FACTORIZATION OF ABSOLUTELY CONTINUOUS POLYNOMIALS

D. ACHOUR, E. DAHIA, P. RUEDA AND E. A. SÁNCHEZ PÉREZ[1]

Abstract. In this paper we study the ideal of dominated $(p;\sigma)$-continuous polynomials, that extend the nowadays well known ideal of p-dominated polynomials to the more general setting of the interpolated ideals of polynomials. We give the polynomial version of Pietsch's factorization Theorem for this new ideal. Although based in [11], our factorization theorem requires new techniques inspired in the theory of Banach lattices.

Introduction

The operator ideal of (p,σ)-absolutely continuous operators was introduced in 1987 in order to analyze super-reflexivity and some other properties of Banach spaces ([20]). This new ideal was created by means of a general interpolation procedure due to Jarchow and Matter ([16]), and must be understood as an ideal located in between absolutely p-summing operators and continuous operators. Matter [20] applied (p,σ)-absolutely continuous operators to obtain a description of operators factoring through super-reflexive Banach spaces. Later, several authors studied factorization properties of this new class of operators, the tensor product representation and found more applications (see for example [1, 17, 18]).

The multi-ideal of $(p;p_1,\ldots,p_m;\sigma)$-absolutely continuous multilinear operators on Banach spaces has been recently defined and characterized by Dahia et al. in [13] as a natural multilinear extension of the classical ideal of $(p;\sigma)$-absolutely continuous linear operators. This multi-ideal has many good properties and extends almost all the ones that are satisfied by the ideals of absolutely p-summing and p-dominated multilinear operators, as inclusion theorems, Pietsch domination theorems, factorization theorems and tensor product representations. On the other hand, in the last ten years a considerable effort has been made in order to increase the knowledge on the polynomials that belong to some operator ideal (see for instance [2, 3, 4, 12, 22] and the references therein). The case of the p-dominated polynomials is particularly relevant and has been intensively studied ([6, 7, 8, 9, 10]).

In this paper we introduce and study the polynomial version of $(p;p_1,\ldots,p_m;\sigma)$-absolutely continuous multilinear operators, that will be called $(p;q;\sigma)$-absolutely continuous polynomials. An especial attention is given to the particular case of dominated $(p;\sigma)$-continuous polynomials. Inspired by the factorization theorem for dominated polynomials

2000 *Mathematics Subject Classification.* Primary 46A32, Secondary 47B10.

Key words and phrases. Absolutely continuous polynomials, Pietsch domination theorem.

1 Corresponding Author. Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia. Tel. 0034963877660. easancpe@mat.upv.es.

[8], we prove a more general factorization scheme for dominated $(p;\sigma)$-continuous polynomials. The factorization in [8] is based in finding a prototype of a p-dominated polynomial with values in a linear subspace of an L_p space, endowed with a suitable norm, through which any p-dominated polynomial factors. The impossibility of keeping the L_p-norm on such a subspace obligates to consider again another suitable norm when working on a factorization diagram for dominated $(p;\sigma)$-continuous polynomials that, far from being a mere adaptation of the previous one, is built by means of convexification techniques. Although the main steps come from [11], the factorization theorem we present here requires new techniques adapted to $(p;\sigma)$-dominated polynomials. To be more precise, as in the above paper, adequate renormed subspaces of L_p spaces are constructed. However, the interpolating nature of this new class of polynomials yields to build them by means of methods that have their roots on the convexification of Banach functions spaces and the interpolation theory. We also pretend to suggest that the basic lines of the procedure shown in [11] could be actually extended, with extra work, to the broad class of ideals $\mathcal{P}$ of polynomials so that the underlying ideal $\mathcal{I}$ of linear operators is characterized by a domination theorem and that any $P \in \mathcal{P}$ can be decomposed as $P = Q \circ u$, where Q is a polynomial and u a linear operator in $\mathcal{I}$.

This paper is organized as follows. In Section 1, we recall some notation and basic facts on sequences spaces and polynomials on Banach spaces. In Section 2, we study and characterize the ideal of $(p;q;\sigma)$-absolutely continuous polynomials. In Section 3 we present a particular case: the dominated $(p;\sigma)$-continuous polynomials, where a factorization should apply. Far from being trivial, the expectations are met. First we establish a domination theorem for such operators similar to one that holds in the m-linear case, comparing also dominated $(p;\sigma)$-continuous and p-dominated polynomials and, in Section 4, we show our main result: the factorization theorem for dominated $(p;\sigma)$-continuous polynomials.

1. Definitions and general results

The definitions and notations used in the paper are, in general, standard. Let $m \in \mathbb{N}$ and $X_1, \ldots, X_m, X, Y, F, G$ be Banach spaces over $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). The space of all continuous m-linear mappings $T : X_1 \times \ldots \times X_m \to Y$ will be denoted by $\mathcal{L}(X_1, \ldots, X_m; Y)$. It becomes a Banach space with the natural norm

$$\|T\| = \sup\left\{\left\|T(x^1, \ldots, x^m)\right\| : \left\|x^j\right\| \leq 1, j = 1, \ldots, m\right\}.$$

In the case $X_1 = \ldots = X_m = X$, we will simply write $\mathcal{L}\left({}^m X; Y\right)$. As usual, $\mathcal{L}(X;Y) := \mathcal{L}({}^1 X; Y)$ is the space of bounded linear operators from X to Y.

Let $1 \leq p < \infty$. We will write $\ell_p^n\left(X\right)$ for the space of all sequences $(x_i)_{i=1}^n$ in X with the norm

$$\|(x_i)_{i=1}^n\|_p = (\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}},$$

and $\ell_{p,\omega}^n\left(X\right)$ for the space of all sequences $(x_i)_{i=1}^n$ in X with the norm

$$\|(x_i)_{i=1}^n\|_{p,\omega} = \sup_{\|\phi\|_{X^*} \leq 1} (\sum_{i=1}^n |\phi(x_i)|^p)^{\frac{1}{p}},$$

where X^* denotes the topological dual of X. The closed unit ball of X will be denoted by B_X. Let $\ell_p\left(X\right)$ be the Banach space of all absolutely p-summable sequences $(x_i)_{i=1}^\infty$ in X

with the norm

$$\|(x_i)_{i=1}^{\infty}\|_p = (\sum_{i=1}^{\infty} \|x_i\|^p)^{\frac{1}{p}}$$

We denote by $\ell_p^{\omega}(X)$ the Banach space of all weakly p-summable sequences $(x_i)_{i=1}^{\infty}$ in X with the norm

$$\|(x_i)_{i=1}^{\infty}\|_{p,\omega} = \sup_{\|\xi\|_{X^*}\leq 1} \|(\xi(x_i))_{i=1}^{\infty}\|_p \,.$$

Note that $\ell_{p,\omega}(X) = \ell_p(X)$ for some $1 \leq p < \infty$ if, and only if, X is finite dimensional.

