Quasi-LDU factorization of nonsingular totally nonpositive matrices ☆

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Abstract

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonsingular totally nonpositive matrix. In this paper we describe some properties of these matrices when $a_{11} = 0$ and obtain a characterization in terms of the quasi-LDU factorization of A, where L is a block lower triangular matrix, D is a diagonal matrix and U is a unit upper triangular matrix.

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1. Introduction

A matrix is called *totally positive (strictly totally positive)* if all its minors are nonnegative (positive). This class of matrices has a wide variety of applications in statistics, economics, computer aided geometric design and others fields, see [10, 16], and is abbreviated as TP (STP). Nevertheless, a new class of matrices that satisfy the opposite property has recently become of interest due to its applications in social and economic problems [2, 17, 19]. If all minors of a matrix are nonpositive (negative) the matrix is called *totally nonpositive (totally negative)* and is abbreviated as t.n.p. (t.n.). They are included in the class of sign regular matrices, which are widely used because of their variation diminishing property (see [1, 6]).

Several authors have studied square and rectangular TP and STP matrices, see [1, 5, 7, 8, 10, 11, 12, 13, 14, 16], obtaining properties, the Jordan structure and characterizations by applying the Gaussian or Neville elimination that allow one to significantly reduce the number of minors to be checked in order to decide if a matrix is TP or STP.

For t.n. matrices, a characterization in terms of the parameters computed from the Neville elimination is obtained in [12] and spectral properties and LDU factorizations are studied in [9].

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The nonsingular t.n.p. matrices with a negative $(1,1)$ entry have been characterized in terms of the factors of their LDU factorization in [3]. This factorization provides a criteria to determine if a matrix is t.n.p. and allows us to reduce the numbers of minors to be checked to decide the total nonpositivity of a nonsingular matrix with a negative $(1,1)$ entry. On the other hand, some properties of nonsingular t.n.p. matrices analogous to those satisfied by nonsingular TP matrices have also been studied [1, 10]. When the $(1,1)$ entry is equal to zero but the (n,n) entry is negative we can obtain a UDL factorization of this nonsingular t.n.p. matrix by permutation similarity.

The rectangular t.n.p. matrices have been studied in [4], obtaining a full rank LDU factorization in echelon form of this class of matrices and other characterization by means of its thin QR factorization. This QR characterization is similar to the one obtained in [5] for rectangular TP matrices and it is an extension of the result for square TP matrices given in [11].

When the nonsingular t.n.p. matrix has the $(1,1)$ and (n,n) entries equal to zero a characterization, in terms of the sign of its minors containing contiguous rows or columns and including the first row or column, respectively, is obtained in [15].

In this paper we characterize the nonsingular t.n.p. matrices with the $(1,1)$ entry equal to zero in terms of a quasi-$\tilde{L}DU$ factorization, where $\tilde{L}$ is a lower block triangular matrix, D is a diagonal matrix and U is a unit upper triangular TP matrix. This result holds when the (n,n) entry is equal to zero or when it is negative but we do not use permutation similarity. We study this characterization for $n \geq 3$, since the case $n = 2$ is trivial. Finally, we prove some properties of this kind of matrices analogous to those satisfied by nonsingular t.n.p. matrices with negative $(1,1)$ entry described in [3].

We follow the notation given in [1]. For $k, n \in \mathbb{N}$, $1 \leq k \leq n$, $\mathcal{Q}_{k,n}$ denotes the set of all increasing sequences of k natural numbers less than or equal to n. If A is an $n \times n$ matrix and $\alpha, \beta \in \mathcal{Q}_{k,n}$, $A[\alpha|\beta]$ denotes the $k \times k$ submatrix of A lying in rows α and columns β. The principal submatrix $A[\alpha|\alpha]$ is abbreviated as $A[\alpha]$. Note that, an $n \times n$ matrix A is a t.n.p. matrix if $\det A[\alpha|\beta] \leq 0$, $\forall \alpha, \beta \in \mathcal{Q}_{k,n}$, $k = 1, 2, \ldots, n$.

Throughout the paper an LDU factorization means the corresponding factorization resulting from Gauss or Neville elimination with no pivoting, where L and U are unit lower- and upper-triangular matrices, respectively, and D is a diagonal matrix.

2. Characterization of nonsingular t.n.p. matrices with the $(1,1)$ entry equal to zero by the quasi-LDU factorization

In this section we derive a characterization of nonsingular t.n.p matrices with the $(1,1)$ entry equal to zero in terms of their quasi-LDU factorization.

Given an $n \times n$, nonsingular t.n.p. matrix $A = (a_{ij})$ it is known that $a_{ij} < 0$ whenever $(i,j) \notin \{(1,1), (n,n)\}$, [18, Theorem 2.1 (i)] and $\det A[1,2,\ldots,k] < 0$, for all $k = 2, 3, \ldots, n$, [15, Theorem 5].

Since A is a nonsingular t.n.p. matrix with $a_{11} = 0$, it is not possible to obtain a LDU factorization with no pivoting. Therefore, we work with a matrix $B = PA$, where P is the permutation matrix $P = [2,\, 1,\, 3,\, \ldots,\, n]$, that is

$$B = PA = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ 0 & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-1} & a_{n-1n} \\ a_{n1} & a_{n2} & \cdots & a_{nn-1} & a_{nn} \end{bmatrix}.$$

Let $n \geq 3$, as $\det B[1, 2, \ldots, k] = -\det A[1, 2, \ldots, k] > 0$, for all $k = 2, 3, \ldots, n$, we can obtain the factorization $B = L_B D_B U_B$ by applying the Gauss elimination process with no pivoting, where

$$L_B = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \hline l_{31} & l_{32} & 1 & \cdots & 0 & 0 \\ l_{41} & l_{42} & l_{43} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ l_{n-11} & l_{n-12} & l_{n-13} & \cdots & 1 & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn-1} & 1 \end{array}\right],$$

$$D_B = \begin{bmatrix} -d_1 & 0 & 0 & \cdots & 0 \\ 0 & -d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix},$$

$$U_B = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n-1} & u_{1n} \\ 0 & 1 & \cdots & u_{2n-1} & u_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & u_{n-1n} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

The matrices U_B, D_B and L_B satisfy the following properties.

