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Research Article

Characterization of the Existence of an N_0-Completion of a Partial N_0-Matrix with an Associated Directed Cycle

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An $n\times n$ matrix is called an N_0-matrix if all its specified principal minors are nonpositive. In the context of partial matrices, a partial matrix is called a partial N_0-matrix if all its specified principal minors are nonpositive. In this paper we characterize the existence of an N_0-matrix completion of a partial N_0-matrix whose associated graph is a directed cycle.

1. Introduction

A *partial matrix* is a rectangular array, some of whose entries are specified while the remaining unspecified entries are free to be chosen (from a certain set). In this paper we are going to work on the set of the real numbers and to assume that all diagonal entries are prescribed. A *completion* of a partial matrix is the matrix resulting from a particular choice of values for the unspecified entries. A *completion problem* asks if we can obtain a completion of a partial matrix with some prescribed properties.

The technics to obtain this completion depend on the pattern of the partial matrix which can be *combinatorially symmetric* (i.e., a_{ij} is specified if and only if a_{ji} is) or *noncombinatorially symmetric*. Here we are going to work with this second class of partial matrices.

A natural way to describe an $n \times n$ partial matrix $A = (a_{ij})$ is via a graph $G_A = (V, E)$, where the set of vertices V is $\{1, 2, \ldots, n\}$ and there is an arc from i to j if and only if position (i, j) of A is specified. In general, a directed graph (resp., nondirected graph) is associated with a noncombinatorially symmetric (resp., combinatorially symmetric) partial matrix. Since all main diagonal entries are specified we omit loops.

A *cycle* in a directed graph G is a sequence of arcs $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_1)$, where $i_k \neq i_l$ for all $k \neq l$.

In the last years many completions problems have been analysed. The completion problem for partial M-matrices, P-matrices, N-matrices, …, has been studied by Johnson, Hogben, Urbano, Mendes, …, among others. See, for instance [1–9] and the references therein.

As a class of square real matrices that contains the N-matrices we define the N_0*-matrices*, $n \times n$ real matrices $A = (a_{ij})$, where all its principal minors are nonpositive. Since N_0-matrices are preserved by principal submatrices we define a *partial* N_0*-matrix* as a partial matrix whose completely specified principal submatrices are N_0-matrices.

In general it is not always true that a partial N_0-matrix has an N_0-matrix completion as the following matrix shows (see [7]):

$$A = \begin{bmatrix} -1 & 1 & -10 & x \\ 2 & -1 & 1 & -100 \\ -0.1 & 10 & -1 & 1 \\ 1 & -10 & 1 & -1 \end{bmatrix}. \tag{1}$$

A is a partial N_0-matrix that has an N_0-matrix completion that leads us to analyze the N_0-matrix completion problem depending on the pattern of the partial matrix. We have studied in [7] when a combinatorially symmetric partial N_0-matrix with no null main diagonal entries such that the graph of its specified entries is a 1-chordal graph or a cycle has an N_0-matrix completion.

In this paper we study the mentioned problem for partial N_0-matrices that can have zeros at the main diagonal and whose associated graph is a directed cycle of length equal to the order of the matrix. In this case we may suppose without loss of generality that these matrices have the form:

$$\begin{bmatrix} a_{11} & a_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & a_{22} & a_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n1} & x_{n2} & x_{n3} & \cdots & a_{nn} \end{bmatrix}, \tag{2}$$

where a_{ij} denotes a specified entry and x_{ij} an unspecified one.

Observe that when we say "$a_{i,i+1}$, $i = 1, 2, \ldots, n$, where the subscripts are expressed module n", we are using the congruence module n; that is, we are considering the entries $a_{12}, a_{23}, \ldots, a_{n-1n}, a_{n1}$.