The space $\ell^{p\sigma}(X)$ of $(p;\sigma)$-weakly summable sequences was introduced in [17] in order to give a characterization of the class of $(p;\sigma)$-absolutely continuous operators (see [17, Theorem 1.7]). Now we recall some properties of this space. Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. Define

$$\delta_{p\sigma}((x_i)_{i=1}^{\infty}) = \sup_{\phi\in B_{X^*}} \left(\sum_{i=1}^{\infty} \left(|\phi(x_i)|^{1-\sigma} \|x_i\|^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}$$

and

$$H_{p;\sigma}(X) = \{(x_i)_{i=1}^{\infty} \subset X : \delta_{p\sigma}((x_i)_{i=1}^{\infty}) < \infty\} \,.$$

It is clear that

$$\|(x_i)_{i=1}^{\infty}\|_{\frac{p}{1-\sigma},\omega} \leq \delta_{p\sigma}((x_i)_{i=1}^{\infty}) \leq \|(x_i)_{i=1}^{\infty}\|_{\frac{p}{1-\sigma}}\,, \quad (x_i)_{i=1}^{\infty} \in H_{p;\sigma}(X). \tag{1}$$

A sequence $(x_i)_{i=1}^{\infty}$ in X is $(p;\sigma)$-weakly summable if it belongs to the vector normed space $\ell^{p\sigma}(X)$ spanned by $H_{p;\sigma}(X)$ with the norm

$$\|(x_i)_{i=1}^{\infty}\|_{p;\sigma} = \inf \sum_{l=1}^{k} \delta_{p\sigma}\left((x_i^l)_{i=1}^{\infty}\right)$$

where the infimum is taken over all representations of $(x_i)_{i=1}^{\infty}$ of the form

$$(x_i)_{i=1}^{\infty} = \sum_{l=1}^{k} \left(x_i^l\right)_{i=1}^{\infty},$$

with $\left(x_i^l\right)_{i=1}^{\infty} \in H_{p;\sigma}(X), k \in \mathbb{N}$. In addition, we have the inclusions

$$\ell_{\frac{p}{1-\sigma}}(X) \subset \ell^{p\sigma}(X) \subset \ell_{\frac{p}{1-\sigma},\omega}(X)\,,$$

with

$$\|(x_i)_{i=1}^{\infty}\|_{\frac{p}{1-\sigma},\omega} \leq \|(x_i)_{i=1}^{\infty}\|_{p;\sigma} \leq \|(x_i)_{i=1}^{\infty}\|_{\frac{p}{1-\sigma}} \quad \text{for all } (x_i)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(X)\,.$$

The completion of $\ell^{p\sigma}(X)$ is denoted by $\hat{\ell}^{p\sigma}(X)$.

A map $P : X \to Y$ is an *m-homogeneous polynomial* if there exists a unique symmetric m-linear operator $\check{P} : X \times \overset{(m)}{\cdots} \times X \to Y$ such that $P(x) = \check{P}\left(x, \overset{(m)}{\ldots}, x\right)$ for every $x \in X$. Both are related by the polarization formula [23, Theorem 1.10]

$$\check{P}\left(x^1, \ldots, x^m\right) = \frac{1}{2^m m!} \sum_{\varepsilon_i=\pm 1} \varepsilon_1 \cdots \varepsilon_m P(\varepsilon_1 x^1 + \cdots + \varepsilon_m x^m), \quad (x^j)_{j=1}^m \subset X.$$

We denote by $\mathcal{P}(^mX;Y)$ the Banach space of all continuous m -homogeneous polynomials from X into Y endowed with the norm

$$\|P\| = \sup\left\{\|P(x)\| : \|x\| \leq 1\right\} = \inf\left\{C : \|P(x)\| \leq C\left\|x\right\|^m, x \in X\right\}.$$

The (finite) linear combinations of the m-homogeneous polynomials $x \to \phi(x)^m y$, where $\phi \in X^*$ and $y \in Y$, are called *polynomials of finite type*. For the general theory of homogeneous polynomials we refer to [14].

Let $\otimes_{\pi_s}^{m,s} X$ denote the m-fold symmetric tensor product of X endowed with the projective s-tensor norm π_s, and $\widehat{\otimes}_{\pi_s}^{m,s} X$ stands for its completion. We use P^L to denote the *linearization* of the polynomial $P \in \mathcal{P}(^mX;Y)$, that is, P^L is a linear operator from $\widehat{\otimes}_{\pi_s}^{m,s} X$ into Y such that $P(x) = P^L(x \otimes \cdots \otimes x)$ for every $x \in X$. The correspondence between a polynomial and its linearization establishes an isometric isomorphism between $\mathcal{P}(^mX;Y)$ and $\mathcal{L}(\widehat{\otimes}_{\pi_s}^{m,s} X;Y)$. For definitions and basic properties of symmetric tensor products, the s-tensor norm and the interplay with homogeneous polynomials we refer to [15].

In this paper we follow the standard definition of ideal of polynomials which can be found for example in [5]. For a fixed ideal of polynomials $\mathcal{Q}$ and $m \in \mathbb{N}$, the class $\mathcal{Q}^m := \cup_{E,F}\mathcal{Q}(^mE;F)$ is called ideal of m-homogeneous polynomials.

Let $m \in \mathbb{N}$ and let $X_1, \ldots, X_m$ and Y be Banach spaces. Let $1 \leq p, p_1, \ldots, p_m < \infty$ with $\frac{1}{p} \leq \frac{1}{p_1} + \ldots + \frac{1}{p_m}$ and $0 \leq \sigma < 1$. The definition of $(p;p_1,\ldots,p_m;\sigma)$-absolutely continuous multilinear operator below was firstly given in [13].

Definition 1.1. A mapping $T \in \mathcal{L}(X_1,\ldots,X_m;Y)$ is $(p;p_1,\ldots,p_m;\sigma)$-*absolutely continuous* if there is a constant $C > 0$ such that

$$\left\|\left(T\left(x_i^1,\ldots,x_i^m\right)\right)_{i=1}^n\right\|_{\frac{p}{1-\sigma}} \leq C\prod_{j=1}^m \delta_{p_j\sigma}\left((x_i^j)_{i=1}^n\right) \tag{2}$$

for all choices of $m \in \mathbb{N}$ and $x_1^j, \ldots, x_n^j \in X_j, (1 \leq j \leq m)$.

The space of all such m-linear operators is denoted by $\mathcal{L}_{as(p;p_1,\ldots,p_m)}^{\sigma}(X_1,\ldots,X_m;Y)$ and is endowed with the norm given by $\|T\|_{\mathcal{L}_{as(p;p_1,\ldots,p_m)}^{\sigma}} = \inf\left\{C > 0 : C \text{ satisfies (2)}\right\}.$

With this notation, $(\mathcal{L}_{as(p;p_1,\ldots,p_m)}^{\sigma}, \|.\|_{\mathcal{L}_{as(p;p_1,\ldots,p_m)}^{\sigma}})$ is a Banach multi-ideal that generalizes the corresponding ideal of linear operators. Indeed, when $m = 1$ and $p_1 = p$, $(\mathcal{L}_{as(p;p)}^{\sigma}, \|.\|_{\mathcal{L}_{as(p;p)}^{\sigma}})$ coincides with the ideal of all $(p;\sigma)$-absolutely continuous linear operators. This multi-ideal has to be thought as an intermediate ideal in between the ideal of all absolutely $(p;p_1,\ldots,p_m)$-summing m-linear operators and the whole class of continuous m-linear mappings. Indeed, both classes are attained for $\sigma = 0$ and $\sigma = 1$ respectively.

In the case that $p_1 = \ldots = p_m = q$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m} = \frac{m}{q}$ we say that T is (q,σ)-*dominated continuous* and we denote the corresponding vector space and norm by $\mathcal{L}_{d,q}^{\sigma}(X_1,...,X_m;Y)$ and $\|.\|_{\mathcal{L}_{d,q}^{\sigma}}$ respectively. In this case, the inequality (2) can be written as

$$\left\|\left(T\left(x_i^1,\ldots,x_i^m\right)\right)_{i=1}^n\right\|_{\frac{q}{m(1-\sigma)}} \leq C\prod_{j=1}^m \delta_{q\sigma}\left((x_i^j)_{i=1}^n\right). \tag{3}$$

2. $(p;q;\sigma)$-Absolutely continuous m-homogeneous polynomials

In this section we define and characterize the notion of $(p;q;\sigma)$-absolutely continuous m-homogeneous polynomials, according to the definition of $(p;p_1,\dots,p_m;\sigma)$-absolutely continuous multilinear operators. We start by presenting the following result which characterizes $(p;p_1,\dots,p_m;\sigma)$-absolutely continuous m-linear operators as those which take adequate $(p;\sigma)$-weakly summable sequences into adequate $\frac{p}{1-\sigma}$-summable sequences as expected. This result will be useful to relate $(p;p_1,\dots,p_m;\sigma)$-absolutely continuous multilinear operators and the corresponding homogeneous polynomials.