Proposition 1. The upper triangular matrix U_B is a TP matrix, with positive entries above the main diagonal.

Proof. Since the entries of the first row of B are negative, we have that $u_{1j} > 0$, for $j = 1, 2, \ldots, n$. From Binet-Cauchy [1], we have for all $\beta \in \mathcal{Q}_{k,n}$, $k =$

$2, 3, \dots, n$, that

$$\det B[1,2,\dots,k|\beta] = \sum_{\gamma\in\mathcal{Q}_{k,n}} \det L_B[1,2,\dots,k|\gamma]\det D_B[\gamma]\det U_B[\gamma|\beta]$$
$$= \det L_B[1,2,\dots,k|1,2,\dots,k]\det D_B[1,2,\dots,k]\det U_B[1,2,\dots,k|\beta]$$
$$= \left(\prod_{i=1}^{k} d_i\right)\det U_B[1,2,\dots,k|\beta] \geq 0,$$

which implies that $\det U_B[1,2,\dots,k|\beta] \geq 0$ for all $\beta \in \mathcal{Q}_{k,n},\ k = 1,2,\dots,n$. Then, U_B is a unit upper triangular TP matrix. As a consequence, for $i = 2,3,\dots,n-1$, and $j = i+1,\dots,n$,

$$\det U_B[1,i|i,j] = u_{1i}u_{ij} - u_{1j} \geq 0 \quad\longrightarrow\quad u_{ij} > 0,$$

that is, U_B has positive entries above the main diagonal. □

Proposition 2. The diagonal matrix D_B has all its diagonal entries positive except for the $(1,1)$ and $(2,2)$ entries, which are negative.

Proof. The $(1,1)$ and $(2,2)$ entries of D_B are $-d_1 = a_{21} < 0$ and $-d_2 = a_{12} < 0$, respectively. The remaining diagonal entries are

$$d_i = \frac{\det B[1,2,\dots,i]}{\det B[1,2,\dots,i-1]} > 0, \quad i = 3,4,\dots,n.$$

□

Proposition 3. The unit lower triangular matrix L_B satisfies,

1) The entries of the first column l_{i1}, with $i = 3,4,\dots,n$, are positive.
2) For all $\alpha \in \mathcal{Q}_{k,n},\ k = 2,3,\dots,n$,

$$\det L_B[\alpha|1,2,\dots,k] = \begin{cases} \geq 0 & \text{if} \quad \alpha_1 = 1,\ \alpha_2 = 2 \\ \leq 0 & \text{if} \quad 1 \text{ or } 2 \notin \alpha \end{cases}$$

Proof.

1) From Binet-Cauchy, for $i = 3,4,\dots,n$, we have

$$\det B[i|1] = -d_1 \det L_B[i|1] = -d_i l_{i1} < 0,$$

then, it follows that the entries in the first column of L are positive except for $l_{21} = 0$.

2) Again, from Binet-Cauchy, for all $\alpha \in \mathcal{Q}_{k,n},\ k = 2,3,\dots,n$, we have

$$\det B[\alpha|1,2,\dots,k] = (-d_1)(-d_2)\prod_{i=3}^{k} d_i \det L_B[\alpha|1,2,\dots,k]$$
$$= \prod_{i=1}^{k} d_i \det L_B[\alpha|1,2,\dots,k] = \begin{cases} \geq 0 & \text{if} \quad \alpha_1 = 1,\ \alpha_2 = 2 \\ \leq 0 & \text{if} \quad 1 \text{ or } 2 \notin \alpha \end{cases}$$

which implies that

$$\det L_B[\alpha|1,2,\ldots,k] = \begin{cases} \geq 0 & \text{if} \quad \alpha_1 = 1,\, \alpha_2 = 2 \\ \leq 0 & \text{if} \quad 1 \text{ or } 2 \notin \alpha \end{cases}$$

□

¿From Proposition 3 the following properties of the matrix L_B can be deduced.

Lemma 1. *The entries of the second column l_{i2}, for $i = 3, 4, \ldots, n$, are non-positive. Moreover, if $l_{32} = 0$ then $l_{i2} = 0$ for $i = 4, 5, \ldots, n$, whereas if $l_{32} < 0$ then $l_{i2} < 0$ for $i = 4, 5, \ldots, n$.*

Proof. Since

$$\det L_B[1, i|1, 2] = l_{i2} \leq 0, \text{ for } i = 3, 4, \ldots, n,$$

the entries in the second column of L_B under the main diagonal are nonpositive. Moreover, since

$$\det L_B[1, 3, i|1, 2, 3] = l_{32}l_{i3} - l_{i2} \leq 0, \quad i = 4, 5, \ldots, n,$$

if $l_{32} = 0$ then $l_{i2} = 0$ for $i = 4, 5, \ldots, n$. In addition, as

$$\det L_B[3, i|1, 2] = l_{31}l_{i2} - l_{i1}l_{32} \leq 0 \quad i = 4, 5, \ldots, n,$$

we deduce that if $l_{32} < 0$ then $l_{i2} < 0$ for $i = 4, 5, \ldots, n$. □

Lemma 2. *The submatrix $S = L_B[1, 3, 4, \ldots, n]$ is a TP matrix with all its entries under the main diagonal positive.*

Proof. Since S is a unit lower triangular matrix to assure that it is t.n.p. we need to prove that

$$\det S[\alpha|1, 2, \ldots, k] \geq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n-1}, \quad k = 1, 2, \ldots, n-1.$$

