemes dinàmics lineals no autònoms sobre certs subconjunts invariants. Tots els resultats d'aquest capítol han sigut publicats en [87].

Summary

The Ph.D. Thesis "Strong mixing measures and invariant sets in linear dynamics" has three differenced parts. Chapter 0 introduces the notation, definitions and the basic results that will be needed troughout the thesis. There is a first part consisting of Chapters 1 and 2, where we study the relation between the Frequent Hypercyclicity Criterion and the existence of strongly-mixing Borel probability measures. A third chapter, where we focus our attention on frequent hypercyclicity for translation C_0-semigroups, and the last part corresponding to Chapters 4 and 5, where we study dynamical properties satisfied by autonomous and non-autonomous linear dynamical systems on certain invariant sets. In what follows, we give a brief description of each chapter:

In Chapter 1, we construct strongly mixing Borel probability T-invariant measures with full support for operators on F-spaces which satisfy the Frequent Hypercyclicity Criterion. Moreover, we provide examples of operators that verify this criterion and we also show that this result can be improved in the case of chaotic unilateral backward shifts. The contents of this chapter have been published in [88] and [12].

In Chapter 2, we show that the Frequent Hypercyclicity Criterion for C_0-semigroups, which was given by Mangino and Peris in [82], ensures the existence of invariant strongly mixing measures with full support. We will provide several examples, that range from birth-and-death models to the Black-Scholes equation, which illustrate these results. All the results of this chapter have been published in [86].

In Chapter 3, we focus our attention on one of the most important tests C_0-semigroups, the translation semigroup. Inspired in the work of Bayart and Ruzsa in [22], where they characterize frequent hypercyclicity of

weighted backward shifts we characterize frequently hypercyclic translation C_0-semigroups on $C_0^\rho(\mathbb{R})$ and $L_p^\rho(\mathbb{R})$. Moreover, we first review some known results on the dynamics of the translation C_0-semigroups. Later we state and prove a characterization of frequent hypercyclicity for weighted pseudo shifts in terms of the weights that will be used later to obtain a characterization of frequent hypercyclicity for translation C_0-semigroups on $C_0^\rho(\mathbb{R})$. Finally we study the case of $L_p^\rho(\mathbb{R})$. We will also establish an analogy between the study of frequent hypercyclicity for the translation C_0-semigroup in $L_p^\rho(\mathbb{R})$ and the corresponding one for backward shifts on weighted sequence spaces. The contents of this chapter have been included in [81].

Chapter 4 is devoted to study hypercyclicity, Devaney chaos, topological mixing properties and strong mixing in the measure-theoretic sense for operators on topological vector spaces with invariant sets. More precisely, we establish links between the fact of satisfying any of our dynamical properties on certain invariant sets, and the corresponding property on the closed linear span of the invariant set, or on the union of the invariant sets. Viceversa, we give conditions on the operator (or C_0-semigroup) to ensure that, when restricted to the invariant set, it satisfies certain dynamical property. Particular attention is given to the case of positive operators and semigroups on lattices, and the (invariant) positive cone. The contents of this chapter have been published in [85].

In the last chapter, motivated by the work of Balibrea and Oprocha [4], where they obtained several results about weak mixing and chaos for nonautonomous discrete systems on compact sets, we study mixing properties for nonautonomous linear dynamical systems that are induced by the corresponding dynamics on certain invariant sets. All the results of this chapter have been published in [87].

Contents

Introduction

This Ph.D Thesis treats different aspects about linear dynamics. In the first part of the thesis we focus our attention on frequent hypercyclicicity and its connection with measurable dynamics, i.e. ergodic theory. Ergodic theory was first used for the dynamics of linear operators by Rudnicki [94] and Flytzanis [56] and during the last years it has deserved special attention thanks to the work of Bayart and Grivaux [18, 17].

In 2005, motivated by Birkhoff's Theorem, Bayart and Grivaux introduced in [18] the notion of frequently hypercyclic operators, trying to quantify how "often" an orbit meets non-empty open sets. Moreover, they also gave the first version of a Frequent Hypercyclicity Criterion which ensures that an operator T defined on an F- space is frequently hypercyclic. Later, some new versions of this criterion were stated such as a probabilistic version given by Grivaux in [66], and the most usual one was given by Bonilla an Grosse-Erdmann in [33].

If an operator T turns out to be ergodic with respect to some T- invariant Borel probability measure with full support, then T is frequently hypercyclic by Birkhoff's Ergodic Theorem. Accordingly, it is desirable to find conditions ensuring the existence of such types of measures.

The starting point of the first chapter is motivated by the work of Bayart and Matheron that gave very general conditions, expressed on eigenvector fields associated to unimodular eigenvalues, under which an operator T admits a T-invariant strongly mixing measure with full support [21]. In order to obtain their results, they work with Gaussian measures.

In our case, we show that under the hypothesis of the Frequent Hypercyclicity Criterion, we can ensure the existence of T-invariant strongly mixing measures with full support on F-spaces. Actually, on the one hand our results can be deduced from [21] in the context of complex Fréchet spaces, and on the other hand we only need rather elementary tools. Also, although our measures are not Gaussian, they have certain different properties which have been used recently by Bayart [15].

Moreover, in this first chapter we provide examples that illustrate our results and we show that they can be improved by obtaining the existence of exact measures in the case of chaotic unilateral weighted backward shifts.

We also show that Devaney chaos is therefore a sufficient condition for the existence of strongly mixing measures within the framework of weighted shift operators on sequence F-spaces, and that in some natural spaces such as ℓ^p it is even a characterization of this fact.

The contents of this chapter have been published in [88] and [12].

Motivated by our work in Chapter 1, we wonder whether if it was also possible to obtain some conditions to ensure the existence of strongly-mixing measures with full suppport for C_0-semigroups. In parallel with the theory for linear operators, since the seminal paper by Desch, Schappacher and Webb [45], many researchers turned their attention to the chaotic behaviour of strongly continuous semigroups. Actually hypercyclicity appears in solution semigroups of evolution problems associated with "birth and death" equations for cell populations, transport equations, first order partial differential equations, Black and Sholes equations, diffusion operators as Ornstein-Uhlenbeck operators [6, 7, 9, 31, 82, 37].

In [82], Mangino and Peris obtained a continuous version of the Frequent Hypercyclicity Criterion based on the Pettis integral. We show that this criterion suffices for the existence of invariant Borel probability measures on X that are strongly mixing and have full support.

In contrast with the chaotic behavior in the topological sense, which is trivial to pass from the discrete to the continuous case, while difficult or false to go in the other direction (see, e.g., [38] for hypercyclicity and frequent hypercyclicity, and [16] for Devaney chaos), the measure-theoretic properties are not trivially passed from the discrete to the continuous case, especially because of the requirement of T_t-invariance for every $t > 0$. This is why we need to construct explicitly the strongly mixing measures for C_0-semigroups, and they cannot be obtained from the main result in [88].

In [82] some conditions, expressed in terms of eigenvector fields for the infinitesimal generator of the C_0-semigroup, were given to ensure that the assumptions of the Frequent Hypercyclicity Criterion are satisfied. In consequence we also obtain the stronger result of existence of invariant strongly mixing measures under the same conditions. A different argument for the existence of invariant strongly mixing measures for C_0-semigroups has been obtained by Bayart and Matheron in [21] under weaker assumptions on the eigenvector fields for the generator.

We finish the chapter by presenting several applications of the previous results to the (chaotic) behaviour of the solution C_0-semigroup to certain linear partial differential equations and infinite systems of linear differential equations. These examples range from birth-and-death models to the Black-Scholes equation.

All the results of this chapter have been published in [86].

In Chapter 3 we focus our attention on one of the most important tests C_0-semigroups, the translation semigroup. The role of "test" class, which is played by weighted shifts in the setting of discrete linear dynamical systems, is covered by translation semigroups in the setting of continuous linear dynamical systems. These semigroups have been widely studied by many authors. In [45] hypercyclic translation semigroups were characterized. Mixing and chaotic properties were studied by Bermúdez et alt. in [24] and by Matsui et alt. in [84], respectively.

Inspired in the work of Bayart and Ruzsa in [22], where frequent hypercyclicity of weighted backward shifts is characterized, we characterize frequently hypercyclic translation C_0-semigroups on $C_0^\rho(\mathbb{R})$ and $L_p^\rho(\mathbb{R})$.

Firstly, we state and prove a characterization of frequent hypercyclicity for weighted pseudo shifts in terms of the weights that will be used later to obtain a characterization of frequent hypercyclicity for translation C_0-semigroups on $C_0^\rho(\mathbb{R})$. Moreover, we prove that in $C_0^\rho(\mathbb{R})$ chaos implies frequent hypercyclicity but the converse is not true.

Finally we study the case of $L_p^\rho(\mathbb{R})$. We complete the results obtained by Mangino in Peris in [82] by showing that for translation C_0-semigroups on $L_p^\rho(\mathbb{R})$ chaos is equivalent to frequent hypercyclicity.

We will also establish an analogy between the study of frequent hypercyclicity for the translation C_0-semigroup and the corresponding one for backward shifts on weighted sequence spaces as Barrachina and Peris studied for distributional chaos in [11].

The contents of this chapter have been included in [81].

Chapter 4 is devoted to study hypercyclicity, Devaney chaos, topological mixing properties and strong mixing in the measure-theoretic sense for operators on topological vector spaces with invariant sets. Although chaotic properties for linear operators are usually considered in the context of F-

spaces, more general topological vector spaces have also attracted the attention in recent years (see, e.g.,[32], [99] and Chapter 12 of [72]).

In the first section of the chapter our framework are operators defined on topological vector spaces. We provide several conditions under which a dynamical property can pass from an invariant set (or a countable family of invariant sets) of the operator to the closure of its linear span (or to the union of the invariant sets). In [13] analogous results have been given for backward shift operators and the specification property.

These results allow us to provide some surprising examples that show the interplay between non-linear finite-dimensional dynamics and linear (infinite-dimensional) dynamics.

There are well-known criteria of chaos [62, 25], mixing [62, 75] and weak mixing [59, 27] properties for operators. In section 4.3 we derive some criteria under which an operator restricted to an invariant set satisfies these properties.

In the last section we will give several criteria for operators and C_0-semigroups that allow certain dynamical properties when restricted to invariant sets. Special attention is devoted to positive operators on Fréchet lattices and C_0-semigroups of positive operators on Banach lattices when the invariant set is the positive cone. By following the ideas that we developed in [88] and [86], respectively, we show that certain "positive" versions of frequent hypercyclicity criteria ensure the existence of T-strongly mixing measures supported on the positive cone of a Fréchet lattice, and the existence of $(T_t)_t$-invariant strongly mixing Borel probability measures supported on the positive cone of a Banach lattice, where T is a positive operator and $(T_t)_{t\geq 0}$ is a C_0-semigroup of positive operators, respectively. In this last case, the results are relevant in connection with applications since, for instance, the chaotic behaviour of certain solutions to differential equations make sense only when they are positive. This provides partial answers to questions of Banasiak, Desch and Rudnicki.

The contents of this chapter have been published in [85].

In the last chapter we study mixing properties (topological mixing and weak mixing of arbitrary order) for nonautonomous linear dynamical systems that are induced by the corresponding dynamics on certain invariant sets. The type of nonautonomous systems that we consider can be defined by a sequence $(T_i)_{i\in\mathbb{N}}$ of linear operators $T_i : X \to X$ on a topological vector space X such that there is an invariant set Y for which the dynamics restricted to Y satisfies certain mixing property. We then obtain the corresponding mixing property on the closed linear span of Y. We also prove that the class of nonautonomous linear dynamical systems that are weakly mixing of order n contains strictly the corresponding class with the weak mixing property of order $n+1$.

Chaotic behaviour for nonautonomous discrete systems has been studied by several authors [35, 36, 77, 78, 76].

While the study of dynamics of nonautonomous discrete systems is usually more complex and demanding than the same studies in the setting of autonomous systems (i.e. systems given by a pair (X, T) where T is a linear operator) such studies became more popular each year. The reason is that, they are more flexible tools for the description of real world processes and also because the variety of dynamical behavior that can be represented by such systems is much richer. Very often nonautonomous discrete dynamical systems arise in a natural way as a solution of differential equation. A particular example is the dynamics of evolution of a population modeled by (one or multi-dimensional) difference equations, like forced Pielou equation, periodically forced Beverton-Holt equation, etc. (see for example [42]).

Our work has its starting point in [4], where Balibrea and Oprocha obtained several results about weak mixing and chaos in nonautonomous discrete systems on compact sets. Some of their results will be used to induce the corresponding dynamical behavior on linear nonautonomous systems. The theory of linear dynamics is well established in the case of iterations of a single operator (autonomous dynamical system). The case of nonautonomous linear dynamics is not yet developed, although a more general concept of universality of a sequence of operators $(T_n)_{n\in\mathbb{N}}$ where the orbits are defined as $\{T_n x \,;\, n \in \mathbb{N}\}$, $x \in X$, has been treated by several authors (See, e.g., [24, 26, 27, 64, 79]).

All the results of this chapter have been published in [87].

Chapter 0

Preliminaries

This chapter is devoted to introduce the notation, definitions and the basic results that we will use throughout the thesis. Most of the results related to linear dynamics can be found in [20] and [72].

0.1 Topological dynamics

Dynamical systems appear naturally in the study of the behavior of evolving systems. Let X be a set of elements that describes the different acceptable states of a system. If $x_n \in X$ is the state of the system at time $n \geq 0$, then its evolution will be given by a linear map $T : X \to X$ such that $x_{n+1} = T(x_n)$.

Definition 0.1.1 (Discrete dynamical system) *Let X be a metric space and let T be a continuous map $T : X \to X$. A* discrete dynamical system *is a pair (X, T). We define the* orbit *of a point $x \in X$ as the set $Orb(x, T) = \{T^n x : n \in \mathbb{N}_0\}$, where T^n denotes the n-th iterate of a map T. We will often simply say that T or $T : X \to X$ is a dynamical system.*

Definition 0.1.2 *Let $S : Y \to Y$ and $T : X \to X$ be dynamical systems.*

1. *Then T is called* quasi-conjugate *to S if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$; that is, the*

following diagram commutes.

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \downarrow\Phi & & \downarrow\Phi \\ X & \xrightarrow{T} & X \end{array}$$

2. *If ϕ can be chosen to be a homeomorphism, then S and T are called* conjugates.

Definition 0.1.3 *We say that a property $\wp$ for dynamical systems is preserved under (quasi-)conjugacy if the following holds: if a dynamical system $S : Y \to Y$ has property $\wp$ then every dynamical system $T : X \to X$ that is (quasi-) conjugate to S also has property $\wp$.*

Definition 0.1.4 *Let $T : X \to X$ be a dynamical system. Then $Y \subset X$ is called T-*invariant *or invariant under T if $T(Y) \subset Y$.*

Definition 0.1.5 *We say that $x \in X$ is a* fixed point for the dynamical system $T : X \to X$ *if $Tx = x$, and we say that $x \in X$ is a* periodic point for the dynamical system T *if $T^n x = x$ for some $n \in \mathbb{N}_0$. The set of all periodic points is denoted by $Per(T)$. If $x \in Per(T)$ then the smallest positive integer n such that $T^n x = x$ is called a* primary period *of x.*

Definition 0.1.6 *Let $T : X \to X$ be a dynamical system. For any pair of nonempty open sets U, V we denote by $N(U, V) = \{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \emptyset\}$. Then we have that (X, T) is:*

(i) topologically transitive *if for any pair of nonempty open sets U,$V \subset X$ $N(U, V) \neq \emptyset$;*

(ii) weakly mixing *if the map $T \times T$ is topologically transitive;*

(iii) mixing *if for any pair of nonempty open sets U,$V \subset X$ $N(U, V)$ is cofinite.*

(iv) topologically ergodic *if for any pair of nonempty open sets U,$V \subset X$ $N(U, V)$ is syndetic, that is, there exists $p \in \mathbb{N}$, such that $\{n, n + 1, \dots, n + p\} \cap N(U, V) \neq \emptyset$ for any $n \in \mathbb{N}_0$.*

A result due to Furstenberg [58] is the following:

Theorem 0.1.7 *Let $T : X \to X$ be a weakly mixing dynamical system. Then the n-fold product $T \times \ldots \times T$ is weakly mixing for each $n \geq 2$.*

Remark 0.1.8 *For any linear dynamical system,*

$$\textit{mixing} \Longrightarrow \textit{topologically ergodic}$$

$$\Longrightarrow \textit{weakly mixing} \Longrightarrow \textit{topologically transitive.}$$

In 1989 Robert L. Devaney proposed the first good definition of chaos; see [46]. This concept reflects the unpredictability of chaotic systems because the definition contains a *sensitive dependence on initial conditions*, i.e.:

Definition 0.1.9 *Let X be a metric space without isolated points. Then the dynamical system $T : X \to X$ is said to have* sensitive dependence on initial conditions *if there exists some $\delta > 0$ such that, for every $x \in X$ and $\varepsilon > 0$, there exists some $y \in X$ with $d(x, y) < \varepsilon$ such that, for some $n \geq 0$, $d(T^n x, T^n y) > \delta$. The number δ is called a* sensitivity constant *for T.*

Definition 0.1.10 (Devaney chaos) *A dynamical system $T : X \to X$ is called chaotic in the sense of* Devaney *if it satisfies the following properties:*

(i) T is topologically transitive,

(ii) $Per(T)$ is dense in X,

(iii) T has sensitive dependence on initial conditions.

However, Banks, Brooks, Cairns, Davis and Stacey proved in 1992 ([10]), that one can drop sensitive dependence from Devaney's definition because it is implied by the other two conditions.

Theorem 0.1.11 ([10]) *Let X be a metric space without isolated points. If a dynamical system $T : X \to X$ is topologically transitive and has a dense set of periodic points then T has sensitive dependence on initial conditions with respect to any metric defining the topology of X.*

Proposition 0.1.12 *The following properties are preserved by quasi-conjugacy:*

(i) Topological transitivity.

(ii) The property of having a dense orbit.

(iii) The property of having a dense set of periodic points.

(iv) Devaney Chaos.

(v) The mixing property.

(vi) The weak-mixing property.

(vii) Topological ergodicity.

0.2 Hypercyclic and chaotic operators

Dynamical systems are defined by continuous maps on metric spaces. For linear dynamical systems, the underlying space must in addition have a linear structure, as is the case for Hilbert spaces and Banach spaces. We will give definitions of linear dynamical systems on spaces of a more general type, topological vector spaces.

Definition 0.2.1 *Let $||.|| : X \to \mathbb{R}_+$ be a functional on a vector space X that satisfies:*

(i) $||x + y|| \leq ||x|| + ||y||$

(ii) $||\lambda x|| \leq ||x|| \quad$ *if* $\quad |\lambda| \leq 1$

(iii) $\lim_{\lambda \to 0} ||\lambda x|| = 0$

(iv) $||x|| = 0$ *implies that* $x = 0$.

Then $||.|| : X \to \mathbb{R}_+$ is called an F-norm. If $(X, ||.||)$ is complete under the induced metric $d(x, y) = ||x - y||$, then X is an F-space.

A particular case of F-spaces are Fréchet spaces.

Definition 0.2.2 *A Fréchet space is a vector space X, endowed with a separating increasing sequence $(p_n)_n$ of seminorms, which is complete under the metric given by*

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, p_n(x - y)).$$

Definition 0.2.3 *Let X and Y be topological vector spaces. Then a continuous linear map $T : X \to Y$ is called an* operator*. The space of all operators is denoted by $L(X, Y)$. If $Y = X$ we say that T is an operator on X, with $L(X) = L(X, X)$.*

A link between chaos theory and linear operator theory was established by Birkhoff's Transitivity Theorem in 1922. In this theorem, he showed that topological transitivity was equivalent to the notion of hypercyclicity that Beauzamy established in 1987.

Definition 0.2.4 ([23]) *An operator $T : X \to X$ is said to be* hypercyclic *if there is some $x \in X$ whose orbit under T is dense in X. In that case, x is called a* hypercyclic vector *for T. The set of hypercyclic vectors is denoted by $HC(T)$.*

Theorem 0.2.5 (Birkhoff Transitivity theorem, [29]) *An operator T is hypercyclic if and only if it is topologically transitive. If one of these conditions holds then, the set $HC(T)$ of hypercyclic vectors is a dense G_δ-set; i.e., $HC(T)$ is a countable intersection of open dense sets.*

In 1991 Godefroy and Shapiro adopted Devaney's definition for linear chaos.

Definition 0.2.6 ([62]) *An operator $T : X \to X$ is called* chaotic in the sense of Devaney *if:*

(i) T is hypercyclic.

(ii) $Per(T)$ is dense in X.

Example 0.2.7 The first examples of hypercyclic operators were found by G.D.Birkhoff in 1929 [30], G.R. Maclane in 1952 [80] and S.Rolewickz in 1969 [91].

(i) (**Birkhoff's operators**) The translation operators given by

$$T_a f(z) = f(z+a), a \neq 0.$$

on the space $H(\mathbb{C})$ of entire functions are hypercyclic for all $a \neq 0$.

(ii) (**MacLane's operator**) The differentiation operator:

$$D : f \to f'$$

on $H(\mathbb{C})$ is hypercyclic.

(iii) (**Rolewicz's operators**) On the spaces $X = \ell^p$, $1 \leq p < \infty$, or $X = c_0$ we consider the multiple

$$T = \lambda B : X \to X, (x_1, x_2, x_3 \ldots) \to \lambda(x_2, x_3, x_4, \ldots)$$

of the backward shift, where $\lambda \in \mathbb{K}$. T is hypercyclic whenever $|\lambda| > 1$.

Moreover, these operators are chaotic. We first need the following results.

Proposition 0.2.8 *Let T be a linear map on a complex vector space X. Then the set of periodic points of T is given by*

$$Per(T) = \text{span}\{x \in X; Tx = e^{\alpha\pi i}x \quad \textit{for} \quad \textit{some} \quad \alpha \in \mathbb{Q}\}$$

Let e_λ denotes the exponential function $e_\lambda(z) = e^{\lambda z}$.

Lemma 0.2.9 *Let $\Lambda \subset \mathbb{C}$ be a set with an accumulation point. Then the set*

$$\text{span}\{e_\lambda; \lambda \in \Lambda\}$$

is dense in $H(\mathbb{C})$.

The lemma allows us to show that Birkhoff's and MacLane's operators are chaotic on $H(\mathbb{C})$.

Example 0.2.10 For the differentiation operator D, any function e_λ is an eigenvector of D to the eigenvalue λ. Thus, since the subspace

$$\text{span}\{e_\lambda; \lambda = e^{\alpha\pi i}, \quad \text{for} \quad \text{some} \quad \alpha \in \mathbb{Q}\}$$

is dense in $H(\mathbb{C})$ by lema 0.2.9, proposition 0.2.8 tells us that $Per(T)$ is dense. Since we already know that D is hypercyclic, it is also chaotic.

For the translation operators T_a, $a \in \mathbb{C}\backslash\{0\}$, any function e_λ is an eigenvector of T_a to the eigenvalue $e^{a\lambda}$. Thus, since the subspace

$$\text{span}\{e_\lambda; e^{a\lambda} = e^{\alpha\pi i}, \quad \text{for} \quad \text{some} \quad \alpha \in \mathbb{Q}\} = \text{span}\{e_\lambda; \lambda = \frac{\alpha}{a}i\pi, \alpha \in \mathbb{Q}\}$$

is also dense in $H(\mathbb{C})$, we conclude as before that each T_a is chaotic.

0.3 Measure-theoretic properties

The theory of dynamical systems has its roots in topological dynamics but there is also a parallel theory of measurable dynamics, which is better known under the name of ergodic theory. Ergodic theory was first used for the dynamics of linear operators by Rudnicki [94] and Flytzanis [56]. During the last years it has deserved special attention thanks to the work of Bayart and Grivaux [17, 18]. For instance, the papers [3, 19, 21, 43, 67, 95] contain recent advances on the subject. In this section we present some basic measure-theoretic properties that linear dynamical systems have. These results can be found in [102].

Let T be an operator on a topological vector space X. In order to look at T from the point of view of ergodic theory we need to define a probability measure μ on X. It is natural to assume that μ is defined on the Borel σ-algebra $\mathfrak{B}(X)$, that is the smallest σ-algebra containing the open subsets of X.

Definition 0.3.1
- *A measure μ has* full support *if for all non-empty open set $U \subset X$ $\mu(U) > 0$.*
- *A measurable map $T : (X, \mathfrak{B}(X), \mu) \to (X, \mathfrak{B}(X), \mu)$ is called a* measure-preserving *transformation, or in other words, μ is T-invariant, if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathfrak{B}(X)$.*
- *$T : (X, \mathfrak{B}(X), \mu) \to (X, \mathfrak{B}(X), \mu)$ is an* E-system *if there exists a T-invariant probability measure μ with full support.*
- *A measurable map $T : (X, \mathfrak{B}(X), \mu) \to (X, \mathfrak{B}(X), \mu)$ is called* ergodic *if it is measure-preserving and satisfies one of the following equivalent conditions:*

  (i) Given any measurable sets A, B with positive measures, one can find an integer $n \geq 0$ such that $T^n(A) \bigcap B \neq \emptyset$;

  (ii) if $A \in \mathfrak{B}(X)$ satisfies $T(A) \subset A$, then $\mu(A)(1 - \mu(A)) = 0$.

  (iii) if $A \in \mathfrak{B}(X)$ satisfies $T^{-1}(A) = A$, then $\mu(A)(1 - \mu(A)) = 0$.
- *A measurable map $T : (X, \mathfrak{B}(X), \mu) \to (X, \mathfrak{B}(X), \mu)$ is said to be* strongly mixing *with respect to μ if*

$$lim_{n\to\infty}\mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \quad (A, B \in \mathfrak{B}(X)).$$

- *T is said to be exact if given $A \in \bigcap_{n=0}^{\infty} T^{-n}\mathfrak{B}(X)$ then $\mu(A)(1 - \mu(A)) = 0$.*

Proposition 0.3.2 *For any operator $T : X \to X$,*

$$exact \Longrightarrow strongly \quad mixing \Longrightarrow ergodic.$$

0.4 Hypercyclic criteria

The main purpose of this section is to show several criteria under which an operator is chaotic, mixing or weakly mixing. In this section we show these criteria. This first criterion is due to Godefroy Shapiro and it is contained implicitly in their paper [62] and was isolated by Bernal [25].

Theorem 0.4.1 (Godefroy-Shapiro criterion, [62]) *Let T be an operator. Suppose that the subspaces*

$$X_0 := span\{x \in X; \quad Tx = \lambda x \quad for\ some \quad \lambda \in \mathbb{K} \quad with \quad |\lambda| < 1\}$$

$$Y_0 := span\{x \in X; \quad Tx = \lambda x \quad for\ some \quad \lambda \in \mathbb{K} \quad with \quad |\lambda| > 1\}$$

are dense in X.
Then T is mixing, and in particular hypercyclic.

If, moreover, X is a complex space and the subspace

$$Z_0 := span\{x \in X; \quad Tx = \lambda x \quad for \quad \lambda \in \mathbb{C}, \quad |\lambda|^n = 1 \quad for\ some \quad n \in \mathbb{N}\}$$

is dense in X, then T is chaotic.

Example 0.4.2 Rolewicz's operators
Let $T = \mu B$, with $|\mu| > 1$, be the multiple of the backward shift on any space $X = \ell^p$, $1 \leq p < \infty$ or $X = c_0$. Let us consider the complex case. One easily determines the eigenvectors of B as the nonzero multiples of the sequences

$$e_\lambda := (\lambda, \lambda^2, \lambda^3, \ldots), |\lambda| < 1$$

with corresponding eigenvalue λ. Therefore, e_λ is an eigenvector of $T = \mu B$ corresponding to the eigenvalue $\mu\lambda$. For any subset Λ of the unit disk that has an accumulation point inside the disk, the set $\text{span}\{e_\lambda; \lambda \in \Lambda\}$ is dense in X. By the Hahn-Banach theorem it suffices to show that any continuous linear functional x^* on X that vanishes on each $e_\lambda, \lambda \in \Lambda$ vanishes on X. Since $x^* \in X^*$, via the canonical representation it is given by a sequence $(y_n)_n \in \ell^q$ for a certain q, with $1 \leq q \leq \infty$, we have that

$$x^*(e_\lambda) =< e_\lambda, x^* >= \sum_{n=1}^{\infty} y_n \lambda^n \quad \text{if} \quad |\lambda| < 1.$$

The identity theorem for holomorphic functions implies that each y_n is zero and therefore $x^* = 0$. In particular, the subspace

$$X_0 = \text{span}\{x \in X; Tx = \eta x \quad \text{for} \quad \eta \in \mathbb{K}, \quad |\eta| < 1\} = \text{span}\{e_\lambda; |\lambda| < \frac{1}{|\mu|}\}$$

is dense in X, as the suspaces Y_0 and Z_0 of the Godefroy- Shapiro criterion; note that $\frac{1}{|\mu|} < 1$. This implies that Rolewicz's operators are mixing and chaotic.

The earliest forms of the Hypercyclicity Criterion were found independently by Kitai [75] and by Gethner and Shapiro [59]. In its general form it is due to Bés and Peris [27].

Theorem 0.4.3 (Kitai's criterion, [75]) *Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$ and a map $S : Y_0 \to Y_0$ such that, for any $x \in X_0$, $y \in Y_0$:*

(i) $T^n x \to 0$,

(ii) $S^n y \to 0$,

(iii) $TSy = y$,

then T is mixing.

Example 0.4.4 (i) (**Rolewicz's operators**) Taking $X_0 = Y_0$ the set of finite sequences, which is dense in X, and for $S : Y_0 \to Y_0$ the map $S = \frac{1}{\lambda}F$ where F is the forward shift operator $F : (x_1 x_2, \dots) \to (0, x_1, x_2, \dots)$ the conditions of Kitai's criterion are clearly satisfied.

(ii) (**MacLane's operators**) In this case we take for $X_0 = Y_0$ the set of polynomials, which is dense in $H(\mathbb{C})$, and for S we consider the integral operator $Sf(z) = \int_0^z f(\zeta)d\zeta$. While conditions (i) and (iii) are obvious, we note that condition (ii) is sufficient to be verified by monomials, and $S^n(z^k) = \frac{k!}{(k+n)!} z^{k+n} \to 0$ as $n \to \infty$, uniformly on compact sets, as required.

(iii) (**Birkhoff's operators**) It is sufficient to prove that $T_1 f(z) = f(z+1)$ on $H(\mathbb{C})$ is mixing. For $X_0 = Y_0$ we choose the set of functions $f_{p,\alpha,\nu} = p(z)e^{-\alpha(z-\nu)^2}$, where p is a polynomial and $\alpha > 0$, $\nu \in \mathbb{N}_0$. Since $f_{p,\alpha,\nu} \to p$ in $H(\mathbb{C})$ as $\alpha \to 0$, this set is dense in $H(\mathbb{C})$. Moreover, for S we consider the translation operator $Sf(z) = f(z-1)$. Now if $z = x+iy$ with $|y| \leq \frac{1}{2}|x|$ then we have that $|e^{-\alpha z^2}| = e^{-\alpha(x^2-y^2)} \leq e^{-\frac{3}{4}\alpha x^2}$. This implies, that for any p, α and ν, $f_{p,\alpha,\nu}(z \pm n) \to 0$ uniformly on compact sets as $n \to \infty$, which shows that conditions (i) and (ii) of Kitai's criterion hold, while condition (iii) is trivial.

Theorem 0.4.5 (Gethner-Shapiro criterion, [59]) *Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$, an increasing sequence $(n_k)_k$ of positive integers, and a map $S : Y_0 \to Y_0$ such that, for any $x \in X_0$, $y \in Y_0$:*

(i) $T^{n_k}x \to 0$,

(ii) $S^{n_k}y \to 0$,

(iii) $TSy = y$,

then T is weakly mixing.

Theorem 0.4.6 (Hypercyclicity criterion, [27]) *Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$, an increasing sequence $(n_k)_k$ of positive integers, and maps $S_{n_k} : Y_0 \to X$, $k \geq 1$ such that, for any $x \in X_0$, $y \in Y_0$:*

(i) $T^{n_k}x \to 0$,

(ii) $S_{n_k}y \to 0$,

(iii) $T^{n_k}S_{n_k}y \to y$,

then T is weakly mixing, and in particular hypercyclic.