Definition 2.1. Let $m\in\mathbb{N}$, $1\le p,q<+\infty$ such that $mp\ge q$ and $0\le\sigma<1$. A polynomial $P\in\mathcal{P}(^mX;Y)$ is called $(p;q;\sigma)$-*absolutely continuous* if there exists a constant $C>0$ such that for every $(x_i)_{i=1}^n\subset X$,

$$\|(P(x_i))_{i=1}^n\|_{\frac{p}{1-\sigma}}\le C.\left(\delta_{q\sigma}((x_i)_{i=1}^n)\right)^m. \tag{4}$$

The space of all such polynomials is denoted by $\mathcal{P}^{\sigma}_{as(p,q)}(^mX;Y)$. It is equipped with the complete norm $\|.\|_{\mathcal{P}^{\sigma}_{as(p,q)}}$, which is computed as the infimum of all constants C such that the inequality (4) holds.

For $\sigma=0$ we have $\mathcal{P}^{0}_{as(p,q)}(^mX;Y)=\mathcal{P}_{p,q}(^mX;Y)$, the space of absolutely $(p;q)$-summing polynomials (see [19]).

Remark 2.2. Let us show some basic ways of constructing polynomials belonging to our new class. Let X, Y and Z be Banach spaces and let $m\in\mathbb{N}$, $1\le p,q<+\infty$ *such that* $mp\ge q$ *and* $0\le\sigma<1$.
(a) *Every m-homogeneous polynomial of finite type from X into Y is $(p;q;\sigma)$-absolutely continuous.* A simple calculation shows this result.
(b) *Let us show a particular example of the case mentioned above. Let $P=id_{\mathbb{K}^m}$ be the polynomial $id_{\mathbb{K}^m}:\mathbb{K}\to\mathbb{K}$ given by $id_{\mathbb{K}^m}(x)=x^m$. The following calculations show that it is $(p;q;\sigma)$-absolutely continuous and that $\|id_{\mathbb{K}^m}\|_{\mathcal{P}^{\sigma}_{as(p,q)}}=1$.* Let $(x_i)_{i=1}^n\subset\mathbb{K}$. By the inequality (1) we can write

$$\begin{aligned}\|(id_{\mathbb{K}^m}(x_i))_{i=1}^n\|_{\frac{p}{1-\sigma}} &= \|(x_i)_{i=1}^n\|^m_{\frac{mp}{1-\sigma}}\\ &\le \|(x_i)_{i=1}^n\|^m_{\frac{q}{1-\sigma},\omega}\\ &\le \left(\delta_{q\sigma}\left((x_i)_{i=1}^n\right)\right)^m.\end{aligned}$$

It follows that $id_{\mathbb{K}^m}\in\mathcal{P}^{\sigma}_{as(p,q)}(\mathbb{K};\mathbb{K})$ and $\|id_{\mathbb{K}^m}\|_{\mathcal{P}^{\sigma}_{as(p,q)}}\le 1$. In fact, it can be easily shown that $\|id_{\mathbb{K}^m}\|_{\mathcal{P}^{\sigma}_{as(p,q)}}\ge\|id_{\mathbb{K}^m}\|=1$.
(c) *Let $Q\in\mathcal{P}(^mX;Y)$ and let $u:Z\to X$ be a $(p;\sigma)$-absolutely continuous linear operator. Then the polynomial $P=Q\circ u$ is $(\frac{p}{m};p;\sigma)$-absolutely continuous and*

$$\|P\|_{\mathcal{P}^{\sigma}_{\frac{p}{m},p}}\le\|Q\|\left(\pi_{p;\sigma}(u)\right)^m.$$

In order to see this, note that if $(z_i)_{i=1}^n\subset Z$, then

$$\|(P(z_i))_{i=1}^n\|_{\frac{p}{m(1-\sigma)}}\le\|Q\|.\|u(z_i)_{i=1}^n\|^m_{\frac{p}{1-\sigma}}\le\|Q\|.\|u\|^m_{\mathcal{L}^{\sigma}_{as(p;p)}}.\left(\delta_{p\sigma}((z_i)_{i=1}^n)\right)^m.$$

As a consequence of parts (a) and (b) of the remark above and the next result — which proof is straightforward using calculation as in Remark 2.2(c)— we obtain that $\left(\mathcal{P}^{\sigma}_{as(p,q)}, \|.\|_{\mathcal{P}^{\sigma}_{as(p,q)}}\right)$ is a normed polynomial ideal.

Proposition 2.3. *(Ideal property). Let $u \in \mathcal{L}(X,G)$ and $v \in \mathcal{L}(F,Y)$. If $P \in \mathcal{P}(^mG,F)$ is $(p;q;\sigma)$-absolutely continuous, then $v \circ P \circ u$ is $(p;q;\sigma)$-absolutely continuous and*

$$\|v \circ P \circ u\|_{\mathcal{P}^{\sigma}_{as(p,q)}} \leq \|v\| \, \|P\|_{\mathcal{P}^{\sigma}_{as(p,q)}} \, \|u\|^m .$$

Although $(p;q;\sigma)$-absolutely continuous polynomials have been introduced independently of $(p;p_1,\ldots,p_m)$-absolutely continuous multilinear mappings, in order to relate both classes we characterize first these classes of non linear operators by means of their summability properties.

As in the classical cases, the natural way of presenting the summability properties of our m-linear operators is by defining the corresponding operator between adequate sequence spaces. An m-linear operator $T \in \mathcal{L}(X_1,\ldots,X_m;Y)$ induces an m-linear operator $\widehat{T}$ mapping $\hat{\ell}^{p_1\sigma}(X_1) \times \ldots \times \hat{\ell}^{p_m\sigma}(X_m)$ into $Y^{\mathbb{N}}$ that is given by

$$\widehat{T}\left((x_i^1)_{i=1}^{\infty},\ldots,(x_i^m)_{i=1}^{\infty}\right) = (T(x_i^1,\ldots,x_i^m))_{i=1}^{\infty}.$$

Proposition 2.4. *For $T \in \mathcal{L}(X_1,\ldots,X_m;Y)$ the following conditions are equivalent:*

(a) *T is $(p;p_1,\ldots,p_m;\sigma)$ -absolutely continuous.*
(b) *If $(x_i^j)_{i=1}^{\infty} \in \ell^{p_j\sigma}(X_j)$, for $j=1,\ldots,m$, then $(T(x_i^1,\ldots,x_i^m))_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(Y)$.*
(c) *The mapping $\widehat{T} : \hat{\ell}^{p_1\sigma}(X_1) \times \ldots \times \hat{\ell}^{p_m\sigma}(X_m) \to \ell_{\frac{p}{1-\sigma}}(Y)$ is well-defined and continuous.*

In this case $\|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}} = \|\widehat{T}\|$.

Proof. It is clear that (c) implies (b) and that (c) implies (a) with $\|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}} \leq \|\widehat{T}\|$. Assume that $T \in \mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}(X_1,\ldots,X_m;Y)$. Note first that if $(x_i^j)_{i=1}^{\infty} \in H_{p_j,\sigma}(X_j)$, $j=1,\ldots,m$ we have

$$\begin{aligned}
\left\|\left(T\left(x_i^1,\ldots,x_i^m\right)\right)_{i=1}^n\right\|_{\frac{p}{1-\sigma}} &\leq \|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}} \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \\
&\leq \|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}} \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^{\infty}).
\end{aligned}$$

for all $n \in \mathbb{N}$. Then $\left\|\left(T\left(x_i^1,\ldots,x_i^m\right)\right)_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma}} \leq \|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}} \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^{\infty})$.