- For all $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \in \mathcal{Q}_{k,n-1}$, with $\alpha_1 = 1$, we have

$$\begin{aligned} &\det S[1, \alpha_2, \ldots, \alpha_k|1, 2, \ldots, k] \\ &= \det L_B[1, \alpha_2 + 1, \ldots, \alpha_k + 1|1, 3, \ldots, k+1] \\ &= \det L_B[1, 2, \alpha_2 + 1, \ldots, \alpha_k + 1|1, 2, 3, \ldots, k+1] \\ &= \frac{1}{\prod_{i=1}^{k+1} d_i} \det B[1, 2, \alpha_2 + 1, \ldots, \alpha_k + 1|1, 2, 3, \ldots, k+1] \geq 0. \end{aligned}$$

- For all $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \in \mathcal{Q}_{k,n-1}$, with $\alpha_1 > 1$, we have

$$\begin{aligned} &\det S[\alpha_1, \alpha_2, \ldots, \alpha_k|1, 2, \ldots, k] \\ &= \det L_B[\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_k + 1|1, 3, \ldots, k+1] \\ &= -\det L_B[2, \alpha_1 + 1\alpha_2 + 1, \ldots, \alpha_k + 1|1, 2, 3, \ldots, k+1] \\ &= -\frac{1}{\prod_{i=1}^{k+1} d_i} \det B[2, \alpha_1 + 1\alpha_2 + 1, \ldots, \alpha_k + 1|1, 2, 3, \ldots, k+1] \geq 0. \end{aligned}$$

Therefore S is a TP matrix. Moreover, since the entries of the first column of S are positive and $\det S[j, i|1, j] \geq 0$, for $i > j$, $i, j = 2, 3, \ldots, n-1$, it follows that all its entries under the main diagonal are positive. $\square$

Taking into account that $A = PB = PL_B D_B U_D = \tilde{L}DU$ and the previous results, the following theorem gives the quasi-LDU factorization of A.

Theorem 1. *Let A be an $n \times n$ nonsingular t.n.p. matrix with $a_{11} = 0$. Then, A has a factorization $\tilde{L}DU$, where U is a unit upper triangular TP matrix with positive entries above the main diagonal, $D = diag(-d_1, -d_2, d_3, \ldots, d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$, and $\tilde{L}$ is the block lower triangular matrix*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \textit{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the entries in the first column of $\tilde{L}_{21}$ are positive, in the second one are nonpositive, $\tilde{L}_{22}$ is unit lower triangular TP matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n}, \; k = 2, 3, \ldots, n.$$

Example 1. *The nonsingular t.n.p. matrix*

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ -14 & -14 & -14 & -14 \\ -14 & -13 & -11 & -7 \\ -28 & -25 & -18 & 0 \end{bmatrix},$$

admits the following factorization,

$$A = \tilde{L}DU \quad = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} -14 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Remark 1. Other properties that verifies the matrix L_B are,

1. From Binet-Cauchy, it is easy to see that
   • The $k \times k$ column initial minors of $L_B[3, 4, \ldots, n|1, 2, 3, \ldots, n-2]$ are nonpositive, for $k \geq 2$.

   • The column initial minors of $L_B[3, 4, \ldots, n|2, 3, \ldots, n-1]$ are nonpositive.
2. The matrix $L_B = P\tilde{L}$ admits the following decomposition,

$$L_B = TQ = \begin{bmatrix} I_{2\times 2} & O \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} I_{2\times 2} & O \\ Q_{21} & Q_{22} \end{bmatrix},$$

with

$$T = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \hline t_{31} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_{n-1,1} & 0 & t_{n-1,3} & \cdots & 1 & 0 \\ t_{n,1} & 0 & t_{n,3} & \cdots & t_{n,n-1} & 1 \end{array}\right],$$

$$Q = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & -q_{32} & 1 & \cdots & 0 & 0 \\ 0 & 0 & q_{43} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & q_{n-1,3} & \cdots & 1 & 0 \\ 0 & 0 & q_{n,3} & \cdots & q_{n,n-1} & 1 \end{array}\right],$$

where $t_{ij} > 0$, for $i = 3, 4, \ldots, n$, and $j = 1, 3, \ldots, i-1$, $q_{32} \geq 0$, $q_{ij} > 0$ for $i = 4, 5, \ldots, n$, and $j = 3, 4, \ldots, i-1$, and the submatrices $T[1, 3, 4, \ldots, n]$ and $Q[3, 4, \ldots, n]$ are TP.

Example 2. *If $L_B = P\tilde{L}$, where $\tilde{L}$ is given in the Example 1, then L_B admits the following factorization,*

$$L_B = P\tilde{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & 1 \end{bmatrix} = TQ \quad \textit{where}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The converse of Theorem 1 is not true in general, as the next example shows.

Example 3. *The matrix*

$$A = \tilde{L}DU = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & 1 \end{bmatrix}}_{\tilde{L}} \underbrace{\begin{bmatrix} -15 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$= \begin{bmatrix} 0 & -2 & -4 & -6 \\ -15 & -15 & -15 & -15 \\ -15 & -13 & -10 & -5 \\ -30 & -24 & -14 & 6 \end{bmatrix},$$

is not t.n.p. although the matrices $\tilde{L}$, D and U satisfy the conditions of Theorem 1.

The following theorem gives a necessary condition for a product $\tilde{L}DU$ to be a t.n.p. matrix with the $(1,1)$ entry equal to zero.

Theorem 2. *Let $A = \tilde{L}DU$ be an $n \times n$ matrix where $a_{nn} \leq 0$, U is a unit upper triangular TP matrix with positive entries above the diagonal, $D = diag(-d_1, -d_2, d_3, \ldots, d_n)$ with $d_i > 0$, $i = 1, 2, \ldots, n$, and $\tilde{L}$ is the block lower triangular matrix*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \textit{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where the entries in the first column of $\tilde{L}_{21}$ are positive, in the second one are nonpositive, $\tilde{L}_{22}$ is unit lower triangular TP matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n}, \ k = 2, 3, \ldots, n.$$

Then, A is a nonsingular t.n.p. matrix.