0.5 Weighted shifts

In this section we include some basic results about weighted shifts, which make up an important class of hypercyclic and chaotic operators. Due to its simple structure, the class of weighted shifts is a favorite testing ground for operator-theorists. Salas([96]) characterized hypercyclic and weakly mixing unilateral and bilateral weighted shifts on ℓ^2 and $\ell^2(\mathbb{Z})$, respectively. The characterizations for more general sequence spaces and chaos characterizations are due to Grosse-Erdmann [70].

Definition 0.5.1 *The basic model of all shifts is the* backward shift

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Another shift is the weighted backward shift *which is defined as:*

$$B_w(x_1, x_2, x_3, \ldots) = (w_2x_2, w_3x_3, w_4x_4, \ldots),$$

where $w = (w_n)_n$ is called a weight sequence*. The weights w_n will be assumed to be non-zero.*

These operators can be defined on an arbitrary *sequence space* X, that is, a linear space of sequences or, in other words, a subspace of $w = \mathbb{K}^{\mathbb{N}}$. Moreover, X should carry a topology that is compatible with the sequence space structure of X. We interpret this as demanding that convergence in X should imply coordinatewise convergence. A Banach (Fréchet, F-) space of this kind is called a Banach (Fréchet, F-) sequence space.

Theorem 0.5.2 *Let X be a Fréchet sequence space in which $(e_n)_n$ (where $e_n = (0, \ldots, 0, \underbrace{1}_{n}, 0, \ldots)$) is a basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:*

(i) B is hypercyclic;

(ii) B is weakly mixing;

(iii) there is an increasing sequence $(n_k)_k$ of positive integers such that $e_{n_k} \to 0$ in X as $k \to \infty$.

Example 0.5.3 Let

$$\ell_p^v = \{(x_n)_n; \sum_{n=1}^{\infty} |x_n|^p v_n < \infty\},$$

with $1 \leq p < \infty$, be a weighted ℓ_p-space, where $v = (v_n)_n$ is a positive weight sequence. Then B is an operator on ℓ_p^v if and only if there is an $M > 0$ such that, for all $x \in \ell_p^v$

$$\left(\sum_{n=1}^{\infty} |x_{n+1}|^p v_n\right)^{\frac{1}{p}} \leq M \left(\sum_{n=1}^{\infty} |x_n|^p v_n\right)^{\frac{1}{p}}$$

which is equivalent to $\sup_{n\in\mathbb{N}} \frac{v_n}{v_{n+1}} < \infty$. Theorem 0.5.2, tells us that hypercyclicity of B is characterized by $\inf_{n\in\mathbb{N}} v_n = 0$.

The same conditions also characterize the continuity and hypercyclicity of the backward shift B on the weighted c_0-space

$$c_0^v = \{(x_n)_n; \lim_{n\to\infty} |x_n| v_n = 0\}.$$

The following Theorem provides a charcaterization of mixing backward shifts.

Theorem 0.5.4 *Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:*

- *B is mixing;*
- *$e_n \to 0$ in X as $n \to \infty$.*

In order to show the following results we first need the definition of unconditional convergence.

Definition 0.5.5 *Let X be a Fréchet space. Then the following assertions are equivalent:*

(i) $\sum_{n=1}^{\infty} x_n$ *is unconditionally convergent;*

(ii) for any 0-1*-sequence* $(\epsilon_n)_n$, $\sum_{n=1}^{\infty} \epsilon_n x_n$ *converges;*

(iii) for any bounded sequence $(\alpha_n)_n$ *of scalars,* $\sum_{n=1}^{\infty} \alpha_n x_n$ *converges;*

(iv) for any $\epsilon > 0$ *there is some* $N \in \mathbb{N}$ *such that for any finite set* $F \subset \{N, N+1, N+2, \ldots\}$ *we have that*

$$\left\| \sum_{n \in F} x_n \right\| < \epsilon;$$

(v) for any $\epsilon > 0$ *there is some* $N \in \mathbb{N}$ *such that for any* 0-1*-sequence* $(\epsilon_n)_n$, $\sum_{n=1}^{\infty} \epsilon_n x_n$ *converges and*

$$\left\| \sum_{n \geq N} \epsilon_n x_n \right\| < \epsilon;$$

(vi) for any $\epsilon > 0$ *there is some* $N \in \mathbb{N}$ *such that whenever* $\sup_{n\geq 1} |\alpha_n| \leq 1$ *then* $\sum_{n=1}^{\infty} \alpha_n x_n$ *converges and*

$$\left\| \sum_{n \geq N} \alpha_n x_n \right\| < \epsilon;$$

Definition 0.5.6 *A sequence $(e_n)_n$ in a Fréchet space X is called an* unconditional basis *if it is a basis such that, for every $x \in X$, the representa-*

tion

$$x = \sum_{n=1}^{\infty} a_n e_n$$

converges unconditionally.

Theorem 0.5.7 *Let X be a Fréchet sequence space in which $(e_n)_n$ is an unconditional basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:*

(i) *B is chaotic;*

(ii) *$\sum_{n=1}^{\infty} e_n$ converges in X;*

(iii) *the constant sequences belong to X;*

(iv) *B has a non-trivial periodic point.*

It is easy to transfer results to weighted shifts by a conjugacy. Let B_w be a weighted shift on some sequence space X. We define v_n by

$$v_n = \left(\prod_{\nu=1}^{n} w_\nu \right)^{-1}, \quad n \geq 1$$

and consider the sequence space

$$X_v = \{(x_n)_n; (x_n v_n)_n \in X\}.$$

The map $\Phi_v : X_v \to X$, $(x_n)_n \to (x_n v_n)_n$ is a vector space isomorphism and $B_w \circ \Phi_v = \Phi_v \circ B$, that is the following diagram commutes:

$$\begin{array}{ccc} X_v & \xrightarrow{B} & X_v \\ {\scriptstyle \phi_v}\downarrow & & \downarrow{\scriptstyle \phi_v} \\ X & \xrightarrow{B_w} & X \end{array}$$

Thus $B_w : X \to X$ and $B : X_v \to X_v$ are conjugate operators.

Theorem 0.5.8 *Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the weighted shift B_w is an operator on X.*

1. *The following assertions are equivalent:*

   (i) *B_w is hypercyclic;*

(ii) B_w *is weakly mixing;*

(iii) there is an increasing sequence $(n_k)_k$ *of positive integers such that*

$$\left(\prod_{\nu=1}^{n_k} w_\nu\right)^{-1} e_{n_k} \to 0$$

in X *as* $k \to \infty$.

2. *The following assertions are equivalent:*

   (i) B_w *is mixing;*

   (ii) we have that

$$\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n \to 0$$

   in X *as* $n \to \infty$*;*

3. *Suppose that the basis* $(e_n)_n$ *is unconditional. Then the following assertions are equivalent:*

   (i) B_w *is chaotic;*

   (ii) the series

$$\sum_{n=1}^{\infty} \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$$

   converges in X*;*

   (iii) the sequence

$$\left(\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1}\right)_n$$

   belongs to X*;*

   (iv) B_w *has a non-trivial periodic point.*

Example 0.5.9 A weighted backward shift B_w is an operator on a sequence space ℓ^p, $1 \leq p < \infty$, or c_0 if and only if the weights w_n are bounded. The respective characterizing conditions for B_w to be hypercyclic mixing or chaotic on ℓ^p are

$$\sup_{n\geq 1} \prod_{\nu=1}^{n} |w_\nu| = \infty, \quad lim_{n\to\infty} \prod_{\nu=1}^{n} |w_\nu| = \infty, \quad \sum_{n=1}^{\infty} \frac{1}{\prod_{\nu=1}^{n} |w_\nu|^p} < \infty.$$

The first condition also characterizes when B_w is hypercyclic on c_0 and the second when it is mixing or equivalently chaotic on c_0. In particular for Rolewicz's operator $T = \lambda B$, $|\lambda| > 1$, we have that $\prod_{\nu=1}^{n} |w_\nu| = \lambda^n$, which implies that this operator is chaotic.

We can also study shifts on sequence spaces indexed over $\mathbb{Z}$. The bilateral backward shift is given by

$$B(x_n)_{n\in\mathbb{Z}} = (x_{n+1})_{n\in\mathbb{Z}}$$

and the bilateral weighted backward shifts are given by

$$B_w(x_n)_{n\in\mathbb{Z}} = (w_{n+1}x_{n+1})_{n\in\mathbb{Z}}$$

where $w = (w_n)_n$ is called a *weight sequence.*

Theorem 0.5.10 *Let X be a Fréchet sequence space in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Suppose that the bilateral shift B is an operator on X.*

1. *The following assertions are equivalent:*

   (i) B is hypercyclic;

   (ii) B is weakly mixing;

   (iii) there is an increasing sequence $(n_k)_k$ of positive integers such that for any $j \in \mathbb{Z}$, $e_{j-n_k} \to 0$ and $e_{j+n_k} \to 0$ in X as $k \to \infty$.

2. *The following assertions are equivalent:*

   - *B is mixing;*
   - *$e_{-n} \to 0$ and $e_n \to 0$ in X as $n \to \infty$.*

3. *The following assertions are equivalent:*

   (i) B is chaotic;

   (ii) $\sum_{n=-\infty}^{\infty} e_n$ converges in X;

   (iii) The constant sequences belong to X;

   (iv) B has a nontrivial periodic point.

Using a suitable conjugacy this result can be generalized immediately to weighted shifts. The conjugacy is given by:

$$\begin{array}{ccc} X_v & \xrightarrow{B} & X_v \\ \phi_v \downarrow & & \downarrow \phi_v \\ X & \xrightarrow{B_w} & X \end{array}$$

where

$$X_v = \{(x_n)_{n\in\mathbb{Z}}; (x_n v_n)_n \in X\}$$

and $\Phi_v : X_v \to X, (x_n)_{n\in\mathbb{Z}} \to (x_n v_n)_{n\in\mathbb{Z}}$ with

$$v_n = \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} \quad \text{for} \quad n \geq 1, v_n = \prod_{\nu=n+1}^{0} w_\nu \quad \text{for} \quad n \leq -1, v_0 = 1.$$

Theorem 0.5.11 *Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Suppose that the weighted shift B_w is an operator on X.*

1. *The following assertions are equivalent:*

   (i) B_w is hypercyclic;

   (ii) B_w is weakly mixing;

   (iii) there is an increasing sequence $(n_k)_k$ of positive integers such that, for any $j \in \mathbb{Z}$

$$\left(\prod_{\nu=j-n_k+1}^{j} w_\nu\right) e_{j-n_k} \to 0 \quad \textit{and} \quad \left(\prod_{\nu=j+1}^{j+n_k} w_\nu\right)^{-1} e_{j+n_k} \to 0$$

   in X as $k \to \infty$.

2. *The following assertions are equivalent:*

   (i) B_w is mixing;

   (ii) we have

$$\left(\prod_{\nu=-n+1}^{0} w_\nu\right) e_{-n} \to 0 \quad \textit{and} \quad \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n \to 0$$

   in X as $n \to \infty$.

3. *Suppose that the basis $(e_n)_n$ is unconditional. Then the following assertions are equivalent:*

   (i) *B_w is chaotic;*

   (ii) *the series*

$$\sum_{n=-\infty}^{0}\left(\prod_{\nu=n+1}^{0} w_\nu\right) e_n + \sum_{n=1}^{\infty}\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$$

   converges in X;

   (iii) *the sequence $(x_n)_{n\in\mathbb{Z}}$ with*

$$x_n = \prod_{\nu=n+1}^{0} w_\nu (n \le 0), x_n = \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} \quad (n \ge 1)$$

   belongs to X;

   (iv) *B_w has a nontrivial periodic point.*

Remark 0.5.12 *A weighted backward shift B_w is an operator on a sequence space $\ell^p(\mathbb{Z})$, $1 \le p < \infty$, if and only if the weights $w_n, n \in \mathbb{Z}$ are bounded. Such an operator is then hypercyclic, mixing or chaotic if and only if the following conditions, respectively, are satisfied. There exists $(n_k)_k$ such that for all $j \in \mathbb{Z}$:*

$$\lim_{k\to\infty} \prod_{\nu=j-n_k+1}^{j} w_\nu = 0 \quad \textit{and} \quad \lim_{k\to\infty} \prod_{\nu=j+1}^{j+n_k} |w_\nu| = \infty;$$

$$\lim_{n\to\infty} \prod_{\nu=-n+1}^{0} w_\nu = 0 \quad \textit{and} \quad \lim_{n\to\infty} \prod_{\nu=1}^{n} |w_\nu| = \infty;$$

$$\sum_{n=0}^{\infty} \prod_{\nu=-n+1}^{0} |w_\nu|^p < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{1}{\prod_{\nu=1}^{n} |w_\nu|^p} < \infty.$$

In particular, a symmetric weight (that is, one with $w_{-n} = w_n$ for all $n \ge 0$) never defines a hypercyclic weighted shift B_w on these spaces.

0.6 Weighted pseudo-shifts

In this section we study a more general kind of operators, the weighted pseudo-shifts operators.

Definition 0.6.1 *Let X, Y be topological sequence spaces over countable sets I and J. Then a continuous linear operator $T : X \to Y$ is called a* weighted pseudo-shift *if there is a sequence $(b_j)_{j\in J}$ of non-zero scalars and an injective mapping $\Phi : J \to I$ such that*

$$T(x_i)_{i\in I} = (b_j x_{\Phi(j)})_{j\in J}$$

for $(x_i) \in X$. We then write $T = T_{b,\Phi}$, and $(b_j)_{j\in J}$ is called the weight sequence.

Remark 0.6.2 *Every unilateral or bilateral weighted backward shift is a weighted pseudo-shift with $b_n = a_{n+1}$ and $\Phi(n) = n+1$, and every bilateral weighted forward shift is a weighted pseudo-shift with $b_n = a_{n-1}$ and $\Phi(n) = n-1$.*

Some well-known results about hypercyclicity of weighted pseudo-shifts will be shown. First of all, we need to recall that the family $(e_i)_{i\in I}$ of unit vectors is called an *M-basis* in a topological sequence space X over I if $span\{e_i : i \in I\}$ is a dense subspace of X [100]. We shall call $(e_i)_{i\in I}$ an *OP-basis* if it is an M-basis and if the family of coordinate projections $x \to x_i e_i (i \in I)$ on X is equicontinuous.

Definition 0.6.3 ([69]) *We recall that a sequence $(T_n)_{n\in\mathbb{N}_0}$ of continuous mappings $T_n : X \to Y$ between topological spaces X and Y is called* universal *if there is an element $x \in X$ such that the set $\{T_n x : n \in \mathbb{N}_0\}$ is dense in Y. The element x is called* universal for $(T_n)_{n\in\mathbb{N}_0}$.

Definition 0.6.4 *A sequence $(\Phi_n)_{n\in\mathbb{N}_0}$ of mappings $\Phi_n : J \to I$ is called a* run-away sequence *if for each pair of finite subsets $I_0 \subset I$ and $J_0 \subset J$ there exists an $n_0 \in \mathbb{N}_0$ such that, for every $n \geq n_0$, $\Phi_n(J_0) \cap I_0 = \emptyset$*

In [70], the author characterizes the universality of sequences of weighted pseudo-shifts.

Theorem 0.6.5 ([70]) *Let X and Y be F-sequence spaces over I and J, respectively, in which $(e_i)_{i\in I}$ and $(e_j)_{j\in J}$ are OP-bases. Let $T_n = T_{b_n,\Phi_n} : X \to Y \quad (n \in \mathbb{N}_0)$ be weighted pseudo-shifts with weights $b_n = (b_{n,j})_{j\in J}$. If $(\Phi_n)_n$ is a run-away sequence, then the following assertions are equivalent:*

(i) *the sequence* (T_n) *has a dense set of universal elements;*

(ii) *there exists an increasing sequence* (n_k) *of positive integers such that*

$$b_{n_k,j}^{-1} e_{\Phi_{n_k}(j)} \to 0 \quad in \quad X, j \in J$$

$$b_{n_k,\Phi_{n_k}^{-1}(i)} e_{\Phi_{n_k}^{-1}(j)} \to 0 \quad in \quad Y, i \in I$$

as $k \to \infty$.

Clearly, an operator $T : X \to X$ is hypercyclic if and only if the sequence $(T^n)_n$ is universal.

Theorem 0.6.6 ([70]) *Let* X *be an* F*-sequence space over* I *in which* $(e_i)_{i \in I}$ *is an* OP*-basis. Let* $T = T_{b,\Phi} : X \to X$ *be a weighted pseudo-shift. Then the following assertions are equivalent:*

(i) T *is hypercyclic;*

(ii) (1) *the mapping* $\Phi : I \to I$ *has no periodic points;*

(2) *there exists an increasing sequence* (n_k) *of positive integers such that, for every* $i \in I$,

$$\left(\prod_{\nu=0}^{n_k-1} b_{\Phi^\nu(i)} \right)^{-1} e_{\Phi^{n_k}(i)} \to 0$$

$$\left(\prod_{\nu=1}^{n_k} b_{\Phi^\nu(i)} \right) e_{\Phi^{n_k}(i)} \to 0$$

in X*, as* $k \to \infty$.

0.7 C_0-semigroups

In this section we study dynamical properties of strongly continuous semigroups of operators on Banach spaces, that is, for C_0-semigroups. They can viewed as the continuous-time analogue of the discrete-time case of iterates of a single operator. All these results about C_0-semigroups can be found in the books of Engel and Nagel ([53] and [52]) and in [72].

Definition 0.7.1 *A one-parameter family* $(T_t)_{t \geq 0}$ *of operators on a Banach space* X *is called a* strongly continuous semigroup of operators *if the following three conditions are satisfied:*

(i) $T_0 = I$

(ii) $T_t T_s = T_{t+s}$ *for all* $t, s \geq 0$

(iii) $\lim_{s\to t} T_s x = T_t x$ *for all* $x \in X$ *and* $t \geq 0$

One also refers to it as a C_0*-semigroup.*

A systematic study of the dynamical properties of semigroups, was started by Desch, Schappacher and Webb [45]. In particular they introduced the notions of hypercyclicity and chaos for semigroups.

Definition 0.7.2 *Let* $(T_t)_{t\geq 0}$ *be a* C_0*-semigroup on* X*.*

(i) The semigroup is hypercyclic *if there is some* $x \in X$ *whose orbit* $Orb(x, T_t) = \{T_t x; t \geq 0\}$ *is dense in* X*. In such a case,* x *is called a* hypercyclic vector *for* $(T_t)_{t\geq 0}$*.*

(ii) The semigroup is called topologically transitive *if for any pair* U, V *of nonempty open sets of* X*, there exists some* $t_0 \geq 0$ *such that* $T_{t_0}(U) \cap V \neq \emptyset$*.*

(iii) The semigroup is mixing *if, for any pair* U, V *of nonempty open sets of* X*, there exists some* $t_0 \geq 0$ *such that* $T_t(U) \cap V \neq \emptyset$ *for all* $t \geq t_0$*.*

(iv) The semigroup is weakly mixing *if* $(T_t \oplus T_t)_{t\geq 0}$ *is topologically transitive on* $X \oplus X$*.*

(v) A point $x \in X$ *is called a* periodic point *of* $(T_t)_{t\geq 0}$ *if there is some* $t_0 > 0$ *such that* $T_{t_0} x = x$*.*

(vi) The semigroup is said to be chaotic *if it is hypercyclic and its set of periodic points is dense in* X*.*

(vii) Let $(S_t)_{t\geq 0}$ *be a* C_0*-semigroup on a Banach space* Y*. Then* $(T_t)_{t\geq 0}$ *is called* quasiconjugate *to* $(S_t)_{t\geq 0}$ *if there exists a continuous map* $\Phi : Y \to X$ *with dense range such that* $T_t \circ \Phi = \Phi \circ S_t$ *for all* $t \geq 0$*. If* Φ *can be chosen to be a homeomorphism then* $(T_t)_{t\geq 0}$ *and* $(S_t)_{t\geq 0}$ *are called* conjugate*.*

Proposition 0.7.3 *Hypercyclicity, mixing, weak mixing and chaos for a* C_0*-semigroup are preserved under quasiconjugacy.*

Now, we give some measure-theoretic properties properties related to C_0-semigroups defined on a probability space $(X, \mathfrak{B}, \mu)$, where X is a Banach space and $\mathfrak{B}$ denotes the σ-algebra of Borel subsets of X.

Definition 0.7.4 *(i) A C_0-semigroup is called* measure-preserving*, or in other words, μ is T_t-invariant, if $\mu(A) = \mu(T_t^{-1}(A))$ for all $t \geq 0$ and for all $A \in \mathfrak{B}$.*

(ii) A C_0-semigroup is strongly mixing with respect to μ *if it is measure-preserving and*

$$\lim_{t\to\infty} \mu(A \cap T_t^{-1}(B)) = \mu(A)\mu(B) \qquad (A, B \in \mathfrak{B}).$$

(iii) A C_0-semigroup is ergodic with respect to μ *if it is measure-preserving and satisfies one of the following equivalent conditions:*

(i) Given any measurable sets A, B with positive measures, there exists $t_0 \geq 0$ such that $T_{t_0}(A) \bigcap B \neq \emptyset$;

(ii) if $A \in \mathfrak{B}(X)$ satisfies $T_t(A) \subset A$ for all $t \geq 0$, then $\mu(A)(1 - \mu(A)) = 0$.

0.7.1 Criteria for hypercyclicity and chaos of C_0-semigroups

The first criteria for hypercyclicicty of C_0-semigroups were found by Desch, Schappacher and Webb [45]. In the form that we give, the Hypercyclicity criterion is due to Conejero and Peris [39] and El Mourchid [49], while the criterion for mixing, is due to Bermúdez, Bonilla, Conejero and Peris [24].

Theorem 0.7.5 (Hypercyclicity Criterion for semigroups ([39],[49])) *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If there are dense subsets $X_0, Y_0 \subset X$, a sequence $(t_n)_n \in \mathbb{R}_+$ with $t_n \to \infty$, and maps $S_{t_n} : Y_0 \to X, n \in \mathbb{N}$, such that, for any $x \in X_0, y \in Y_0$,*

(i) $T_{t_n} x \to 0$,

(ii) $S_{t_n} y \to 0$,

(iii) $T_{t_n} S_{t_n} y \to y$,

then $(T_t)_{t\geq 0}$ is weakly mixing, and in particular hypercyclic.

If in the Hypercyclicity criterion one has convergence along the whole real line then we obtain a criterion for mixing.

Theorem 0.7.6 ([24]) *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If there are dense subsets $X_0, Y_0 \subset X$, and maps $S_t : Y_0 \to X, t \geq 0$, such that, for any $x \in X_0, y \in Y_0$,*

(i) $T_t x \to 0$,

(ii) $S_t y \to 0$,

(iii) $T_t S_t y \to y$,

then $(T_t)_{t\geq 0}$ is mixing.

Now let $(T_t)_{t\geq 0}$ be an arbitrary C_0-semigroup on X. It can be shown that

$$Ax := \lim_{t\to 0} \frac{1}{t}(T_t x - x)$$

exists on a dense subspace of X; the set of these x, the domain of A is denoted by $D(A)$. Then A, or rather $(A, D(A))$, is called the *infinitesimal generator of the semigroup.* Moreover $T_t(D(A)) \subset D(A)$ with $AT_t x = T_t Ax$, for every $t \geq 0$ and $x \in D(A)$, see for instance [104]. Another important property is provided by the point spectral mapping theorem for semigroups. If X is a complex Banach space then, for every $x \in D(A)$ and $\lambda \in \mathbb{C}$,

$$Ax = \lambda x \Rightarrow T_t x = e^{\lambda t} x$$

for every $t \geq 0$.

Sometimes the Hypercyclicity Criterion is hard to be applied. In many situations we can obtain the infinitesimal generator of a semigroup although we do not have the explicit representation of its operators. Desch, Schappacher, and Webb gave a criterion which permits us to state Devaney chaos (and hypercyclicity) of a C_0-semigroup in terms of the abundance of eigenvectors of the infinitesimal generator [45].

Theorem 0.7.7 ([45]) *Let X be a complex separable Banach space, and $(T_t)_{t\geq 0}$ a C_0-semigroup on X with generator $(A, D(A))$. Assume that there exists an open connected subset U and weakly holomorphic functions $f_j : U \to X, j \in J$, such that*

(i) $U \bigcap i\mathbb{R} \neq \emptyset$,

(ii) $f_j(\lambda) \in Ker(\lambda I - A)$ *for every* $\lambda \in U; j \in J$,

(iii) for any $x^* \in X^*$, if $\langle f_j(\lambda), x^* \rangle = 0$ for all $\lambda \in U$ and $j \in J$ then $x^* = 0$,

then the semigroup $(T_t)_{t\geq 0}$ *is mixing and chaotic.*

A more general version of this criterion can be found in [50].

Theorem 0.7.8 *Let* X *be a complex separable Banach space, and* $(T_t)_{t\geq 0}$ *a* C_0*-semigroup on* X *with generator* $(A, D(A))$*. Assume that there are* $a < b$ *and continuous functions* $f_j : [a, b] \to X$, $j \in J$, *such that*

(i) $f_j(s) \in Ker(isI - A)$ *for every* $s \in [a, b]$, $j \in J$,

(ii) $\mathrm{span}\{f_j(s); s \in [a, b], j \in J\}$ *is dense in* X,

then the semigroup $(T_t)_{t\geq 0}$ *is mixing and chaotic.*

0.8 Frequent Hypercyclicity

The concept of frequent hypercyclicity was introduced by Bayart and Grivaux [18] inspired by Birkhoff's Ergodic Theorem.

Theorem 0.8.1 (Birkhoff's Ergodic Theorem, [28]) *Let* T *be an operator on a Fréchet space* X *ergodic respect to* μ *then, for any* μ*-integrable function* f *on* X*, its time average with respect to* T *coincides with its space average; more precisely*

$$\frac{1}{N+1}\sum_{n=0}^{N} f(T^n x) \to \int_X f d\mu$$

for μ*-almost all* $x \in X$ *as* $N \to \infty$.

First of all we recall the following definition:

Definition 0.8.2 *The* lower density *of a subset* $A \subset \mathbb{N}_0$ *is defined as*

$$\underline{dens}(A) = \liminf_{N\to\infty} \frac{card\{0 \leq n \leq N; n \in A\}}{N+1}.$$

Definition 0.8.3 *An operator* T *on a Fréchet space* X *is called* frequently hypercyclic *if there is some* $x \in X$ *such that, for any nonempty open subset* U *of* X,

$$\underline{dens}\{n \in \mathbb{N}_0; T^n x \in U\} > 0.$$

In this case, x *is called a* frequently hypercyclic vector *for* T*. The set of frequently hypercyclic vectors of* T *is denoted by* $FHC(T)$*.*

Proposition 0.8.4 *A vector x is frequently hypercyclic for T if and only if, for any nonempty open subset U of X, there is a strictly increasing sequence $(n_k)_k$ of positive integers such that*

$$T^{n_k}x \in U \quad \textit{for all} \quad k \in \mathbb{N}, \textit{and} \quad n_k = O(k).$$

By contrast, T is hypercyclic if and only if the same is true for some $(n_k)_k$, not necessarily of order $O(k)$.

Definition 0.8.5 *We recall that a sequence $(T_n)_n$ of continuous mappings between topological spaces X and Y is* called frequently universal *if there exists $x \in X$ such that for every non-empty open set $U \subseteq Y$,*

$$\underline{dens}\{n \in \mathbb{N}_0 \, : \, T_n x \in U\} > 0.$$

In this case, x is called a frequently universal vector *for $(T_n)_{n\in\mathbb{N}_0}$.*

The first ones that used ergodic theory for the dynamics of linear operators were Rudnicki [94] and Flytzanis [56]. The notion of frequent hypercyclicity was extended to C_0-semigroups in [3]. We recall the corresponding notion of lower density for a subset of $\mathbb{R}_+$.

Definition 0.8.6 *The lower density of a measurable set $M \subset \mathbb{R}_+$ is defined by*

$$\underline{Dens}(M) := \liminf_{N\to\infty} \frac{\lambda(M \bigcap [0, N])}{N},$$

where λ is the Lebesgue measure on $\mathbb{R}_+$.

Definition 0.8.7 *A C_0-semigroup $(T_t)_{t\geq 0}$ is said to be* frequently hypercyclic *if there exists $x \in X$ such that $\underline{Dens}(\{t \in \mathbb{R}_+ \; ; \; T_t x \in U\}) > 0$ for any non-empty open set $U \subset X$.*

Proposition 0.8.8 *Frequent hypercyclicity is preserved by quasiconjugacy.*

0.9 Lattices

In this section we remind some basic definitions about lattices, that will be very useful in chapter 4. All these results can be found in [57] and [97].

Definition 0.9.1 *A* lattice *is a non-empty set M with an order $\leq$ such that every pair of elements $x, y \in M$ has both a supremum and an infimum.*

Definition 0.9.2 *An* ordered vector space *is a real vector space E which is also an ordered space with the linear and order structures such that:*

- *If $x, y, z \in E$ and $x \leq y$ then $x + z \leq y + z$.*
- *If $x, y \in E$, $x \leq y$ and $0 \leq \alpha \in \mathbb{R}$ then $\alpha x \leq \alpha y$.*

The set $E^+ = \{x \in E : x \geq 0\}$ is termed the positive cone in E and its elements are termed positive. An ordered vector space which is also a lattice is a vector lattice.

Definition 0.9.3 *If E and F are vector lattices then $T : E \to F$ is positive if $x \geq 0$ implies $Tx \geq 0$.*

Definition 0.9.4 *(i) A* normed lattice *is a normed space which is also a vector lattice in which $x \leq y$ implies $\|x\| \leq \|y\|$. A normed lattice which is also a Banach space is called a* Banach lattice.

(ii) A Fréchet lattice *is a Fréchet space that is a vector lattice and carries an increasing sequence of seminorms $(p_n)_n$ such that $x \leq y$ implies $p_n(x) \leq p_n(y)$ for all $n \in \mathbb{N}$.*

0.10 Pettis Integral

In this section we recall the main definitions and results about Pettis integrability. The proofs of all these results can be found in [47] for the case of a finite measure space, but they easily extend to σ-finite measure spaces. Let X be a Banach space and (Ω, μ) a σ-finite measure space.

Definition 0.10.1 *(i) A function $f : \Omega \to X$ is said to be* weakly μ-measurable *if the scalar function $\varphi \circ f$ is μ-measurable for every $\varphi \in X^*$, where X^* denotes the topological dual of X*

(ii) f is said to be μ-measurable *if there exists a sequence $(f_n)_n$ of simple functions such that $\lim_{n\to\infty} \|f_n - f\| = 0$ μ-a.e.*

Lemma 0.10.2 (Dunford's lemma) *Let f be weakly μ-measurable and $\varphi \circ f \in L_1(\Omega, \mu)$ for every $\varphi \in X^*$, then for every measurable $E \subseteq \Omega$ there exists $x_E \in X^{**}$ such that*

$$x_E(\varphi) = \int_E \varphi \circ f d\mu,$$

for every $\varphi \in X^*$.

Definition 0.10.3 *(i) If $f : \Omega \to X$ is weakly μ-measurable and $\varphi \circ f \in L_1(\Omega, \mu)$ for every $\varphi \in X^*$, then f is called* Dunford integrable*. The Dunford integral of f over a measurable $E \subseteq \Omega$ is defined by the element $x_E \in X^{**}$ such that $x_E(\varphi) = \int_E \varphi \circ f d\mu$, for every $\varphi \in X^*$.*

(ii) In the case that $x_E \in X$ for every measurable E, then f is said to be Pettis integrable *and x_E is called the Pettis integral of f over E, which is denoted by $(P) - \int_E f d\mu$.*

(iii) If $\|f\|$ is integrable on Ω, then f is said to be Bochner integrable *on Ω.*

Clearly the Dunford and Pettis integrals coincide if X is a reflexive space, and if f is *Bochner integrable*, then it is Pettis integrable. Some basic and useful results to characterize Pettis integral are the following:

Theorem 0.10.4 *If f is Pettis integrable, then for every sequence $(E_n)_n$ of disjoint measurable sets in Ω*

$$\int_{\bigcup_{n\in\mathbb{N}} E_n} f d\mu = \sum_{n\in\mathbb{N}} \int_{E_n} f d\mu,$$

where the series converges unconditionally.

As a consequence,

Corollary 0.10.5 *If $f : [0, +\infty[\to X$ is Pettis integrable on $[0, +\infty[$, then for every $\varepsilon > 0$ there exists $N > 0$ such that for every compact set $K \subset [N, +\infty)$*

$$\left\| \int_K f(t)dt \right\| < \varepsilon.$$

Chapter 1

Strong mixing measures for linear operators and frequent hypercyclicity

1.1 Introduction

In this chapter we construct strongly mixing invariant measures with full support for operators on F-spaces which satisfy the Frequent Hypercyclicity Criterion. In order to obtain this results we need to use ergodic theory (see section 0.3). The contents of this chapter have been published in [88].