Now let $(x_i^j)_{i=1}^{\infty} \in \ell^{p_j\sigma}(X_j)$, $j=1,\ldots,m$. For each $j=1,\ldots,m$ and $\varepsilon > 0$, there exists $(x_i^{j,l})_{i=1}^{\infty} \in H_{p_j,\sigma}(X_j)$ such that

$$(x_i^j)_{i=1}^{\infty} = \sum_{l=1}^{k_j} (x_i^{j,l})_{i=1}^{\infty} \quad \text{and} \quad \sum_{l=1}^{k_j} \delta_{p_j\sigma}\left((x_i^{j,l})_{i=1}^{\infty}\right) \leq \varepsilon + \left\|(x_i^j)_{i=1}^{\infty}\right\|_{p_j,\sigma}.$$

So we have

$$\begin{aligned}\left\|\left(T\left(x_i^1,\ldots,x_i^m\right)\right)_{i=1}^\infty\right\|_{\frac{p}{1-\sigma}} &\le \sum_{l_1=1}^{k_1}\cdots\sum_{l_m=1}^{k_m}\left\|\left(T\left(x_i^{1,l_1},\ldots,x_i^{m,l_m}\right)\right)_{i=1}^\infty\right\|_{\frac{p}{1-\sigma}}\\ &\le \|T\|_{\mathcal{L}^\sigma_{as(p;p_1,\ldots,p_m)}}\left(\sum_{l_1=1}^{k_1}\delta_{p_1\sigma}\left((x_i^{1,l_1})_{i=1}^\infty\right)\right)\cdots\left(\sum_{l_m=1}^{k_m}\delta_{p_m\sigma}\left((x_i^{m,l_m})_{i=1}^\infty\right)\right)\\ &\le \|T\|_{\mathcal{L}^\sigma_{as(p;p_1,\ldots,p_m)}}\left(\varepsilon+\left\|(x_i^1)_{i=1}^\infty\right\|_{p_1,\sigma}\right)\cdots\left(\varepsilon+\|(x_i^m)_{i=1}^\infty\|_{p_m,\sigma}\right).\end{aligned}$$

Since this holds for all $\varepsilon>0$, we obtain

$$\left\|\left(T\left(x_i^1,\ldots,x_i^m\right)\right)_{i=1}^\infty\right\|_{\frac{p}{1-\sigma}}\le\|T\|_{\mathcal{L}^\sigma_{as(p;p_1,\ldots,p_m)}}\prod_{j=1}^m\left\|(x_i^j)_{i=1}^\infty\right\|_{p_j,\sigma}.$$

Then it follows that $T:\ell^{p_1\sigma}(X_1)\times\ldots\times\ell^{p_m\sigma}(X_m)\to\ell_{\frac{p}{1-\sigma}}(Y)$ is well-defined and continuous with norm $\le\|T\|_{\mathcal{L}^\sigma_{as(p;p_1,\ldots,p_m)}}$. Its continuous extension to $\hat{\ell}^{p_1\sigma}(X_1)\times\ldots\times\hat{\ell}^{p_m\sigma}(X_m)$ coincides with the mapping $\widehat{T}:\hat{\ell}^{p_1\sigma}(X_1)\times\ldots\times\hat{\ell}^{p_m\sigma}(X_m)\to\ell_{\frac{p}{1-\sigma}}(Y)$ already defined and we see that (a) implies (b) and (a) implies (c) with $\|\widehat{T}\|\le\|T\|_{\mathcal{L}^\sigma_{as(p;p_1,\ldots,p_m)}}$.
An appeal to the Closed Graph Theorem shows that (b) implies (c). Actually the Closed Graph Theorem is used to show that $\widehat{T}$ is separately continuous, hence continuous (see [17, Theorem 1.7 (2⇒1)]). □

Adapting the proof of Proposition 2.4, we easily get the characterization of $(p;q;\sigma)$-absolutely continuous polynomials by means of transformations of vector valued sequence spaces.

Proposition 2.5. *Let $m\in\mathbb{N}$, $1\le p,q<+\infty$ such that $mp\ge q$ and $0\le\sigma<1$. Let $P\in\mathcal{P}(^mX;Y)$. Then the polynomial P is $(p;q;\sigma)$-absolutely continuous if and only if $(P(x_i))_{i=1}^\infty\in\ell_{\frac{p}{1-\sigma}}(Y)$ for every $(x_i)_{i=1}^\infty\in\ell^{q\sigma}(X)$.*

The above characterization allows to relate the properties of the $(p;q;\sigma)$-absolutely continuous polynomials with the ones of their corresponding symmetric multilinear maps.

Corollary 2.6. *Let $P\in\mathcal{P}(^mX;Y)$. Then $P\in\mathcal{P}^\sigma_{as(p,q)}(^mX;Y)$ if and only if $\check{P}\in\mathcal{L}^\sigma_{as(p;q,\ldots,q)}(^mX;Y)$.*

Proof. Assume that $\check{P}\in\mathcal{L}^\sigma_{as(p;q,\ldots,q)}(^mX;Y)$. For each sequence $(x_i)_{i=1}^n$ in X we have

$$\|(P(x_i))_{i=1}^n\|_{\frac{p}{1-\sigma}}=\left\|\left(\check{P}(x_i,\overset{(m)}{\ldots},x_i\right)_{i=1}^n\right\|_{\frac{p}{1-\sigma}}\le\|\check{P}\|_{\mathcal{L}^\sigma_{as(p;q,\ldots,q)}}.[\delta_{q\sigma}((x_i)_{i=1}^n)]^m.$$

It follows that $P\in\mathcal{P}^\sigma_{as(p,q)}(^mX;Y)$. Conversely, let $P\in\mathcal{P}^\sigma_{as(p,q)}(^mX;Y)$ and $(x_i^j)_{i=1}^\infty\in\ell^{q\sigma}(X), j=1,\ldots,m$. By the polarization formula we have

$$\left(\check{P}\left(x_i^1,\overset{(m)}{\ldots},x_i^m\right)\right)_{i=1}^\infty=\frac{1}{2^m m!}\sum_{\varepsilon_j=\pm1}\varepsilon_1\ldots\varepsilon_m\left(P(\varepsilon_1x_i^1+\ldots+\varepsilon_mx_i^m)\right)_{i=1}^\infty. \tag{5}$$

Now, for every choice $\varepsilon_1, \ldots, \varepsilon_m = \pm 1$ of signs we get $\left(\varepsilon_1 x_i^1 + \ldots + \varepsilon_m x_i^m\right)_{i=1}^{\infty} \in \ell^{q\sigma}(X)$. As in the proof of Proposition 2.4, (a)⇒(b), we obtain

$$\left\| \left(P(\varepsilon_1 x_i^1 + \ldots + \varepsilon_m x_i^m)\right)_{i=1}^{\infty} \right\|_{\frac{p}{1-\sigma}} \leq \|P\|_{\mathcal{P}^{\sigma}_{as(p,q)}} . \left\| (\varepsilon_1 x_i^1 + \ldots + \varepsilon_m x_i^m)_{i=1}^{\infty} \right\|_{q,\sigma}^m < \infty.$$

This implies $\left(P(\varepsilon_1 x_i^1 + \ldots + \varepsilon_m x_i^m)\right)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(Y)$. It follows from (5) that

$$\left(\check{P} \left(x_i^1, \overset{(m)}{\ldots}, x_i^m \right) \right)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(Y),$$

and by Proposition 2.4 this shows that $\check{P} \in \mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}(^m X; Y)$. □

Let us see that this large class of summing polynomials also satisfies an inclusion theorem.