Proof. From the structure of the matrices $\tilde{L}$, D and U, and the product $\tilde{L}DU$, we obtain easily that $a_{11} = 0$, $a_{1n} < 0$ and $a_{n1} < 0$. Then, by [15, Theorem 5] we know that A is a nonsingular t.n.p. matrix if the following inequalities hold,

$$\begin{aligned} \det A[\alpha|1, 2, \ldots, k] &\leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n}, \quad k = 1, 2, \ldots, n \\ \det A[1, 2, \ldots, k|\beta] &\leq 0, \quad \forall \beta \in \mathcal{Q}_{k,n}, \quad k = 1, 2, \ldots, n \\ \det A[1, 2, \ldots, k] &< 0, \quad k = 2, 3, \ldots, n. \end{aligned}$$

In order to verify that A satisfies these inequalities we consider the matrix $B = PA$ and its factorization $B = L_B D_B U_B$, where $L_B = P\tilde{L}$, $D_B = D$ and $U_B = U$.

(a) The principal minors are negative for all $k = 2, 3, \dots, n$, that is,

$$\begin{aligned}
\det A[1,2,\dots,k] &= -\det B[1,2,\dots,k] \\
&= -\sum_{\gamma\in\mathcal{Q}_{k,n}} \det L_B[1,2,\dots,k|\gamma]\det(D_B U_B)[\gamma|1,2,\dots,k] \\
&= -(-d_1)(-d_2)d_3\dots d_k < 0.
\end{aligned}$$

(b) The row initial minors are nonpositive for all $\beta \in \mathcal{Q}_{k,n}$, $k = 1, 2, \dots, n$.

$$\begin{aligned}
\det A[1,2,\dots,k|\beta] &= -\det B[1,2,\dots,k|\beta] \\
&= -\sum_{\gamma\in\mathcal{Q}_{k,n}} \det L_B[1,2,\dots,k|\gamma]\det(D_B U_B)[\gamma|\beta] \\
&= -(-d_1)(-d_2)d_3\dots d_k \det U_B[1,2,\dots,k|\beta] \\
&= -(-d_1)(-d_2)d_3\dots d_k \det U[1,2,\dots,k|\beta] \leq 0.
\end{aligned}$$

(c) The column initial minors are nonpositive for all $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_k\} \in \mathcal{Q}_{k,n}$, $k = 1, 2, \dots, n$. We distinguish the following cases,

(c1) $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 \geq 3$,

$$\begin{aligned}
&\det A[1,2,\alpha_3,\dots,\alpha_k|1,2,\dots,k] = -\det B[1,2,\alpha_3,\dots,\alpha_k|1,2,\dots,k] \\
&= -\sum_{\gamma\in\mathcal{Q}_{k,n}} \det L_B[1,2,\alpha_3,\dots,\alpha_k|\gamma]\det(D_B U_B)[\gamma|1,2,\dots,k] \\
&= -\left(\prod_{i=1}^{k} d_i\right)\det L_B[\alpha_3,\dots,\alpha_k|3,\dots,k] \\
&= -\left(\prod_{i=1}^{k} d_i\right)\det \tilde{L}[\alpha_3,\dots,\alpha_k|3,\dots,k] \leq 0.
\end{aligned}$$

(c2) $\alpha_1 = 1$, $\alpha_2 \geq 3$,

$$\begin{aligned}
&\det A[1,\alpha_2,\dots,\alpha_k|1,2,\dots,k] = \det B[1,\alpha_2,\dots,\alpha_k|1,2,\dots,k] \\
&= \sum_{\gamma\in\mathcal{Q}_{k,n}} \det L_B[1,\alpha_2,\dots,\alpha_k|\gamma]\det(D_B U_B)[\gamma|1,2,\dots,k] \\
&= \left(\prod_{i=1}^{k} d_i\right)\det L_B[\alpha_2,\dots,\alpha_k|2,\dots,k] \\
&= \left(\prod_{i=1}^{k} d_i\right)\det \tilde{L}[\alpha_2,\dots,\alpha_k|2,\dots,k] \\
&= \left(\prod_{i=1}^{k} d_i\right)\det \tilde{L}[2,\alpha_2,\dots,\alpha_k|1,2,\dots,k] \leq 0.
\end{aligned}$$

(c3) $\alpha_1 = 2$ and $\alpha_2 \geq 3$,

$$\begin{aligned}
&\det A[2, \alpha_2, \ldots, \alpha_k|1, 2, \ldots, k] = \det B[2, \alpha_2, \ldots, \alpha_k|1, 2, \ldots, k] \\
&= \sum_{\gamma \in \mathcal{Q}_{k,n}} \det L_B[2, \alpha_2, \ldots, \alpha_k|\gamma] \det(D_B U_B)[\gamma|1, 2, \ldots, k] \\
&= \det L_B[2, \alpha_2, \ldots, \alpha_k|1, 2, \ldots, k](-d_1)(-d_2)d_3 \ldots d_k \\
&= -\left(\prod_{i=1}^{k} d_i\right) \det L_B[\alpha_2, \alpha_3, \ldots, \alpha_k|1, 3, \ldots, k] \\
&= -\left(\prod_{i=1}^{k} d_i\right) \det \tilde{L}[\alpha_2, \alpha_3, \ldots, \alpha_k|1, 3, \ldots, k] \\
&= \left(\prod_{i=1}^{k} d_i\right) \det \tilde{L}[1, \alpha_2, \alpha_3, \ldots, \alpha_k|1, 2, 3, \ldots, k] \leq 0.
\end{aligned}$$