Given a matrix A of size $n \times n$ the submatrix lying in rows α and columns β, $\alpha, \beta \subseteq N = \{1, 2, \ldots, n\}$ is denoted by $A[\alpha \mid \beta]$ and the principal submatrix $A[\alpha \mid \alpha]$ is abbreviated to $A[\alpha]$.

We denote $\overline{a} = 1$ if $a = 0$ and $\overline{a} = a$ if $a \neq 0$.

In the next section we introduce necessary and sufficient conditions in order to guarantee the existence of an N_0-matrix completion of a partial N_0-matrix whose associated graph is a directed cycle.

2. Completion of Partial N_0-Matrices

It is easy to prove that N_0-matrices as well as partial N_0-matrices satisfy the following properties.

Proposition 1. *Let $A = (a_{ij})$ be an $n \times n$ N_0-matrix.*

(1) *If P is a permutation matrix, then PAP^T is an N_0-matrix.*

(2) *If D is a positive diagonal matrix, then DA and AD are N_0-matrices.*

(3) *If D is a nonsingular diagonal matrix, then DAD^{-1} is an N_0-matrix.*

(4) *If $a_{ii} \neq 0$, $i = 1, 2, \ldots, n$, then $a_{ij} \neq 0$, for all $i, j \in \{1, 2, \ldots, n\}$.*

(5) *Every principal submatrix of A is an N_0-matrix.*

In [7] the authors proved that any $n \times n$ N_0-matrix with no null diagonal entries is diagonally similar to an N_0-matrix in the set:

$$S_n = \left\{A = \left(a_{ij}\right) : \operatorname{sign}\left(a_{ij}\right) = (-1)^{i+j+1}, \forall i, j\right\}. \tag{3}$$

But since there are N_0-matrices with some entries equal to zero we need to introduce the following set:

$$wS_n = \left\{A = \left(a_{ij}\right) : a_{ij} = 0 \text{ or } \operatorname{sign}\left(a_{ij}\right) = (-1)^{i+j+1}, \forall i, j\right\}. \tag{4}$$

We also extend the definition of wS_n matrices to partial matrices; that is, PwS_n consists of the $n \times n$ partial matrices $A = (a_{ij})$ such that if $a_{ij} \neq 0$ then $\operatorname{sign}(a_{ij}) = (-1)^{i+j+1}$, for all specified entry (i, j), $i, j \in \{1, 2, \ldots, n\}$. The following matrix B is an example of a matrix of PwS_4,

$$B = \begin{bmatrix} -1 & 1 & -10 & x \\ 2 & y & 0 & -100 \\ z & 10 & -1 & 1 \\ 1 & -10 & 1 & -1 \end{bmatrix}. \tag{5}$$

The following results, consequence of Proposition 1, allow us to transform a partial N_0-matrix $A = (a_{ij})$, whose associated graph is a directed cycle, into a matrix whose diagonal nonzero values are -1; the nonzero elements of the first upperdiagonal are 1 and the element in position $(n, 1)$ is $\overline{a}_{12}\overline{a}_{23} \cdots \overline{a}_{n1}$.

Proposition 2. *Let $A = (a_{ij})$ be an $n \times n$ partial N_0-matrix. There exists a positive diagonal matrix D such that matrix $B = DA = (b_{ij})$ is also an N_0-partial matrix with b_{ii} equal to -1 or to zero, for all $i = 1, 2, \ldots, n$.*

Proof. It suffices to consider $D = \{d_1, d_2, \ldots, d_n\}$ defined $d_i = -1/a_{ii}$ if $a_{ii} \neq 0$ and $d_i = 0$ if $a_{ii} = 0$. □

Proposition 3. *Let $A = (a_{ij})$ be an $n \times n$ partial N_0-matrix, whose associated graph is a directed cycle, such that a_{ii} is -1 or 0, for all $i = 1, 2, \ldots, n$. Then there exists a diagonal matrix D such that $DAD^{-1} = (c_{ij})$ is a partial N_0-matrix with $c_{ii} = a_{ii}$ for all $i = 1, 2, \ldots, n$, $c_{ii+1} = 1$ or zero for all $i = 1, 2, \ldots, n-1$ and $c_{n1} = \overline{a}_{12}\overline{a}_{23} \cdots \overline{a}_{n1}$.*