Proposition 2.7. *(Inclusion theorem). Let $1 \leq p \leq q < \infty$ and $1 \leq p_1 \leq q_1 < \infty$ be such that $\frac{m}{p_1} - \frac{1}{p} \leq \frac{m}{q_1} - \frac{1}{q}$. Then $\mathcal{P}^{\sigma}_{p,p_1}(^m X; Y) \subset \mathcal{P}^{\sigma}_{q,q_1}(^m X; Y)$. Moreover, we have $\|.\|_{\mathcal{P}^{\sigma}_{q,q_1}} \leq \|.\|_{\mathcal{P}^{\sigma}_{p,p_1}}$.*

Proof. By the monotonicity of the ℓ_s-norms we may assume that $\frac{m}{p_1} - \frac{1}{p} = \frac{m}{q_1} - \frac{1}{q}$. Considering $1 \leq r, r_1 < \infty$ with $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$, $\frac{1}{r_1} + \frac{1}{q_1} = \frac{1}{p_1}$ it follows that $\frac{m}{r_1} = \frac{1}{r}$. Take P in $\mathcal{P}^{\sigma}_{p,p_1}(^m X; Y)$ and a sequence $(x_i)_{i=1}^n$ in X. Observe that for $\lambda_i = \|P(x_i)\|^{\frac{q}{r_1}}$, $(i = 1, \ldots, n)$ we have

$$\|P(\lambda_i x_i)\|^{\frac{p}{1-\sigma}} = \|P(x_i)\|^{\frac{q}{1-\sigma}}.$$

Using Hölder's inequality we obtain

$$\begin{aligned}
\left(\sum_{i=1}^n \|P(x_i)\|^{\frac{q}{(1-\sigma)}} \right)^{\frac{1-\sigma}{p}} &= \left(\sum_{i=1}^n \|P(\lambda_i x_i)\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\leq \|P\|_{\mathcal{P}^{\sigma}_{p,p_1}} \left[\sup_{\phi \in B_{X^*}} \left(\sum_{i=1}^n \left(\lambda_i |\phi(x_i)|^{1-\sigma} \|x_i\|^{\sigma} \right)^{\frac{p_1}{1-\sigma}} \right)^{\frac{1-\sigma}{p_1}} \right]^m \\
&\leq \|P\|_{\mathcal{P}^{\sigma}_{p,p_1}} \left(\sum_{i=1}^n |\lambda_i|^{\frac{r_1}{1-\sigma}} \right)^{\frac{m(1-\sigma)}{r_1}} . (\delta_{q_1 \sigma}((x_i)_{i=1}^n)^m \\
&= \|P\|_{\mathcal{P}^{\sigma}_{p,p_1}} \left(\sum_{i=1}^n \|P(x_i)\|^{\frac{q}{1-\sigma}} \right)^{\frac{1-\sigma}{r}} . (\delta_{q_1 \sigma}((x_i)_{i=1}^n)^m .
\end{aligned}$$

Since $\frac{1-\sigma}{p} - \frac{1-\sigma}{r} = \frac{1-\sigma}{q}$, we obtain the following inequality, which proves the result.

$$\|(P(x_i))_{i=1}^n\|_{\frac{q}{1-\sigma}} \leq \|P\|_{\mathcal{P}^{\sigma}_{p,p_1}} . [\delta_{q_1 \sigma}((x_i)_{i=1}^n]^m .$$

□

3. Dominated (p,σ)-Continuous Polynomials

A relevant special case of $(p;q;\sigma)$-absolutely continuous polynomial is when we have $mp = q$. In this situation —and following the standard notations in similar cases—, we will call the mappings dominated $(p;\sigma)$-continuous polynomials, and we will denote the corresponding vector space and norm by $\mathcal{P}^{\sigma}_{d,p}(^mX;Y)$ and $\|.\|_{\mathcal{P}^{\sigma}_{d,p}}$, respectively, for $p \geq m$. Actually, we have $\mathcal{P}^{\sigma}_{d,p}(^mX;Y) = \mathcal{P}^{\sigma}_{\frac{p}{m},p}(^mX;Y)$, i.e. a polynomial $P \in \mathcal{P}(^mX;Y)$ is *dominated* $(p;\sigma)$*-continuous* if there is a constant $C > 0$ such that for every $(x_i)_{i=1}^n \subset X$ we have

$$\|P(x_i)_{i=1}^n\|_{\frac{p}{m(1-\sigma)}} \leq C. \left[\delta_{p\sigma}\left((x_i)_{i=1}^n\right)\right]^m. \tag{6}$$

Notice that for $m = 1$ we recover also the ideal of (p,σ)-absolutely continuous linear operators. When $\sigma = 0$, $\mathcal{P}^0_{d,p}(^mX;Y)$ is the space of all p-dominated m-homogeneous polynomials, which is denoted simply by $\mathcal{P}_{d,p}(^mX;Y)$. The definition and some fundamental results on p-dominated homogeneous polynomials between Banach spaces can be found in [5], [19] or [22].

Remark 3.1. From Corollary 2.6, [13, Theorem 3.6] and [5, Proposition 9] it follows that the decomposition in part (c) of Remark 2.2 actually characterizes dominated $(p;\sigma)$-continuous polynomials. Indeed, $P \in \mathcal{P}^{\sigma}_{d,p}(^mX;Y)$ if and only if there is a Banach space Z, a $(p;\sigma)$-absolutely continuous linear operator $u : X \to Z$ and a polynomial $Q \in \mathcal{P}(^mZ;Y)$ such that $P = Q \circ u$. This factorization will be used several times and is essential for our purposes of getting a Pietsch type factorization theorem for dominated $(p;\sigma)$-continuous polynomials.

It is well known that $(\mathcal{P}_{d,p}, \|.\|_{d,p})$ is a Banach ideal of polynomials if $p \geq m$. Although a domination theorem for dominated polynomials follows easily as in the linear case, to get the corresponding factorization theorem requires new techniques that use mainly symmetric tensor products and an adequate representation of these spaces. This has been done in [8] and [11], where it is shown that any p-dominated polynomial factors through a canonical prototype of a p-dominated polynomial in the spirit of Pietsch's classical result. Our aim is to obtain a domination/factorization result for the larger class of dominated $(p;\sigma)$-continuous polynomials. We will see that new constructions of renormed subspaces of L_p spaces different from the ones used in [8] and [11] are required.

Theorem 3.2. *Let $m \in \mathbb{N}$ and $1 \leq p < \infty$. An m-homogeneous polynomial $P \in \mathcal{P}(^mX;Y)$ is dominated $(p;\sigma)$-continuous if and only if there is a regular Borel probability measure μ on B_{X^*} (with the weak* topology) and a constant $C > 0$ such that for all $x \in X$*

$$\|P(x)\| \leq C\,\|x\|^{m\sigma} \left(\int_{B_{X^*}} |\phi(x)|^p \, d\mu(\phi)\right)^{\frac{m(1-\sigma)}{p}}. \tag{7}$$

Moreover, in this case $\|P\|_{\mathcal{P}^{\sigma}_{d,p}} = \inf\{C > 0 : C \text{ satisfies (7)}\}$.

Proof. First let us assume that (7) holds with C and μ as described. A direct calculation shows easily that

$$\|(P(x_i))_{i=1}^n\|_{\frac{p}{m(1-\sigma)}} \leq C. \left[\delta_{p\sigma}\left((x_i)_{i=1}^n\right)\right]^m$$

for every $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subset X$. Hence $P \in \mathcal{P}_{d,p}^{\sigma}(^m X;Y)$ and $\|P\|_{\mathcal{P}_{d,p}^{\sigma}} \le C$.
For the converse, take $P \in \mathcal{P}_{d,p}^{\sigma}(^m X;Y)$. By Remark 3.1 there is a Banach space Z, a $(p;\sigma)$-absolutely continuous linear operator $u : X \to Z$ and a polynomial $Q \in \mathcal{P}(^m Z;Y)$ such that $P = Q \circ u$. The domination theorem for $(p;\sigma)$-absolutely continuous linear operators [20] ensures that there exists a regular Borel probability measure μ on B_{X^*} and a constant $C > 0$ such that

$$\|u(x)\| \le C\|x\|^{\sigma} \left(\int_{B_{X^*}} |\phi(x)|^p d\,\mu \right)^{\frac{1-\sigma}{p}}$$

for all $x \in X$. Then

$$\|P(x)\| = \|Q \circ u(x)\| \le \|Q\|\|u(x)\|^m \le \|Q\|C^m\|x\|^{m\sigma} \left(\int_{B_{X^*}} |\phi(x)|^p d\,\mu \right)^{\frac{m(1-\sigma)}{p}}$$

for all $x \in X$. □

Any regular Borel probability measure μ on B_{X^*}, with the weak* topology that satisfies (7) is called a *Pietsch measure* for P.

An inclusion between the classes of the dominated $(p;\sigma)$-continuous polynomials and the p-dominated polynomials follows easily from the definitions.