(c4) $\alpha_1 \geq 3$,

$$\begin{aligned}
&\det A[\alpha_1, \alpha_2, \ldots, \alpha_k|1, 2, \ldots, k] = \det B[\alpha_1, \alpha_2, \ldots, \alpha_k|1, 2, \ldots, k] \\
&= \sum_{\gamma \in \mathcal{Q}_{k,n}} \det L_B[\alpha_1, \alpha_2, \ldots, \alpha_k|\gamma] \det(D_B U_B)[\gamma|1, 2, \ldots, k] \\
&= \left(\prod_{i=1}^{k} d_i\right) \det L_B[\alpha_1, \alpha_2, \ldots, \alpha_k|1, 2, \ldots, k] \\
&= \left(\prod_{i=1}^{k} d_i\right) \det \tilde{L}[\alpha_1, \alpha_2, \ldots, \alpha_k|1, 2, \ldots, k] \leq 0,
\end{aligned}$$

which concludes the prove. □

Combining Theorems 1 and 2 we obtain the following result.

Theorem 3. *Let A be an $n \times n$ nonsingular matrix with $a_{11} = 0$, $a_{nn} \leq 0$. Then, A is t.n.p. if and only if A has a factorization $\tilde{L}DU$, where U is a unit upper triangular TP matrix with positive entries above the main diagonal, $D = diag(-d_1, -d_2, d_3, \ldots, d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$, and $\tilde{L}$ is the block lower triangular matrix*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where the entries in the first column of $\tilde{L}_{21}$ are positive, in the second one are nonpositive, $\tilde{L}_{22}$ is unit lower triangular TP matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n}, \; k = 2, 3, \ldots, n.$$

Remark 2. The quasi-LDU factorization obtained in Theorem 3 provides a criteria to determine if a matrix with a zero $(1,1)$ entry is t.n.p. The number of minors to be computed to decide the total nonpositivity is equal to the ones to be checked in [15] but their computation is easier because the matrices are triangular, block triangular and diagonal.

Otherwise, this factorization allows us to directly obtain nonsingular t.n.p. matrices of any size.

Remark 3. (1) Let C be an $n \times n$ matrix with $c_{nn} \neq 0$ and let A be the matrix

$$A = C - c_{nn}E_{nn} \longrightarrow C = A + c_{nn}E_{nn},$$

where E_{nn} is the $n \times n$ matrix whose only nonzero element is 1 in position (n,n). We know that

$$\begin{aligned} \det C[\alpha|\beta] &= \det A[\alpha|\beta], \ \forall \alpha,\beta \in \mathcal{Q}_{k,n}, \ k = 1,\ldots,n, \ \ n \notin \alpha \cap \beta \\ \det C[\alpha,n|\beta,n] &= \det A[\alpha,n|\beta,n] + c_{nn} \det C[\alpha|\beta]. \end{aligned}$$

If C is a nonsingular t.n.p. matrix, then the nonsingular matrix A with $a_{nn} = 0$ is t.n.p. Moreover, if $C = \tilde{L}DU$, where $D = \operatorname{diag}(-d_1, -d_2, d_3, \ldots, d_n)$ with $d_i > 0$, for $i = 1,2,\ldots,n$, then $A = \tilde{L}\bar{D}U$, where $\bar{D} = \operatorname{diag}(-d_1, -d_2, d_3, \ldots, d_n - c_{nn})$ with $d_i > 0$, for $i = 1,2,\ldots,n-1$, and $d_n - c_{nn} > 0$.

(2) Let A be an $n \times n$ nonsingular t.n.p. matrix with $a_{11} = a_{nn} = 0$. Then, there exists an $x > 0$ such that the matrix $A_x = A - xE_{nn}$ is also a nonsingular t.n.p. matrix.
¿From matrix $B = PA$ we construct for all $x > 0$ the matrix

$$B_x = B - xE_{nn}.$$

Using the factorization $B = L_B D_B U_B$ we have

$$\begin{aligned} B_x &= B - xE_{nn} = L_B D_B U_B - xE_{nn} \\ &= \begin{bmatrix} L_1 & 0 \\ l_1 & 1 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & d_n \end{bmatrix} \begin{bmatrix} U_1 & u_1 \\ 0 & 1 \end{bmatrix} - x \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} L_1 & 0 \\ l_1 & 1 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & d_n - x \end{bmatrix} \begin{bmatrix} U_1 & u_1 \\ 0 & 1 \end{bmatrix} \\ &= L_B D'_B U_B = L_{B_x} D_{B_x} U_{B_x}. \end{aligned}$$

It is not difficult to prove that $b_{x_{nn}} = b_{nn} - x$, so if $b_{nn} = 0$, B_x has negative (n,n) entry. Since $L_{B_x} = L_B$ and $U_{B_x} = U_B$, if we take $0 < x < d_n$, D_{B_x} is a positive diagonal matrix except for the two first negative entries. Therefore, by Theorem 2 the matrix $A_x = PB_x$ is a nonsingular t.n.p. matrix with the $(1,1)$ entry equal to zero and the remainder negative.

3. Some properties of nonsingular t.n.p. matrices

All nonsingular t.n.p. matrices verify the following two properties.

Proposition 4. Let A be an $n \times n$ nonsingular t.n.p. matrix. Then, for $k = 2, 3, \ldots, n$, one of the following conditions holds,

(i) $\det A[1, \alpha_2, \alpha_3, \ldots, \alpha_k | 1, 2, \ldots, k] < 0$,

(ii) If $\det A[1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k] = 0$ then $\det A[\alpha_2, \alpha_3, \ldots, \alpha_k | 2, \ldots, k] = 0$.