Proof. It suffices to consider $D = \operatorname{diag}(1, \overline{a}_{12}, \overline{a}_{12}\overline{a}_{23}, \ldots, \overline{a}_{12}\overline{a}_{23} \cdots \overline{a}_{n-1n})$. □

Therefore, if $A = (a_{ij})$ is an $n \times n$ partial N_0-matrix whose associated graph is a directed cycle, we will assume, without loss of generality, that A has the following structure: -1 or zeros on the main diagonal, 1's or zeros in the first upper diagonal and $\overline{a}_{12}\overline{a}_{23} \cdots \overline{a}_{n1}$ in position $(n, 1)$.

The following theorem characterizes the PwS_n matrices as an intermediate step to obtain the desired completion. It can be easily obtained from the transformations of Propositions 2 and 3.

Theorem 4. *Let A be an $n \times n$ partial matrix, n even (resp., odd), whose associated graph is a directed cycle. If all entries a_{ii+1}, $i = 1, 2, \ldots, n$, where the indices are expressed module n, are nonzero the matrix A is diagonally similar to an element of PwS_n if and only if $a_{12}a_{23} \cdots a_{n1} > 0$ (resp., $a_{12}a_{23} \cdots a_{n1} < 0$).*

Now we analyze the existence of an N_0-matrix completion of a partial N_0-matrix with an associated directed cycle, by distinguishing between matrices with no null main diagonal entries and matrices with some null values in the main diagonal.

Theorem 5. *Let A be an $n \times n$ partial N_0-matrix, with nonzero main diagonal entriessuch that its associated graph is a directed* *cycle. The following statements are equivalent:*

(1) $a_{12}a_{23}\cdots a_{n1} > 0$ *if n is even* ($a_{12}a_{23}\cdots a_{n1} < 0$ *if n is odd*),

(2) *A is diagonally similar to an element of* PwS_n,

(3) *there exists an* N_0*-matrix completion of A.*

Proof. Observe that from (4) of Proposition 1, we have that all the specified entries are nonzero. Then, from commentary after Proposition 3, we assume that all the elements in the main diagonal are −1 and the first upper diagonal is formed by 1's.

Let us suppose that n is even; the case n odd is analogous. Since the upper diagonal and the element in position $(n, 1)$ are nonzero, by applying Theorem 4, the condition $a_{12}a_{23}\cdots a_{n1} > 0$ is equivalent to item 2.

Now, we assume that the second statement is true. We consider $A' = (a'_{ij})$, where $a'_{ij} = a_{ij}$ if a_{ij} is a specified value of A, $a'_{i+1i} = 1$ for $i = 1, 2, \ldots, n$, where subscripts are expressed module n and $a'_{n1} = 1/(a_{12}a_{23}\cdots a_{n1})$. Then A' is an $n \times n$ partial N_0-matrix, with nonzero main diagonal entries such that its associated graph is a nondirected cycle. Theorem 4.3 of [7] assures that A', and therefore A has an N_0-matrix completion.

Finally, from the note after Proposition 1, the third statement implies the second one. □