Proposition 3.3. *Let $1 \le p < \infty$ and $0 \le \sigma < 1$. Then $\mathcal{P}_{d,\frac{p}{(1-\sigma)}}(^m X;Y) \subset \mathcal{P}_{d,p}^{\sigma}(^m X;Y)$. Consequently, $\mathcal{P}_{d,p}(^m X;Y) \subset \mathcal{P}_{d,p}^{\sigma}(^m X;Y)$.*

Proof. Let $P \in \mathcal{P}_{d,\frac{p}{(1-\sigma)}}(^m X;Y)$. Let $(x_i)_{i=1}^n$ be a sequence in X. Using inequality (1) we have

$$\left(\sum_{i=1}^n \|P(x_i)\|^{\frac{p}{m(1-\sigma)}} \right)^{\frac{m(1-\sigma)}{p}} \le \|P\|_{d,\frac{p}{(1-\sigma)}} \left[\|(x_i)_{i=1}^n\|_{\frac{p}{1-\sigma},\omega} \right]^m \le \|P\|_{d,\frac{p}{(1-\sigma)}} \left[\delta_{p\sigma}((x_i)_{i=1}^n \right]^m .$$

Then $P \in \mathcal{P}_{d,p}^{\sigma}(^m X;Y)$ and $\|P\|_{\mathcal{P}_{d,p}^{\sigma}} \le \|P\|_{d,\frac{p}{(1-\sigma)}}$. Hence $\mathcal{P}_{d,\frac{p}{(1-\sigma)}}(^m X;Y) \subset \mathcal{P}_{d,p}^{\sigma}(^m X;Y)$. Since $p \le \frac{p}{1-\sigma}$ it follows that $\mathcal{P}_{d,p}(^m X;Y) \subset \mathcal{P}_{d,\frac{p}{(1-\sigma)}}(^m X;Y)$ (see [22]). Hence the inclusion $\mathcal{P}_{d,p}(^m X;Y) \subset \mathcal{P}_{d,p}^{\sigma}(^m X;Y)$ is proved. □

By [4, Example 1] there is a m-dominated polynomial $P \in \mathcal{P}(^m X;Y)$, $m \ge 2$, which is not weakly compact. Then Proposition 3.3 gives the existence of a dominated (m,σ)-continuous polynomial which is not weakly compact.

Remark 3.4. In general, $\mathcal{P}_{d,p}^{\sigma} \ne \mathcal{P}_{d,\frac{p}{(1-\sigma)}}$. Let us show an example of a polynomial belonging to $\mathcal{P}_{d,p}^{\sigma}$ that is not in $\mathcal{P}_{d,\frac{p}{(1-\sigma)}}$. Let $L^1 := L^1[0,1]$ or $L^1 := \ell^1$, and L^2 the corresponding Hilbert space. We know by [13, Example 3.8] that there is a *symmetric* bilinear operator $T : L^2 \times L^2 \to L^1$ such that

$$T \in \mathcal{L}_{as(1;2,2)}^{\sigma}(^2 L^2; L^1) \quad \text{but} \quad T \notin \mathcal{L}_{as(\frac{1}{(1-\sigma)};\frac{2}{(1-\sigma)},\frac{2}{(1-\sigma)})}(^2 L^2; L^1).$$

Then, by Corollary 2.6 and [22, Theorem 6], the polynomial $\hat{T} \in \mathcal{P}(^2 L^2; L^1)$ associated to T satisfies that $\hat{T} \in \mathcal{P}_{d,2}^{\sigma}(^2 L^2; L^1)$, but $\hat{T} \notin \mathcal{P}_{d,\frac{2}{1-\sigma}}(^2 L^2; L^1)$.

4. The factorization theorem

As in the case of the p-dominated multilinear operators, we will show that there is a factorization theorem that characterizes when a polynomial is dominated $(p;\sigma)$-continuous. In fact, this theorem presents the prototype of dominated $(p;\sigma)$-continuous polynomial, i.e. the polynomial belonging to this class through which each polynomial of the class factors. The ideas for proving the factorization follows the lines of the one that are used in [11] but there are some meaningful differences based on the fact that the domination for the dominated $(p;\sigma)$-continuous polynomials is not based in a norm but in some interpolated expression between the norm of X and the one of $L^p(\mu)$. To deal with, we use techniques inspired on the convexification of Banach lattices.

If X is a Banach space and $m \in \mathbb{N}$, we define the m-homogeneous polynomial $\Delta\colon X \longrightarrow C(B_{X^*})$; $\Delta(x)(\varphi) = \varphi(x)^m$. We consider the restriction δ of its linearization to the m-fold symmetric tensor product $\otimes_{\pi_s}^{m,s} X$. So defined, δ is a linear operator $\delta\colon \otimes_{\pi_s}^{m,s} X \longrightarrow C(B_{X^*})$ given by $\delta(x \otimes \cdots \otimes x)(\varphi) = \varphi(x)^m$, $x \in X$, $\varphi \in B_{X^*}$. By Lemma 4.1. in [8], this map is injective. To simplify the notation, sometimes we shall write $\otimes_m x := x \otimes \cdots \otimes x$. Let δ_m stand for the canonical m-homogeneous polynomial from X to $\otimes_{\pi_s}^{m,s} X$ defined by $\delta_m(x) = \otimes_m x$. Let $i_X : X \to C(B_{X^*})$ be the canonical isometric inclusion given by the evaluation. Given μ a Borel measure on B_{X^*}, $j_p : C(B_{X^*}) \to L_p(\mu)$ denotes the canonical map. On $j_p \circ i_X(X)$ consider the seminorm

$$\|j_p \circ i_X(x)\| := \inf \left\{ \sum_{j=1}^{n} \|x_j\|^{\sigma} \|j_p \circ i_X(x_j)\|_{L_p(\mu)}^{1-\sigma} : x = \sum_{j=1}^{n} x_j, x_j \in X,\ n \in \mathbb{N} \right\}$$

Consider the relation $j_p \circ i_X(x) \equiv j_p \circ i_X(y)$ if and only if $\|i_X(x-y)\| = 0$ and denote by $L_{p,\sigma}(\mu)$ the quotient space and by $j_{p,\sigma} : i_X(X) \to L_{p,\sigma}(\mu)$ the quotient map. Then $\|\cdot\|$ becomes a norm on $L_{p,\sigma}(\mu)$ that we shall call $\|\cdot\|_{L_{p,\sigma}}$. Its completion can be identified with the real interpolation space $(X, L_p(\mu))_{1-\sigma,1}$ (see [13, Section 3] for more details on this space).

Following a general construction that is well-known for the case of Banach function spaces (see for example [24, Ch.2]), we can define what we call the m-th power $L_{p,\sigma}(\mu)_{[m]}$ of $L_{p,\sigma}(\mu)$. However, notice that in this case this new space is not a Banach function space. It is a linear space that is defined as the linear span of all polynomials of the form $(j_{\frac{p}{m}} \circ i_X(x))^m$ for x being an element of X, i.e.

$$\begin{aligned} L_{p,\sigma}(\mu)_{[m]} &:= \left\{ h \in L_{\frac{p}{m}}(\mu) : h = \sum_{j=1}^{n} \lambda_j j_{\frac{p}{m}} \circ \Delta(x_j), x_j \in X, \lambda_j \in \mathbb{K}, n \in \mathbb{N} \right\} \\ &= \left\{ h \in L_{\frac{p}{m}}(\mu) : h = \sum_{j=1}^{n} \lambda_j j_{\frac{p}{m}} (i_X(x_j)^m), x_j \in X, \lambda_j \in \mathbb{K}, n \in \mathbb{N} \right\}. \end{aligned}$$