Proof. If $\det A[1, \alpha_2, \alpha_3, \ldots, \alpha_k | 1, 2, \ldots, k] < 0$ the result holds. Now, suppose that $\det A[1, \alpha_2, \alpha_3, \ldots, \alpha_k | 1, 2, \ldots, k] = 0$ and $\det A[\alpha_2, \ldots, \alpha_k | 2, \ldots, k] < 0$. Since A is nonsingular there exists $t,\ k < t \leq n$, such that

$$\det A[1, \alpha_2, \alpha_3, \ldots, \alpha_k | 2, \ldots, k, t] < 0.$$

Since

$$\begin{aligned}
&\det A[1, \alpha_2, \ldots, \alpha_k | 2, \ldots, k, t] \\
&= \det \begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1k} & a_{1t} \\ a_{\alpha_2 2} & a_{\alpha_2 3} & \cdots & a_{\alpha_2 k} & a_{\alpha_2 t} \\ a_{\alpha_3 2} & a_{\alpha_3 3} & \cdots & a_{\alpha_3 k} & a_{\alpha_3 t} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{\alpha_k 2} & a_{\alpha_k 3} & \cdots & a_{\alpha_k k} & a_{\alpha_k t} \end{bmatrix} \\
&= \det \begin{bmatrix} 0 & 0 & \cdots & 0 & \tilde{a}_{1t} \\ a_{\alpha_2 2} & a_{\alpha_2 3} & \cdots & a_{\alpha_2 k} & a_{\alpha_2 t} \\ a_{\alpha_3 2} & a_{\alpha_3 3} & \cdots & a_{\alpha_3 k} & a_{\alpha_3 t} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{\alpha_k 2} & a_{\alpha_k 3} & \cdots & a_{\alpha_k k} & a_{\alpha_k t} \end{bmatrix} \\
&= (-1)^{1+k} (\tilde{a}_{1t}) \det A[\alpha_2, \alpha_3, \ldots, \alpha_k | 2, 3, \ldots, k] < 0,
\end{aligned}$$

we obtain that $(-1)^{1+k}(\tilde{a}_{1t}) > 0$.

Again, since A is nonsingular there exists l, such that $\alpha_i < l < \alpha_{i+1},\ \ i = 1, 2, \ldots, k$, with $\alpha_1 = 1,\ \alpha_{k+1} = n$, so that

$$\det A[\alpha_2, \ldots, \alpha_i, l, \alpha_{i+1}, \ldots, \alpha_k | 1, 2, \ldots, k] < 0.$$

From

$$\begin{aligned}
&\det A[\alpha_2, \ldots, \alpha_i, l, \alpha_{i+1}, \ldots, \alpha_k | 1, 2, \ldots, k] \\
&= \begin{bmatrix} a_{\alpha_2 1} & a_{\alpha_2 2} & \cdots & a_{\alpha_2 k-1} & a_{\alpha_2 k} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{\alpha_i 1} & a_{\alpha_i 2} & \cdots & a_{\alpha_i k-1} & a_{\alpha_i k} \\ a_{l1} & a_{l2} & \cdots & a_{lk-1} & a_{lt} \\ a_{\alpha_{i+1} 1} & a_{\alpha_{i+1} 2} & \cdots & a_{\alpha_{i+1} k-1} & a_{\alpha_{i+1} k} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{\alpha_k 1} & a_{\alpha_k 2} & \cdots & a_{\alpha_k k-1} & a_{\alpha_k t} \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} 0 & a_{\alpha_2 2} & \cdots & a_{\alpha_2 k-1} & a_{\alpha_2 k} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{\alpha_i 2} & \cdots & a_{\alpha_i k-1} & a_{\alpha_i k} \\ \tilde{a}_{l1} & a_{l2} & \cdots & a_{lk-1} & a_{lt} \\ 0 & a_{\alpha_{i+1} 2} & \cdots & a_{\alpha_{i+1} k-1} & a_{\alpha_{i+1} k} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{\alpha_k 2} & \cdots & a_{\alpha_k k-1} & a_{\alpha_k t} \end{bmatrix}$$

$$= (-1)^{i+1}(\tilde{a}_{l1}) \det A[\alpha_2, \alpha_3, \ldots, \alpha_k | 2, 3, \ldots, k] < 0,$$

we have that $(-1)^{i+1}(\tilde{a}_{l1}) > 0$. Therefore

$$\begin{aligned} &\det A[1, \alpha_2, \ldots, \alpha_i, l, \alpha_{i+1}, \ldots, \alpha_k | 1, 2, \ldots, k, t] \\ &= (-1)^{1+k+1}(\tilde{a}_{1t}) \det A[\alpha_2, \ldots, \alpha_i, l, \alpha_{i+1}, \ldots, \alpha_k | 1, 2, \ldots, k] \\ &= (-1)^{1+k+1}(\tilde{a}_{1t})(-1)^{i+1}(\tilde{a}_{l1}) \det A[\alpha_2, \alpha_3, \ldots, \alpha_k | 2, \ldots, k] \\ &= -\left\{ \underbrace{(-1)^{1+k}(\tilde{a}_{1t})}_{>0} \underbrace{(-1)^{i+1}(\tilde{a}_{l1})}_{>0} \underbrace{\det A[\alpha_2, \alpha_3, \ldots, \alpha_k | 2, \ldots, k]}_{<0} \right\} > 0, \end{aligned}$$

which is absurdum because A is t.n.p. □

Proposition 5. Let A be an $n \times n$ nonsingular t.n.p. matrix. Then, for $k = 2, 3, \ldots, n$, one of the following conditions holds,
(i) $\det A[1, 2, \ldots, k | 1, \beta_2, \beta_3, \ldots, \beta_k] < 0$,
(ii) If $\det A[1, 2, \ldots, k | 1, \beta_2, \beta_3, \ldots, \beta_k] = 0$ then $A[2, \ldots, k | \beta_2, \beta_3, \ldots, \beta_k] = 0$.