First of all, we recall the different versions of the Frequent Hypercyclicity Criterion. Bayart and Grivaux gave the definition of a frequently hypercyclic operator (see 0.8) but they also gave the first version of the Frequent Hypercyclicity Criterion.

Theorem 1.1.1 ([18]) *Let T be an operator on a separable F-space X and d a translation invariant metric which makes it complete. If there is a dense sequence $(x_l)_{l\geq 1}$ of vectors of X and a map S defined on X such that,*

(i) $\sum_{k\geq 1} d(T^k x_l, 0)$ *is convergent for every* $l \geq 1$,

(ii) $\sum_{k\geq 1} d(S^k x_l, 0)$ *is convergent for every* $l \geq 1$, *and*

(iii) $TS = I$,

then T is frequently hypercyclic.

Another (probabilistic) version of it was given by Grivaux .

Theorem 1.1.2 ([66]) *Let $(\Omega, \mathfrak{F}, \mu)$ be a probabilistic space, and let $(g_k)_{k\geq 0}$ be a sequence of independent real-valued standard Gaussian random variables. Let T be an operator on a infinite dimensional real or complex Banach space X. If there is a dense sequence $(x_l)_{l\geq 1}$ of vectors of X and a map S defined on X such that,*

(i) for every $l \geq 1$, the series $\sum_{k\geq 1} g_k(\omega)T^k x_l$ converges almost everywhere,

(ii) for every $l \geq 1$, the series $\sum_{k\geq 1} g_k(\omega)S^k x_l$ converges almost everywhere, and

(iii) $TS = I$,

then T is frequently hypercyclic.

In order to obtain our main result we will consider the formulation of Bonilla and Grosse-Erdmann for operators on separable F-spaces.

Theorem 1.1.3 ([33]) *Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ converges unconditionally,

(ii) $\sum_{n=0}^{\infty} S_n x$ converges unconditionally, and

(iii) $T^n S_n x = x$ and $T^m S_n x = S_{n-m} x$ if $n > m$,

then T is frequently hypercyclic.

This theorem allows us to show that some of the classical hypercyclic operators such as MacLane's and Rolewicz's operators are frequently hypercyclic.

Example 1.1.4 (**MacLane's operator**) The differentiation operator D on $H(\mathbb{C})$ is frequently hypercyclic. Let X_0 be the set of polynomials and S the operator $Sf(z) = \int_0^z f(\zeta)d\zeta$. Condition (i) of the Frequent Hypercyclicity Criterion is satisfied since any finite series converges unconditionally, and (iii) is trivial. For (ii) we need only consider the monomial, for which we find that $\sum_{n=0}^{\infty} S^n(z^k) = k! \sum_{n=0}^{\infty} \frac{1}{(k+n)!} z^{k+n}$, which converges uniformly and unconditionally on any compact set.

1.2 Invariant measures and the frequent hypercyclicity criterion

In this section under the hypothesis of Bonilla and Grosse-Erdmann criterion we derive a stronger result by showing that a T-invariant mixing measure can be obtained.

For the existence of strong mixing measures with full support, certain Cantor subsets of $\mathbb{N}^N$, with either $N = \mathbb{N}$ or $N = \mathbb{Z}$, will be needed. Actually they will be of the form $C = \prod_{n \in N} F_n$, where the cardinalities of the finite sets F_n tend to infinity as $n \to \infty$.

From now on, T will be an operator defined on a separable F-space X.

We are now ready to present our main result.

The idea behind the proof is to construct, given T an operator on a separable F-space X satisfying the hypothesis of Theorem 1.1.3,

1. a "model" probability space $(Z, \overline{\mu})$ and
2. a Borel measurable map $\Phi : Z \to X$ with dense range,

where

- σ is the Bernoulli shift defined as

$$\sigma(\dots, n_{-1}, n_0, n_1, \dots) = (\dots, n_0, n_1, n_2, \dots),$$

- $Z \subset \mathbb{N}^{\mathbb{Z}}$ is a σ-invariant subset of the space $\mathbb{N}^{\mathbb{Z}}$ of bilateral sequences with the product topology,
- $\overline{\mu}$ is a σ^{-1}-invariant strongly mixing measure with full support, and
- $\Phi\sigma^{-1} = T\Phi$ on Z.

As a consequence, the Borel probability measure μ on X defined by $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$, $A \in \mathfrak{B}(X)$, is T-invariant, strongly mixing and has full support.

Theorem 1.2.1 *Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*

(ii) $\sum_{n=0}^{\infty} S_n x$ *converges unconditionally, and*

(iii) $T^n S_n x = x$ *and* $T^m S_n x = S_{n-m} x$ *if* $n > m$,

then there is a T-invariant strongly mixing Borel probability measure μ on X with full support.

Proof.

We suppose $X_0 = \{x_n \ ; \ n \in \mathbb{N}\}$ with $x_1 = 0$ and $S_n 0 = 0$ for all $n \in \mathbb{N}$. Let $(U_n)_n$ be a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$. By (i) and (ii), there exists an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k > N_n} T^k x_{m_k} \in U_{n+1} \text{ and } \sum_{k > N_n} S_k x_{m_k} \in U_{n+1},$$
$$\text{if } m_k \leq 2l, \text{ for } N_l < k \leq N_{l+1}, \ l \geq n. \tag{1.1}$$

Actually, this is a consequence of the completeness of X and the fact that, for each 0-neighbourhood U and for all $l \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that $\sum_{k \in F} T^k x \in U$ and $\sum_{k \in F} S_k x \in U$ for any finite subset $F \subset [N, +\infty)$ and for each $x \in \{x_1, \ldots, x_{2l}\}$.

Indeed, by the definition of unconditional convergence 0.5.5, for each $\epsilon_n = \frac{1}{(2n)2^{n+2}}$, there exists N_n such that:

$$\left\| \sum_{k \in F \subset [N_n, \infty)} T^k x \right\| < \epsilon_n \text{ and } \left\| \sum_{k \in F \subset [N_n, \infty)} S_k x \right\| < \epsilon_n$$

for every finite set F and every $x \in \{x_1, \ldots, x_{2n}\}$. Let $||.||$ denotes the F-norm defined on the space. Without loss of generality we can select the sequence $(N_n)_n$ such that $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$.

In 1.1 we have that:

$$\sum_{k>N_n} T^k x_{m_k} = \sum_{j=n}^{\infty} \sum_{N_j<k\leq N_{j+1}} T^k x_{m_k} \quad \text{and}$$
$$\sum_{k>N_n} S_k x_{m_k} = \sum_{j=n}^{\infty} \sum_{N_j<k\leq N_{j+1}} S_k x_{m_k} \tag{1.2}$$

So we have for each $j \geq n$:

$$\left\| \sum_{N_j<k\leq N_{j+1}} T^k x_{m_k} \right\| \leq \left\| \sum_{N_j<k\leq N_{j+1}} T^k x_1 \right\|$$
$$+ \quad \ldots + \left\| \sum_{N_j<k\leq N_{j+1}} T^k x_{2j} \right\| \leq \left(\sum_{i=1}^{2j} \epsilon_j \right) \leq \frac{1}{2^{j+2}}$$

Analogously, we get :

$$\left\| \sum_{N_j<k\leq N_{j+1}} S_k x_{m_k} \right\| \leq \frac{1}{2^{j+2}}.$$

Finally in 1.2:

$$\left\| \sum_{k>N_n} T^k x_{m_k} \right\| \leq \sum_{j=n}^{\infty} \frac{1}{2^{j+2}} = \frac{1}{2^{n+1}} \quad \text{and}$$
$$\left\| \sum_{k>N_n} S_k x_{m_k} \right\| \leq \sum_{j=n}^{\infty} \frac{1}{2^{j+2}} = \frac{1}{2^{n+1}}$$

and we conclude the result in 1.1.

1.-The model probability space $(Z, \overline{\mu})$.

We define the space $K = \prod_{k\in\mathbb{Z}} F_k$, which is compact when endowed with the product topology inherited from $\mathbb{N}^{\mathbb{Z}}$, where

$$F_k = \{1, \ldots m\} \text{ if } N_m < |k| \leq N_{m+1},\ m \in \mathbb{N},\ \text{and } F_k = \{1\},\ \text{if } |k| \leq N_1.$$

Let $K(s) := \sigma^s(K)$, $s \in \mathbb{Z}$, where $\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ is the backward shift. $K(s)$ is a subspace of $\mathbb{N}^{\mathbb{Z}}$, $s \in \mathbb{Z}$.

We consider in $\mathbb{N}^{\mathbb{Z}}$ the product measure $\overline{\mu} = \bigotimes_{k\in\mathbb{Z}} \overline{\mu_k}$, where $\overline{\mu_k}(\{n\}) = p_n$ for all $n \in \mathbb{N}$ and $\overline{\mu_k}(\mathbb{N}) = \sum_{n=1}^{\infty} p_n = 1$, $k \in \mathbb{Z}$. The values of $p_n \in]0,1[$ are selected such that, if

$$\beta_j := \left(\sum_{i=1}^{j} p_i\right)^{N_{j+1}-N_j}, \ j \in \mathbb{N}, \text{ then } \prod_{j=1}^{\infty} \beta_j > 0.$$

Let $Z = \bigcup_{s\in\mathbb{Z}} K(s)$, which is a countable union of Cantor sets, invariant under the shift, and satisfies

$$\begin{aligned}\overline{\mu}(Z) \geq \overline{\mu}(K) &= \prod_{|k|\leq N_1} \overline{\mu}_k(\{1\}) \prod_{l=1}^{\infty} \left(\prod_{N_l<|k|\leq N_{l+1}} \overline{\mu}_k(\{1,\ldots,l\}) \right) \\ &= p_1^{2N_1+1} \left(\prod_{l=1}^{\infty} \beta_l\right)^2 > 0.\end{aligned}$$

It is well-known [102] that $\overline{\mu}$ is a σ^{-1}-invariant strongly mixing Borel probability measure. Since $\sigma(Z) = Z$, Z has positive measure, and every strong mixing measure is ergodic (0.3.2), we necessarily have that $\overline{\mu}(Z) = 1$.

2.-The map Φ.

Given $s \in \mathbb{Z}$ we define the map $\Phi : K(s) \to X$ by

$$\Phi((n_k)_{k\in\mathbb{Z}}) = \sum_{k<0} S_{-k} x_{n_k} + x_{n_0} + \sum_{k>0} T^k x_{n_k}. \tag{1.3}$$

Φ is well-defined since, given $(n_k)_{k\in\mathbb{Z}} \in K(s)$ and for $l \geq |s|$, we have $n_k \leq 2l$ if $N_l < |k| \leq N_{l+1}$, which shows the convergence of the series in (1.3) by (1.1).

$\Phi|_{K(s)}$ is also continuous for each $s \in \mathbb{Z}$. Indeed, let $(\alpha(j))_j$ be a sequence of elements of $K(s)$ that converges to $\alpha \in K(s)$ and fix any $n \in \mathbb{N}$ with $n > |s|$. We will find $n_0 \in \mathbb{N}$ such that $\Phi(\alpha(j)) - \Phi(\alpha) \in U_n$ for $j \geq n_0$. To do this, by definition of the topology in $K(s)$ there exists $n_0 \in \mathbb{N}$ such that

$$\alpha(j)_k = \alpha_k \quad \text{if} \quad |k| \leq N_{n+1} \text{ and } j \geq n_0.$$

By (1.1) we have

$$\Phi(\alpha(j)) - \Phi(\alpha) = \sum_{k<-N_{n+1}} S_{-k}(x_{\alpha(j)_k} - x_{\alpha_k}) + \sum_{k>N_{n+1}} T^k(x_{\alpha(j)_k} - x_{\alpha_k}) \in U_n$$

for all $j \geq n_0$. This shows the continuity of $\Phi : K(s) \to X$ for every $s \in \mathbb{Z}$.

The map Φ is then well-defined on Z, and $\Phi : Z \to X$ is measurable (i.e., $\Phi^{-1}(A) \in \mathfrak{B}(Z)$ for every $A \in \mathfrak{B}(X)$).

3.-The measure μ on X.

Let us define $L(s) := \Phi(K(s))$, $s \in \mathbb{Z}$, and $Y := \bigcup_{s\in\mathbb{Z}} L(s) = \Phi(Z)$ is a T-invariant Borel subset of X because it is a countable union of Cantor sets and $\Phi\sigma^{-1} = T\Phi$. Indeed, let $(n_k)_k$ be a sequence of $K(s)$, then:

$$T\Phi((n_k)_k) = \sum_{k<0} TS_{-k}x_{n_k} + Tx_{n_0} + \sum_{k>0} T^{k+1}x_{n_k} =$$
$$\sum_{k<-1} S_{-k-1}x_{n_k} + x_{n_{-1}} + \sum_{k\geq 0} T^{k+1}x_{n_k} = \Phi\sigma^{-1}((n_k)_k).$$

We then define on X the measure $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$ for all $A \in \mathfrak{B}(X)$. Since we know that $\overline{\mu}$ is strongly mixing on Z we have that μ is well-defined and it is a T-invariant strongly mixing Borel probability measure. It is T-invariant because given $A \in \mathfrak{B}(X)$, we have that $\mu(T^{-1}(A)) = \overline{\mu}(\Phi^{-1}T^{-1}(A)) = \overline{\mu}(\sigma\Phi^{-1}(A)) = \overline{\mu}(\Phi^{-1}(A)) = \mu(A)$. Finally, it is strongly-mixing, because given $A, B \in \mathfrak{B}(X)$ we have:

$$\begin{aligned}
&\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \lim_{n\to\infty} \overline{\mu}(\Phi^{-1}(A \cap T^{-n}(B)))\\
&= \lim_{n\to\infty} \overline{\mu}(\Phi^{-1}(A) \cap \Phi^{-1}(T^{-n}(B))) = \lim_{n\to\infty} \overline{\mu}(\Phi^{-1}(A) \cap \sigma^n(\Phi^{-1}(B)))\\
&= \overline{\mu}(\Phi^{-1}(A))\overline{\mu}(\Phi^{-1}(B)) = \mu(A)\mu(B).
\end{aligned}$$

The proof is completed by showing that μ has full support. Indeed, given a non-empty open set U in X, we pick $n \in \mathbb{N}$ satisfying $x_n + U_n \subset U$. Thus, $\mu(U)$ is greater than :

$$\begin{aligned}
&\mu(\{x = x_n + \sum_{k>N_n} T^k x_{m_k} + \sum_{k>N_n} S_k x_{m_k} \ ; \ m_k \leq 2l; N_l < k \leq N_{l+1},\ l \geq n\})\\
&\geq \overline{\mu}_0(\{n\}) \prod_{0<|k|\leq N_n} \overline{\mu}_k(\{1\}) \prod_{l=n}^{\infty} \left(\prod_{N_l<|k|\leq N_{l+1}} \overline{\mu}_k(\{1,\ldots,2l\}) \right)\\
&\geq p_n p_1^{2N_n} \left(\prod_{l=n}^{\infty} \beta_{2l} \right)^2 > 0
\end{aligned}$$

and we obtain that μ has full support. $\square$

Recently, Bayart and Matheron gave very general conditions expressed on eigenvector fields associated to unimodular eigenvalues under which an operator T admits a T-invariant mixing measure [21]. A $\mathbb{T}$-eigenvectors field for T is a map $E : \wedge \to X$ defined on some set $\wedge \subset \mathbb{T}$, such that

$$TE(\lambda) = \lambda E(\lambda)$$

for every $\lambda \in \wedge$.

A complex Borel measure μ on $\mathbb{T}$ is said to be Rajchman, if $\lim_{|n|\to+\infty} \hat{\mu}(n) = 0$, where $\hat{\mu}(n) = \int e^{-inx} d\mu, n \in \mathbb{Z}$. A set $\wedge \subset \mathbb{T}$ is called a set of *extended uniqueness* if for every positive Rajchman measure μ we have $\mu(\wedge) = 0$. Recall also that a closed set $\wedge \subset \mathbb{T}$ is *perfect* if it has no isolated points or, equivalently, if $V \cap \wedge$ is uncountable for any open set $V \subset \mathbb{T}$ sucht that $V \cap \wedge \neq \emptyset$. Analogously, a closed set $\wedge \subset \mathbb{T}$ is $\mathcal{U}_0$*-perfect* if $V \cap \wedge$ is not a set of extended uniqueness for any open set V such that $V \cap \wedge \neq \emptyset$.
Given any property $\wp$ relative to measure-preserving transformations, an operator T has property $\wp$ in the Gaussian sense if there exists some Gaussian probability measure μ on X with full support with respect to which T has $\wp$.

Theorem 1.2.2 ([21]) *Let X be a separable complex Fréchet space, and let T be an operator on X. Assume that one has at hand a family of continuous $\mathbb{T}$-eigenvectors fields $(E_i)_{i\in I}$ for T, where $E_i : \wedge_i \to X$ is defined on some closed set $\wedge_i \subset \mathbb{T}$, such that span$(\cup E_i(\wedge_i))$ is dense in X.*

- *If each $\wedge_i$ is a perfect set, then T is weakly mixing in the Gaussian sense.*
- *If each $\wedge_i$ is a $\mathcal{U}_0$-perfect set, then T is strongly mixing in the Gaussian sense.*

Indeed, the argument of É. Matheron is the following:

Let $T : X \to X$ be an operator on a separable complex Fréchet space X satisfying the hypothesis of the Frequent Hypercyclicity Criterion given in Theorem 1.2.1, and suppose $X_0 = \{x_n \ ; \ n \in \mathbb{N}\}$. We define the following family of continuous $\mathbb{T}$-eigenvector fields for T

$$E_m(\lambda) = \sum_{n\in\mathbb{N}_0} \lambda^{-n} T^n x_m + \sum_{n\in\mathbb{N}} \lambda^n S_n x_m, \quad \lambda \in \mathbb{T}, \quad m \in \mathbb{N}.$$

They span X since, for any functional x^* that vanishes on $E_m(\lambda)$ for each $\lambda \in \mathbb{T}$ and $m \in \mathbb{N}$, the equality $\langle x^*, E_m(\lambda)\rangle = 0$ for fixed m and for all $\lambda \in \mathbb{T}$ implies that $\langle x^*, T^n x_m\rangle = 0$ for every $n \in \mathbb{N}_0$. Thus, $\langle x^*, x_m\rangle = 0$ for each $m \in \mathbb{N}$, and by density $x^* = 0$.

Actually, on the one hand our results can be deduced from [21] in the context of complex Fréchet spaces, and on the other hand we only need rather elementary tools.

1.3 Consequences

The previous theorem can be applied to different classes of operators. A distinguished one is the class of weighted shifts on F-sequence spaces.

Corollary 1.3.1 *Let $T : X \to X$ be a chaotic bilateral weighted backward shift on an F-sequence space X in which $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis. Then there exists a T-invariant strongly mixing Borel probability measure on X with full support.*

Proof. First of all, since T is a chaotic weighted backward shift by 0.5.11, the series

$$\sum_{n=-\infty}^{0}\left(\prod_{\nu=n+1}^{0} w_\nu\right) e_n + \sum_{n=1}^{\infty}\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$$

converges unconditionally in X. We choose as X_0 as the set of finite sequences, which is dense by assumption, and for $S_n = {F_w}^n$, where $F_w(x_n)_{n\in\mathbb{Z}} = \left(\frac{1}{w_n} x_{n-1}\right)_{n\in\mathbb{Z}}$. It is clear that $B_w \circ F_w = Id$.

By linearity we need to check hypothesis (i) and (ii) of theorem 1.2.1 for the sequences $e_k, k \in \mathbb{Z}$. But then:

$$\sum_{n=0}^{\infty} F_w^n e_k = \sum_{n=0}^{\infty} \frac{e_{k+n}}{w_{k+1}\cdots w_{k+n}} = \left(\prod_{\nu=1}^{k} w_\nu\right)\sum_{n=0}^{\infty}\left(\prod_{\nu=1}^{k+n} w_\nu\right)^{-1} e_{k+n}$$

and

$$\sum_{n=0}^{\infty} B_w^n e_k = \sum_{n=0}^{\infty} e_{k-n}(w_k \cdots w_{k-n+1}) = \sum_{n=-\infty}^{0} e_{k+n}\left(\prod_{\nu=k+n+1}^{k} w_\nu\right),$$

and by hypothesis these series converge unconditionally. The hypotheses of theorem 1.2.1 are satisfied and then, there exists a B_w-invariant strongly mixing Borel probability measure on X with full support. □

Remark 1.3.2 The preceding result can be improved if T is a unilateral weighted backward shift operator on a sequence F-space. In that case, there exists a T-invariant exact Borel probability measure on X with full support.

Theorem 1.3.3 *Let $T : X \to X$ be a chaotic unilateral weighted backward shift on an F-sequence space X in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. Then there exists a T-invariant exact measure on X with full support.*

Proof.

We fix a countable set $M = \{z_n \ ; \ n \in \mathbb{N}\}$ of pairwise different scalars which form a dense set in $\mathbb{K}$ with $z_1 = 0$, and a basis $(U_n)_n$ of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$. We will consider as U_n the 0-neighbourhood of radius $\frac{1}{2^n}$. Again, since T is chaotic, $\sum_{n=1}^{\infty} \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$ converges unconditionally (see 0.5.8), so there exists an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} \alpha_k \left(\prod_{\nu=1}^{k} w_\nu\right)^{-1} e_k \in U_{n+1},$$

$$\text{if } \alpha_k \in \{z_1, \dots, z_{2m}\}, \text{ for } N_m < k \leq N_{m+1}, \ m \geq n. \tag{1.4}$$

Indeed, by the definition of unconditional convergence (0.5.5), for each $\epsilon_n = \frac{1}{(1+\sum_{i=1}^{2n}|z_i|)2^{n+2}}$,$n \in \mathbb{N}$, there exists N_n such that:

$$\left\| \sum_{k\in F\subset[N_n,\infty)} \left(\prod_{\nu=1}^{k} w_\nu\right)^{-1} e_k \right\| < \epsilon_n$$

for every finite set F. Let $||.||$ denotes the F-norm defined on the space. Without loss of generality we can select the sequence $(N_n)_n$ such that $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$.

In 1.4 we have that:

$$\sum_{k>N_n} \alpha_k \left(\prod_{\nu=1}^{k} w_\nu\right)^{-1} e_k = \sum_{j=n}^{\infty} \sum_{N_j<k\leq N_{j+1}} \alpha_k \left(\prod_{\nu=1}^{k} w_\nu\right)^{-1} e_k \tag{1.5}$$

So we have for each $j \geq n$:

$$\left\| \sum_{N_j<k\leq N_{j+1}} \alpha_k \left(\prod_{\nu=1}^{k} w_\nu \right)^{-1} e_k \right\| \leq |z_1| \left\| \sum_{\alpha_k=z_1, N_j<k\leq N_{j+1}} \left(\prod_{\nu=1}^{k} w_\nu \right)^{-1} e_k \right\|$$

$$+ \quad \ldots + |z_{2j}| \left\| \sum_{\alpha_k=z_{2j}, N_j<k\leq N_{j+1}} \left(\prod_{\nu=1}^{k} w_\nu \right)^{-1} e_k \right\| \leq \left(\sum_{i=1}^{2j} |z_i| \right) \epsilon_j \leq \frac{1}{2^{j+2}}$$

Finally in 1.5:

$$\left\| \sum_{k>N_n} \alpha_k \left(\prod_{\nu=1}^{k} w_\nu \right)^{-1} e_k \right\| \leq \sum_{j=n}^{\infty} \frac{1}{2^{j+2}} = \frac{1}{2^{n+1}}.$$

and we conclude the result in 1.4.

We define the compact space $K = \prod_{k\in\mathbb{N}} F_k$ where

$F_k = \{1, \ldots m\}$ if $N_m < k \leq N_{m+1},\ m \in \mathbb{N},$ and $F_k = \{1\},$ if $1 \leq k \leq N_1$.

Let $K(s) := \sigma^s(K),\ s \in \mathbb{N}_0$, where $\sigma : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is the backward shift. $K(s)$ is a subspace of $\mathbb{N}^{\mathbb{N}}$, $s \in \mathbb{N}_0$.

We consider in $\mathbb{N}^{\mathbb{N}}$ the product probability measure $\overline{\mu} = \bigotimes_{k\in\mathbb{N}} \overline{\mu_k}$, where $\overline{\mu_k}(\{n\}) = p_n$ for all $n \in \mathbb{N}$ and $\overline{\mu_k}(\mathbb{N}) = \sum_{n=1}^{\infty} p_n = 1,\ k \in \mathbb{N}$. The values of $p_n \in]0,1[$ are selected such that, if

$$\beta_j := \left(\sum_{i=1}^{j} p_i \right)^{N_{j+1}-N_j},\ j \in \mathbb{N},\ \text{then}\ \prod_{j=1}^{\infty} \beta_j > 0.$$

Let $Z = \bigcup_{s\in\mathbb{N}_0} K(s)$, which is a countable union of Cantor sets, invariant under the shift, and satisfies

$$\begin{aligned} \overline{\mu}(Z) \geq \overline{\mu}(K) &= \prod_{k=1}^{N_1} \overline{\mu}_k(\{1\}) \prod_{l=1}^{\infty} \left(\prod_{N_l<k\leq N_{l+1}} \overline{\mu}_k(\{1,\ldots,l\}) \right) \\ &= p_1^{N_1} \left(\prod_{l=1}^{\infty} \beta_l \right) > 0. \end{aligned}$$

It is known [102] that $\overline{\mu}$ is a σ-invariant exact Borel probability measure. We set $Z = \bigcup_{s\geq 0} K(s)$. Since $\sigma(Z) = Z$, Z has positive measure and every exact measure is ergodic 0.3.2, we necessarily have that $\overline{\mu}(Z) = 1$.

Now we define the map $\Phi : K(s) \to X$ given by

$$\Phi((n_k)_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} \alpha_{n_k} \left(\prod_{\nu=1}^{k} w_\nu \right)^{-1} e_k.$$

Φ is well-defined for each $s \geq 0$ by 1.4. Indeed, given $(n_k)_{k\in\mathbb{N}} \in K(s)$ and for $l \geq s$, we have $n_k \leq 2l$ if $N_l < k \leq N_{l+1}$, which shows the convergence of the series $\sum_{k=1}^{\infty} \alpha_{n_k} \left(\prod_{\nu=1}^{k} w_\nu \right)^{-1} e_k$ by (1.4).

$\Phi|_{K(s)}$ is also continuous for each $s \in \mathbb{N}_0$. Indeed, let $(\rho(j))_j$ be a sequence of elements of $K(s)$ that converges to $\rho \in K(s)$ and fix any $n \in \mathbb{N}$ with $n > s$. We will find $n_0 \in \mathbb{N}$ such that $\Phi(\rho(j)) - \Phi(\rho) \in U_n$ for $j \geq n_0$. To do this, by definition of the topology in $K(s)$ there exists $n_0 \in \mathbb{N}$ such that

$$\rho(j)_k = \rho_k \quad \text{if} \quad 1 \leq k \leq N_{n+1} \text{ and } j \geq n_0.$$

By (1.4) we have

$$\Phi(\rho(j)) - \Phi(\rho) = \sum_{k > N_{n+1}} \alpha_{\rho(j)_k} \left(\prod_{\nu=1}^{k} w_\nu \right)^{-1} e_k - \sum_{k > N_{n+1}} \alpha_{\rho_k} \left(\prod_{\nu=1}^{k} w_\nu \right)^{-1} e_k.$$

And $\Phi(\rho(j)) - \Phi(\rho) \in U_n$ for all $j \geq n_0$. This shows the continuity of $\Phi : K(s) \to X$ for every $s \in \mathbb{N}_0$.

The map Φ is then well-defined on Z, and $\Phi : Z \to X$ is measurable (i.e., $\Phi^{-1}(A) \in \mathfrak{B}(Z)$ for every $A \in \mathfrak{B}(X)$).

$L(s) := \Phi(K(s))$ is compact in X, $s \geq 0$, and $Y := \bigcup_{s\geq 0} L(s) = \Phi(Z)$ is a B_ω-invariant Borel subset of X because it is a countable union of Cantor subsets of X and $\Phi\sigma = B_\omega \Phi$. Indeed, let $(n_k)_k$ be a sequence of $K(s)$, then:

$$\Phi\sigma((n_k)_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} \alpha_{n_{k+1}} \left(\prod_{\nu=1}^{k} w_\nu \right)^{-1} e_k = B_\omega \Phi((n_k)_{k\in\mathbb{N}}).$$

We then define on X the measure $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$ for all $A \in \mathfrak{B}(X)$. As in Theorem 1.2.1, we conclude that μ is well-defined on X, it is B_ω-invariant and exact. Indeed, given $A \in \bigcap_{n=0}^{\infty} B_\omega^{-n} \mathfrak{B}(X)$, we have for each $n \in \mathbb{N}_0$ that $A = B_\omega^{-n}(A_n)$, with $A_n \in \mathfrak{B}(X)$. Then we have that

$$\Phi^{-1}(A) \in \bigcap_{n=0}^{\infty} \sigma^{-n} \mathfrak{B}(\Phi^{-1}(X)).$$

By the exactness of μ, necessarily $\overline{\mu}(\Phi^{-1}(A)) = 0$ or $\overline{\mu}(\Phi^{-1}(A)) = 1$ and then $\mu(A) = 0$ or 1.

Now it only remains to show that μ has full support. Indeed, given a non-empty open set U, we pick $y = \Phi((n_k)_k) \in Y$ such that $y + U_n \subset U$ and then $\mu(U)$ is greater than:

$$\mu \quad \{y + \sum_{k \geq N_n} \frac{\alpha_k}{\prod_{\nu=1}^{k} w_\nu} e_k : \alpha_k \in \{z_1, \ldots z_{2m}\}, N_m < k \leq N_{m+1}, m \geq n\}$$

$$\geq \prod_{k=1}^{N_n} \overline{\mu}_k(\{n_k\}) \prod_{l=n}^{\infty} \left(\prod_{N_l < k \leq N_{l+1}} \overline{\mu}_k(\{1, \ldots, 2l\}) \right) \geq \prod_{k=1}^{N_n} p_{n_k} \left(\prod_{l=n}^{\infty} \beta_{2l} \right) > 0.$$

$\square$

Devaney chaos is therefore a sufficient condition for the existence of strongly mixing measures within the framework of weighted shift operators on sequence F-spaces. In some natural spaces it is even a characterization of this fact. For instance, F. Bayart and I. Z. Ruzsa [22] recently proved that weighted shift operators on ℓ^p, $1 \leq p < \infty$, are frequently hypercyclic if, and only if, they are Devaney chaotic. It turns out that this is equivalent to the existence of an invariant strongly mixing Borel probability measure with full support on ℓ^p. Also, for the space ω, every weighted shift operator is chaotic [70]. In particular, for the unilateral case we obtain exact measures.

Example 1.3.4 Every unilateral weighted backward shift operator on $\omega = \mathbb{K}^{\mathbb{N}}$ admits an invariant exact Borel probability measure with full support on ω.

Chapter 2

Strong mixing measures for C_0-semigroups and frequent hypercyclicity

2.1 Introduction

In this chapter, our purpose is to obtain a very effective and general method to prove that certain C_0-semigroups admit invariant strongly mixing measures.

More precisely, we show that the Frequent Hypercyclicity Criterion for C_0-semigroups ensures the existence of invariant strongly mixing measures with full support. We will provide several examples, that range from birth-and-death models to the Black-Scholes equation, which illustrate these results.

As we mentioned in Chapter 1, there exists a criterion that gives a sufficient condition to ensure when an operator is frequently hypercyclic. In the case of C_0- semigroups there also exists a continuous version of this criterion.

In [82], Mangino and Peris obtained a sufficient condition for frequent hypercyclicity. This frequent hypercyclicity criterion is based on the Pettis integral.

Theorem 2.1.1 ([82]) *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If there exist $X_0 \subset X$ dense in X and maps $S_t : X_0 \to X_0$, $t > 0$, such that* *(i)* $T_tS_tx = x, T_tS_rx = S_{r-t}x, t > 0, r > t > 0$ *for all* $x \in X_0$,

(ii) $t \to T_tx$ *is Pettis integrable on* $[0, \infty)$ *for all* $x \in X_0$,

(iii) $t \to S_tx$ *is Pettis integrable on* $[0, \infty)$ *for all* $x \in X_0$,

then $(T_t)_{t\geq 0}$ *is frequently hypercyclic.*

Our purpose is to show that this criterion suffices for the existence of invariant Borel probability measures on X that are strongly mixing and have full support.