Consider the m-homogeneous polynomial $Q : j_{p,\sigma} \circ i_X(X) \to L_{p,\sigma}(\mu)_{[m]}$ given by

$$Q(j_{p,\sigma} \circ i_X(x)) := j_{\frac{p}{m}} \circ \Delta(x),$$

and let Q^L be its linearization. Given $h = \sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(x_j) \in L_{p,\sigma}(\mu)_{[m]}$ we will denote by θ the tensor $\theta := \sum_{j=1}^n \lambda_j \otimes_m x_j$. Let $T\colon \otimes^{n,s} E \longrightarrow \otimes^{n,s} j_p \circ i_X(E)$ be the linear operator given by $T(\otimes_m x) = \otimes_m j_p(i_X(x))$ for every $x \in E$. For each $h \in L_{p,\sigma}(\mu)_{[m]}$ define

$$\pi_{p;\sigma,m}(h) \quad := \quad \inf\left\{\sum_{j=1}^n |\lambda_j| \|j_{p,\sigma} \circ i_X(x_j)\|^m_{L_{p,\sigma}} : T(\theta) = \sum_{i=1}^n \lambda_i \otimes_m j_p \circ i_X(x_i)\right\}$$

$$= \inf\left\{\sum_{i=1}^n |\lambda_i| \cdot \left(\sum_{k=1}^{n_i} \|x_k^i\|^\sigma \|j_p \circ i_X(x_k^i)\|^{1-\sigma}_{L_p}\right)^m : T(\theta) = \sum_{i=1}^n \lambda_i \otimes_m (j_p \circ i_X(\sum_{k=1}^{n_i} x_k^i))\right\}$$

The equalities

$$Q^L \circ T(\theta) = j_{\frac{p}{m}} \circ \delta(\theta) = j_{\frac{p}{m}} \circ \delta\left(\sum_{j=1}^n \lambda_j \otimes_m (\sum_{k=1}^{n_j} x_k^j)\right) = \sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(\sum_{k=1}^{n_j} x_k^j) = h$$

yield that $\pi_{p;\sigma,m}(h)$ is well defined.

Proposition 4.1. *If $p \geq m$ then $\pi_{p;\sigma,m}$ is a norm on $L_{p,\sigma}(\mu)_{[m]}$.*

Proof. Easy calculations prove that $\pi_{p;\sigma,m}$ satisfies the axioms of norm. The only one that requires some attention is that $\pi_{p;\sigma,m}(h) = 0$ implies $h = 0$. Following [11], on the space $X^{\frac{p}{m}} := j^m_{\frac{p}{m}} \circ i_X(X) \subseteq L_{\frac{p}{m}}(\mu)$ a norm is defined by

$$\pi_{\frac{p}{m}}((j_{\frac{p}{m}} \circ \delta)(\theta)) := \inf\left\{\sum_{i=1}^n |\lambda_i| \cdot \|(j_{\frac{p}{m}} \circ \delta) \otimes_m x_i)\|_{L_{\frac{p}{m}}}\right\},$$

where the infimum is taken over all representations of $T(\theta) \in \otimes^{m,s}_{\pi_s} j_p \circ i_X(X)$ of the form $T(\theta) = \sum_{i=1}^n \lambda_i \otimes_m (j_p \circ i_X(x_i))$ with $n \in \mathbb{N}$, $\lambda_i \in \mathbb{K}$ and $x_i \in X$. We have that

$$\begin{aligned}
\pi_{\frac{p}{m}}((j_{\frac{p}{m}} \circ \delta)(\theta)) &= \inf\left\{\sum_{i=1}^n |\lambda_i| \cdot \|(j_{\frac{p}{m}} \circ \delta)(\otimes_m x_i)\|_{L_{\frac{p}{m}}} : T(\theta) = \sum_{i=1}^n \lambda_i \otimes_m (j_p \circ i_X(x_i))\right\} \\
&= \inf\left\{\sum_{i=1}^n |\lambda_i| \cdot \|(j_p \circ i_X(x_i)\|^m_{L_p} : T(\theta) = \sum_{i=1}^n \lambda_i \otimes_m (j_p \circ i_X(x_i))\right\} \\
&\leq \inf\left\{\sum_{i=1}^n |\lambda_i| \cdot \left(\sum_{k=1}^{n_i} \|j_p \circ i_X(x_k^i)\|_{L_p}\right)^m : T(\theta) = \sum_{i=1}^n \lambda_i \otimes_m (j_p \circ i_X(\sum_{k=1}^{n_i} x_k^i))\right\} \\
&\leq \inf\left\{\sum_{i=1}^n |\lambda_i| \cdot \left(\sum_{k=1}^{n_i} \|x_k^i\|^\sigma \|j_p \circ i_X(x_k^i)\|^{1-\sigma}_{L_p}\right)^m : T(\theta) = \sum_{i=1}^n \lambda_i \otimes_m (j_p \circ i_X(\sum_{k=1}^{n_i} x_k^i))\right\} \\
&= \pi_{p;\sigma,m}(h)
\end{aligned}$$

Therefore, if $\pi_{p;\sigma,m}(h) = 0$ then $\pi_{\frac{p}{m}}((j_{\frac{p}{m}} \circ \delta)(\theta)) = 0$. Hence $(j_{\frac{p}{m}} \circ \delta)(\theta) = 0$ and so $h = 0$. $\square$

Next proposition shows that Q^L is an isometric isomorphism between $\hat{\otimes}^{m,s}_{\pi_s} j_{p,\sigma} \circ i_X(X)$ and the completion of the space $(L_{p,\sigma}(\mu)_{[m]}, \pi_{p;\sigma,m})$.

Proposition 4.2. *The completion of the space* $(L_{p,\sigma}(\mu)_{[m]}, \pi_{p;\sigma,m})$ *is isometrically isomorphic to* $\hat{\otimes}^{m,s}_{\pi_s} j_{p,\sigma} \circ i_X(X)$.

Proof. Consider Q^L restricted to $\otimes^{m,s}_{\pi_s} j_{p,\sigma} \circ i_X(X)$, that is $Q^L : \otimes^{m,s}_{\pi_s} j_{p,\sigma} \circ i_X(X) \to (L_{p,\sigma}(\mu)_{[m]}, \pi_{p;\sigma,m})$. Let us see first that Q^L is onto. Given $h = \sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(x_j) \in L_{p,\sigma}(\mu)_{[m]}$, let $\theta = \sum_{j=1}^n \lambda_j x_j \otimes \cdots \otimes x_j$. Then

$$\begin{aligned} Q^L(T(\theta)) &= \sum_{j=1}^n \lambda_j Q^L(\otimes_m j_{p,\sigma} \circ i_X(x_j)) = \sum_{j=1}^n \lambda_j Q(j_{p,\sigma} \circ i_X(x_j)) \\ &= \sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(x_j) = h \end{aligned}$$

From the definitions of the norms it follows that $\pi_{p;\sigma,m}(j_{\frac{p}{m}} \circ i_X(x)) = \pi_s(T(\theta))$ and so Q^L is an isometry. Therefore, the extension of Q^L to the completions is the required isometric isomorphism. □

To simplify the notation we shall use $(L_{p,\sigma}(\mu)_{[m]}, \pi_{p;\sigma,m})$ for its completion too. Let us define a polynomial which shall play the role of the canonical prototype of dominated $(p;\sigma)$-continuous m-homogeneous polynomial through which any other polynomial of the class must factor. Define $j_{p;\sigma,m} := Q \circ j_{p,\sigma} : i_X(X) \to L_{p,\sigma}(\mu)_{[m]}$. For each $x \in X$,

$$j_{p;\sigma,m}(i_X(x)) = Q \circ j_{p,\sigma}(i_X(x)) = j_{\frac{p}{m}} \circ \Delta(x) = (j_{\frac{p}{m}} \circ i_X(x))^m$$

and so $j_{p;\sigma,m}$ can be identified with the restriction of $j^m_{\frac{p}{m}}$ to $i_X(X)$ and with values in $L_{p,\sigma}(\mu)_{[m]}$.