Proof. The proof is similar to that of Proposition 4. □

From Propositions 4 and 5 we obtain the relationship between the nonsingular t.n.p. matrices with the $(1, 1)$ entry equal to zero and the nonsingular t.n.p. matrices with negative $(1, 1)$ entry.

Proposition 6. Let A be an $n \times n$ nonsingular t.n.p. matrix with $a_{11} = 0$. Then, there exists $\epsilon_0 > 0$ such that $\forall \epsilon < \epsilon_0$ the matrix $A_\epsilon = -\epsilon E_{11} + A$, where E_{11} is the $n \times n$ matrix whose only nonzero entry is 1 in position $(1, 1)$, is a nonsingular t.n.p. matrix.

Proof. The matrix A_ϵ verifies

$$\det A_\epsilon[\alpha|\beta] = \det A[\alpha|\beta] \leq 0, \quad \text{if} \quad 1 \notin \alpha \cap \beta,$$

therefore for all $\alpha \in \mathcal{Q}_{k,n}$, with $\alpha_1 \geq 2$, and for all $\beta \in \mathcal{Q}_{k,n}$, with $\beta_1 \geq 2$, we have

$$\begin{aligned} \det A_\epsilon[\alpha_1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k] &= \det A[\alpha_1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k] \leq 0, \\ \det A_\epsilon[1, 2, \ldots, k | \beta_1, \beta_2, \ldots, \beta_k] &= \det A[1, 2, \ldots, k | \beta_1, \beta_2, \ldots, \beta_k] \leq 0. \end{aligned}$$

If $1 \in \alpha \cap \beta$, for $k = 2, 3, \ldots, n$, we have

$$\begin{aligned} \det A_\epsilon[1, 2, \ldots, k] &= -\epsilon \det A_\epsilon[2, \ldots, k] + \det A[1, 2, \ldots, k] \\ &= -\epsilon \det A[2, \ldots, k] + \det A[1, 2, \ldots, k], \end{aligned}$$

since $\det A[1, 2, \ldots, k] < 0$ and $\det A[2, \ldots, k] < 0$, if $\epsilon_k = \dfrac{\det A[1, 2, \ldots, k]}{\det A[2, \ldots, k]}$, for all positive $\epsilon < \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ we have that $\det A_\epsilon[1, 2, \ldots, k] < 0$.

Moreover

$$\begin{aligned} &\det A_\epsilon[1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k] \\ &\quad= -\epsilon \det A_\epsilon[\alpha_2, \ldots, \alpha_k | 2, \ldots, k] + \det A[1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k] \\ &\quad= -\epsilon \det A[\alpha_2, \ldots, \alpha_k | 2, \ldots, k] + \det A[1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k]. \end{aligned}$$

If $\det A[1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k] = 0$, then $\det A[\alpha_2, \ldots, \alpha_k | 2, \ldots, k] = 0$ applying Proposition 4. Then, $\det A_\epsilon[1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k] = 0$ for all $\epsilon > 0$.

If $\det A[1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k] < 0$ and $\det A[\alpha_2, \ldots, \alpha_k | 2, \ldots, k] < 0$, provided that

$$\epsilon < \epsilon_{\alpha,k} = \frac{\det A[1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k]}{\det A[\alpha_2, \ldots, \alpha_k | 2, \ldots, k]},$$

we have $\det A_\epsilon[1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k] < 0$.

Therefore, for all $\epsilon < \min\{\epsilon_{\alpha,k},\ \alpha \subset \{2, 3, \ldots, n\},\ k = 2, 3, \ldots, n\}$ it is satisfied that

$$\det A_\epsilon[1, \alpha_2, \ldots, \alpha_k | 1, 2, \ldots, k] \leq 0.$$

Similarly, we have for all $\epsilon < \min\{\epsilon_{\beta,k},\ \beta \subset \{2, 3, \ldots, n\},\ k = 2, 3, \ldots, n\}$

$$\det A_\epsilon[1, 2, \ldots, k | 1, \beta_2, \ldots, \beta_k] \leq 0.$$

If $\epsilon_0 < \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \epsilon_{\alpha,k}, \epsilon_{\beta,k}, \alpha \subset \{2, 3, \ldots, n\}, \beta \subset \{2, 3, \ldots, n\},\ k = 2, 3, \ldots, n\}$ we have for all $\epsilon \leq \epsilon_0$ that A_ϵ is a nonsingular t.n.p. matrix. □

Proposition 7. Let C be an $n \times n$ nonsingular t.n.p. matrix with $c_{11} < 0$. Then, $A = C - c_{11}E_{11}$, is a nonsingular t.n.p. matrix with the $(1, 1)$ entry equal to zero.

Proof. Since C is a nonsingular t.n.p. matrix, for $k = 1, 2, \ldots, n$, the following inequalities hold

$$\begin{aligned} \det C[\alpha | 1, 2, \ldots, k] &\leq 0, & \forall \alpha &\in \mathcal{Q}_{k,n} \\ \det C[1, 2, \ldots, k | \beta] &\leq 0, & \forall \beta &\in \mathcal{Q}_{k,n} \\ \det C[1, 2, \ldots, k] &< 0. \end{aligned}$$

Matrix A is also a nonsingular t.n.p. matrix because it verifies,

- $\det A = \underbrace{\det C}_{<0} - \underbrace{c_{11} \det C[2, 3, \ldots, n]}_{<0} < 0 \quad \Longrightarrow \quad \det A < 0,$

- if $1 \notin \alpha \cap \beta$ we have that $\det A[\alpha|\beta] = \det C[\alpha|\beta] \leq 0$,
- if $1 \in \alpha \cap \beta$, $\det A[1, \alpha_2, \ldots, \alpha_k|1, \beta_2, \ldots, \beta_k] =$
  $\det C[1, \alpha_2, \ldots, \alpha_k|1, \beta_2, \ldots, \beta_k] - c_{11} \det C[\alpha_2, \ldots, \alpha_k|\beta_2, \ldots, \beta_k] \leq 0$.