In contrast with the chaotic behavior in the topological sense, which is trivial to pass from the discrete to the continuous case, while difficult or false to go in the other direction (see, e.g., [38] for hypercyclicity and frequent hypercyclicity, and [16] for Devaney chaos), the measure-theoretic properties are not trivially passed from the discrete to the continuous case, especially because of the requirement of T_t-invariance for every $t > 0$. This is why we need to construct explicitly the strongly mixing measures for C_0-semigroups, and they cannot be obtained from the main result in [88]. All the results of this chapter have been published in [86].

2.2 Invariant measures and the frequent hypercyclicity criterion

Now, we are allowed to present our main result.

Theorem 2.2.1 *Let* $(T_t)_{t\geq 0}$ *be a* C_0*-semigroup on a separable Banach space* X*. If there exist,* $X_0 \subset X$ *dense in* X *and maps* $S_t : X_0 \to X_0$, $t > 0$ *such that :*

(i) $T_tS_tx = x, T_tS_rx = S_{r-t}x, t > 0, r > t > 0$ *for all* $x \in X_0$,

(ii) $t \to T_tx$ *is Pettis integrable on* $[0, \infty)$ *for all* $x \in X_0$,

(iii) $t \to S_tx$ *is Pettis integrable on* $[0, \infty)$ *for all* $x \in X_0$,

then there is a $(T_t)_{t\geq 0}$*-invariant strongly mixing Borel probability measure* μ *on* X *with full support.*

The idea behind the proof is to construct, given a C_0-semigroup $(T_t)_{t\geq 0}$ on a separable Banach space X satisfying the hypothesis of Theorem 2.1.1,

1. a "model" probability space $(Z, \overline{\mu})$ and
2. a Borel measurable map $\Phi : Z \to X$ with dense range,

where

- $(R_t)_{t\in\mathbb{R}}$ is the translation group defined as $R_t f(x) = f(x-t)$,
- $Z \subset C(\mathbb{R})$ is a $(R_t)_{t\in\mathbb{R}}$-invariant subset of the space $C(\mathbb{R})$ of continuous functions on the real line, endowed with its natural Fréchet space compact-open topology,
- $\overline{\mu}$ is a $(R_t)_{t\in\mathbb{R}}$-invariant strongly mixing measure with full support, and
- $\Phi R_t = T_t \Phi$ on Z for all $t \geq 0$.

As a consequence, the Borel probability measure μ on X defined by $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$, $A \in \mathfrak{B}(X)$, is $(T_t)_{t\geq 0}$-invariant, strongly mixing, and has full support. Proving the measure μ is $(T_t)_{t\geq 0}$-invariant and strongly mixing is simple. Showing μ has full support will take a little work.

Proof. We suppose $X_0 = \{x_n; n \in \mathbb{N}\}$ with $x_1 = 0$. Let $U_n = B(0, \frac{1}{2^n})$, the open ball of radius $1/2^n$ centered at 0. By conditions (ii) and (iii) we can obtain an increasing sequence $\{N_n\}_n \in \mathbb{N}$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that, for any sequence $(C_k)_k$ of mutually disjoint compact sets with $C_k \subset [k/2, +\infty[$, $k \in \mathbb{N}$, we have that

$$\text{if } m_k \leq 2l, \text{ for } N_l < k \leq N_{l+1}, \quad l \geq n, \quad n \in \mathbb{N},$$

then

$$\sum_{k\geq N_n} \int_{C_k} T_t x_{m_k} dt \in U_{n+1} \text{ and } \sum_{k\geq N_n} \int_{C_k} S_t x_{m_k} dt \in U_{n+1}. \tag{2.1}$$

Indeed, by 0.10.5 for each $\epsilon_n = \frac{1}{2n2^n}$,$n \in \mathbb{N}$, there exists N_n such that for every compact set $K \subset [N_n, \infty)$,

$$\left\| \int_K T_t x \right\| < \epsilon_n \quad \text{and} \quad \left\| \int_K S_t x \right\| < \epsilon_n$$

for every $x \in \{x_1, \ldots, x_{2n}\}$. Without loss of generality we can select the sequence $(N_n)_n$ such that $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$.

In 2.1 we have that:

$$\sum_{k>N_n}\int_{C_k} T_t x_{m_k} = \sum_{j=n}^{\infty}\sum_{N_j<k\le N_{j+1}}\int_{C_k} T_t x_{m_k} \quad \text{and}$$

$$\sum_{k>N_n}\int_{C_k} S_t x_{m_k} = \sum_{j=n}^{\infty}\sum_{N_j<k\le N_{j+1}}\int_{C_k} S_t x_{m_k}$$

So we have for each $j \geq n$:

$$\left\|\sum_{N_j<k\le N_{j+1}}\int_{C_k} T_t x_{m_k}\right\| \le \left\|\sum_{N_j<k\le N_{j+1}}\int_{C_k} T_t x_1\right\|$$

$$+ \quad \ldots + \left\|\sum_{N_j<k\le N_{j+1}}\int_{C_k} T_t x_{2j}\right\| \le \left(\sum_{i=1}^{2j}\epsilon_j\right) \le \frac{1}{2^{j+2}}$$

Analogously we get:

$$\left\|\sum_{N_j<k\le N_{j+1}}\int_{C_k} S_t x_{m_k}\right\| \le \frac{1}{2^{j+2}}.$$

Finally in 2.2:

$$\left\|\sum_{k>N_n}\int_{C_k} T_t x_{m_k}\right\| \le \sum_{j=n}^{\infty}\frac{1}{2^{j+2}} = \frac{1}{2^{n+1}} \quad \text{and}$$

$$\left\|\sum_{k>N_n}\int_{C_k} S_t x_{m_k}\right\| \le \sum_{j=n}^{\infty}\frac{1}{2^{j+2}} = \frac{1}{2^{n+1}}$$

and we conclude the result in 2.1.

1.-The model probability space $(Z, \overline{\mu})$.

First of all, we define the following set $A \subset C(\mathbb{R})$ of continuous functions: $f \in A$ if there exist a sequence $(s_i)_{i\in\mathbb{Z}}$ of real numbers such that

(a) $\ldots s_{-4} < s_{-2} < 0 \le s_0 < s_2 < s_4 < \ldots,$

(b) $|s_{2i+2} - s_{2i} - 1| \le \frac{1}{2}$, and

(c) $s_{2i+1} = (s_{2i} + s_{2i+2})/2$, $i \in \mathbb{Z}$;

and a sequence of natural numbers $(n_i)_{i\in\mathbb{Z}}$ such that

(d) $f(s_{2i}) = n_i$,

(e) $f(s_{2i+1}) = 0$, and

(f) $f''|_{]s_i,s_{i+1}[} \equiv 0$ for all $i \in \mathbb{Z}$.

We write $f_{(s_{2k},n_k)_k}$ to denote the continuous function f associated with sequences $(s_{2k})_{k\in\mathbb{Z}}$ and $(n_k)_{k\in\mathbb{Z}}$ given above.

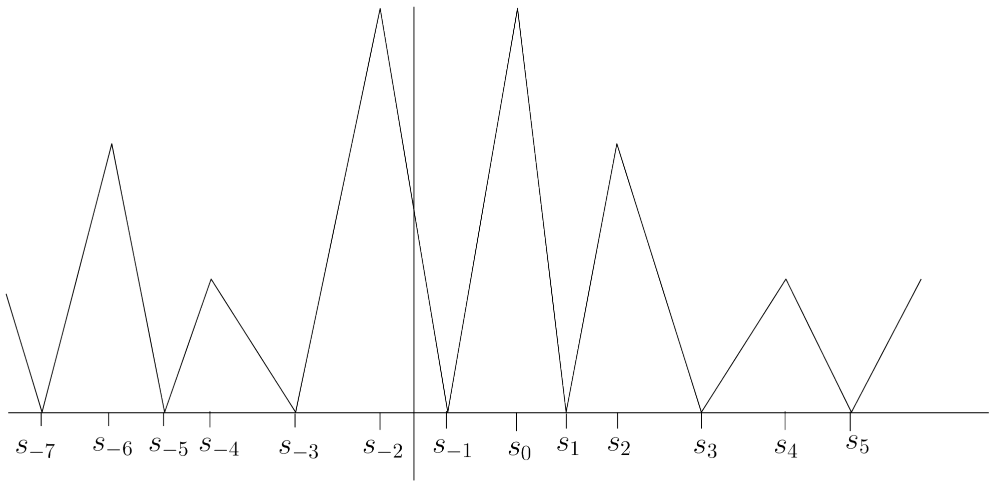

Figure 2.1: Graph of a typical function $f \in A$

A is a closed subset of $C(\mathbb{R})$, where $C(\mathbb{R})$ is endowed with the compact-open topology (that is, the topology of uniform convergence on compact subsets of $\mathbb{R}$), therefore a complete separable metric space.
Indeed, if $(f_j)_j$ is a sequence of functions in A that converges to certain $f \in C(\mathbb{R})$, then each f_j has associated sequences $(s_{2k}(j))_{k\in\mathbb{Z}}$ and $(n_k(j))_{k\in\mathbb{Z}}$ satisfying conditions (a)–(f) above. From the convergence with respect to the compact-open topology we deduce that there exist the limits $\lim_j s_{2k}(j)$ and $\lim_j n_k(j)$ for each $k \in \mathbb{Z}$.
Now, we either have $\lim_j s_{-2}(j) < 0$, so that $s_{2k} := \lim_j s_{2k}(j)$ and $n_k := \lim_j n_k(j)$, $k \in \mathbb{Z}$, define the sequences that make $f \in A$, or we have $\lim_j s_{-2}(j) = 0$, that yields $s_{2k} := \lim_j s_{2k-2}(j)$ and $n_k := \lim_j n_{k-1}(j)$, $k \in \mathbb{Z}$, the defining sequence for $f \in A$.

We will introduce a measure on A. Let us consider λ the Lebesgue measure on $\mathbb{R}$, and let p be the probability measure defined on $\mathbb{N}$ such that $p(\{j\}) = p_j$, with $0 < p_j < 1$, $p(\mathbb{N}) = \sum_{j=1}^{\infty} p_j = 1$ and, if

$$\beta_j := \left(\sum_{i=1}^{j} p_i \right)^{N_{j+1}-N_j}, j \in \mathbb{N}, \quad \text{then} \quad \prod_{j=1}^{\infty} \beta_j > 0. \tag{2.2}$$

We define the map $\Psi : A \to (\mathbb{R} \times \mathbb{N})^{\mathbb{Z}}$ given by $\Psi(f_{(s_{2j},n_j)_{j\in\mathbb{Z}}}) = (s_{2j}, n_j)_{j\in\mathbb{Z}}$. The map Ψ is continuous on $A \setminus A_0$, where

$$A_0 := \{f = f_{(s_{2j},n_j)_j} \in A \ ; \ s_0 = 0\}.$$

Let $(f_{(s_{2j}^k,n_j^k)_j})_k$ be a sequence that converges to $f_{(s_{2j},n_j)_j} \in A$ with $s_0 > 0$. Then, for any compact set $C \subset \mathbb{R}$, we have that

$$\lim_{k\to\infty} \sup_{x\in C} \left| f_{(s_{2j}^k,n_j^k)_j}(x) - f_{(s_{2j},n_j)_j}(x) \right| = 0.$$

In particular, for any $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that,

$$\text{if } |j| \leq N \text{ and } k \geq k_0, \text{ then } n_j^k = n_j \text{ and } |s_{2j}^k - s_{2j}| < \varepsilon. \tag{2.3}$$

Then we have that $(s_{2j}^k, n_j^k)_j)_k$ converges to $(s_{2j}, n_j)_j$ and this shows the continuity of Ψ on A.
Analogously, Ψ is also continuous on A_0, thus Ψ is Borel measurable.

Let $\Pi_n : (\mathbb{R} \times \mathbb{N})^{\mathbb{Z}} \to (\mathbb{R} \times \mathbb{N})^{2n+1}$ be the projection onto the corresponding coordinate space centered at index 0 and define the measure $\widetilde{\lambda}_n = (\lambda \otimes p)^{2n+1}$ on $\Pi_n(\Psi(A))$. We have

$$\begin{aligned} \Pi_1(\Psi(A)) = \Big\{ &((s_{-2}, n_{-1}), (s_0, n_0), (s_2, n_1)) \in (\mathbb{R} \times \mathbb{N})^3 \ ; \\ &s_0 \geq 0, \ s_{-2} < 0, \ \frac{1}{2} \leq s_{2i} - s_{2i-2} \leq \frac{3}{2}, \ i = 0, 1 \Big\}, \end{aligned}$$

and its associated measure is

$$\begin{aligned} \widetilde{\lambda}_1(\Pi_1(\Psi(A))) = &\int_{-\frac{1}{2}}^{0} \left(\int_{s_{-2}+\frac{1}{2}}^{s_{-2}+\frac{3}{2}} \left(\int_{s_0+\frac{1}{2}}^{s_0+\frac{3}{2}} ds_2 \right) ds_0 \right) ds_{-2} \\ &+ \int_{-\frac{3}{2}}^{-\frac{1}{2}} \left(\int_{0}^{s_{-2}+\frac{3}{2}} \left(\int_{s_0+\frac{1}{2}}^{s_0+\frac{3}{2}} ds_2 \right) ds_0 \right) ds_{-2} = \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Analogously, $\widetilde{\lambda}_n(\Pi_n(\Psi(A))) = 1$ for all $n \in \mathbb{N}$.

$$\widetilde{\lambda}_n(\Pi_n(\Psi(A))) = \int_{s_{-2n-2}+\frac{1}{2}}^{s_{-2n-2}+\frac{3}{2}} \cdots \left(\int_{-\frac{1}{2}}^{0} \left(\int_{s_{-2}+\frac{1}{2}}^{s_{-2}+\frac{3}{2}} \left(\int_{s_0+\frac{1}{2}}^{s_0+\frac{3}{2}} \ldots ds_2 \right) ds_0 \right) ds_{-2} \right) ds_{-2n}$$
$$+ \int_{s_{-2n-2}+\frac{1}{2}}^{s_{-2n-2}+\frac{3}{2}} \cdots \left(\int_{-\frac{3}{2}}^{\frac{1}{2}} \left(\int_{0}^{s_{-2}+\frac{3}{2}} \left(\int_{s_0+\frac{1}{2}}^{s_0+\frac{3}{2}} \ldots ds_2 \right) ds_0 \right) ds_{-2} \right) ds_{-2n} = 1.$$

Let $\mathfrak{B}_n$ be the σ-algebra of Borel subsets of $\Pi_n(\Psi(A))$. We consider $\mathfrak{A} := \bigcup_{n\in\mathbb{N}} \Pi_n^{-1}(\mathfrak{B}_n)$, which is an algebra consisting of Borel subsets of $\Psi(A)$ since $\Pi_n^{-1}(\mathfrak{B}_n) \subset \Pi_{n+1}^{-1}(\mathfrak{B}_{n+1})$ for all $n \in \mathbb{N}$. Also, the σ-algebra generated by $\mathfrak{A}$ coincides with the family $\widetilde{\mathfrak{B}}$ of all Borel subsets of $\Psi(A)$ because $\mathfrak{A}$ contains the open subsets of $\Psi(A)$.
For each $n \in \mathbb{N}$ and $B \in \Pi_n^{-1}(\mathfrak{B}_n)$, we define $\widetilde{\lambda}(B) = \widetilde{\lambda}_n(\Pi_n(B))$. $\widetilde{\lambda}$ is a well-defined probability measure on $\mathfrak{A}$ with full support since $\widetilde{\lambda}_n(\Pi_n(B)) = \widetilde{\lambda}_{n+1}(\Pi_{n+1}(B))$ for every $B \in \Pi_n^{-1}(\mathfrak{B}_n)$, $n \in \mathbb{N}$. There is a unique extension of $\widetilde{\lambda}$ to $\widetilde{\mathfrak{B}}$, for which we keep the same notation (see, e.g., [73]).
Now, since A is a complete separable metric space and $\Psi : A \to (\mathbb{R} \times \mathbb{N})^{\mathbb{Z}}$ is an injective measurable map, we have that the family of Borel sets of A equals $\Psi^{-1}(\widetilde{\mathfrak{B}})$ (see, e.g., Corollary 3.3 in [73]), and we obtain that $\overline{\mu} := \widetilde{\lambda} \circ \Psi$ is a Borel probability measure on A with full support.

Moreover, A is R_t-invariant for any $t \in \mathbb{R}$, where $(R_t)_{t\in\mathbb{R}}$ is the translation C_0-group, since given $f_{(s_{2j},n_j)_j} \in A$ we have that $R_t(f_{(s_{2j},n_j)_j}) = f_{(t+s_{2j+2k},n_{j+k})_j} \in A$, where

$$k := \min\{j \in \mathbb{Z} \ ; \ t + s_{2j} \geq 0\}. \tag{2.4}$$

The definition of $\overline{\mu}$ easily yields that $\overline{\mu}$ is $(R_t)_{t\in\mathbb{R}}$-invariant.

We also note that $\overline{\mu}$ is strongly mixing with respect to the translation C_0-group $(R_t)_{t\in\mathbb{R}}$. Actually, it suffices to prove it on a basis of open sets of A.

Let us define, for each

$$\alpha = ((s_{2j})_{j=-n}^{m}, (n_j)_{j=-n}^{m}, \varepsilon) \in \mathbb{R}^{n+m+1} \times \mathbb{N}^{n+m+1} \times]0, 1/4[$$

with $s_{-2n} < \cdots < s_{-2} < 0 \leq s_0 < s_2 < \cdots < s_{2m}$, $\frac{1}{2} \leq s_{2j+2} - s_{2j} \leq \frac{3}{2}$, $j = -n, \ldots, m-1$, the set

$A_\alpha = \{f \in A \ ; \ \exists t_{2j} \in]s_{2j} - \varepsilon, s_{2j} + \varepsilon[$ with $f(t_{2j}) = n_j, \ f(t_{2j+1}) = 0$ for

$t_{2j+1} := \dfrac{t_{2j}+t_{2j+2}}{2}$, $j=-n,\ldots,m-1$, $f''|_{]t_i,t_{i+1}[} \equiv 0$, $i=-2n,\ldots,2m-1\}$.

They form a basis of open sets in A as a topological subspace of $C(\mathbb{R})$ endowed with the compact-open topology. Let A_α and $A_{\alpha'}$ be two elements from the above basis, where

$$\alpha = ((s_{2j})_{j=-n}^{m}, (n_j)_{j=-n}^{m}, \varepsilon) \text{ and } \alpha' = ((s'_{2j})_{j=-n'}^{m'}, (n'_j)_{j=-n'}^{m'}, \varepsilon').$$

If t is large enough then $[s_{-n}-\varepsilon, s_m+\varepsilon] \bigcap [t+s'_{-n'}-\varepsilon', t+s'_{m'}+\varepsilon'] = \emptyset$ and because of the definition of $\overline{\mu}$ and the empty intersection of the previous intervals we have:

$$\overline{\mu}(A_\alpha \cap R_t(A_{\alpha'})) = \overline{\mu}(A_\alpha)\overline{\mu}(A_{\alpha'}).$$

Let us consider the subset of A given by

$$H = \{f_{(s_{2k},n_k)_k} \in A \ ; \ n_k = f(s_{2k}) \in \{1,\ldots,m\} \text{ if } N_m < |k| \leq N_{m+1},$$

$$m \in \mathbb{N}, \ f(s_{2k}) = 1 \text{ for } |k| \leq N_1\}.$$

Clearly, H is a closed subset of A which is bounded in $C(\mathbb{R})$. An easy argument shows that the subsets of A that are bounded in $C(\mathbb{R})$ are relatively compact, thus H is compact.

Let $(f_{(s^k_{2j},n^k_j)_j})_k$, be a sequence in H and fix $n \in \mathbb{N}$. We will show that, there exists a Cauchy subsequence $(f_{(s^{k_i}_{2j},n^{k_i}_j)_j})_{k_i}$ with respect to the uniform convergence in $[-n,n]$. By the selection of $(s_{2j})_j$, we have that $s_{2j}(k) \notin [-n,n]$ for every $j \in \mathbb{Z}$ with $|j| > 2n+1$ and for each $k \in \mathbb{N}$. We select an increasing sequence $(k_i)_i$ in $\mathbb{N}$ such that there exists

$$s_{2j} = \lim_{i\to\infty} s_{2j}(k_i)$$

and

$$n_j = \lim_{i\to\infty} n_j(k_i),$$

for $|j| \leq 2n+2$. We conclude, by definition of the elements of A that $(f_{(s^{k_i}_{2j},n^{k_i}_j)_j})_{k_i}$ is a Cauchy sequence with respect to the uniform convergence in $[-n,n]$. Since $n \in \mathbb{N}$ was arbitrary, we have found a subsequence of $(f_{(s^k_{2j},n^k_j)_j})_k$ that is Cauchy in H, therefore convergent since H is closed, and we get that H is compact.

Let $Z = \bigcup_{t\in\mathbb{R}} R_t(H) = \bigcup_{j\in\mathbb{Z}} R_j(H)$, therefore a Borel subset of A.

This last equality is clear because

$$R_t(f_{(s_{2j},n_j)_j}) = f_{(t+s_{2j+2k},n_{j+k})_j} = f_{([t]+h+s_{2j+2k},n_{j+k})_j} = R_{[t]}(f_{(h+s_{2j},n_j)_j})$$

where

$$k := \min\{j \in \mathbb{Z} \; ; \; t + s_{2j} \geq 0\} \tag{2.5}$$

and $t = [t] + h$, where $[t]$ denotes the integer part of t.

We easily get

$$\overline{\mu}(Z) \geq \overline{\mu}(H) = (p_1)^{2N_1+1} \left(\prod_{l=1}^{\infty} \beta_l \right)^2 > 0.$$

The last inequality is obtained by using Fubini's theorem again.

Since Z is R_t-invariant, it has positive measure and $\overline{\mu}$ is strongly mixing and then ergodic, $\overline{\mu}(Z) = 1$.

2.-The map Φ.

We define the map $\Phi : Z \to X$ by

$$\begin{aligned} \Phi(f_{(s_{2j},n_j)_j}) &= \sum_{j \leq -2} \int_{s_{2j}}^{s_{2j+2}} S_{-t} x_{n_j} dt + \int_{s_{-2}}^{0} S_{-t} x_{n_{-1}} dt \\ &+ \int_0^{s_0} T_t x_{n_{-1}} dt + \sum_{j \geq 0} \int_{s_{2j}}^{s_{2j+2}} T_t x_{n_j} dt. \end{aligned} \tag{2.6}$$

Φ is well defined since, given $f_{(s_{2j},n_j)_j} \in R_{t_0}(H)$, $t_0 \in \mathbb{R}$, and for $l \geq |t_0|$, we have that $n_k \leq 2l$ if $N_l < |k| \leq N_{l+1}$, which shows the convergence of the series in (2.6) by (2.1).

Let us see that $T_a \circ \Phi = \Phi \circ R_a$ for any $a > 0$. We will distinguish two cases:

Case 1 $s_{-2} < -a < 0$:

$$
\begin{aligned}
&T_a \circ \Phi(f_{(s_{2j},n_j)_j}) = \\
&\quad \sum_{j\le -2}\int_{s_{2j}}^{s_{2j+2}} T_a S_{-t} x_{n_j} + \int_{s_{-2}}^{0} T_a S_{-t} x_{n_{-1}} + \int_{0}^{s_0} T_a T_t x_{n_{-1}} \\
&\quad + \sum_{j\ge 0}\int_{s_{2j}}^{s_{2j+2}} T_a T_t x_{n_j} = \sum_{j\le -2}\int_{s_{2j}}^{s_{2j+2}} S_{-(t+a)} x_{n_j} + \int_{s_{-2}}^{-a} T_a S_{-t} x_{n_{-1}} \\
&\quad + \int_{-a}^{0} T_a S_{-t} x_{n_{-1}} + \int_{0}^{s_0} T_{t+a} x_{n_{-1}} + \sum_{j\ge 0}\int_{s_{2j}}^{s_{2j+2}} T_{t+a} x_{n_j} \\
&\quad = \sum_{j\le -2}\int_{s_{2j}+a}^{s_{2j+2}+a} S_{-t} x_{n_j} + \int_{s_{-2}}^{-a} S_{-t-a} x_{n_{-1}} + \int_{-a}^{0} T_{a+t} x_{n_{-1}} \\
&\quad\; + \int_{a}^{s_0+a} T_t x_{n_{-1}} + \sum_{j\ge 0}\int_{s_{2j}+a}^{s_{2j+2}+a} T_t x_{n_j} = \sum_{j\le -2}\int_{a+s_{2j}}^{a+s_{2j+2}} S_{-t} x_{n_j} \\
&\quad\; + \int_{a+s_{-2}}^{0} S_{-t} x_{n_{-1}} + \int_{0}^{a+s_0} T_t x_{n_{-1}} + \sum_{j\ge 0}\int_{a+s_{2j}}^{a+s_{2j+2}} T_t x_{n_j} \\
&\quad\; = \Phi(f_{(a+s_{2j},n_j)_j}) = \Phi \circ R_a(f_{(s_{2j},n_j)_j}).
\end{aligned}
$$

since, in this case, $0 = \min\{j \in \mathbb{Z}\ ;\ a + s_{2j} \geq 0\}$.

Case 2 $s_{2k} < -a \leq s_{2k+2}$, for some $k \in \mathbb{Z}^-$, $k \leq -2$:

$$
\begin{aligned}
&T_a \circ \Phi(f_{(s_{2j},n_j)_j}) = \\
&\quad \sum_{j\leq -2} \int_{s_{2j}}^{s_{2j+2}} T_a S_{-t} x_{n_j} + \int_{s_{-2}}^{0} T_a S_{-t} x_{n_{-1}} + \int_{0}^{s_0} T_{a+t} x_{n_{-1}} \\
&\quad + \sum_{j\geq 0} \int_{s_{2j}}^{s_{2j+2}} T_{a+t} x_{n_j} = \sum_{j<k} \int_{s_{2j}}^{s_{2j+2}} S_{-t-a} x_{n_j} + \int_{s_{2k}}^{-a} T_a S_{-t} x_{n_k} \\
&\quad + \int_{-a}^{s_{2k+2}} T_a S_{-t} x_{n_k} + \sum_{k<j<-1} \int_{s_{2j}}^{s_{2j+2}} T_{a+t} x_{n_j} + \int_{s_{-2}+a}^{a} T_t x_{n_{-1}} \\
&\quad + \int_{a}^{s_0+a} T_t x_{n_{-1}} + \sum_{j\geq 0} \int_{a+s_{2j}}^{a+s_{2j+2}} T_t x_{n_j} = \sum_{j<k} \int_{s_{2j}+a}^{s_{2j+2}+a} S_{-t} x_{n_j} \\
&\quad + \int_{s_{2k}+a}^{0} S_{-t} x_{n_k} + \int_{-a}^{s_{2k+2}} T_{a+t} x_{n_k} + \sum_{k<j<-1} \int_{s_{2j}+a}^{s_{2j+2}+a} T_t x_{n_j} \\
&\quad + \int_{s_{-2}+a}^{a} T_t x_{n_{-1}} + \int_{a}^{s_0+a} T_t x_{n_{-1}} + \sum_{j\geq 0} \int_{a+s_{2j}}^{a+s_{2j+2}} T_t x_{n_j} = \\
&\quad \sum_{j<k} \int_{a+s_{2j}}^{a+s_{2j+2}} S_{-t} x_{n_j} + \int_{a+s_{2k}}^{0} S_{-t} x_{n_k} + \int_{0}^{a+s_{2k+2}} T_t x_{n_k} \\
&\quad + \sum_{j>k} \int_{a+s_{2j}}^{a+s_{2j+2}} T_t x_{n_j} = \Phi(f_{(a+s_{2j+2k+2},n_{j+k+1})_j}) = \Phi \circ R_a(f_{(s_{2j},n_j)_j})
\end{aligned}
$$

since, in this case, $k+1 = \min\{j \in \mathbb{Z} \,;\, a + s_{2j} \geq 0\}$.

Also, Φ is continuous almost everywhere on $R_{t_0}(H)$ for any $t_0 \in \mathbb{R}$. Indeed, let $(f_{(s_{2j}^k,n_j^k)_j})_k$ be a sequence in $R_{t_0}(H)$ that converges to $f_{(s_{2j},n_j)_j} \in R_{t_0}(H)$ with $s_0 > 0$. Then, for any compact set $C \subset \mathbb{R}$, we have that

$$\lim_{k\to\infty} \sup_{x\in C} \left| f_{(s_{2j}^k,n_j^k)_j}(x) - f_{(s_{2j},n_j)_j}(x) \right| = 0.$$

In particular, for any $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that,

$$\text{if } |j| \leq N \text{ and } k \geq k_0, \text{ then } n_j^k = n_j \text{ and } |s_{2j}^k - s_{2j}| < \varepsilon. \tag{2.7}$$

Fix $n > |t_0|$ and $N = N_n$. Let $\varepsilon > 0$ such that $\|\int_I S_{-t} x_{n_j} dt\| + \|\int_J T_t x_{n_j} dt\| < (3(N+1)2^{n+1})^{-1}$ whenever $I \subset]-\infty, 0]$ and $J \subset [0, +\infty[$ are intervals of length less than ε and $|j| \le N$. By (2.7) and (2.1), there exists an integer k_0 such that for every $k \ge k_0$,

$$\left\|\Phi(f^k_{(s^k_{2j},n^k_j)_j}) - \Phi(f_{(s_{2j},n_j)})\right\| \le \left\|\sum_{j<-N_n}\int_{s^k_{2j}}^{s^k_{2j+2}} S_{-t}x^k_{n_j}\right\|$$
$$+\left\|\sum_{j>N_n}\int_{s^k_{2j}}^{s^k_{2j+2}} T_t x^k_{n_j}\right\| + \left\|\sum_{j<-N_n}\int_{s_{2j}}^{s_{2j+2}} S_{-t}x_{n_j}\right\|$$
$$+\left\|\sum_{j>N_n}\int_{s_{2j}}^{s_{2j+2}} T_t x_{n_j}\right\| + \sum_{-N_n\le j\le -2}\left\|\int_{\min(s^k_{2j},s_{2j})}^{\max(s^k_{2j},s_{2j})} S_{-t}x_{n_{2j}}\right\|$$
$$+\sum_{-N_n\le j\le -2}\left\|\int_{\min(s^k_{2j+2},s_{2j+2})}^{\max(s^k_{2j+2},s_{2j+2})} S_{-t}x_{n_j}\right\| + \left\|\int_{\min(s^k_{-2},s_{-2})}^{\max(s^k_{-2},s_{-2})} S_{-t}x_{n_{-1}}\right\|$$
$$+\left\|\int_{\min(s^k_0,s_0)}^{\max(s^k_0,s_0)} T_t x_{n_{-1}}\right\| + \sum_{0\le j\le N_n}\left\|\int_{\min(s^k_{2j},s_{2j})}^{\max(s^k_{2j},s_{2j})} T_t x_{n_j}\right\|$$
$$+\sum_{0\le j\le N_n}\left\|\int_{\min(s^k_{2j+2},s_{2j+2})}^{\max(s^k_{2j+2},s_{2j+2})} T_t x_{n_j}\right\| <$$
$$\frac{1}{2^{n+1}}+\frac{1}{2^{n+1}}+\frac{1}{2^{n+1}}+\frac{1}{2^{n+1}}+\frac{1}{3(N_n+1)2^{n+1}}+\frac{2(N_n-2)}{3(N_n+1)2^{n+1}}$$
$$+\frac{2(N_n+1)}{3(N_n+1)2^{n+1}} = \frac{1}{2^{n-1}}+\frac{1}{2^n}.$$

This shows the continuity almost everywhere of $\Phi : R_t(H) \to X$ for every $t \in \mathbb{R}$. The map Φ is well-defined on Z, and $\Phi : Z \to X$ is Borel measurable.

3.-The measure μ on X.