Proposition 4.3. *The m-homogeneous polynomial $j_{p;\sigma,m}$ is dominated $(p;\sigma)$-continuous.*

Proof. Given $x_1, \ldots, x_n \in X$,

$$\begin{aligned} \left(\sum_{i=1}^n \pi_{p;\sigma,m}(j_{p;\sigma,m}(i_X(x_i)))^{\frac{p}{m(1-\sigma)}}\right)^{\frac{m(1-\sigma)}{p}} &= \left(\sum_{i=1}^n \pi_{p;\sigma,m}((j_{\frac{p}{m}} \circ i_X(x_i))^m)^{\frac{p}{m(1-\sigma)}}\right)^{\frac{m(1-\sigma)}{p}} \\ &\leq \left(\sum_{i=1}^n \|(j_{p,\sigma} \circ i_X(x_i)\|^{\frac{p}{1-\sigma}}_{L_{p,\sigma}}\right)^{\frac{m(1-\sigma)}{p}} \\ &= \left(\sum_{i=1}^n \|(j_p \circ i_X(x_i)\|^{\frac{p}{1-\sigma}}_{L_{p,\sigma}}\right)^{\frac{m(1-\sigma)}{p}} \\ &\leq C[\delta_{p\sigma}((i_X(x_i)_{i=1}^n)]^m. \end{aligned}$$

□

To get the factorization of dominated $(p;\sigma)$-continuous polynomials through $j_{p;\sigma,m}$ we need some preliminary results.

Although the following proposition pretends to generalize [11, Proposition 3.4], it only applies for some specific Pietsch measures of a dominated $(p;\sigma)$-continuous m-homogeneous polynomial P and a different proof is required. From Remark 3.1, P can be written as $P = Q \circ u$, where u is a $(p;\sigma)$-absolutely continuous linear operator from X into some Banach space Z and $Q : Z \to Y$ is a continuous m-homogeneous polynomial. An easy calculation shows that any Pietsch measure μ for u is a Pietsch measure for P. In the following result we are considering μ such a measure.

Proposition 4.4. *Let $P \in \mathcal{P}^{\sigma}_{d,p}(^mX;Y)$. If $x, y \in X$ are such that $j_{p,\sigma} \circ i_X(x) = j_{p,\sigma} \circ i_X(y)$ then $P(x) = P(y)$.*

Proof. By Remark 3.1, there exist a Banach space Z, a $(p;\sigma)$-absolutely continuous linear operator $u : X \to Z$ and a polynomial $Q \in \mathcal{P}(^mZ;Y)$ such that P can be written as $P = Q \circ u$. Let μ be a Pietsch measure for u with constant C. Therefore, $\check{P} = \check{Q} \circ (u, \ldots, u)$. Take $\epsilon > 0$. The equality $j_{p,\sigma} \circ i_X(x) = j_{p,\sigma} \circ i_X(y)$ says that $\|j_{p,\sigma} \circ i_X(x-y)\| = 0$. Then, there exist $x_1, \ldots, x_n \in X$ such that $x - y = \sum_{k=1}^{n} x_k$ and

$$\sum_{k=1}^{n} \|x_k\|^{\sigma} \|j_p \circ i_X(x_k)\|^{1-\sigma}_{L_p(\mu)} < \epsilon. \tag{8}$$

It follows that

$$\begin{aligned}
\|P(x) - P(y)\| &= \|\check{P}(x, \ldots, x) - \check{P}(y, \ldots, y)\| = \|\sum_{j=1}^{m} \check{P}(y^{j-1}, x - y, x^{n-j})\| \\
&\leq \sum_{j=1}^{m} \sum_{k=1}^{n} \|\check{P}(y^{j+1}, x_k, x^{n-j})\| = \sum_{j=1}^{m} \sum_{k=1}^{n} \|\check{Q}(u(y)^{j+1}, u(x_k), u(x)^{n-j})\| \\
&\leq \sum_{j=1}^{m} \sum_{k=1}^{n} \|\check{Q}\| \|u(y)\|^{j+1} \|u(x_k)\| \|u(x)\|^{n-j} \\
&\leq \|\check{Q}\| \sum_{j=1}^{m} \|u(y)\|^{j+1} \|u(x)\|^{n-j} C \sum_{k=1}^{n} \|x_k\|^{\sigma} \|j_p \circ i_X(x_k)\|^{1-\sigma}_{L_p(\mu)} \\
&< C\|\check{Q}\| \sum_{j=1}^{m} \|u(y)\|^{j+1} \|u(x)\|^{n-j} \epsilon
\end{aligned}$$

As $\epsilon > 0$ is arbitrary we conclude that $P(x) = P(y)$. □

We present now the factorization theorem for dominated $(p;\sigma)$-continuous polynomials.

Theorem 4.5. *Let $m \in \mathbb{N}$ and $p \geq m$. A polynomial $P \in \mathcal{P}(^mX;Y)$ is dominated $(p;\sigma)$-continuous if and only if there exist a regular Borel probability measure μ on B_{X^*}, with the weak* topology, and a continuous linear operator $v : (L_{p,\sigma}(\mu)_{[m]}, \pi_{p;\sigma,m}) \to Y$ such that* *the following diagram commutes*

$$\begin{tikzcd}
X \arrow[r, "P"] \arrow[d, "i_X"] & Y \\
i_X(X) \arrow[r, "j_{p;\sigma,m}"] \arrow[d] & L_{p,\sigma}(\mu)_{[m]} \arrow[u, "v"] \arrow[d] \\
C(B_{X^*}) & L_{\frac{p}{m}}(\mu)
\end{tikzcd} \tag{9}$$

Proof. Assume that $P \in \mathcal{P}^{\sigma}_{d,p}({}^m X; Y)$. By Remark 3.1, there exist a Banach space Z, a $(p;\sigma)$-absolutely continuous linear operator $u : X \to Z$ and a polynomial $Q \in \mathcal{P}({}^m Z; Y)$ such that P can be written as $P = Q \circ u$. Let μ be a Pietsch measure for u. For each $x \in X$ define $R(j_{p,\sigma} \circ i_X(x)) := P(x)$. By Proposition 4.4 the map $R : j_{p,\sigma} \circ i_X(X) \to Y$ is a well-defined m-homogeneous polynomial. Let $Q^L : \otimes^{m,s}_{\pi_s} j_{p,\sigma} \circ i_X(X) \to L_{p,\sigma}(\mu)_{[m]}$ be the surjective isometric isomorphism given by Proposition 4.2. Taking into account the commutative diagrams

$$\begin{tikzcd}
X \arrow[rrr, "P"] \arrow[dr, "\Delta"] \arrow[dd, "i_X"] & & & Y \\
& \Delta(X) \arrow[dr, "j_{\frac{p}{m}}"] & & \\
i_X(X) \arrow[rr, "j_{p;\sigma,m}"] \arrow[d, "j_{p,\sigma}"] & & L_{p,\sigma}(\mu)_{[m]} \arrow[uur, "v"] & \\
j_{p,\sigma} \circ i_X(X) \arrow[urr, "Q"] \arrow[rrr] & & & \hat{\otimes}^{m,s}_{\pi_s} j_{p,\sigma} \circ i_X(X) \arrow[ul, "Q^L"] \arrow[uuu, "R^L"']
\end{tikzcd}$$

let us define $v := R^L \circ (Q^L)^{-1}$. It is clear that, so defined, v is continuous and closes the diagram (9).

The converse follows from Proposition 4.3 and the ideal property. □

D. Achour acknowledges with thanks the support of the Ministére de l'Enseignament Supérieur et de la Recherche Scientifique (Algeria) under project PNR 8-U28-6543. E. Dahia acknowledges with thanks the support of the Ministére de l'Enseignament Supérieur et de la Recherche Scientifique (Algeria) under grant 170/PGRS/C.U.K.M(2012) for short term stage. P. Rueda acknowledges with thanks the support of the Ministerio de Ciencia e Innovación (Spain) MTM2011-22417. E.A. Sánchez Pérez acknowledges with thanks the support of the Ministerio de Economía y Competitividad (Spain) MTM2012-36740-C02-02.

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D. Achour, University of M'sila , Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces, 28000 M'sila, Algeria.
E-mail address: `dachourdz@yahoo.fr`

E. Dahia, University of M'sila , Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces, 28000 M'sila, Algeria.
E-mail address: `hajdahia@gmail.com`

P. Rueda, Departamento de Análisis Matemático, Universidad de Valencia, 46100 Burjasot Valencia. Spain.
E-mail address: pilar.rueda@uv.es

E. A. Sánchez Pérez, Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia. Spain.
E-mail address: easancpe@mat.upv.es