□

As a consequence of Propositions 6 and 7 we deduced the following results. The analogous ones for nonsingular t.n.p. matrices with $a_{11} < 0$ can be found in [3].

Proposition 8. If A is an $n \times n$ nonsingular t.n.p. matrix with $a_{11} = 0$, then $\det A[1, \alpha] < 0$ for all $\alpha \subset \{2, 3, \ldots, n\}$.

Proposition 9. Let A be an $n \times n$ nonsingular t.n.p. matrix with $a_{11} = 0$. Then, $\det A[\alpha] < 0$, for all $\alpha \in \mathcal{Q}_{k,n}$, $k = 1, 2, \ldots, n$, except for $k = 1$ and $\alpha = 1$ or $\alpha = n$.

4. Another quasi-LDU factorization of nonsingular t.n.p. matrices with the $(1,1)$ entry equal to zero using the transpose

In Section 2, given the nonsingular t.n.p. matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ with $a_{11} = 0$, from the factorization $L_B D_B U_B$ of $B = PA$, where P is the permutation matrix $P = [2,\ 1,\ 3,\ \ldots,\ n]$, we have obtained the factorization $A = \tilde{L} D U$, where U is a unit upper triangular TP matrix with positive entries above the main diagonal, $D = \operatorname{diag}(-d_1,\ -d_2,\ d_3,\ \ldots,\ d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$, and $\tilde{L}$ is the block lower triangular matrix

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where the entries in the first column of $\tilde{L}_{21}$ are positive, in the second one are nonpositive, $\tilde{L}_{22}$ is unit lower triangular TP matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n},\ k = 2, 3, \ldots, n.$$

Now, consider the matrix $C = PA^T$. Since A^T is nonsingular t.n.p. matrix with $A^T(1,1) = 0$, by applying Theorem 1 to A^T we obtain a factorization $A^T = \tilde{L}_{A^T} D_{A^T} U_{A^T}$, where U_{A^T} is a unit upper triangular TP matrix with positive entries above the main diagonal, $D_{A^T} = \operatorname{diag}(-d_{A_1^T},\ -d_{A_2^T},\ d_{A_3^T},\ \ldots,\ d_{A_n^T})$ with $d_{A_i^T} > 0$, for $i = 1, 2, \ldots, n$, and $\tilde{L}_{A^T}$ is the block lower triangular matrix

$$\tilde{L}_{A^T} = \begin{bmatrix} \tilde{L}_{A_{11}^T} & O \\ \tilde{L}_{A_{21}^T} & \tilde{L}_{A_{22}^T} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{A_{11}^T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where the entries in the first column of $\tilde{L}_{A_{21}^T}$ are positive, in the second one are nonpositive, $\tilde{L}_{A_{22}^T}$ is unit lower triangular TP matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}_{A^T}[\alpha|1, 2, \ldots, k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n},\ k = 2, 3, \ldots, n.$$

From this factorization we obtain that

$$A = \left(\tilde{L}_{A^T} D_{A^T} U_{A^T}\right)^T = U_{A^T}^T D_{A^T}^T \tilde{L}_{A^T}^T = L\bar{D}\tilde{U},$$

where $L = U_{A^T}^T$, $\tilde{U} = \tilde{L}_{A^T}^T$ and $\bar{D} = D_{A^T}^T$ (note that $\bar{D} = PDP$). Then, we can give the following result.

Theorem 4. *Let A be an $n \times n$ nonsingular t.n.p. matrix with $a_{11} = 0$, $a_{nn} \leq 0$. Then, A is t.n.p. if and only if A has a factorization $L\bar{D}\tilde{U}$, where L is a unit lower triangular TP matrix with positive entries under the main diagonal, $\bar{D} = diag(-\bar{d}_1, -\bar{d}_2, \bar{d}_3, \ldots, \bar{d}_n)$ with $\bar{d}_i > 0$, for $i = 1, 2, \ldots, n$, and $\tilde{U}$ is the block upper triangular matrix*

$$\tilde{U} = \begin{bmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ O & \tilde{U}_{22} \end{bmatrix}, \quad \textit{with} \quad \tilde{U}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where the entries in the first row of $\tilde{U}_{12}$ are positive, in the second one are nonpositive, $\tilde{U}_{22}$ is unit upper triangular TP matrix with positive entries above the main diagonal, and such that

$$\det \tilde{U}[1, 2, \ldots, k|\beta] \leq 0, \quad \forall \beta \in \mathcal{Q}_{k,n}, \ k = 2, 3, \ldots, n.$$

Example 4. *The nonsingular t.n.p. matrix*

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ -15 & -15 & -15 & -15 \\ -15 & -14 & -12 & -8 \\ -30 & -27 & -20 & -3 \end{bmatrix},$$

admits the following factorizations.

$$A = \tilde{L}DU = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} -15 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A = L\bar{D}\tilde{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 15 & 1 & 0 & 0 \\ 14 & 1 & 1 & 0 \\ 27 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. Conclusions

Nonsingular t.n.p. matrices with a negative $(1,1)$ entry have been characterized in terms of the factors of their LDU factorization. This factorization provides a criteria to determine if a matrix is t.n.p. and moreover, it is a useful tool to easily obtain nonsingular t.n.p. matrices of any size. Nevertheless, when the $(1, 1)$ entry is equal to zero the LDU factorization resulting from Gauss or Neville elimination with no pivoting does not exist. In this case, we have obtained a quasi-LDU factorization which allows us to determine if a given matrix with the $(1, 1)$ entry equal to zero is t.n.p. and to construct this kind of matrices of any size.

Consequently, with this factorization we have completely characterized the nonsingular t.n.p. matrices.

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