$L(s) := \Phi(R_s(H))$ is a countable union of compact sets in X for each $s \in \mathbb{R}$. Indeed,

$$\Phi(R_s(H)) =$$
$$\Phi(\{f_{(s_{2j},n_j)} \in R_s(H)\ ;\ s_0 = 0\}) \cup \bigcup_{n\in\mathbb{N}} \Phi(\{f_{(s_{2j},n_j)} \in R_s(H)\ ;\ s_0 \ge \frac{1}{n}\}).$$

We then have that $Y := \bigcup_{s\in\mathbb{R}^+} L(s) = \bigcup_{n\in\mathbb{N}} L(n)$ is a countable union of compact sets, thus a T_t-invariant Borel subset of X because $\Phi\circ R_t = T_t\circ\Phi$, $t>0$.

We then define in X the measure $\mu(B) = \overline{\mu}(\Phi^{-1}(B))$ for all $B\in\mathfrak{B}(X)$. Obviously, μ is well-defined and it is a $(T_t)_t$-invariant strongly mixing Borel probability measure. The proof is completed by showing that μ has full support. In the proof of Theorem 2.2 in [82] it was shown that, for $u_k := \int_0^1 T_t x_k dt$, $k\in\mathbb{N}$, the set $\{u_k\ ;\ k\in\mathbb{N}\}$ is dense in X. Thus, given a non-empty open set U in X, we pick $n\in\mathbb{N}$ and $\varepsilon>0$ satisfying

$$\int_{s_0}^{s_2} T_t x_n dt + U_n \subset U$$

for any $s_0\in[0,\varepsilon]$, $s_2\in[1,1+\varepsilon]$. Together with (2.1), this implies

$$\begin{aligned}
&\mu(U) \ge \mu\big(\{\Phi(f_{(s_{2j},n_j)_j})\ ;\ f_{(s_{2j},n_j)_j}\in Z,\ s_0\in[0,\varepsilon],\ s_2\in[1,1+\varepsilon],\\
&\quad n_0=n,\ n_k=1 \text{ if } 0<|k|\le N_n,\ n_k\le 2l,\ \text{for } N_l<|k|\le N_{l+1},\ l\ge n\}\big)\\
&\ge \dots\int_{-\frac{3}{2}}^{\varepsilon-\frac{3}{2}} p_1\left(\int_0^{s_{-2}+\frac{3}{2}} p_n\left(\int_1^{1+\varepsilon} p_1\dots ds_2\right) ds_0\right)\dots\\
&+\dots\int_{\varepsilon-\frac{3}{2}}^{-\frac{1}{2}} p_1\left(\int_0^{\varepsilon} p_n\left(\int_1^{1+\varepsilon} p_1\dots ds_2\right) ds_0\right) ds_{-2}\dots\\
&+\dots\int_{-\frac{1}{2}}^{\varepsilon-\frac{1}{2}} p_1\left(\int_{s_{-2}+\frac{1}{2}}^{\varepsilon} p_n\left(\int_1^{1+\varepsilon} p_1\dots ds_2\right) ds_0\right) ds_{-2}\dots\\
&\ge \varepsilon^2 p_n(p_1)^{2N_n}\prod_{l=n}^{\infty}\left(\prod_{N_l<|k|\le N_{l+1}}\sum_{r=1}^{2l} p_r\right) \ge \varepsilon^2 p_n(p_1)^{2N_n}\prod_{l=n}^{\infty}(\beta_{2l})^2 > 0
\end{aligned}$$

□

Remark 2.2.2 There exists an alternative way of defining the measure on the space of continuous functions, by using Brownian motions (for more details see [92],[93]). We denote by $\mathfrak{B} = \mathfrak{B}(C([0,\infty)))$, the σ-algebra of Borel subsets of $C([0,\infty))$. Let ω_t, $t\ge 0$, be a Brownian motion defined on a probability space $(\Omega,\mathfrak{F},\mu)$. Assume that the sample functions of ω_t are continuous. Setting $\xi_t = e^t\omega_{e^{-2t}}$ for $t\ge 0$, then ξ_t is a stationary Gaussian process with mean value $E\xi_t = 0$ and correlation function $E\xi_t\xi_{t+h} = e^{-|h|}$. Then the measure on $\mathfrak{B} = \mathfrak{B}(C([0,\infty)))$ induced by ξ_t is strongly mixing with full support. The details of the construction of the measure can be found in [93].

In Corollary 2.3 in [82] some conditions, expressed in terms of eigenvector fields for the infinitesimal generator of the C_0-semigroup, were given which ensure that the assumptions of Theorem 2.2.1 are satisfied. In consequence we also obtain the stronger result of existence of invariant strongly mixing measures under the same conditions. A different argument for the existence of invariant strongly mixing measures for C_0-semigroups has been obtained in [21] under weaker assumptions on the eigenvectors fields for the generator.

Corollary 2.2.3 *Let X be a separable complex Banach space and let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X with generator A. Assume that there exists a family $(f_j)_{j\in\Gamma}$ of locally bounded measurable maps $f_j : I_j \to X$ such that I_j is an interval in $\mathbb{R}$, $f_j(I_j) \subset D(A)$, where $D(A)$ denotes the domain of the generator, $Af_j(t) = itf_j(t)$ for every $t \in I_j$, $j \in \Gamma$ and $\operatorname{span}\{f_j(t) : j \in \Gamma,\ t \in I_j\}$ is dense in X. If either*

(i) $f_j \in C^2(I_j, X)$, $j \in \Gamma$
or

(ii) X *does not contain* c_0 *and* $\langle \varphi, f_j\rangle \in C^1(I_j)$, $\varphi \in X'$, $j \in \Gamma$,

then there is a $(T_t)_{t\geq 0}$-invariant strongly mixing Borel probability measure μ on X with full support.

2.3 Applications

In this section we will present several applications of the previous results to the (chaotic) behavior of the solution C_0-semigroup to certain linear partial differential equations and infinite systems of linear differential equations.

Example 2.3.1 Let us consider the following linear perturbation of the one-dimensional Ornstein-Uhlenbeck operator

$$\mathcal{A}_\alpha u = u'' + bxu' + \alpha u,$$

where $\alpha \in \mathbb{R}$, with domain

$$D(\mathcal{A}_\alpha) = \left\{ u \in L^2(\mathbb{R}) \cap W^{2,2}_{\mathrm{loc}}(\mathbb{R}) \ ;\ \mathcal{A}_\alpha u \in L^2(\mathbb{R}) \right\}.$$

We know that, if $\alpha > b/2 > 0$, then the semigroup generated by $\mathcal{A}_\alpha$ in $L^2(\mathbb{R})$ is chaotic [37] and frequently hypercyclic [82]. Actually, it was shown that the C_0-semigroup satisfies the hypothesis of Corollary 2.2.3 [82]. Therefore, we also obtain that it admits an invariant strongly mixing measure with full support.

Example 2.3.2 Rudnicki [95] recently showed the existence of invariant strongly mixing measures for some C_0-semigroups generated by a partial differential equation of population dynamics. More precisely, he reduced the equation to

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = au(t,x) + bu(t,2x),$$

whose formal solution, given the initial condition $u(0,x) = u_0(x)$, is

$$u(t,x) := e^{at} \sum_{n=0}^{\infty} \frac{(bt)^n}{n!} u_0(2^n e^{-t} x).$$

He considered the space

$$X = X_{\alpha,\beta} := \left\{ u \in C(]0,\infty[) \ ; \ \lim_{x\to 0} x^\alpha |u(x)| = 0, \ \lim_{x\to\infty} x^\beta |u(x)| = 0 \right\}$$

endowed with the norm $\|u\| := \sup_{x\in]0,\infty[} |u(x)|\rho(x)$, where $\rho(x) = x^\alpha$ if $x \le 1$ and $\rho(x) = x^\beta$ if $x > 1$. If $2^a b \log 2 < e^{-1}$, $\beta < \log_2 b + \log_2(\log 2)$, and $\alpha > \alpha_0$, where α_0 satisfies $(a+\alpha_0)2^{\alpha_0} = b$, then there exists a Borel strongly mixing probability measure μ on X with full support which is invariant under the solution C_0-semigroup generated by the above equation as it is proved in Theorem 1 of [95]. Actually, this fact was shown by reducing the problem to the translation flow $(R_t)_{t\in\mathbb{R}}$ on the space

$$Y := \left\{ g \in C(\mathbb{R}) \ ; \ \lim_{|x|\to\infty} \frac{g(x)}{x} = 0 \right\},$$

of weighted continuous functions with the norm

$$\|g\|_Y = \sup_{x\in\mathbb{R}} \frac{|g(x)|}{1+|x|}.$$

The corresponding generator is $A = D$, the derivative operator. We can apply directly our Corollary 2.2.3 to the map $f : \mathbb{R} \to Y$ given by $[f(t)](x) := e^{itx}$, which is a C^2-map, and we obtain the same result since $\operatorname{span}\{f(t) \ ; \ t \in \mathbb{R}\}$ is the set of trigonometric polynomials, which is dense in Y.

Example 2.3.3 The chaotic behaviour associated to birth-and-death processes has been widely studied by Banasiak et al [5, 6, 7, 9]. We will consider three cases that are shown to admit invariant strongly mixing measures.

1. In [9], Banasiak and Moszynski studied the following “birth-and-death” model with constant coefficients:

$$\begin{aligned} \frac{df_1}{dt} &= (\mathcal{L}f)_1 = && af_1 + df_2, \\ \frac{df_n}{dt} &= (\mathcal{L}f)_n = && bf_{n-1} + af_n + df_{n+1}, \qquad n \geq 2. \end{aligned} \tag{2.8}$$

   Among other things, they studied the chaotic behaviour of the solution C_0-semigroup.

   Theorem 2.3.4 ([9]) *Let $a,\ b,\ d \in \mathbb{R}$ satisfy $0 < |b| < |d|$ and $|a| < |b+d|$. Then the solution C_0-semigroup to the Cauchy problem (2.8) is Devaney chaotic on ℓ^p.*

   Actually, to show this result they used a spectral criterion (see [8] and [45]) which is less general than the criterion of Corollary 2.2.3. In consequence, we obtain that the solution C_0-semigroup to the Cauchy problem (2.8) admits an invariant strongly mixing measure on ℓ^p with full support.

2. In [2], Aroza and Peris studied the same model with coefficients depending on $\mathbb{N}$,

$$\begin{aligned} \frac{df_1}{dt} &= a_1 f_1 + d_1 f_2, \\ \frac{df_n}{dt} &= b_n f_{n-1} + a_n f_n + d_n f_{n+1}, \qquad n \geq 2. \end{aligned} \tag{2.9}$$

   with $a_n, b_n, d_n \in \mathbb{R}$ and the infinite matrix

$$\mathcal{L} = \begin{pmatrix} a_1 & d_1 & & & \\ b_2 & a_2 & d_2 & & \\ & b_3 & a_3 & d_3 & \\ & & b_4 & a_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

   They intended to obtain sub-chaos (i.e., Devaney chaos on a subspace) results for birth-and-death type models with proliferation in a wide range of coefficients depending on $\mathbb{N}$. They considered the Banach space $X = X(\gamma)$ on which the operator associated with $\mathcal{L}$ generates a C_0-semigroup. Given $1 \leq p < \infty$, let

$$X(\gamma) := \left\{ f \in \ell^p : \mathcal{L}^n f \in \ell^p, \forall n \in \mathbb{N}, ||f|| := \sum_{n=0}^{\infty} ||\mathcal{L}^n f||_p \gamma^{-n} < \infty \right\}.$$

If the sequences $(a_n)_n, (b_n)_n$ and $(d_n)_n$ are bounded, $\mathcal{L}$ has an associated bounded operator $\mathcal{S}_p$ on ℓ^p, with spectral radius $r(\mathcal{S}_p) < \infty$, and $X(\gamma) = \ell^p$ for $\gamma > r(\mathcal{S}_p)$. If any of the sequences $(a_n)_n, (b_n)_n$ and $(d_n)_n$ is unbounded, we have that the operator $\mathcal{S}_X$ associated with $\mathcal{L}$ is a bounded operator on X and, therefore, it generates a C_0-semigroup $\mathcal{T}_X$ on X. They obtained the following result:

Theorem 2.3.5 ([2]) *Let $(a_n), (b_n)$ and $(d_n)_n$ be sequences of real numbers such that $d_n \neq 0$ for all $n \in \mathbb{N}$, $1 \leq p < \infty$, and $\gamma > 0$. Assume that either*

1. $\lim_{n\to\infty} a_n = a, \lim_{n\to\infty} b_n = b, \lim_{n\to\infty} d_n = d \neq 0$ *with* $|b| < |d|$ *and* $|a| < |b + d|$ *or*
2. $\lim_{n\to\infty} \frac{a_n}{d_n} = \alpha, \lim_{n\to\infty} \frac{b_n}{d_n} = \beta, \lim_{n\to\infty} d_n = \infty$ *with* $\alpha^2 \neq 4\beta$, $|\beta| < 1$ *and* $|\alpha| < |1 + \beta|$

then the C_0-semigroup $\mathcal{T}_X$ is sub-chaotic on $X(\gamma)$. Moreover, in case 1, $\mathcal{S}_p$ generates a sub-chaotic C_0-semigroup $\mathcal{T}_p$ on ℓ^p.

Actually, to show this result they proved that the solution C_0-semigroup satisfies the spectral criterion of [8], in particular the conditions of Corollary 2.2.3 on a certain subspace Y. Thus, we obtain that the corresponding solution C_0-semigroup admits an invariant strongly mixing measure μ on $X(\gamma)$ whose support is Y.

3. Let us consider the death model with variable coefficients

$$\begin{cases} \frac{\partial f_n}{\partial t} = -\alpha_n f_n + \beta_n f_{n+1}, & n \geq 1, \\ f_n(0) = a_n, & n \geq 1 \end{cases} \tag{2.10}$$

where $(\alpha_n)_n$ and $(\beta_n)_n$ are bounded positive sequences and $(a_n)_n \in \ell^1$ is a real sequence. Considering $X = \ell^1$, and the map A given by

$$Af = (-\alpha_n f_n + \beta_n f_{n+1})_n \text{ for } f = (f_n)_n \in X,$$

since A is a bounded operator on ℓ^1, it generates a C_0-semigroup $(T_t)_{t\geq 0}$ which is solution of (2.10). It is shown in [72], that if

$$\sup_{n\geq 1} \alpha_n < \liminf_{n\to\infty} \beta_n$$

then the semigroup $(T_t)_{t\geq 0}$ satisfies the hypothesis of the spectral criterion [45], and then we can ensure the existence of an invariant strongly mixing measure with full support on X.

Example 2.3.6 Let us consider the solution semigroup $(e^{tA})_{t\geq 0}$ of the hyperbolic heat transfer equation problem:

$$\begin{cases} \tau\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}, \\ u(0,x)=\varphi_1(x), \quad x\in\mathbb{R}, \\ \frac{\partial u}{\partial t}(0,x)=\varphi_2(x), \quad x\in\mathbb{R} \end{cases} \tag{2.11}$$

where φ_1 and φ_2 represent the initial temperature and the initial variation of temperature, respectively, $\alpha>0$ is the thermal diffusivity, and $\tau>0$ is the thermal relaxation time. We can represent it as a C_0-semigroup on the product of a certain function space with itself. We set $u_1=u$ and $u_2=\frac{\partial u}{\partial t}$. Then the associated first-order equation is :

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau}\frac{\partial^2}{\partial x^2} & \frac{-1}{\tau}I \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \\ \begin{pmatrix} u_1(0,x) \\ u_2(0,x) \end{pmatrix}=\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x\in\mathbb{R} \end{cases} \tag{2.12}$$

We fix $\rho>0$ and consider the space

$$X_\rho=\left\{f:\mathbb{R}\to\mathbb{C}; f(x)=\sum_{n=0}^{\infty}\frac{a_n\rho^n}{n!}x^n, (a_n)_{n\geq 0}\in c_0\right\}$$

endowed with the norm $||f||=\sup_{n\geq 0}|a_n|$.

Since

$$A:=\begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau}\frac{\partial^2}{\partial x^2} & \frac{-1}{\tau}I \end{pmatrix}. \tag{2.13}$$

is an operator on $X:=X_\rho\oplus X_\rho$, we have that $(e^{tA})_{t\geq 0}$ is the C_0-semigroup solution of 2.11. We know from [40] and [72] that, given α, τ and ρ such that $\alpha\tau\rho>2$, the solution semigroup $(e^{tA})_{t\geq 0}$ defined on $X_\rho\oplus X_\rho$ satisfies the hypothesis of the spectral criterion [45], and we conclude the existence of an invariant strongly mixing measure with full support on $X_\rho\oplus X_\rho$.

Example 2.3.7 In [31], Black and Scholes proved that under some assumptions about the market, the value of a stock option $u(x,t)$, as a function of the current value of the underlying asset $x\in\mathbb{R}^+=[0,\infty)$ and time, satisfies the final value problem:

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru & \text{in } \mathbb{R}^+ \times [0,T] \\ u(0,T) = 0 & \text{for } t \in [0,T] \\ u(x,T) = (x-p)^+ & \text{for } x \in \mathbb{R}^+ \end{cases}$$

where $p > 0$ represents a given strike price, $\sigma > 0$ is the volatility and $r > 0$ is the interest rate. Let $v(x,t) = u(x, T-t)$, then it satisfies the forward Black-Scholes equation defined for all time $t \in \mathbb{R}^+$ by

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv & \text{in } \mathbb{R}^+ \times \mathbb{R}^+ \\ v(0,T) = 0 & \text{for } t \in \mathbb{R}^+ \\ v(x,0) = f(x) & \text{for } x \in \mathbb{R}^+ \end{cases}$$

with

$$f(x) = (x-p)^+ = \begin{cases} x-p & \text{if } x > p \\ 0 & \text{if } x \le p. \end{cases}$$

In order to express this problem in an abstract form , we define $D_\nu = \nu x \frac{\partial}{\partial x}$, where $\nu = \frac{\sigma}{\sqrt{2}}$ and $\mathcal{B} = (D_\nu)^2 + \gamma(D_\nu) - rI$, with $\gamma = \frac{r}{\nu} - \nu$. Then the problem can be reformulated as:

$$\begin{cases} \frac{\partial v}{\partial t} = \mathcal{B}v, \\ v(0,T) = 0, \\ v(x,0) = f(x) \quad \text{for } x \in \mathbb{R}^+. \end{cases}$$

Recently in [63], the authors gave a simple explicit representation of the solution of the Black-Scholes equation and this representation holds in the spaces $Y^{s,\tau}$. Let

$$Y^{s,\tau} = \left\{ u \in C((0,\infty)) \; ; \; \lim_{x\to\infty} \frac{u(x)}{1+x^s} = 0, \quad \lim_{x\to 0} \frac{u(x)}{1+x^{-\tau}} = 0 \right\}$$

be endowed with the norm

$$||u||_{Y^{s,\tau}} = \sup_{x>0} \left| \frac{u(x)}{(1+x^s)(1+x^{-\tau})} \right|.$$

It is shown that the C_0-semigroup solution of the Black-Scholes equation can be represented by $T_t := f(D_\nu)$, where

$$f(z) = e^{tg(z)} \text{ with } g(z) = z^2 + \gamma z - r \text{ and } D_\nu = \nu x \frac{\partial}{\partial x}.$$

For more information and details see [31].

In [51], it is proved that the Black-Scholes semigroup is strongly continuous and chaotic for $s > 1, \tau \geq 0$ with $s\nu > 1$. We will see that, with a little more work, the Black-Scholes semigroup satisfies the spectral criterion in [45] under the same restrictions on the parameters and, therefore, the hypothesis of Corollary 2.2.3.

Let $s > \frac{1}{\nu}$, $0 < \nu < 1$ and $s > 1$. Let $S_s = \{\lambda \in \mathbb{C} \; ; \; 0 < Re\lambda < s\nu\}$. By Lemma 3.5 in [51], we have that $g(S_s) \cap i\mathbb{R} \neq \emptyset$. Then there exists an open ball $U \subset g(S_s)$ such that $U \cap i\mathbb{R} \neq \emptyset$ and such that $U \cap \mathbb{R} = \emptyset$. In particular, we find an inverse g^{-1} well defined (and holomorphic) on U.
We set $F = L \circ g^{-1}$, $F : U \to Y^{s,\tau}$, where $L : S_s \to Y^{s,\tau}$ is defined as $L(\lambda) = h_{\frac{\lambda}{\nu}}$, with $h_\lambda(x) = x^\lambda$. It is clear that F is weakly holomorphic since L is weakly holomorphic [51].
Finally, $AF(\lambda) = g\left(\nu \frac{g^{-1}(\lambda)}{\nu}\right) F(\lambda) = \lambda F(\lambda)$ for any $\lambda \in U$, where $(A, D(A))$ is the generator of the Black-Scholes semigroup, and the equality $\langle F(\lambda), \psi \rangle = 0$ for a fixed $\psi \in (Y^{s,\tau})^*$ and for every $\lambda \in U$ necessarily implies $\psi = 0$. All the details are proved in Theorem 3.6 in [51].
Thus, the spectral criterion in [45] is satisfied and the Black-Scholes semigroup admits an invariant strongly mixing Borel probability measure on $Y^{s,\tau}$ with full support by Corollary 2.2.3.

Chapter 3

Frequently hypercyclic translation C_0-semigroups

This chapter is divided in three sections. In the first one we review some known results on the dynamics of the translation C_0-semigroups, later we state and prove a characterization of frequent hypercyclicity for weighted pseudo-shifts in terms of the weights that will be used in the last section to obtain a characterization of frequent hypercyclicity for translation C_0-semigroups on $C_0^\rho(\mathbb{R})$. Finally, in the third one, we establish a characterization of frequently hypercyclic translation C_0-semigroups on $C_0^\rho(\mathbb{R})$ and $L_p^\rho(\mathbb{R})$. Moreover, we establish an analogy between the study of frequent hypercyclicity for the translation C_0-semigroup and the corresponding one for backward shifts on weighted sequence spaces. The contents of this chapter have been included in [81].

For linear discrete dynamical systems, shift operators on sequence spaces are one of the most important test operators. In the continuous case this role is played by the translation semigroup. Firstly, let us introduce the spaces of functions where we are going to consider translation C_0-semigroups. These spaces are denoted by $L_p^\rho(\mathbb{R})$, with $1 \leq p < \infty$ and $C_0^\rho(\mathbb{R})$ and ρ is an admissible weight function.

Definition 3.0.8 ([45]) *We recall that by an* admissible weight function *on $\mathbb{R}$, we mean a measurable funtion $\rho : \mathbb{R} \to \mathbb{R}$ such that:*

- $\rho(t) > 0$ *for all* $t \in \mathbb{R}$ *and*

- *there exist* $M \geq 1$ *and* $\omega \in \mathbb{R}$ *such that* $\rho(\tau) \leq Me^{\omega t}\rho(\tau + t)$ *for all* $\tau \in \mathbb{R}$ *and all* $t > 0$.

We recall the following useful property for admissible weight functions.

Lemma 3.0.9 ([45]) *Let* ρ *be an admissible weight function on* $\mathbb{R}$. *For each* $l > 0$ *there are constants* $0 < A < B$ *such that for each* $\sigma \in \mathbb{R}$ *and each* $\tau \in [\sigma, \sigma + l]$, *we have* $A\rho(\sigma) \leq \rho(\tau) \leq B\rho(\sigma + l)$.

We consider the following function spaces

$$L_p^\rho(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \ ; \ f \text{ is measurable and } \|f\|_p < \infty\},$$

where $\|f\|_p = (\int_{-\infty}^{\infty} |f(t)|^p \rho(t) dt)^{\frac{1}{p}}$ and

$$C_0^\rho(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \ ; \ f \text{ is continuous and } \lim_{x \to \infty} f(x)\rho(x) = 0\},$$

with $\|f\|_\infty = \sup_{t \in \mathbb{R}} f(t)\rho(t)$. If X is any of the spaces above, the *translation semigroup* $(T_t)_{t \geq 0}$ defined as $T_t f(x) = f(x + t)$ is a well defined C_0-semigroup .

3.1 Existing results on the dynamics of translation C_0-semigroups

In this section, we have compiled some characterizations of hypercyclic, mixing and Devaney chaotic translation C_0-semigroups in terms of the weight function in order to establish a relation between these concepts and being frequently hypercyclic.

Theorem 3.1.1 ([45]) *Let* X *be one of the spaces* $L_p^\rho(\mathbb{R})$ *or* $C_0^\rho(\mathbb{R})$ *with* ρ *an admissible weight function. The translation* C_0*-semigroup* $(T_t)_{t \geq 0}$ *is hypercyclic on* X *if and only if for all* $\sigma \in \mathbb{R}$

$$\liminf_{x \to \infty} \rho(x + \sigma) = \liminf_{x \to \infty} \rho(-x + \sigma) = 0.$$

Theorem 3.1.2 ([24]) *Let* X *be one of the spaces* $L_p^\rho(\mathbb{R})$ *or* $C_0^\rho(\mathbb{R})$ *with* ρ *an admissible weight function. The translation* C_0*-semigroup* $(T_t)_{t \geq 0}$ *is mixing on* X *if and only if*

$$\lim_{x \to \infty} \rho(x) = \lim_{x \to \infty} \rho(-x) = 0.$$

Theorem 3.1.3 ([84]) *Let X be the space $C_0^\rho(\mathbb{R})$ with ρ an admissible weight function. The translation C_0-semigroup $(T_t)_{t\geq 0}$ is chaotic on X if and only if*

$$\lim_{x\to\infty}\rho(x)=\lim_{x\to\infty}\rho(-x)=0.$$

Theorem 3.1.4 ([84]) *Let X be the space $L_p^\rho(\mathbb{R})$ with ρ an admissible weight function. The translation C_0-semigroup $(T_t)_{t\geq 0}$ is chaotic on X if and only if for every $\epsilon,\sigma>0$ there exists $P>0$:*

$$\sum_{n\in\mathbb{Z}\setminus\{0\}}^{\infty}\rho(\sigma+nP)<\epsilon.$$

Theorem 3.1.5 ([44]) *Let X be the space $L_p^\rho(\mathbb{R})$ with ρ an admissible weight function. The following are equivalent:*

(i) The translation C_0-semigroup $(T_t)_{t\geq 0}$ is chaotic on X;

(ii) $\int_{-\infty}^{\infty}\rho(s)ds<\infty$;

(iii) $\sup\{\nu\in\mathbb{R};\int_{-\infty}^{\infty}e^{\nu s}\rho(s)ds<\infty\}>0$;

(iv) T_1 has a non-trivial periodic point;

(v) T_1 is chaotic.

Theorem 3.1.6 ([82]) *Let X be the space $L_p^\rho(\mathbb{R})$ with ρ an admissible weight function. The translation C_0-semigroup $(T_t)_{t\geq 0}$ is chaotic on X if and only if it satisfies the Frequent Hypercyclicity Criterion for semigroups.*

Theorem 3.1.7 ([82]) *Let X be the space $C_0^\rho(\mathbb{R})$ with ρ an admissible weight function. If $\int_{-\infty}^{\infty}\rho(s)ds<\infty$, then the translation C_0-semigroup $(T_t)_{t\geq 0}$ satisfies the Frequent Hypercyclicity Criterion for semigroups.*

Proposition 3.1.8 ([82]) *Let X be the space $L_p^\rho([0,\infty))$ with ρ an admissible weight function and $(T_t)_{t\geq 0}$ the translation C_0-semigroup on X. If $(T_t)_{t\geq 0}$ is frequently hypercyclic, then for every $\epsilon>0$ there exists a sequence $(n_k)_k\in\mathbb{N}$ with positive lower density such that $\sum_{k>i}\rho(n_k-n_i)<\epsilon$ for all $i\in\mathbb{N}$. Moreover, ρ is bounded.*
If $(T_t)_{t\geq 0}$ is a frequently hypercyclic translation semigroup on $C_0^\rho([0,\infty))$ then for every $\epsilon>0$ there exists a sequence $(n_k)_k\in\mathbb{N}$ with positive lower density such that $\rho(n_k-n_i)<\epsilon$ for all $i\in\mathbb{N}$ and $k>i$.

3.2 Frequently hypercyclic weighted pseudo-shifts

The concept of weighted pseudo-shift was introduced in section 0.6. We will be interested in weighted pseudo-shifts acting on spaces of vanishing sequences. More precisely, given a countable set I, we consider the space

$$c_0(I) = \{(x_i)_{i\in I} : \forall \epsilon > 0, \exists \quad J \subset I, J \quad \text{finite}: \ \forall i \in I \setminus J \quad |x_i| < \epsilon\},$$

endowed with the norm $||(x_i)_{i\in I}|| = \sup_{i\in I} |x_i|$.

The first result that we prove is a characterization of frequently universal sequences of weighted pseudo-shifts on $c_0(I)$. In order to prove this, we will follow the idea of Bayart and Ruzsa in [22] for proving frequent hypercyclicity of weighted backward shifts on $c_0(\mathbb{Z})$. We first recall the characterization they obtained.

Theorem 3.2.1 *Let $(w_i)_{i\in\mathbb{Z}}$ be a bounded and bounded below sequence of positive integers. Then B_w is frequently hypercyclic on $c_0(\mathbb{Z})$ if and only if there exist (for all) a sequence $(M(p))_{p\in\mathbb{N}}$ of positive real numbers tending to ∞ and a sequence $(E_p)_{p\in\mathbb{N}}$ of subsets of $\mathbb{Z}_+$ such that*

(a) *For any $p \geq 1$, $\underline{dens}(E_p) > 0$.*

(b) *For any $p, q \geq 1$, $p \neq q$, $(E_p + [-p, p]) \cap (E_q + [-q, q]) = \emptyset$.*

(c) $\lim_{n\to\infty, n\in E_p} w_1 \dots w_n = \infty$.

(d) *For any $p, q \geq 1$, $n \in E_p$, $m \in E_q$, $n \neq m$:*

$$\begin{cases} w_1 \dots w_{m-n} \geq M(p)M(q), & \text{if} \quad m > n \\ w_{(m-n)+1} \dots w_{-1}w_0 \leq \frac{1}{M(p)M(q)}, & \text{if} \quad m < n. \end{cases} \tag{3.1}$$

Now, we obtain a characterization for weighted pseudo-shifts on $c_0(I)$.

Theorem 3.2.2 *Let $(T_n)_n$ be a sequence of weighted pseudo-shifts on $c_0(I)$ defined by $T_n[(x_i)_{i\in I}] = (b_i^n x_{\phi_n(i)})_{i\in I}$, where b_i^n are positive real numbers. Assume that:*

(i) *$(\phi_n)_n$ is a run-away sequence, i.e. for each pair of finite subsets $I_0, J_0 \subset I$ there exists an $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $\phi_n(J_0) \cap I_0 = \emptyset$,*

(ii) *there exists* $\rho > 1$ *such that* $\frac{1}{\rho^{|n-m|}} \leq \frac{b_s^n}{b_t^m}$ *for all* $n, m \in \mathbb{N}$, $s, t \in I$ *such that* $\phi_n(s) = \phi_m(t)$,

(iii) *there exists* $g : I \to \mathbb{R}$ *such that* $|n-m| \leq |g(s)-g(t)|$ *for all* $n, m \in \mathbb{N}$, $s, t \in I$ *such that* $\phi_n(s) = \phi_m(t)$.

Then $(T_n)_n$ *is frequently universal on* $c_0(I)$ *if and only if there exist a sequence* $(M(p))_{p\in\mathbb{N}}$ *of positive real numbers tending to* ∞, *a sequence* $(E_p)_{p\in\mathbb{N}}$ *of subsets of* $\mathbb{N}$, *and an increasing sequence* $(W_p)_{p\in\mathbb{N}}$ *of finite subsets of* I *with* $I = \bigcup_{p=1}^{\infty} W_p$, *such that:*

(a) *For any* $p \geq 1$, $\underline{\text{dens}}(E_p) > 0$.

(b) *For any* $p, q \geq 1$, $p \neq q$, $n \in E_p, m \in E_q$, $\phi_n(W_p) \cap \phi_m(W_q) = \emptyset$.

(c) *For every* $p \geq 1$ *and every* $s \in W_p$: $\lim_{n\to\infty, n\in E_p} b_s^n = \infty$.

(d) *For any* $p, q \geq 1$, $n \in E_p$, $m \in E_q$, $n \neq m$, $t \in W_q$ *and* $s \in I$ *such that* $\phi_n(s) = \phi_m(t)$:

$$\frac{b_s^n}{b_t^m} \leq \frac{1}{M(p)M(q)}.$$

Proof.

"$\Rightarrow$": Let $x \in c_0(I)$ be a frequently universal vector for $(T_n)_n$. Let $(\alpha_p)_{p\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers such that $\alpha_1 = 2$ and for all $p \geq 2$, $\alpha_p > 4\alpha_{p-1}\rho^{2\Psi(p)}$, where $\Psi(p) = \max\{|g(t)| : t \in W_p\}$ and define

$$E_p = \left\{ n \in \mathbb{N} : ||T_n x - \alpha_p \sum_{i\in W_p} e_i|| < \frac{1}{p} \right\}.$$

Clearly $\underline{\text{dens}}(E_p) > 0$. In order to prove (b), fix $p \neq q$, with $p < q$, $n \in E_p, m \in E_q$ and assume by contradiction, that there exist $s \in W_p$ and $t \in W_q$ such that $\phi_n(s) = \phi_m(t)$. The s-th coefficient of $T_n x$ is $b_s^n x_{\phi_n(s)}$, then

$$|b_s^n x_{\phi_n(s)}| \leq \|T_n x - \alpha_p \sum_{i\in W_p} e_i\| + \alpha_p \| \sum_{i\in W_p} e_i\| < \frac{1}{p} + \alpha_p < 2\alpha_p.$$

The t-th coefficient of $T_m x$ is $b_t^m x_{\phi_m(t)}$ and

$$|b_t^m x_{\phi_m(t)}| \geq \alpha_q - |b_t^m x_{\phi_m(t)} - \alpha_q| \geq \alpha_q - \|T_n x - \alpha_p \sum_{i\in W_p} e_i\| \geq \alpha_q - \frac{1}{q} \geq \frac{\alpha_q}{2}. \tag{3.2}$$

Then we get

$$\frac{1}{\rho^{2\Psi(q)}} \leq \frac{1}{\rho^{|n-m|}} \leq \frac{|b_s^n x_{\phi_n(s)}|}{|b_t^m x_{\phi_m(t)}|} \leq 2\alpha_p \frac{2}{\alpha_q} \leq 4\frac{\alpha_{q-1}}{\alpha_q}$$

, contraddicting the choice of $(\alpha_p)_p$.

Now let $p \geq 1$ and $s \in W_p$; for every $n \in E_p$ the s-th coefficient of $T_n x$ is $b_s^n x_{\phi_n(s)}$ and, with the same argument as in (3.2), we get that its modulus is greater or equal than $\frac{\alpha_p}{2}$. Let $M > 0$. Given $\epsilon = \frac{\alpha_p}{2M}$, since $x \in c_0(I)$, there exists $J \subset I$ finite such that $|x_i| < \epsilon$ for all $i \in I \setminus J$. As ϕ_n is a run-away sequence there exists $n_0 \in \mathbb{N}$ such that for all $n \in E_p$, $n > n_0$ and for all $s \in W_p$, $\phi_n(s) \notin J$, and then $|x_{\phi_n(s)}| < \epsilon$. As a result, for all $n \in E_p$, $n \geq n_0$:

$$|b_s^n| \geq \frac{\alpha_p}{2|x_{\phi_n(s)}|} \geq \frac{\alpha_p}{2\epsilon} = M.$$

So, we have proved (c).

Finally, let $n \in E_p$, $m \in E_q, t \in W_q$, $s \in I$ such that $\phi_n(s) = \phi_m(t)$; then by (b) $s \notin W_p$ and

$$\frac{b_s^n}{b_t^m} = \frac{|b_s^n x_{\phi_n(s)}|}{|b_t^m x_{\phi_m(t)}|} \leq \frac{1}{p}\frac{2}{\alpha_q} \leq \frac{1}{p}\frac{1}{q}.$$

Hence (d) holds with $M(p) = p$.

"$\Leftarrow$": Observe that if properties (a) to (d) hold true for some sequence $(M(p))$, then they are also satisfied for any subsequence of it, passing to a subsequence of $(E_p)_p$ if necessary. Therefore we may assume that, for any $p \geq 1$, $M(p) \geq \rho^{4p}$. Moreover, observe that $E_p \cap E_q = \emptyset$ if $p \neq q$. Indeed, assume $p < q$; if there exists $n \in E_p \cap E_q$, then for any $s \in W_p \subset W_q$, one would have $\phi_n(s) \in \phi_n(W_p) \cap \phi_n(W_q)$, contraddicting (b).

We set

$$E'_p = E_p \setminus \bigcup_{s \in W_p} \{n \in \mathbb{N} : b_s^n \leq \rho^{4p}\}.$$

By (c), E'_p is a cofinite subset of E_p, hence $\underline{\text{dens}}(E'_p) > 0$. If $E'_p = \{n_k^p \mid k \in \mathbb{N}\}$, where $(n_k^p)_k$ is an increasing sequence natural numbers, we consider $F_p = \{n^p_{(2[\Psi(p)]+3)k} \mid k \in \mathbb{N}\}$ where $[\Psi(p)]$ is the integer part of $\Psi(p)$. F_p has positive lower density and moreover the distance between two different elements of F_p is greater than $2\Psi(p)$.

Let $(y^p)_{p\geq 0}$ be a dense sequence in $c_0(I)$ such that $supp(y^p) \subset W_p$ and $||y^p|| < \rho^p$. We define $x \in \mathbb{R}^I$ by setting

$$x_i = \begin{cases} \frac{1}{b_s^n} y^p(s), & \text{if} \quad i = \phi_n(s), n \in F_p, s \in W_p \\ \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.3}$$

This definition is well posed, because if $i = \phi_n(s) = \phi_m(t)$, with $n \in F_p$, $s \in W_p$, $m \in F_q$, $t \in W_q$, then, by (b), $p = q$ and, by the assumption (iii), $|n - m| \leq |g(s) - g(t)| \leq 2\Psi(p)$, hence, by the definition of F_p, $n = m$ and hence $s = t$, by the injectivity of ϕ_n.

It holds that that $x \in c_0(I)$. Indeed, given $\epsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that for $p \geq p_0$ and $n \in F_p, s \in W_p$:

$$|x_i| \leq \frac{\rho^p}{\rho^{4p}} \leq \epsilon.$$

If $p \leq p_0$:

$$|x_i| \leq \frac{\rho^{p_0}}{b_s^n} \to 0, n \to \infty.$$

We finally show that x is a frequently hypercyclic vector by proving that for all $p \geq 1, n \in F_p$, $||T_n x - y^p|| < \epsilon(p)$ with $\epsilon(p) \to 0$ as $p \to \infty$. We have that

$$||T_n x - y^p|| = \sup_{s \notin W_p} |b_s^n x_{\phi_n(s)}|.$$

The terms which appear in the modulus do not vanish if and only if $\phi_n(s) = \phi_m(t), m \in F_q, t \in W_q$. It holds that $n \neq m$, otherwise $n \in F_p \cap F_q$, hence $p = q$ and, by the inyectivity of ϕ_n, $s = t \in W_p = W_q$, while $s \notin W_p$. Hence, we can apply (d) to get that

$$|b_s^n x_{\phi_n(s)}| = \left| \frac{b_s^n}{b_t^m} y^q(t) \right| \leq \frac{\rho^q}{M(p)M(q)} \leq \frac{\rho^q}{\rho^p \rho^q} = \frac{1}{\rho^p}.$$

□

As a corollary, we obtain a characterization of frequent hypercyclicity for weighted backward shifts operators defined on $c_0(I)$, in the case that $I \subseteq \mathbb{R}$ is any countable set such that $I + \mathbb{Z} \subset I$, $I = \bigcup_{p=1}^{\infty} W_p$, where $(W_p)_p$ is an increasing sequence of finite subsets.

Corollary 3.2.3 *Let $(w_i)_{i\in I}$ be a bounded and bounded below sequence of positive integers. The operator $T : c_0(I) \to c_0(I)$ defined by $T(x_i)_{i\in I} = (w_{i+1}x_{i+1})_{i\in I}$ is frequently hypercyclic on $c_0(I)$ if and only if there exist (for all) a sequence $(M(p))_{p\in\mathbb{N}}$ of positive real numbers tending to ∞ and a sequence $(E_p)_{p\in\mathbb{N}}$ of subsets of $\mathbb{Z}_+$ such that*

(a) *For any $p \geq 1$, $\underline{dens}(E_p) > 0$.*

(b) *For any $p, q \geq 1$, $p \neq q$, $(E_p + W_p) \cap (E_q + W_q) = \emptyset$.*

(c) $\lim_{n\to\infty, n\in E_p, s\in W_p} w_{s+1} \dots w_{s+n} = \infty$.

(d) *For any $p, q \geq 1$, $n \in E_p$, $m \in E_q$, $n \neq m$ and $t \in W_q$:*

$$\frac{w_{m-n+t+1} \dots w_{m+t}}{w_{t+1} \dots w_{t+m}} \leq \frac{1}{M(p)M(q)}.$$

Proof. This corollary is a particular case of Theorem 3.2.2 when we consider $T_n = T^n$ with

$$T(x_i)_{i\in I} = (w_{i+1}x_{i+1})_{i\in I},$$

$b_s^n = w_{s+1}w_{s+2} \dots w_{s+n}$, $\phi(s) = s + 1$, $\phi_n = \phi^n$ and $g : I \to \mathbb{R}$ defined by $g(s) = s$. □

Remark 3.2.4 *Observe that condition (d) is equivalent to say that for any $p, q \geq 1$, $n \in E_p$, $m \in E_q$, $n \neq m$ and $t \in W_q$:*

$$\begin{cases} w_{t+1} \dots w_{t+m-n} \geq M(p)M(q), & \textit{if} \quad m > n \\ w_{t+(m-n)+1} \dots w_{t-1}w_t \leq \frac{1}{M(p)M(q)}, & \textit{if} \quad m < n. \end{cases} \tag{3.4}$$

and we obtain similar conditions as the ones obtained in 3.2.1.

3.3 Frequently hypercyclic translation semigroups

Our main purpose is to obtain a characterization of frequent hypercyclicity for translation semigroups on $C_0^\rho(\mathbb{R})$ and $L_p^\rho(\mathbb{R})$.

To treat the case of continuous functions, we will first need to recall some known results about the construction of a Schauder basis in $C_0(\mathbb{R})$, referring for more details to [98].

Let $\widetilde{D}$ be the set of dyadic numbers, that is $\widetilde{D} = \bigcup_{n=0}^{\infty} D_n$ where $D_0 = \{0, 1\}$ and, if $n \geq 1$,

$$D_n = \left\{ \frac{2k-1}{2^n} \,:\, k = 1, \ldots, 2^{n-1} \right\}.$$

For any $\tau \in D_n$, set $\tau^- = \tau - 2^{-n}$ and $\tau^+ = \tau + 2^{-n}$.

Let $\phi(x) = \max(0, 1 - |x|)$, $x \in \mathbb{R}$ and define $\phi_{k+\tau}(x) = \phi(2^n(x - k - \tau))$ where $k \in \mathbb{Z}, \tau \in D_n, \tau \neq 1$. Observe that $\phi_{k+\tau}(x) = \phi_\tau(x-k)$ where ϕ_τ is the Faber-Schauder dyadic function with peak at τ.

Set $I = \mathbb{Z} + \widetilde{D}$ and consider the partition $I = V_0 \cup V_1 \cup \ldots$ where $V_0 = \{0, 1\}$, and

$$V_n = \{-n + h + D_h | h = 1, \ldots, n\} \cup \{h + D_{n-h} | h = 0, 1, \ldots, n\}. \tag{3.5}$$

We define an order on I assuming that the elements of V_k are earlier that the elements of V_n if $0 \leq k < n$, and within each V_n keep the usual order.

The system $(\phi_i)_{i \in I}$, is a Schauder basis in $C_0(\mathbb{R})$. More precisely, if $f \in C_0(\mathbb{R})$ then $f = \sum_{k+\tau \in \mathbb{Z}+\widetilde{D}} a_{k+\tau}\phi_{k+\tau}$ where $a_k = f(k)$, for $k \in \mathbb{Z}$ and $a_{k+\tau} = f(k+\tau) - \frac{1}{2}(f(k+\tau^-) + f(k+\tau^+))$ for $k \in \mathbb{Z}, \tau \in \widetilde{D}$.

Lemma 3.3.1 *Let ρ be an admissible weight function on $\mathbb{R}$ such that $\rho(x) = \rho([x])$ for any $x \in \mathbb{R}$ and let $T_1 : C_0^\rho(\mathbb{R}) \to C_0^\rho(\mathbb{R})$ be the translation operator defined as $T_1 f(x) = f(x+1)$. Then the weighted backward shift operator $B_w : c_0(\mathbb{Z} + \widetilde{D}) \to c_0(\mathbb{Z} + \widetilde{D})$ defined by*

$$B_w((x_{k+\tau})_{k+\tau \in \mathbb{Z}+D}) = (w_{k+\tau}x_{k+\tau+1})_{k+\tau \in \mathbb{Z}+D}$$

where $w_{k+\tau} = \frac{\rho(k)}{\rho(k+1)}$, $k + \tau \in \mathbb{Z} + \widetilde{D}$ is quasi conjugated to T_1.

Proof.

Given $f \in C_0^\rho(\mathbb{R})$, we define $Q(f(x)) = (a_{k+\tau})_{n+\tau \in \mathbb{Z}+\widetilde{D}}$ where

$$f(x)\rho(x) = \sum_{k+\tau \in \mathbb{Z}+\widetilde{D}} a_{k+\tau}\phi_{k+\tau}(x).$$

Clearly $Q : C_0^\rho(\mathbb{R}) \to c_0(\mathbb{Z} + \widetilde{D})$ is a continuous linear operator and

$$B_w \circ Q(f) = B_w(a_{k+\tau})_{k+\tau \in \mathbb{Z}+\widetilde{D}} = \left(\frac{\rho(k)}{\rho(k+1)} a_{k+\tau+1} \right)_{k+\tau \in \mathbb{Z}+\widetilde{D}}.$$

On the other hand, $Q \circ T_1(f(x)) = Q(f(x+1)) = (b_{k+\tau})_{k+\tau\in\mathbb{Z}+\widetilde{D}}$, where $f(x+1)\rho(x) = \sum_{k+\tau\in\mathbb{Z}+\widetilde{D}} b_{k+\tau}\phi_{k+\tau}(x)$. We have that:

$$b_{k+\tau} = \begin{cases} f(k+1)\rho(k) = a_{k+1}\frac{\rho(k)}{\rho(k+1)}, \quad \text{if} \quad \tau = 0 \\ \\ (f(k+1+\tau) - \frac{1}{2}(f(k+1+\tau^-) + f(k+1+\tau^+))\rho(k) \\ \\ = a_{k+\tau+1}\frac{\rho(k)}{\rho(k+1)}, \quad \text{if} \quad \tau \neq 0. \end{cases} \tag{3.6}$$

for $k \in \mathbb{Z}, \tau \in \widetilde{D}$, taking into account that $\rho(k+\tau) = \rho(k+\tau^-) = \rho(k+\tau^+) = \rho(k)$ for all $k \in \mathbb{Z}, \tau \in \widetilde{D}$. Then

$$Q \circ T_1(f(x)) = \left(\frac{\rho(k)}{\rho(k+1)} a_{k+\tau+1}\right)_{k+\tau\in\mathbb{Z}+\widetilde{D}} = B_w \circ Q(f).$$

So the diagram

$$\begin{array}{ccc} C_0^\rho(\mathbb{R}) & \xrightarrow{T_1} & C_0^\rho(\mathbb{R}) \\ \downarrow Q & & \downarrow Q \\ c_0(\mathbb{Z}+\widetilde{D}) & \xrightarrow{B_w} & c_0(\mathbb{Z}+\widetilde{D}) \end{array}$$

is commutative and we conclude the result. □

Theorem 3.3.2 *Let $(T_t)_{t\geq 0}$ be the translation semigroup on $C_0^\rho(\mathbb{R})$, where ρ is an admissible weight function and $\sup_{k\in\mathbb{Z}} \frac{\rho(k+1)}{\rho(k)} < \infty$. $(T_t)_{t\geq 0}$ is frequently hypercyclic on $C_0^\rho(\mathbb{R})$ if and only if there exist a sequence $(M(p))_{p\in\mathbb{N}}$ of positive real numbers tending to ∞ and a sequence $(E_p)_{p\in\mathbb{N}}$ of subsets of $\mathbb{Z}_+$ such that:*

(a) For any $p \geq 1$, $\underline{dens}(E_p) > 0$.

(b) For any $p, q \geq 1$, $p \neq q$, $(E_p + W_p) \cap (E_q + W_q) = \emptyset$.

(c) $\lim_{n\to\infty, n\in E_p, k\in[-p,p+1]} \rho(k+n) = 0$.

(d) For any $p, q \geq 1$, for any $n \in E_p$ and any $m \in E_q$, $n \neq m$ and for all $k \in [-q, q+1]$:

$$\rho(k+m-n+1) \leq \frac{1}{M(p)M(q)}, \tag{3.7}$$

where $W_p = \bigcup_{k=0}^{p} V_k$ *defined in* (3.5).

Proof. $(T_t)_t$ is frequently hypercyclic if and only if T_1 is frequently hypercylic [38, 82]. Let us point out that if ρ is an admissible weight function then $\sup \frac{\rho(k)}{\rho(k+1)} < \infty$.

By hypothesis we have $\sup \frac{\rho(k+1)}{\rho(k)} = M < \infty$, then there exist constants $0 < A < B$ such that

$$A\rho(k) \leq \rho(x) \leq B\rho(k+1) \leq BM\rho(k).$$

Then if we define $\tilde{\rho}(x) = \rho(x)$ for $x \in [k, k+1[$, there exist constants $M_1, M_2 > 0$ such that

$$M_1 ||f||_\infty^{\tilde{\rho}} \leq ||f||_\infty^{\rho} \leq M_2 ||f||_\infty^{\tilde{\rho}}.$$

We conclude the result combining Corollary 3.2.3 and Lemma 3.3.1. □

Remark 3.3.3 Let ρ be an admissible weight function on $\mathbb{R}$ such that $\sup_{k\in\mathbb{Z}} \frac{\rho(k+1)}{\rho(k)} < \infty$ and set $w_k = \frac{\rho(k)}{\rho(k+1)}$, $k \in \mathbb{Z}$. If B_w is frequently hypercyclic on $c_0(\mathbb{Z})$, the corresponding translation semigroup is frequently hypercyclic on $C_0^\rho(\mathbb{R})$.

Let (E_p) be a sequence of subsets of $\mathbb{N}$ such that for any $p, q \geq 1$, $p \neq q$, $(E_p + [-p, p]) \cap (E_q + [-q, q]) = \emptyset$.

Choosing $F_p = E_{p+1}$, we get that $(F_p + W_p) \cap (F_q + W_q) = \emptyset$ if $p \neq q$, where the sets W_p are defined as in the assumptions of Theorem 3.3.2. Indeed, if $n \in F_p$, $s \in [-p, p+1]$, $\sigma \in \widetilde{D}$, $\sigma = \frac{2u-1}{2^h}$, $m \in F_q$, $t \in [-q, q+1]$, $\tau \in \widetilde{D}$, $\tau = \frac{2v-1}{2^k}$ are such that

$$n + s + \sigma = m + t + \tau,$$

we have that $\tau - \sigma \in \mathbb{Z}$. Thus straightforward calculations give that $h = k$ and $|u - v| = a2^{h-1}$ with $a \in \mathbb{Z}_+$.
On the other hand $|u - v| < 2^{h-1}$, hence $a = 0$. Therefore $\tau = \sigma$ and so $n + s = m + t$. Now the assertion follows by the properties of the set E_p.

As an immediate consequence using Theorem 3.2.1 hypothesis (c) and (d) of Theorem 3.3.2 are verifed and we get that if ρ is an admissible weight function on $\mathbb{R}$ such that $\sup_{k\in\mathbb{Z}} \frac{\rho(k+1)}{\rho(k)} < \infty$ and we set $w_k = \frac{\rho(k)}{\rho(k+1)}$, $k \in \mathbb{Z}$, if B_w is frequently hypercyclic on $c_0(\mathbb{Z})$, the corresponding translation semigroup is frequently hypercyclic on $C_0^\rho(\mathbb{R})$.

Proposition 3.3.4 *Let $(T_t)_{t\geq 0}$ be a mixing (equivalently chaotic) translation C_0-semigroup on $C_0^\rho(\mathbb{R})$. Then $(T_t)_{t\geq 0}$ is frequently hypercyclic.*

Proof. As it is shown in 3.1.3 and 3.1.2, chaos and mixing are equivalent properties for the translation C_0-semigroup on $C_0^\rho(\mathbb{R})$, and this happens if and only if $\lim_{x\to\infty}\rho(x) = \lim_{x\to\infty}\rho(-x) = 0$.
Consider a sequence (E_p) of subsets of $\mathbb{Z}_+$ such that for any $p \geq 1$, $\underline{dens}(E_p) > 0$ and for any $p, q \geq 1$, $p \neq q$, $(E_p + [-p,p]) \cap (E_q + [-q,q]) = \emptyset$. (see e.g. the constructions in [20]). It is clear that hypothesis (c) of 3.3.2 will be verified trivially, while (b) is satisfied by Remark 3.3.3.
Moreover, given $n \in E_p, m \in E_q$ and $k \in [-q, q+1]$, we can define for each $i \in \mathbb{N}$:

$$M(i) = \min_{k\in[-q,q+1]} \left\{ \frac{1}{\sup_{|n|\geq i}\{\rho(k+n+1)\}} \right\}.$$

It is clear that for $n \in E_p, m \in E_q$, $|m-n| \geq \max(p,q)$, and

$$\rho(k+m-n+1) \leq \sup_{|s|\geq 1}\rho(k+s+1) \leq \tfrac{1}{M(i)}, m < n, i = p, q, \tag{3.8}$$

So, we have for each $k \in [q, q+1]$:

$$\rho(k+m-n+1) \leq \tfrac{1}{\sqrt{M(p)}}\tfrac{1}{\sqrt{M(q)}}, \tag{3.9}$$

and hypothesis (d) is satisfied by the sequence $M'(p) = \sqrt{M(p)}$ and therefore $(T_t)_{t\geq 0}$ is frequently hypercyclic. □

Remark 3.3.5 The converse of the previous proposition does not hold. Indeed, let $(w_k)_{k\in\mathbb{Z}}$ be one of the sequence of weights constructed in [22, 19] such that B_w is frequently hypercylic on $c_0(\mathbb{Z})$ and $w_1 \dots w_k = 1$ for infinitely many k. Define $\rho(k) = (w_1 \dots w_{k-1})^{-1}$ if $k \geq 1$ and $\rho(k) = w_k \cdot w_{k+1} \cdot \dots \cdot w_0$ if $k \leq 0$ and $\rho(x) = \rho([x])$ for any $x \in \mathbb{R}$. Then, by Remark 3.3.3, the translation semigroup is frequently hypercyclic on $C_0^\rho(\mathbb{R})$, while clearly it is not mixing, since $\rho(k) = 1$ for infinitely many k.

Now we will be devoted to characterize frequent hypercyclicity for translation semigroups on $L_p^\rho(\mathbb{R})$. In order to do this, we will establish a relation between the discrete and the continuous case. The relation between the discrete and continuous case for Devaney chaos was studied in [16] and distributional chaos in [11].

First of all, we recall the characterization of frequent hypercyclicity for weighted backward shifts on ℓ_p given in [22].

Theorem 3.3.6 *Let $1 \le p < \infty$ and let $w = (w_n)_{n\in\mathbb{Z}}$ be a bounded sequence of positive real numbers. Then B_w is frequently hypercyclic on $\ell_p(\mathbb{Z})$ if and only if the series $\sum_{k\ge 1} \frac{1}{(w_1 \dots w_k)^p}$ and $\sum_{k<0}(w_{-1}\dots w_k)^p$ are convergent.*

The following lemma follows immediately by the conjugacy of the backward shift on ℓ_p^v and the weighted backward shift B_w on ℓ_p where $w_k = \left(\frac{v_{k-1}}{v_k}\right)^{\frac{1}{p}}$, $k \in \mathbb{Z}$ and Theorem 3.3.6.

Lemma 3.3.7 *Let $v = (v_k)_{k\in\mathbb{Z}}$ be a sequence of strictly positive weights such that $(\frac{v_{k-1}}{v_k})_k$ is bounded. Then the backward shift operator B is frequently hyperyclic on ℓ_p^v if and only if $\sum_{k\in\mathbb{Z}} v_k < \infty$.*

Proof.

Let us define $T : \ell_p \to \ell_p^v$ as $T(x_k)_k = \left(\dfrac{x_k}{v_k^{p^{-1}}}\right)_k$,

$$T^{-1}\circ B \circ T(x_k)_n = T^{-1}\circ B\left(\dots, \frac{x_{-1}}{v_{-1}^{p^{-1}}}, \underbrace{\frac{x_0}{v_0^{p^{-1}}}}_{0}, \frac{x_1}{v_1^{p^{-1}}}, \dots\right) =$$

$$\left(\dots, \frac{v_{-1}^{p^{-1}} x_0}{v_0^{p^{-1}}}, \underbrace{\frac{v_0^{p^{-1}} x_1}{v_1^{p^{-1}}}}_{0}, \frac{v_1^{p^{-1}} x_2}{v_2^{p^{-1}}}, \dots\right) = B_w(x_n)_n$$

where $w_k = \left(\frac{v_{k-1}}{v_k}\right)^{\frac{1}{p}}$. We have that B is frequently hyperyclic in ℓ_p^v if and only if B_w is frequently hypercyclic. By 3.3.6, this happens if and only if $\sum_{k\ge 1} \frac{1}{(w_1 w_2 \dots w_k)^p} < \infty$ and $\sum_{k<0}(w_{-1}\dots w_k)^p < \infty$. As a result, B is frequently hyperyclic on ℓ_p^v if and only if $\sum_{k\in\mathbb{Z}} v_k < \infty$. □

Theorem 3.3.8 *Let $(T_t)_{t\geq 0}$ be the translation semigroup defined on $L_p^\rho(\mathbb{R})$. If $(T_t)_{t\geq 0}$ is frequently hypercyclic, then the backward shift operator B is frequently hypercyclic on ℓ_p^v, where $v_k = \rho(k)$ for all $k \in \mathbb{Z}$.*

Proof. Since ρ is an admissible function by 3.0.9 there exists $A, B \geq 0$ such that for all $t \in [k, k+1]$, $A\rho(k) \leq \rho(t) \leq B\rho(k+1)$. If $(T_t)_{t\geq 0}$ is frequently hypercyclic, then T_1 is frequently hypercyclic [38]. Hence there exists $f \in L_p^\rho$ such that for all $g \in L_p^\rho$ and for all $\epsilon > 0$, $\underline{dens}\{n \in \mathbb{N} : ||T_1^n f - g|| < \epsilon\} > 0$. Since $f \in L_p^\rho$ we have that $|f|\rho^{\frac{1}{p}} \in L_p([k, k+1]) \subset L_1([k, k+1])$ for every $k \in \mathbb{Z}$. Being ρ a strictly positive continuous function we get that $f \in L_1([k, k+1])$ for all $k \in \mathbb{Z}$. Therefore we can define $x_k = \int_k^{k+1} f(t)dt$ for all $k \in \mathbb{Z}$. We have that:

$$\sum_{k\in\mathbb{Z}} |x_k|^p \rho(k) = \sum_{k\in\mathbb{Z}} \left|\int_k^{k+1} f(t)dt\right|^p \rho(k) \leq \sum_{k\in\mathbb{Z}} \int_k^{k+1} |f(t)|^p \rho(k)dt \leq$$

$$\frac{1}{A}\sum_{k\in\mathbb{Z}} \int_k^{k+1} |f(t)|^p \rho(t)dt = \frac{1}{A}||f||_p < \infty.$$

So $x = (x_k)_{k\in\mathbb{Z}} \in \ell_v^p$ with $v_k = \rho(k)$.
Let $y = (0, \ldots, y_{-N}, \ldots, y_0, \ldots, y_M, \ldots, 0)$ and let $\epsilon > 0$.
Set $g = \sum_{k=-N}^{M} y_k \chi_{[k,k+1]} \in L_p^\rho(\mathbb{R})$. We show that:

$$\{n \in \mathbb{N} : ||T_1^n f - g|| < A^{\frac{1}{p}}\epsilon\} \subset \{n \in \mathbb{N} : ||B^n x - y|| < \epsilon\}$$

and therefore

$$\underline{dens}\{n \in \mathbb{N} : ||B^n x - y|| < \epsilon\} > 0$$

because f is a frequently hyperciclic vector. We have:

$$||B^n x - y||^p = \sum_{k\in\mathbb{Z}} |x_{n+k} - y_k|^p \rho(k) \leq \frac{1}{A}\sum_{k\in\mathbb{Z}} \int_k^{k+1} |f(t+n) - g(t)|^p \rho(t) \leq \frac{1}{A} A\epsilon^p.$$

By the density of finite sequences in ℓ_p^v we get that B is frequently hypercyclic. □

Finally we are able to characterize frequently hypercyclic translation semigroups in $L_p^\rho(\mathbb{R})$.

Proposition 3.3.9 *Let $(T_t)_{t\geq 0}$ be the translation semigroup defined on $L_p^\rho(\mathbb{R})$ and ρ and admissible weight function on $\mathbb{R}$. The following assertions are equivalent:*

(1) $(T_t)_{t\geq 0}$ *is frequently hypercyclic.*

(2) $\sum_{k\in\mathbb{Z}} \rho(k) < \infty$.

(3) $\int_{-\infty}^{\infty} \rho(t)dt < \infty$.

(4) $(T_t)_{t\geq 0}$ *is chaotic.*

(5) $(T_t)_{t\geq 0}$ *satisfies the Frequent Hypercyclicity Criterion.*

Proof. Observe that $\left(\frac{\rho(k-1)}{\rho(k)}\right)_k$ is bounded by the admissibility of the function ρ. By Theorem 3.3.8 and Lemma 3.3.7, (1) $\Longrightarrow$ (2), while (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (1) are proved in [82]. □

Chapter 4

Chaotic behavior on invariant sets of linear operators

4.1 Introduction

In this chapter we study hypercyclicity, Devaney chaos, topological mixing properties and strong mixing in the measure-theoretic sense for operators on topological vector spaces with invariant sets. More precisely, our purpose is to establish links between the fact of satisfying any of our dynamical properties on certain invariant sets, and the corresponding property on the closed linear span of the invariant set, or on the union of the invariant sets. Viceversa, we give conditions on the operator (or C_0-semigroup) to ensure that, when restricted to the invariant set, it satisfies certain dynamical property. Particular attention is given to the case of positive operators and semigroups on lattices, and the (invariant) positive cone. We also present examples that illustrate these results.

Although chaotic properties for linear operators are usually considered in the context of F-spaces, more general topological vector spaces have also attracted the attention in recent years (see, e.g.,[32], [99] and Chapter 12 of [72]). In the first section we deal with operators T on general topological vector spaces X. We will provide several conditions under which a dynamical property can pass from an invariant set (or a countable family of invariant sets) of the operator to the closure of its linear span (or to the union of the invariant sets). In [13] analogous results have been given for backward shift operators and the specification property. Some examples include an interplay between finite-dimensional and infinite-dimensional dynamics.

In the last section we will give several criteria for operators and C_0-semigroups that allow certain dynamical properties when restricted to invariant sets. Special attention is devoted to positive operators on Fréchet lattices and C_0-semigroups of positive operators on Banach lattices when the invariant set is the positive cone. In this case the results are relevant in connection with applications since, for instance, the chaotic behavior of certain solutions to differential equations make sense only when they are positive. This provides partial answers to questions of Banasiak, Desch and Rudnicki.

The contents of this chapter have been published in [85].

4.2 Topological dynamics and invariant sets

In this section we plan to study dynamical properties, in the topological sense, of operators that admit invariant sets, supposed that the corresponding properties are satisfied when the operators are restricted to the invariant sets. We will show that these properties can be extended to the closed span of the corresponding invariant sets.

Our first result is rather general and does not need linearity. Although it is almost immediate, we include a proof of it from the sake of completeness.

Proposition 4.2.1 *Let $T : X \to X$ be an operator, $(K_n)_n$ an increasing sequence of T-invariant sets, and $Y = \overline{\bigcup_{n=1}^{\infty} K_n}$. Then:*

(i) If $T|_{K_n}$ is transitive for all $n \in \mathbb{N}$ then $T : Y \to Y$ is transitive.

(ii) If $T|_{K_n}$ is mixing for all $n \in \mathbb{N}$ then $T : Y \to Y$ is mixing.

(iii) If $T|_{K_n}$ is weakly-mixing for all $n \in \mathbb{N}$ then $T : Y \to Y$ is weakly-mixing.

(iv) If $T|_{K_n}$ is chaotic for all $n \in \mathbb{N}$ then $T : Y \to Y$ is chaotic.

(v) If $T|_{K_n}$ is topologically ergodic for all $n \in \mathbb{N}$ then $T : Y \to Y$ is topologically ergodic.

Proof. It is sufficient to show that $T|_{\bigcup_{n=1}^{\infty} K_n}$ satisfies these properties since the property trivially extends to Y by density.

(i) Let U, V be non-empty open sets of $\bigcup_{n=1}^{\infty} K_n$. Then there exist U', V' non-empty open sets of X such that

$$U = U' \cap (\bigcup_{n=1}^{\infty} K_n) \neq \emptyset \quad \text{and} \quad V = V' \cap (\bigcup_{n=1}^{\infty} K_n) \neq \emptyset.$$

Since U and V are non-empty there exist $n_1, n_2 \in \mathbb{N}$ such that $U' \cap K_{n_1} \neq \emptyset$ and $V' \cap K_{n_2} \neq \emptyset$. Without loss of generalization, suppose that $n_1 \leq n_2$. Thus, $K_{n_1} \subset K_{n_2}$, and then $U' \cap K_{n_2} \neq \emptyset$.

By hypothesis $T\mid_{K_{n_2}}$ is transitive and there exists an $n \in \mathbb{N}$ such that

$$T^n(U' \cap K_{n_2}) \cap (V' \cap K_{n_2}) \neq \emptyset.$$

Since

$$U' \cap K_{n_2} \subseteq U' \cap (\bigcup_{n=1}^{\infty} K_n) \quad \text{and} \quad V' \cap K_{n_2} \subseteq V' \cap (\bigcup_{n=1}^{\infty} K_n),$$

then

$$T^n(U' \cap (\bigcup_{n=1}^{\infty} K_n)) \cap (V' \cap (\bigcup_{n=1}^{\infty} K_n)) = T^n(U) \cap V \neq \emptyset.$$

(ii) Let U, V be non-empty open sets of $\bigcup_{n=1}^{\infty} K_n$. Then there exist U', V' non-empty open sets of X such that

$$U = U' \cap (\bigcup_{n=1}^{\infty} K_n) \neq \emptyset \quad \text{and} \quad V = V' \cap (\bigcup_{n=1}^{\infty} K_n) \neq \emptyset.$$

Since U and V are non-empty there exist $n_1, n_2 \in \mathbb{N}$ such that $U' \cap K_{n_1} \neq \emptyset$ and $V' \cap K_{n_2} \neq \emptyset$. Suppose that $n_1 \leq n_2$ then $K_{n_1} \subset K_{n_2}$ and $U' \cap K_{n_2} \neq \emptyset$.

By hypothesis $T\mid_{K_{n_2}}$ is mixing and there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$T^n(U' \cap K_{n_2}) \cap (V' \cap K_{n_2}) \neq \emptyset.$$

Since

$$U' \cap K_{n_2} \subseteq U' \cap (\bigcup_{n=1}^{\infty} K_n) \quad \text{and} \quad V' \cap K_{n_2} \subseteq V' \cap (\bigcup_{n=1}^{\infty} K_n),$$

we have

$$T^n(U'\cap(\bigcup_{n=1}^{\infty} K_n))\cap(V'\cap(\bigcup_{n=1}^{\infty} K_n)) = T^n(U)\cap V \neq \emptyset \quad \text{for} \quad \text{all} \quad n \geq n_0.$$

(iii) Let U_1, U_2, V_1, V_2 be non-empty open sets of $\bigcup_{n=1}^{\infty} K_n$. Then there exist U_1', U_2', V_1', V_2' non-empty open sets of X such that

$$U_i = U_i' \cap (\bigcup_{n=1}^{\infty} K_n) \neq \emptyset \quad \text{and} \quad V_i = V_i' \cap (\bigcup_{n=1}^{\infty} K_n) \neq \emptyset$$

for $i = 1, 2$.

Since U_i and V_i are non-empty there exist $n_1, n_2, n_3, n_4 \in \mathbb{N}$ such that $U_1' \cap K_{n_1} \neq \emptyset$ $U_2' \cap K_{n_2} \neq \emptyset$, $V_1' \cap K_{n_3} \neq \emptyset$ and $V_2' \cap K_{n_4} \neq \emptyset$.

Let $n_0 = max\{n_1, n_2, n_3, n_4\}$; so, $K_{n_i} \subset K_{n_0}$.

Then $U_i' \cap K_{n_0} \neq \emptyset$ and $V_i \cap K_{n_0} \neq \emptyset$. By hypothesis, $T \mid_{K_{n_0}}$ is weakly-mixing, so there exists an $n \in \mathbb{N}$ such that

$$T^n(U_1' \cap K_{n_0}) \cap (V_1' \cap K_{n_0}) \neq \emptyset \quad \text{and} \quad T^n(U_2' \cap K_{n_0}) \cap (V_2' \cap K_{n_0}) \neq \emptyset.$$

Then

$$T^n(U_i' \cap (\bigcup_{n=1}^{\infty} K_n)) \cap (V_i' \cap (\bigcup_{n=1}^{\infty} K_n)) = T^n(U_i) \cap V_i \neq \emptyset$$

for $i = 1, 2$.

(iv) By part (i), it follows that $T : \bigcup_{n=1}^{\infty} K_n \to \bigcup_{n=1}^{\infty} K_n$ is transitive.

On the other hand the set of periodic points $\overline{Per(T \mid_{K_n})} \subset \overline{Per(T \mid_{\bigcup_{n=1}^{\infty} K_n})}$ for each $n \in \mathbb{N}$, then the set $Per(T \mid_{\bigcup_{n=1}^{\infty} K_n})$ will be dense in X and then $T \mid_{\bigcup_{n=1}^{\infty} K_n}$ is chaotic.

(v) Let U, V be non-empty open sets of $\bigcup_{n=1}^{\infty} K_n$. Then there exist U', V' non-empty open sets of X such that

$$U = U' \cap (\bigcup_{n=1}^{\infty} K_n) \neq \emptyset \quad \text{and} \quad V = V' \cap (\bigcup_{n=1}^{\infty} K_n) \neq \emptyset.$$

Since U and V are non-empty there exist $n_1, n_2 \in \mathbb{N}$ such that $U' \cap K_{n_1} \neq \emptyset$ and $V' \cap K_{n_2} \neq \emptyset$. Suppose that $n_1 \leq n_2$, then $K_{n_1} \subset K_{n_2}$ and $U' \cap K_{n_2} \neq \emptyset$.

By hypothesis $T\mid_{K_{n_2}}$ is topologically ergodic, then $N(U' \cap K_{n_2}, V' \cap K_{n_2})$ is syndetic and $N(U' \cap K_{n_2}, V' \cap K_{n_2}) \subset N(U, V)$, which implies that $N(U, V)$ is syndetic too and $T|_Y$ is topologically ergodic.

□

If we have a T-invariant subset $K \subset X$ which is absolutely convex, then $nK \subset mK$ when $n \leq m$ and $\mathrm{span}(K) = \bigcup nK$. Therefore, an easy application of Proposition 4.2.1 yields the following result.

Corollary 4.2.2 *Let $T : X \to X$ be an operator and let K be an absolutely convex T-invariant set such that $T|_K$ is transitive (respectively weakly-mixing, mixing, chaotic, topologically ergodic), then $T|_{\overline{\mathrm{span}(K)}}$ is transitive (respectively weakly-mixing, mixing, chaotic, topologically ergodic). In particular, if $\overline{\mathrm{span}(K)} = X$, then the property is inherited by T on the whole space X.*

Absolute convexity of the invariant set is not needed if we assume, at least, the weak mixing property for $T|_K$. A version of the following result, except chaos, will be given for non-autonomous dynamical systems in the last chapter.

Theorem 4.2.3 *Let $T : X \to X$ be an operator and let K be a T-invariant set such that $0 \in K$. Then:*

(i) If $T|_K$ is weakly mixing, then $T|_{\overline{\mathrm{span}(K)}}$ is weakly mixing.

(ii) If $T|_K$ is mixing, then $T|_{\overline{\mathrm{span}(K)}}$ is mixing.

(iii) If $T|_K$ is weakly mixing and chaotic, then $T|_{\overline{\mathrm{span}(K)}}$ is weakly mixing and chaotic.

Proof.

(i) It will be sufficient to prove that $T|_{\mathrm{span}(K)}$ is weakly mixing. Let $U_j, V_j \subset \mathrm{span}(K)$, $j = 1, 2$, be non-empty open sets. We fix $\lambda_{i,j}, \lambda'_{i,j} \in \mathbb{K}$ and $x_{i,j}, x'_{i,j} \in K$, $i = 1, \ldots, n$, $j = 1, 2$, such that $x_j := \sum_{i=1}^{n} \lambda_{i,j} x_{i,j} \in$

U_j and $x'_j := \sum_{i=1}^n \lambda'_{i,j} x'_{i,j} \in V_j$, $j = 1, 2$. Let $U_{i,j}, V_{i,j}, W \subset K$ be relatively open sets in K with $0 \in W$, $x_{i,j} \in U_{i,j}$, $x'_{i,j} \in V_{i,j}$, $i = 1, \dots, n$, $j = 1, 2$, and

$$\sum_{i=1}^n \lambda_{i,j} U_{i,j} + \sum_{i=1}^n \alpha_i W \subset U_j, \quad \sum_{i=1}^n \lambda'_{i,j} V_{i,j} + \sum_{i=1}^n \alpha_i W \subset V_j,$$

for any $\alpha_i \in \{\lambda_{1,1}, \dots, \lambda_{n,2}, \lambda'_{1,1}, \dots, \lambda'_{n,2}\}$, $i = 1, \dots, n$. Since $T|_K$ is weakly mixing, by Furstenberg's result [58], there are $y_{i,j} \in U_{i,j}$, $z_{i,j} \in W$, and $n \in \mathbb{N}$ such that $T^n y_{i,j} \in W$, and $T^n z_{i,j} \in V_{i,j}$, $i = 1, \dots, n$, $j = 1, 2$. By the above selection, $y_j := \sum_{i=1}^n (\lambda_{i,j} y_{i,j} + \lambda'_{i,j} z_{i,j}) \in U_j$ and $T^n y_j \in V_j$, $j = 1, 2$.

(ii) Let $U, V \subset \text{span}(K)$, be non-empty open sets. We fix $\lambda_i, \lambda'_i \in \mathbb{K}$ and $x_i, x'_i \in K$, $i = 1, \dots, n$, such that $x := \sum_{i=1}^n \lambda_i x_i \in U$ and $x' := \sum_{i=1}^n \lambda'_i x'_i \in V$. Let $U_i, V_i, W \subset K$ be relatively open sets in K with $0 \in W$, $x_i \in U_i$, $x'_i \in V_i$, $i = 1, \dots, n$, and

$$\sum_{i=1}^n \lambda_i U_i + \sum_{i=1}^n \alpha_i W \subset U, \quad \sum_{i=1}^n \lambda'_i V_i + \sum_{i=1}^n \alpha_i W \subset V,$$

for any $\alpha_i \in \{\lambda_1, \dots, \lambda_n, \lambda'_1, \dots, \lambda'_n\}$, $i = 1, \dots, n$. Since $T|_K$ is mixing, there are $y_i \in U_i$, $z_i \in W$, and $n_0 \in \mathbb{N}$ such that $T^n y_i \in W$, and $T^n z_i \in V_i$, $i = 1, \dots, n$, for all $n \geq n_0$. By the above selection, $y := \sum_{i=1}^n (\lambda_i y_i + \lambda'_i z_i) \in U$ and $T^n y \in V$.

(iii) It is sufficient to show that $\overline{Per(T|_{\text{span}(K)})} = \text{span}(K)$ in order to prove that $T|_{\overline{\text{span}(K)}}$ is chaotic. Let $x \in \text{span}(K)$ and let U be a neighbourhood of x, with $x = \sum_{i=1}^m \alpha_i x_i$, and $x_i \in K$. Let $U_1, U_2 \dots U_n$ be neighbourhoods of x_i respectively such that $\sum_{i=1}^n \alpha_i U_i \subset U$. For each x_i there exists $y_i \in Per(T|_K) \cap U_i$. Let us denote by $n = m.c.m.\{n_i; i = 1, \dots, n\}$ where n_i is the period of y_i for all $i = 1, \dots, n$. We have that $T^n(\sum_{i=1}^m \alpha_i y_i) = \sum_{i=1}^m \alpha_i y_i$, that is $y = \sum_{i=1}^m \alpha_i y_i \in Per(T|_{\text{span}(K)}) \cap U$.

□

Actually, we know that every chaotic operator T on a general topological vector space is weakly mixing as it is proved in corollary 3 in [71].

The previous Theorem allows us to provide some surprising examples that show the interplay between non-linear finite-dimensional dynamics and linear (infinite-dimensional) dynamics. The first example is inspired by [90], where a procedure known as Carleman linearization is indicated. The second example was given by Feldman [54], who showed that there exists a universal chaotic operator "containing" the dynamics of every continuous map on a compact metric space.
We also consider in the third example the so-called Lipschitz-free Banach spaces (or Arens-Eells spaces) generated by a metric space (see [61],[74]). Note that free spaces were also used in [99] in the context of hypercyclicity. Finally, the Frobenius-Perron operator on the space of measures is also considered in [14].

Examples 4.2.4 *(1)* ***Logistic map***
Let $p : [0,1] \to [0,1]$ be the logistic polynomial $p(x) := 4x(1-x)$, which is chaotic and mixing ([72]). We will embed $[0,1]$ in a locally convex space X via a map ϕ, and we will give an operator $T : X \to X$ such that $T \circ \phi = \phi \circ p$ and $\overline{\mathrm{span}(\phi[0,1])} = X$. An application of Theorem 4.2.3 will yield that T is mixing and chaotic. To do so we set

$$X = \{(x_i)_i \in \mathbb{C}^{\mathbb{N}} \ ; \ \exists r > 0 \ \textit{such that} \ \sup_{i \in \mathbb{N}} |x_i| \, r^i < \infty\}.$$

The space X can be identified with $\mathcal{H}(0)$, the space of holomorphic germs at 0, if we associate to each function the coefficients of its Taylor expansion. X is endowed with its natural inductive topology. We refer the reader to, e.g., [48] for the details.

We define the embedding $\phi : I \to X$ as $\phi(x) = (x, x^2, x^3, \dots)$. It is clear that ϕ is injective and it is well defined. Given $x \in [0,1]$, we fix $r \leq 1$ and

$$||\phi(x)|| = \sup_{i \in \mathbb{N}} |x|^i r^i \leq |x| r < \infty.$$

The operator $T : X \to X$ is defined by

$$T(x_1, x_2, \dots)_k = 4^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} x_{j+k}, \quad k \in \mathbb{N}.$$

The selection of the sequence space X easily gives that T is a well-defined operator on X. Indeed, let $(x_j)_j \in X$, that is there exists $R > 0$ such that $|x_{j+k}| \leq R^{k+j}$, for all $j, k \in \mathbb{N}$. Then, we have that $|T(x_1, x_2, \dots)_k| \leq 4^k (R + R^2)^k$, for all $k \in \mathbb{N}$.

Also, a simple computation shows that $T \circ \phi = \phi \circ p$. *Let* $Y := \phi[0,1]$. *We observe that* $\operatorname{span}(Y)$ *is dense in* X *by the Hahn-Banach theorem. Indeed, since the dual of* X *is*

$$X^* = \{(y_i)_i \in \mathbb{C}^{\mathbb{N}} \ ; \ \sum_{i=1}^{\infty} |y_i| \, R^i < \infty \text{ for all } R > 0\},$$

which can be identified with the space of entire functions, we have that $\langle \phi(x), (y_i)_i \rangle = \sum_i y_i x^i = 0$ *for some* $(y_i)_i \in X^*$ *and for all* $x \in I$, *implies* $y_i = 0$ *for every* $i \in \mathbb{N}$. *The hypothesis of Theorem 4.2.3 are satisfied, and hence* T *is mixing and chaotic. This example can be generalized to many classes of maps that satisfy certain chaotic properties on subsets of* $\mathbb{R}$ *or* $\mathbb{C}$.

(2) ***Universal Hilbert-space operator***
In [54] Feldman constructed a Hilbert space operator which is universal, in the sense that it "represents" all possible dynamics on a compact metric space.

Let $f : M \to M$ *be a continuous map on a compact metric space* M *for which we additionally suppose that there exists* $z \in M$ *such that* $f(z) = z$. *Given a countable dense subset* $\{x_n \ ; n \in \mathbb{N}\}$ *of* M, *we fix* $h : M \to \ell^2$ *defined by*

$$h(x) = \sum_{i=1}^{\infty} \frac{d(x, x_i) - d(z, x_i)}{2^i} e_i, \quad x \in M,$$

where $(e_i)_i$ *is the canonical basis of* ℓ^2. *It is clear that h is well defined, because given* $x \in M$ $h(x) < \sum_{i=1}^{\infty} \frac{d(x,z)}{2^i} e_i \in \ell^2$. *It is continuous because given* $(y_n)_n$ *such that* $y_n \to y$, *we have that* $||h(y_n) - h(y)|| < \sum_{i=1}^{\infty} \frac{|d(y_n,y)|^2}{2^{i+1}} < \epsilon$. *Let* $X := \ell^2(\ell^2)$ *and* $\Phi : M \to X$ *defined by*

$$\Phi(x) = \left(h(x), \frac{h(f(x))}{2}, \frac{h(f^2(x))}{2^2}, \dots \right), \quad x \in M.$$

Φ *is well defined. Notice that* $x, h \circ f(x), h \circ f^2(x) \dots$ *are all vectors in* ℓ^2 *and since* $Orb(h \circ f, x) \subset h(f(M))$ *and* $h(f(M))$ *is a bounded set in* ℓ^2 *it follows easily that* $||\Phi(x)|| < \infty$. *Thus* Φ *does map* M *into* X. *It is clear that it is injective and let us see that* Φ *is continuous. So, suppose that* $x_0 \in M$ *and* $\epsilon > 0$. *Let* $d > 0$ *be the diameter of* $h(f(M))$, *that is* $\|h(f(x)) - h(f(y))\| \leq d$ *for all* $x, y \in M$. *Now we choose an* $m \geq 1$ *such that* $\sum_{k=m}^{\infty} \frac{d^2}{4^k} \leq \frac{\epsilon^2}{4}$. *Since* $h, h \circ f^1, h \circ f^2, \dots, h \circ f^m$ *are all continuous at* x_0, *there exists* $\delta > 0$ *such that if* $\|x - x_0\| < \delta$, *then* $\|h(f^k(x)) - h(f^k(x_0))\| < \frac{\epsilon}{2\sqrt{(m+1)}}$ *for all*

$k \in \{0, \dots, m\}$. *Thus if* $\|x - x_0\| < \delta$, *then*

$$\|\Phi(x) - \Phi(x_0)\|^2 = \sum_{k=0}^{m} \frac{\|h(f^k(x)) - h(f^k(x_0))\|^2}{4^k} + \sum_{k=m+1}^{\infty} \frac{\|h(f^k(x)) - h(f^k(x_0))\|^2}{4^k}$$
$$\leq \sum_{k=0}^{m} \frac{\epsilon^2}{4(m+1)} + \sum_{k=m+1}^{\infty} \frac{d^2}{4^k} \leq \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{2} < \epsilon^2.$$

Thus $\|\Phi(x) - \Phi(x_0)\| < \epsilon$. *Hence* Φ *is continuous. Also, since it is clear that* $\|\Phi(x) - \Phi(y)\| \geq \|h(x) - h(y)\| \geq \frac{d(x,y)}{2}$ *we have that* $\Phi^{-1} : \Phi(M) \to M$ *is also continuous. Thus* Φ *is an homeomorphism onto its image. If we set* $T : X \to X$, $T(v_1, v_2, \dots) := (2v_2, 2v_3, \dots)$, *then* f *and* $T|_K$ *are topologically conjugated via* Φ,*that is* $\Phi \circ f = T|_K \circ \Phi$, *where* $K := \Phi(M)$. *By Theorem 4.2.3 we obtain that* $T|_{\overline{\mathrm{span}(K)}}$ *is weakly mixing (respectively, mixing, weakly mixing and chaotic) if* f *is so.*

*(3)**Lipschitz-free spaces***
Given a metric space (K, d) *with a distinguished point* $0 \in K$, *one can consider the space of Lipschitz maps on* K *that annihilate on* 0

$$\mathrm{Lip}_0(K) = \{f : K \to \mathbb{R} \ ; \ f(0) = 0, \ f \ \textit{Lipschitz}\}$$

endowed with the norm

$$\|f\|_L := \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} \ ; \ x \neq y \in K\right\}.$$

Let $\delta : K \to \mathrm{Lip}_0(K)^*$ *be the evaluation map* $\langle \delta_x, f \rangle = f(x)$, $x \in K$. δ *is an isometry and the* Lipschitz-free Banach space *generated by* K *is*

$$\mathcal{F}(K) := \overline{\mathrm{span}}\{\delta_x \ ; \ x \in K\}.$$

Actually, $\mathcal{F}(K)$ *is a predual of* $\mathrm{Lip}_0(K)$. *Moreover, if* $L : K \to K$ *is a Lipschitz map with* $L(0) = 0$, *then it induces an operator* T_L *on* $\mathcal{F}(K)$ *such that* $T_L\delta = \delta L$(*[103],[61]). Theorem 4.2.3 yields that, when* L *is weakly mixing (respectively, mixing, weakly mixing and chaotic), so is* T_L.

(4) ***Frobenius-Perron operator***
Let $T : K \to K$ *be a continuous map defined on a compact metric space* K. *Let us denote by* $M(K)$ *the space of Borel probability measures defined on* K *endowed with the weak topology.* T *induces a map* $T_M : M(K) \to M(K)$,

defined by $(T_M\mu)(A) = \mu T^{-1}(A)$ *(* $\mu \in M(K), A \in \mathfrak{B}(K)$ *Borel set).* T_M *is known as the* Frobenius-Perron operator *on measures associated with* T. *Actually, by Riesz representation theorem,* T_M *is the adjoint operator of the composition operator* $f \mapsto f \circ T$ *on the space of continuous functions* $C(K)$*. From now on we will consider that* T_M *is defined on* $C(K)^*$.

Suppose that T *admits a fixed point* $x \in K$*, and let us consider the translated compact set* $K' = M(K) - \delta_x$*, where* δ_x *is the Dirac measure associated with* x*. It is clear that* $0 \in K'$*, and* K' *is* T_M*-invariant, due to the fact that* $T(x) = x$*. In [14], the authors prove that if* T *is mixing or weakly mixing then so is* $T_M|_{M(K)}$*. It is easy to see, that if* $T_M|_{M(K)}$ *satisfies one of the previous properties, then so does* $T_M|_{K'}$*.*

Finally, we have that $\overline{\mathrm{span}(K')} = \{\mu \in C(K)^* \ ; \ <1, \mu>= 0\}$*, where* 1 *is the constant function on* K*. By Theorem 4.2.3,* $T_M|_{\overline{\mathrm{span}(K')}}$ *is also mixing or weakly mixing, respectively.*

Theorem 4.2.5 *Let* $T : X \to X$ *be an operator and let* Y *be an absolutely convex* T*-invariant set such that* (Y, T) *is an* E*-system and* $\overline{\mathrm{span}(Y)} = X$*, then* T *is topologically ergodic.*

Proof. Every E-system is topologically ergodic (see [60]), so (Y, T) is topologically ergodic, and by corollary 4.2.2, T is topologically ergodic. □

The following result shows that one can even improve, in some sense, the dynamical properties of the operator from the invariant sets to the corresponding closure of their union.

Theorem 4.2.6 *Let* $T : X \to X$ *be an operator and* $(K_n)_n$ *an increasing sequence of* T*-invariant bounded sets such that* $T|_{K_n}$ *is topologically transitive and* $\overline{\bigcup_{n=1}^{\infty} K_n} = X$*. Then* T *is weakly mixing.*

Proof. We will apply the following result which can be found in [71] (see [65] for the original version on Banach spaces):
If $T : X \to X$ is a transitive operator such that there exists a dense subset $X_0 \subset X$ with $\overline{\mathrm{Orb}(x, T)}$ bounded for all $x \in X_0$, then T is weakly-mixing.

By Proposition 4.2.1, $T : X \to X$ is transitive. If we take $X_0 = \bigcup_{n=1}^{\infty} K_n$, then every $x \in X_0$ has a bounded orbit, and we conclude the result.
□

Let (Y, T) be a dynamical system with Y compact and let $\mathcal{U}$ be a finite cover. We let $r(\mathcal{U})$ denote the minimal cardinality of a subcover of $\mathcal{U}$ and set $c(n) = c(\mathcal{U}, n) := r(\mathcal{U}_0^n)$ where, $\mathcal{U}_0^n = \mathcal{U} \cup \mathbb{T}^{-1}\mathcal{U} \cup \ldots \cup T^{-n}\mathcal{U}$. We call $c(\triangle, \mathcal{U})$ the *complexity function* of the cover $\mathcal{U}$. Given (X, T) a dynamical system we will define the *complexity function* $c(X)$ of X as the supremum of all the complexity functions of the covers of all invariant compact sets contained in X.

Theorem 4.2.7 *Let $T : X \to X$ be an operator and let $Y \subset X$ be a compact T-invariant set such that $T|_Y$ is sensitive to initial conditions, then the complexity function of X is unbounded.*

Proof. As (Y, T) is sensitive to initial conditions, it is not equicontinuous and by lemma 6.1 of [60], for all open cover $\mathcal{U}$ of Y, $C(\mathcal{U}, n)$ is unbounded. So the complexity function of X is unbounded. □

4.3 Dynamics on invariant sets and positive operators

There are well-known criteria of chaos, mixing and weak mixing properties for operators (section 0.4). Our next goal is to derive some criteria under which an operator restricted to an invariant set is mixing or weakly mixing.

Proposition 4.3.1 *Let $T : X \to X$ be an operator and let K be a T-invariant set. If there are dense subsets $X_0, Y_0 \subset K$, an increasing sequence $(n_k)_k$ of positive integers, and a sequence of maps $S_{n_k} : Y_0 \to X$, $k \in \mathbb{N}$, such that, for any $x \in X_0$, $y \in Y_0$,*

(i) $T^{n_k}x \to 0$,

(ii) $S_{n_k}y \to 0$,

(iii) $T^{n_k}S_{n_k}y \to y$,

(iv) for all $x \in X_0$ and $y \in Y_0$ there exists a k_0 such that $x + S_{n_k}y \in K$ for all $k \geq k_0$,

then $T|_K$ is weakly mixing. Moreover, if $n_k = k$ for all $k \in \mathbb{N}$, then $T|_K$ is mixing.

Proof. Let U_1, U_2, V_1 and V_2 be non-empty open sets of K. By assumption we can find vectors $x_j \in U_j \cap X_0$ and $y_j \in V_j \cap Y_0$, j=1,2. Then by (i) and

(iii),

$$T^{n_k}(x_j + S_{n_k} y_j) \to y_j, j = 1, 2.$$

It follows from (ii) and (iv) that, for sufficiently large k, $x_j + S_{n_k} y_j \in U_j$ for $j = 1, 2$. This shows that $T|_K$ is weakly mixing.

The mixing case is analogous. Let U and V be non-empty open sets of K. By assumption we can find vectors $x \in U \cap X_0$ and $y \in V \cap Y_0$. Then by (i) and (iii),

$$T^k(x + S_k y) \to T^k x + y.$$

It follows from (ii) and (iv) that there exists an k_0 such that for $k \geq k_0$, $x + S_k y \in U$. This proves that $T|_K$ is mixing. □

The following two examples illustrate how the previous result can be applied.

Example 4.3.2 We consider a hypercyclic weighted backward shift $T := B_w$ on ℓ^p. We define

$$K = \{x \in \ell^p \quad ; \quad |x_k| \prod_{j=1}^{k} w_j \leq 1, \forall k \geq 1\},$$

with $w_1 = 1$, which is T-invariant. Let $X_0 = Y_0$ be the space of finite sequences in K. If we consider the weighted forward shift $S : Y_0 \to Y_0$ with

$$S(x_1, x_2, \ldots) = (0, w_2^{-1} x_1, w_3^{-1} x_2, \ldots),$$

Since B_w is hypercyclic, there is an increasing sequence of integers $(n_k)_k$ such that $\lim_{k\to\infty} \prod_{j=1}^{n_k} w_j = +\infty$ (see 0.5.8). We fix $S_{n_k} = S^{n_k}$, $k \in \mathbb{N}$. It is clear that $TSy = y$ for all $y \in Y_0$, and that $T^{n_k} x \to 0$, $x \in X_0$, $S_{n_k} y \to 0$, $y \in Y_0$, so that (i), (ii) and (iii) in Proposition 4.3.1 are satisfied. With respect to condition (iv), we just have to observe that, if $x, y \in K$ have disjoint supports, then $z := x + y \in K$. Thus, given $x \in X_0$ and $y \in Y_0$, for sufficiently large k we get that x and $S_{n_k} y$ have disjoint support, so $x + S_{n_k} y \in K$. We conclude that $T|_K$ is weakly mixing.

Example 4.3.3 We consider a hypercyclic weighted backward shift $T := B_w$ on ℓ^p and the subset K defined as

$$K = \{x \in \ell^p \quad ; \quad |x_k| \prod_{j=1}^{k} w_j \in \{0, 1\}, \forall k \geq 1\}.$$

Let $X_0 = Y_0$ be the space of finite sequences in K. By following the proof of the previous example we clearly have that all the conditions in Proposition 4.3.1 are satisfied, and $T|_K$ is weakly mixing.

Remark 4.3.4 It is worth noting that the fourth condition of Proposition 4.3.1 is necessary. Suppose, for instance, that $T := B_w$ is a chaotic weighted backward shift on ℓ^p and let us define K' as

$$K' = \{x \in \ell^p \quad ; \quad \sum_{k\geq 1} \left(\prod_{j=1}^{k} w_j\right)^p |x_k|^p \leq 1, \forall k \geq 1\},$$

with $w_1 = 1$. It is clear that K' is T-invariant. Although the first three conditions are satisfied, $T|_{K'}$ is not even transitive. Given $x \in K'$, since T is chaotic we have $\lim_{k\to\infty} \prod_{j=1}^{k} w_j = \infty$ (for more details see 0.5.8) and there exists $\lambda > 0$ such that $\prod_{j=1}^{k} w_j \geq \lambda > 0$ for all $k \in \mathbb{N}$. Then

$$\lambda^p ||T^n x||^p \leq \sum_{k\geq 1} \left(\prod_{j=1}^{k+n} w_j\right)^p |(T^n x)_k|^p = \sum_{k>n} \left(\prod_{j=1}^{k} w_j\right)^p |x_k|^p \to 0,$$

which excludes the possibility of x having dense orbit in K' for every $x \in K'$.

In the case that X is a complex space, we can offer a sufficient condition for the chaotic behavior of $T|_K$ on an invariant set K.

Corollary 4.3.5 *Let $T : X \to X$ be an operator with X complex, and let $K \subset X$ be a T-invariant set. If $T|_K$ satisfies conditions of Proposition 4.3.1 and the subset*

$$\operatorname{span}\{x \in X \; ; \; Tx = \lambda x \quad \textit{for some} \quad \lambda \in \mathbb{C} \quad \lambda^n = 1 \quad \textit{for some} \quad n \in \mathbb{N}\} \cap K$$

is dense in K, then $T|_K$ is chaotic.

Proof. By 0.2.8 the set before corresponds to the set of periodic points of an operator, then if this set is dense and $T|_K$ is transitive, we have that $T|_K$ is chaotic. □

As a direct consequence of Proposition 4.3.1 we have:

Proposition 4.3.6 *Let $T : X \to X$ be a positive operator defined on a Fréchet lattice X. If there are dense subsets X_0, $Y_0 \subset X^+$, an increasing sequence of integers $(n_k)_k$, and a sequence of maps $S_{n_k} : Y_0 \to X^+$, $k \in \mathbb{N}$, such that, for any $x \in X_0$, $y \in Y_0$,* *(i)* $T^{n_k}x \to 0$,

(ii) $S_{n_k}y \to 0$,

(iii) $T^{n_k}S_{n_k}y \to y$,

then $T|_{X^+}$ *is weakly mixing. If, moreover,* $n_k = k$ *for all* $k \in \mathbb{N}$, *then* $T|_{X^+}$ *is mixing.*

Example 4.3.7 Given a weighted backward shift $T := B_w$ on $X := \ell^p$ with $\lim_{k\to\infty} \prod_{j=1}^{k} w_j = +\infty$, we consider $X_0 = Y_0$ be the space of finite sequences in X^+ and the weighted forward shift $S : Y_0 \to Y_0$, $S(x_1, x_2, \ldots) = (0, w_2^{-1}x_1, w_3^{-1}x_2, \ldots)$. If we set $S_k = S^k$, $k \in \mathbb{N}$, then we have that all the conditions in Proposition 4.3.6 are satisfied, so $T|_{X^+}$ is mixing.

By following the ideas that we developed in [88] and [86], respectively, we show that certain "positive" versions of frequent hypercyclicity criteria ensure the existence of T-strongly mixing measures supported on the positive cone of a Fréchet lattice, and the existence of $(T_t)_t$-invariant strongly mixing Borel probability measures supported on the positive cone of a Banach lattice, where T is a positive operator and $(T_t)_{t\geq 0}$ is a C_0-semigroup of positive operators, respectively. Combined with the previous results, we have at the same time mixing and chaos, in the topological sense, on the positive cone.

Theorem 4.3.8 *Let* T *be a positive operator on a separable Fréchet lattice* X. *If there are, a dense subset* X_0 *of* X^+, *and a sequence of maps* $S_n : X_0 \to X^+$, $n \in \mathbb{N}$, *such that, for each* $x \in X_0$,

(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*

(ii) $\sum_{n=0}^{\infty} S_n x$ *converges unconditionally,*

(iii) $T^n S_n x = x$ *and* $T^m S_n x = S_{n-m}x$ *for* $n > m$,

then $T|_{X^+}$ *is mixing, chaotic, and there is a* T*-invariant strongly mixing Borel probability measure* μ *on* X^+ *whose support is equal to* X^+.

Proof. The fact that $T|_{X^+}$ is mixing is a consequence of Proposition 4.3.6. Concerning chaos, we just need to observe that, for each $x \in X_0$ and $k \in \mathbb{N}$, we can construct the vector

$$y_k := \sum_{n=0}^{\infty} S_{nk}x + x + \sum_{n=0}^{\infty} T^{nk}x \in X^+.$$

By (i) and (ii) both series converge and by (iii) we have that $T^k y_k = y_k$, so that each y_k is a periodic point for T. Moreover, it follows from (i) and (ii) that $y_k \to x$ as $k \to \infty$. Since X_0 is dense we get that the set of periodic points is dense and as $T|_{X^+}$ satisfies the Frequent Hypercyclicity Criterion it is hypercyclic and hence $T|_{X^+}$ is chaotic. Finally, for the existence of the strong mixing measure, we just have to follow step by step the proof of Theorem 1.2.1 in Chapter 1. □

Before giving the result for C_0-semigroups, we would like to recall the concept of *sub-chaos* of Banasiak and Moszyński [8], which requires the existence of a $(T_t)_t$-invariant subspace $Y \subset X$ such that $(T_t|_Y)_{t\geq 0}$ is chaotic (see also [9],[2]). Here we are interested in the case of the invariant positive cone for semigroups of positive operators, mainly because of its applications to certain differential equations.

Theorem 4.3.9 *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup of positive operators on a separable Banach lattice X. If there exist $X_0 \subset X^+$ dense in X^+ and maps $S_t : X_0 \to X^+$, $t > 0$ such that*

- *$T_t S_t x = x, T_t S_r x = S_{r-t} x, t > 0, r > t > 0,$*
- *$t \to T_t x$ is Pettis integrable in $[0, \infty)$ for all $x \in X_0$,*
- *$t \to S_t x$ is Pettis integrable in $[0, \infty)$ for all $x \in X_0$,*

then $(T_t|_{X^+})_{t\geq 0}$ is mixing, each operator $T_t|_{X^+}$ with $t > 0$ is chaotic, and there is a $(T_t)_t$-invariant strongly mixing Borel probability measure μ on X^+ whose support is equal to X^+.

Proof. For the mixing property of $(T_t)_{t\geq 0}$, one can easily check hypotheses of theorem 0.7.6 for $(T_t|_{X^+})_{t\geq 0}$. Given $t > 0$, we can show that $T_t|_{X^+}$ is chaotic by following exactly the same argument as in Proposition 2.6 of [82]. Also, the existence of the strong mixing measure supported on X^+ is a consequence of the proof of Theorem 2.2.1 in chapter 2. □

Remark 4.3.10 It seems that the above criteria for the existence of mixing measures supported on the positive cone can only be derived from the type of constructions given in [88] and [86] since other general criteria, like the ones in ([17],[21]), are for complex spaces and depend strongly on the existence of certain eigenvectors to unimodular eigenvalues, which cannot induce measures supported on the positive cone. Also, the measures obtained in [17, 21] are Gaussian, so their support is a closed subspace which can never be included in the positive cone. Recent results on the existence of invariant measures with ergodic properties in linear dynamics can be found in [68].

We finally present examples of some C_0-semigroups of positive operators where we can apply Theorem 4.3.9.

Example 4.3.11 Let $X = L_p^\rho([0,\infty))$, and we consider the translation C_0-semigroup $(T_t)_{t\geq 0}$ defined by $T_t f(x) = f(x+t)$. We assume that $\int_0^\infty \rho(s)ds < \infty$. In these spaces, $X^+ = \{f \in X \ ; \ f(x) \geq 0, \text{ for all } x \in [0,\infty)\}$. Let X_0 be the positive span of the space generated by the characteristic functions of bounded intervals of $[0,\infty[$, which is clearly dense in X^+. By Propositions 3.3 and 3.4 in [82], we have that the translation semigroup satisfies the hypothesis of Theorem 4.3.9 and, in particular, we can define a T_t-invariant strongly mixing Borel probability measure μ on X^+ whose support is X^+.

Example 4.3.12 In [101] Takeo studied the solution semigroup $(T_t)_{t\geq 0}$ on X, a certain function space defined on an interval I of $\mathbb{R}$, associated with the following partial equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} + h(x)u, \\ u(0,x) &= f(x), \end{aligned} \tag{4.1}$$

where h is a bounded function on I and $f \in X$. We consider $X = L_p([0,\infty),\mathbb{R})$. The semigroup $(T_t)_{t\geq 0}$ defined as

$$T_t f(x) = e^{\int_x^{x+t} h(s)ds} f(x+t)$$

is the solution semigroup to the equation (4.1). Now we consider the translation semigroup $(\widetilde{T}_t)_{t\geq 0}$ on $\widetilde{X} = L_p^\rho([0,\infty),\mathbb{R})$, where ρ is the admissible weight function $e^{-\frac{1}{p}\int_0^x h(s)ds}$. The operator $\phi : \widetilde{X} \to X$ given by $\phi(g)(x) = \rho(x)g(x)$, for $g \in \widetilde{X}$ and for $x \in I$ is an isometric isomorphism and $\phi \circ \widetilde{T}_t = T_t \circ \phi$ (for more details see [101]). Since $\rho(x) > 0$, it is clear that $\phi : \widetilde{X}^+ \to X^+$ is also an isomorphism. We have that, if $h(t) = \frac{a}{t+1}$ with $a > 1/p$, then $\int_0^\infty \rho(s)ds < \infty$ [101], thus by Example 4.3.11 $(\widetilde{T}_t)_{t\geq 0}$ satisfies satisfies the hypothesis of Theorem 4.3.9, and so does $(T_t)_{t\geq 0}$ by conjugacy.

Example 4.3.13 We consider $X = \{f \in C([0,1],\mathbb{R}) : f(0) = 0\}$ with the sup norm. In [83] (see also [1] for more general recent results) the authors consider the following initial value problem of a partial differential equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \gamma x \frac{\partial u}{\partial x} + h(x)u, \\ \\ u(0,x) = f(x) \end{cases} \tag{4.2}$$

where $\gamma < 0$, $h \in C([0,1],\mathbb{R})$ and $f \in X$. Then the solution semigroup $(T_t)_{t\geq 0}$, $T_t f(x) = e^{\int_0^t h(e^{\gamma(t-s)}x)ds} f(e^{\gamma t}x)$ to the equation (4.2) is a strongly continuous semigroup on X. We will see that if $\min\{h(x) : x \in [0,1]\}$ is positive, then $(T_t)_{t\geq 0}$ satisfies the conditions of Theorem 4.3.9. Indeed, let $X_0 = \{f \in C([0,1],\mathbb{R}^+) \ ; \ f(x) = 0, \forall x \in [0,\epsilon], \text{ for some } \epsilon > 0\}$. It is clear that X_0 is dense in X^+. We define the family of maps $S_t : X^+ \to X^+$ as

$$S_t f(x) = e^{-\int_0^t h(e^{-\gamma s}x)ds} f(e^{-\gamma t}x).$$

Given $r \geq t$, we have that

$$T_t(S_r f)(x) = e^{\int_0^t h(e^{\gamma(t-s)}x)ds} S_r f(e^{\gamma t}x)$$

$$= e^{\int_0^t h(e^{\gamma(t-s)}x)ds} e^{-\int_0^r h(e^{-\gamma s}e^{\gamma t}x)} f(e^{\gamma(t-r)}x) = S_{r-t}f(x).$$

It remains to check that $t \mapsto S_t f$ and $t \mapsto T_t f$ are Pettis integrable on $[0,+\infty)$ for all $f \in X_0$. Actually, we will show that they are Bochner integrable on $[0,+\infty)$. In the first case, it is sufficient to prove that $\int_0^\infty \|S_t f\| dt < \infty$ for a given $f \in X_0$, we denote by $M = \max\{f(x) \ ; \ x \in [0,1]\}$. Thus

$$\int_0^\infty \sup_{x\in[0,1]} |e^{-\int_0^t h(e^{-\gamma s}x)ds} f(e^{-\gamma t}x)| dt \leq M \int_0^\infty \sup_{x\in[0,1]} e^{-\int_0^t h(e^{-\gamma s}x)ds} dt$$

$$\leq \int_0^\infty e^{-\int_0^t a ds} dt = \int_0^\infty e^{-at} < \infty,$$

where $a = \min\{h(x) \ ; \ x \in [0,1]\} > 0$.

For the second case, given $f \in X_0$ and $\epsilon > 0$ such that $f|_{[0,\epsilon]} \equiv 0$, there exists $t_0 > 0$ such that $e^{\gamma t} < \epsilon$ for all $t \geq t_0$. Then we have $T_t f = 0$ for all $t \geq t_0$, and $\int_0^\infty \|T_t f\| dt = \int_0^{t_0} \|T_t f\| dt < \infty$, which concludes the result.

Chapter 5

Mixing properties for nonautonomous linear dynamics

In this chapter we study mixing properties (topological mixing and weak mixing of arbitrary order) for nonautonomous linear dynamical systems that are induced by the corresponding dynamics on certain invariant sets. The type of nonautonomous systems considered here can be defined by a sequence $(T_i)_{i\in\mathbb{N}}$ of linear operators $T_i : X \to X$ on a topological vector space X such that there is an invariant set Y for which the dynamics restricted to Y satisfies certain mixing property. We then obtain the corresponding mixing property on the closed linear span of Y. We also prove that the class of nonautonomous linear dynamical systems that are weakly mixing of order n contains strictly the corresponding class with the weak mixing property of order $n+1$. All the results of this chapter have been published in [87].

5.1 Introduction

Some basic definitions related to nonautonomous systems are the following:

Definition 5.1.1 *(i) Given a sequence of operators $T_i : X \to X$, $i \in \mathbb{N}$, defined on a topological vector space X we consider the corresponding* nonautonomous discrete system *(NDS)* $(X, T_\infty) = (X, (T_n \circ T_{n-1} \ldots \circ T_1)_{n\in\mathbb{N}})$

(ii) *The orbit of an element by a NDS is denoted by* $\mathrm{Orb}(x, T_\infty) = \{T^{(k)}x \ ; \ k \geq 0\}$, $x \in X$, *where* $T^{(k)} := T_k \circ \cdots \circ T_1$, $k \in \mathbb{N}$, $T^{(0)} = Id_X$.

(iii) $Y \subset X$ *is an* invariant set *for the NDS* (X, T_∞) *if* $T_n(Y) \subset Y$ *for all* $n \in \mathbb{N}$.

(iv) *A NDS* (X, T_∞) *is* weakly mixing *of order* n *if, for any nonempty open sets* $U_1, \ldots, U_n$, $V_1, \ldots, V_n$ *and for any* $N > 0$ *there is* $k > N$ *such that* $T^{(k)}(U_i) \bigcap V_i \neq \emptyset$ *for* $i = 1, \ldots, n$. *If* (X, T_∞) *is weakly mixing of order* n *for every* $n \geq 2$ *then we say that it is weakly mixing of all orders.*

(v) (X, T_∞) *is said to be* mixing *if for any nonempty open sets* $U, V \subset X$ *there exists* $N > 0$ *such that* $T^{(k)}(U) \bigcap V \neq \emptyset$ *for all* $k \geq N$.

Very recently, Balibrea and Oprocha [4] obtained several results about weak mixing and chaos in nonautonomous discrete systems on compact sets. Some of their results will be used to induce the corresponding dynamical behavior on linear nonautonomous systems. The theory of linear dynamics is well established in the case of iterations of a single operator (autonomous dynamical system). The case of nonautonomous linear dynamics is not yet developed, although a more general concept of universality of a sequence of operators $(T_n)_{n\in\mathbb{N}}$ where the orbits are defined as $\{T_n x \ ; \ n \in \mathbb{N}\}$, $x \in X$, has been treated by several authors (See, e.g., [24, 26, 27, 64, 79]).

A type of linear universality which has attracted the attention in recent years is the dynamics of tuples of operators introduced by Feldman [55]. More precisely, given a commuting tuple $(T_1, \ldots, T_n)$ of operators defined on a certain topological vector space X, he studied the existence of (somewhere) dense orbits $\{(T_n^{k_n} \circ \cdots \circ T_1^{k_1})x \ ; \ k_i \geq 0\}$. The subsystems that correspond to increasing sequences in $\mathbb{N}^n$, with its natural order, can be written as nonautonomous discrete systems.

We will essentially follow the notation of [4] and we will denote a NDS as (X, T_∞). These notions can be extended naturally to a system $(X, (T_k)_k)$ of sequences of maps $T_k : X \to X$, $k \in \mathbb{N}$, by substituting $T^{(k)}$ by T_k.

5.2 Mixing properties on linear NDS induced by invariant sets

The purpose of this section is, given a linear NDS (X, T_∞), where X is a topological vector space, with an invariant set $Y \subset X$, to obtain mixing properties on the closure of $\mathrm{span}(Y)$, the linear span of Y, induced by the corresponding ones in $(Y, T_\infty|_Y)$. Actually, the main result in this section will be given for sequences of operators, so that we will obtain as a consequence the results for linear NDS and for tuples of operators.

Theorem 5.2.1 *Let X be a topological vector space and let the system $(X, (T_n)_n)$, where $\{T_n : X \to X \ ; \ n \in \mathbb{N}\}$ is a sequence of operators such that $T_n(Y) \subset Y$ for every $n \in \mathbb{N}$ and for certain $Y \subset X$ with $0 \in Y$. We consider $Z := \overline{\mathrm{span}(Y)}$.*

(i) If $(Y, (T_n|_Y)_n)$ is weakly mixing of all orders then $(Z, (T_n|_Z)_n)$ is also weakly mixing of all orders.

(ii) If $(Y, (T_n|_Y)_n)$ is mixing then $(Z, (T_n|_Z)_n)$ is also mixing.

Proof.

(i) Suppose then that $(Y, (T_n|_Y)_n)$ is weakly mixing of all orders. Given any $m \in \mathbb{N}$, we have to show that $(Z, (T_n|_Z)_n)$ is weakly mixing of order m.
Let $U_j, V_j \subset Z$ be nonempty open sets, $j = 1, \dots m$. We find $n \in \mathbb{N}$, $\alpha_{i,j}, \beta_{i,j} \subset \mathbb{K}$ and $u_{i,j}, v_{i,j} \in Y$, $i = 1, \dots, n$, $j = 1, \dots, m$, such that $u_j := \sum_{i=1}^n \alpha_{i,j} u_{i,j} \in U_j$ and $v_j := \sum_{i=1}^n \beta_{i,j} v_{i,j} \in V_j$, $j = 1, \dots m$.
There are nonempty open sets $U_{i,j}, V_{i,j}, W \subset Y$ with $0 \in W$, $u_{i,j} \in U_{i,j}$, $v_{i,j} \in V_{i,j}$, $i = 1, \dots, n$, $j = 1, \dots, m$, such that

$$\sum_{i=1}^n \alpha_{i,j} U_{i,j} + \sum_{i=1}^n \gamma_i W \subset U_j, \qquad \sum_{i=1}^n \beta_{i,j} V_{i,j} + \sum_{i=1}^n \gamma_i W \subset V_j$$

for any $\gamma_i \in \{\alpha_{i,j}, i = 1, \dots, n, j = 1, \dots, m\} \cup \{\beta_{i,j}, i = 1, \dots, n, j = 1, \dots, m\}$, $i = 1, \dots, n$, $j = 1, \dots, m$.
Since $(Y, (T_n|_Y)_n)$ is weakly mixing of all orders there are $y_{i,j} \in U_{i,j}$, $w_{i,j} \in W$, and $k \in \mathbb{N}$ such that $T_k(y_{i,j}) \in W$ and $T_k(w_{i,j}) \in V_{i,j}$, $i = 1, \dots, n$, $j = 1, \dots, m$.
The above conditions yield that $y_j := \sum_{i=1}^n (\alpha_{i,j} y_{i,j} + \beta_{i,j} w_{i,j}) \in U_j$ and $T_k y_j \in V_j$, $j = 1, \dots, m$.

(ii) Let $U, V \subset Z$ be nonempty open sets. We find $n \in \mathbb{N}$, $\alpha_i, \beta_i \subset \mathbb{K}$ and $u_i, v_i \in Y$, $i = 1, \ldots, n$, such that $u := \sum_{i=1}^n \alpha_i u_i \in U$ and $v := \sum_{i=1}^n \beta_i v_i \in V$. There are nonempty open sets $U_i, V_i, W \subset Y$ with $0 \in W$, $u_i \in U_i$, $v_i \in V_i$, $i = 1, \ldots, n$, such that

$$\sum_{i=1}^n \alpha_i U_i + \sum_{i=1}^n \gamma_i W \subset U, \quad \sum_{i=1}^n \beta_i V_i + \sum_{i=1}^n \gamma_i W \subset V$$

for any $\gamma_i \in \{\alpha_i, i = 1, \ldots, n\} \cup \{\beta_i, i = 1, \ldots, n\}$, $i = 1, \ldots, n$. Since $(Y, (T_n|_Y)_n)$ is mixing, there exists $N > 0$, such that for all $k \geq N$:

$$T_k(U_i) \cap W \neq \emptyset \quad \text{and} \quad T_k(W) \cap V_i \neq \emptyset.$$

For each $k \geq N$, there are $y_i \in U_i$, $w_i \in W$, such that $T_k(y_i) \in W$ and $T_k(w_i) \in V_i$, $i = 1, \ldots, n$. The above conditions yield that $y_k := \sum_{i=1}^n (\alpha_i y_i + \beta_i w_i) \in U$ and $T_k y_k \in V$, then $T_k(U) \cap V \neq \emptyset$ for all $k \geq N$.

□

The result for linear NDS follows now from Theorem 5.2.1.

Corollary 5.2.2 *Let X be a topological vector space and let (X, T_∞) be a linear NDS with an invariant set $Y \subset X$ such that $0 \in Y$. We consider $Z := \overline{\operatorname{span}(Y)}$.*

(i) If $(Y, T_\infty|_Y)$ is weakly mixing of all orders then $(Z, T_\infty|_Z)$ is also weakly mixing of all orders.

(ii) If $(Y, T_\infty|_Y)$ is mixing then $(Z, T_\infty|_Z)$ is also mixing.

Now we recall the notion of somewhere and everywhere dense sets.

Definition 5.2.3 *A set is called* somewhere dense *if its closure contains a nonempty open set. Moreover, a set is said to be* everywhere dense *if its closure is the whole space.*

In [55] examples were given of somewhere dense orbits for tuples of operators that are not dense.

Example 5.2.4 ([55]) For each $1 \leq i \leq n$ let A_i be the $n * n$ diagonal matrix with ones in the main diagonal except in the (i, i) position which is 2. Also let B_i be the $n * n$ diagonal matrix with ones in the main diagonal except in the (i, i) position which is $\frac{1}{3}$. Then the $2n$-tuple $T = (A_1, \dots, A_n, B_1, \dots, B_n)$ on $\mathbb{R}^n$ has somewhere dense orbits which are not dense.

Moreover, sufficient conditions under which a somewhere dense orbit under a tuple of operators must be everywhere dense were obtained.

Theorem 5.2.5 ([55]) *Let $T = (T_1, \dots, T_n)$ be a tuple of operators on a real or complex locally convex space X. If A^* has no eigenvalues for every $A \in \{(T_n^{k_n} \circ \cdots \circ T_1^{k_1}); k_i \geq 0\}$, then any orbit of T that is somewhere dense in X will be dense in X. In particular, if T is a n-tuple of matrices on $\mathbb{C}^k$, then every somewhere dense orbit of T must be dense in $\mathbb{C}^k$.*

We recall that, for linear autonomous systems, no extra assumptions are needed to show that somewhere dense orbits are everywhere dense, as it was shown by Bourdon and Feldman [34] answering a question in [89] (see also [41] for the corresponding version for C_0-semigroups).

Theorem 5.2.6 (Bourdon-Feldman, [34]) *Let T be an operator on a Fréchet space X and $x \in X$. If $\mathrm{Orb}(x, T)$ is somewhere dense in X, then it is dense in X.*

Now we derive new conditions implying that, when there is a somewhere dense orbit, it must be everywhere dense.

Corollary 5.2.7 *Let $T = (T_1, \dots, T_n)$ be a commuting tuple of operators defined on a topological vector space X. Let $x \in X$ such that $\mathrm{Orb}(x, T) := \{(T_n^{k_n} \circ \cdots \circ T_1^{k_1})x \ ; \ k_i \geq 0$ for all $i\}$ is somewhere dense in X. Let $(R_n)_{n \in \mathbb{N}}$ be an enumeration of $\{T_n^{k_n} \circ \cdots \circ T_1^{k_1} \ ; \ k_i \geq 0$ for all $i\}$ and let $Y := \overline{\mathrm{Orb}(x, T)}$. If $(Y, (R_n|_Y)_n)$ is weakly mixing of all orders then $\mathrm{Orb}(x, T)$ is everywhere dense.*

Proof. By Theorem 5.2.1, $(X, (R_n)_n)$ is weakly mixing of all orders since $\mathrm{span}(Y) = X$ because Y contains a non-empty open set. In particular, given an arbitrary non-empty open set $V \subset X$ and a non-empty open set $U \subset \overline{\mathrm{Orb}(x, T)}$, there exists $k \in \mathbb{N}$ such that $R_k(U) \cap V \neq \emptyset$. By continuity, we find a non-empty open set $\tilde{U} \subset U$ such that $R_k(\tilde{U}) \subset V$. Let $j_1, \dots, j_n \geq 0$ such that $(T_n^{j_n} \circ \cdots \circ T_1^{j_1})x \in \tilde{U}$, and $j'_1, \dots, j'_n \geq 0$ with $R_k = T_n^{j'_n} \circ \cdots \circ T_1^{j'_1}$.

For $k_i := j_i + j_i'$, $i = 1, \dots, n$, we get $(T_n^{k_n} \circ \cdots \circ T_1^{k_1})x \in V$, so $\mathrm{Orb}(x, T)$ is everywhere dense. $\square$

The following example links the nonlinear dynamics in dimension 1 with the linear infinite-dimensional dynamics. The idea follows what is called Carleman linearization, and it is inspired in [90].

Example 5.2.8 Let $\{p_n : I \to I \ ; \ n \in \mathbb{N}\}$ be a sequence of polynomials on an interval I that contains 0 such that $p_n(0) = 0$, $n \in \mathbb{N}$, and the corresponding generated NDS (I, p_∞) is weakly mixing of order 3. By [4, Thm 11] we know that (I, p_∞) is weakly mixing of all orders.

We will embed (I, p_∞) in a linear NDS (X, T_∞) via a map ϕ such that $T_n \circ \phi = \phi \circ p_n$ for every $n \in \mathbb{N}$ and $\overline{\mathrm{span}(\phi(I))} = X$. To do so we set

$$X = \{(x_i)_i \in \mathbb{C}^{\mathbb{N}} \ ; \ \exists r > 0 \text{ such that } \sup_i |x_i| \, r^i < \infty\}.$$

X is endowed with the natural topology as inductive limit. We refer the reader to, e.g., [48] for the details. This space has also been considered in 1 of 4.2.4.

We define the embedding $\phi : I \to X$ as $\phi(x) = (x, x^2, x^3, \dots)$. Given $n \in \mathbb{N}$, we set the operator $T_n : X \to X$ such that the k-th coordinate of $T_n x$ is

$$T_n(x_1, x_2, \dots)_k = \sum_{j=k}^{km_n} \alpha_{k,j} x_j, \quad k \in \mathbb{N}, \quad x = (x_1, x_2, \dots) \in X,$$

where $m_n = \deg(p_n)$ and $p_n(x)^k = \sum_{j=k}^{km_n} \alpha_{k,j} x^j$. The selection of the sequence space X easily gives that T_n is a well-defined operator on X.

Let $(x_j)_j \in X$, that is, there exists $r > 0$ such that $||(x_j)_j||_r = \sup_{i \in \mathbb{N}} |x_j| \, r^j < \infty$. Then we have by the multinomial formula there exists a positive constant A such that $|\alpha_{k,j}| < A^k$ for all $k, j \in \mathbb{N}$. Now let us distinguish two cases:

- When $r < 1$ we take $s = \frac{r^{m_n}}{A}$ and we check that $T_n((x_j)_j) \in X$:

$$\begin{aligned}\sup_{k\in\mathbb{N}} |T_n(x_1, x_2, \dots)_k| s^k &\leq \sup_{k\in\mathbb{N}} \sum_{j=k}^{km_n} |\alpha_{k,j}||x_j| s^k \\ &\leq ||(x_j)_j||_r \sup_{k\in\mathbb{N}} \sum_{j=k}^{km_n} r^{km_n - j} < C||(x_j)_j||_r.\end{aligned}$$

- When $r \geq 1$ we take $s = \frac{1}{2A}$ and we check that $T_n((x_j)_j) \in X$:

$$\begin{aligned}\sup_{k\in\mathbb{N}} |T_n(x_1, x_2, \dots)_k| s^k &\leq \sup_{k\in\mathbb{N}} \sum_{j=k}^{km_n} |\alpha_{k,j}||x_j| s^k \\ &\leq ||(x_j)_j||_r \sup_{k\in\mathbb{N}} \sum_{j=k}^{km_n} \frac{1}{r^j 2^k} < C||(x_j)_j||_r.\end{aligned}$$

Then we have that T_n is a well-defined operator.

Also, a simple computation shows that $T_n \circ \phi = \phi \circ p_n$. Let $Y := \phi(I)$. We observe that $\operatorname{span}(Y)$ is dense in X by the Hahn-Banach theorem. Indeed, since the dual of X is

$$X^* = \{(y_i)_i \in \mathbb{C}^{\mathbb{N}} \; ; \; \sum_{i=1}^{\infty} |y_i| \, R^i < \infty \text{ for all } R > 0\},$$

which can be identified with the space of entire functions, we have that $\langle \phi(x), (y_i)_i \rangle = \sum_i y_i x^i = 0$ for some $(y_i)_i \in X^*$ and for all $x \in I$, implies $y_i = 0$ for every $i \in \mathbb{N}$ because an entire function that is annihilated on a set with accumulation points should be identically 0. The hypotheses of Corollary 5.2.2 are satisfied and (X, T_∞) is weakly mixing of all orders.

5.3 Weak mixing property of different orders

In this last section we will prove that it is possible to obtain examples of linear NDS which show the strict inclusion of the different orders for the weak mixing property. This fact contrasts with the case of nonautonomous interval maps, where it was shown that there are examples which are weakly mixing of order 2 which are not weakly mixing of order 3 as it is shown in Theorem 9 in [4], but once an interval NDS is weakly mixing of order 3, then it follows immediately that it is of arbitrary order $n \geq 2$ as it is proved in Theorem 11 in [4].

Theorem 5.3.1 *Given any $n \geq 2$ there is a linear NDS (ℓ^2, T_∞) defined on the Hilbert space ℓ^2 which is weakly mixing of order n, but it is not weakly mixing of order $n+1$.*

Proof. We consider an arbitrary mixing operator on ℓ^2 like, for instance, the weighted backward shift $T := 2B$, $T(x_1, x_2, \dots) = (2x_2, 2x_3, \dots)$. Since every mixing map is weakly mixing of all orders, given $n \in \mathbb{N}$, let $(w_1, \dots, w_n) \in \ell^2 \times \cdots \times \ell^2$ be a vector whose orbit is dense in $\ell^2 \times \cdots \times \ell^2$ for the operator $T \times \cdots \times T$.

For any $k \geq 0$, let $X_k := \operatorname{span}\{T^k w_1, \dots, T^k w_n\}$ and let $P_k : \ell^2 \to X_k$ be the corresponding orthogonal projection. We observe that $\dim(X_k) = n$ for every $k \geq 0$ since, otherwise, there would be $k_0 \geq 0$ such that $\dim(X_k) \leq \dim(X_{k_0}) < n$ for all $k \geq k_0$, which avoids the fact that $\{(T^k w_1, \dots, T^k w_n) \ ; \ k \geq k_0\}$ is dense in the n-product of ℓ^2. We set $T_1 = P_0$ and $T_{k+1} = P_k \circ T$, $k \in \mathbb{N}$.

The linear NDS (ℓ^2, T_∞) is clearly not weakly mixing of order $n+1$. Indeed, let $V_1, \dots, V_{n+1}$ be non-empty open sets of ℓ^2 such that any $n+1$-tuple $(v_1, \dots, v_{n+1}) \in V_1 \times \cdots \times V_{n+1}$ are linearly independent. Then, since $(T_k \circ \cdots \circ T_1)(\ell^2)$ is n-dimensional, it cannot intersect all the V_i's, $i = 1, \dots, n+1$, and therefore (ℓ^2, T_∞) is not weakly mixing of order $n+1$.

On the other hand, given any collection $U_i, V_i \subset \ell^2$ of non-empty open sets, $i = 1, \dots, n$, since P_0 is an orthogonal projection, thus an open mapping, we find vectors $u_i \in U_i$, $i = 1, \dots, n$, such that

$$\{T_1 u_1, \dots, T_1 u_n\} = \{P_0 u_1, \dots, P_0 u_n\}$$

is linearly independent. Let $P_0 u_i = \sum_{j=1}^n \alpha_{i,j} w_j$, $i = 1, \dots, n$. By definition of the T_k's we obtain

$$(T_{k+1} \circ \cdots \circ T_1) u_i = \sum_{j=1}^n \alpha_{i,j} T^k w_j, \quad i = 1, \dots, n,$$

for all $k \in \mathbb{N}$. We consider the matrix $A := (\alpha_{i,j})_{i,j}$, which is invertible since the $P_0 u_i$'s are linearly independent. Let $B = A^{-1} = (\beta_{i,j})_{i,j}$. We fix

$v_i \in V_i$, $i = 1, \ldots, n$, and a 0-neighbourhood W such that

$$v_i + \sum_{j=1}^{n} \alpha_{i,j} W \subset V_i, \quad i = 1, \ldots, n.$$

By the selection of the w_i's, there is $k \in \mathbb{N}$ such that $T^k w_i \in \sum_{j=1}^{n} \beta_{i,j} v_j + W$, $i = 1, \ldots, n$. Therefore, since $B = A^{-1}$, we have

$$T_1^{(k+1)} u_i = \sum_{j=1}^{n} \alpha_{i,j} T^k w_j \in v_i + \sum_{j=1}^{n} \alpha_{i,j} W \subset V_i, \quad i = 1, \ldots, n,$$

and we conclude that (ℓ^2, T_∞) is weakly mixing of order n. □

Remark 5.3.2 It is known that a system $(T_n)_n$ of commuting operators that is weakly mixing of order 2 is necessarily weakly mixing of all orders [26, 27].
In particular, sequences of operators generated by commuting tuples of operators [55] and discretizations of C_0-semigroups of operators, which are weakly mixing of order 2, immediately happen to be weakly mixing of all orders.
This means that there is no hope to find examples like the one in Theorem 5.3.1 within this framework. We do not know whether it is possible to obtain this type of counterexamples for non artificially constructed linear NDS.

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