Now, it arises the question about establishing an analogous result to Theorem 5, when zero entries appear in the main diagonal. The answer is negative since if we admit a zero diagonal element and a zero entry in the upper diagonal, there exist matrices in PwS_n, n is even, such that $\overline{a}_{12}\overline{a}_{23}\cdots\overline{a}_{n1}$ is negative, but that admits an N_0-matrix completion. For example, matrix $A = (a_{ij})$ defined

$$A = \begin{bmatrix} 0 & 0 & x_{13} & x_{14} \\ x_{21} & -1 & 1 & x_{24} \\ x_{31} & x_{32} & -1 & 1 \\ -1 & x_{42} & x_{43} & -1 \end{bmatrix} \quad (6)$$

is diagonally similar to an element of PwS_4 by using $D = \operatorname{diag}(-1, 1, 1, 1)$ and it has an N_0-matrix completion, A_c, although $\overline{a}_{12}\overline{a}_{23}\overline{a}_{34}\overline{a}_{41}$ is negative,

$$A_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{bmatrix}. \quad (7)$$

The following results characterize this type of matrices. Note that, if there are some null main diagonal entries, the existence of an N_0-matrix completion implies that if $a_{ii}a_{i+1i+1} \neq 0$, then $a_{ii+1} \neq 0$. So, we add this condition as a hypothesis. In addition, recall that we are going to assume that $a_{ii} = -1$ or zero for all $i \in \{1, 2, \ldots, n\}$; the entries in the first upper diagonal are 1 or zero and the value of the element in position $(n, 1)$ is $\overline{a}_{12}\overline{a}_{23}\cdots\overline{a}_{n1}$.

Theorem 6. *Let A be an $n \times n$, n even (resp., odd), $n \geq 3$, partial N_0-matrix with some null main diagonal entries, whose associated graph is a directed cycle. Let one suppose that if $a_{ii}a_{i+1i+1} \neq 0$ for all $i = 1, 2, \ldots, n$, where the indices are expressed module n, then $a_{ii+1} \neq 0$. If there exists $a_{ii+1} = 0$, $i \in \{1, 2, \ldots, n\}$, where the indices are expressed module n, then there exists an N_0-matrix completion.*

Proof. Let $N_p = \{i_1, i_2, \ldots, i_p\}$ be, with $i_k < i_{k'}$ for all $1 \leq k < k' \leq p$, the corresponding indices to the negative diagonal values of matrix A and $N_l = \{i_{p+1}, \ldots, i_{p+l}\}$, where $p + l = n$ and $i_h < i_{h'}$ for all $p + 1 \leq h < h' \leq p + l$, the corresponding ones to the zero diagonal entries. Since there is i, $i \in \{1, 2, 3, \ldots, n-1\}$, such that $a_{ii+1} = 0$, the diagonal similarity allows us to assume, without loss of generality, that $a_{n1} = 1$ if p is even and that $a_{n1} = -1$ if p is odd. Let B be the matrix PAP^T where P is the permutation matrix $P = [e_{i_{p+1}}, e_{i_{p+2}}, \ldots, e_{i_{p+l}}, e_{i_1}, e_{i_2}, \ldots, e_{i_p}]$, being e_k the canonical vector for all $k \in \{1, 2, \ldots, n\}$. Consider B partitioned in a 2×2 block matrix, where B_{11} is of size $l \times l$ and B_{22} is $p \times p$.

Note that elements of the first upper diagonal a_{ii+1}, $i \in \{1, 2, \ldots, n-1\}$, are moved to blocks B_{kl}, $k, l \in \{1, 2\}$, depending on the value of a_{ii} and a_{i+1i+1}: if both entries are zero, after the permutation P a_{ii+1} will be in B_{11}; if $a_{ii} = 0$ and $a_{ii+1} = 1$ the element a_{ii+1} will be in B_{12}; if $a_{ii} = -1$ and $a_{ii+1} = 0$ then a_{ii+1} will be in B_{21} and in other cases a_{ii+1} will be in B_{22}. This is shown in Table 1(a). In Table 1(b) we can see the position that the element a_{n1} of A will occupy after the permutation.

Then we can be sure that each of the l first lines of the permutated matrix B will have as maximum a nonzero value and the last p ones will have exactly one −1 and only another nonzero value as maximum.

We complete with zeros the unspecified entries of blocks B_{11}, B_{12}, and B_{21}. In order to complete B_{22} we distinguish two cases.

(a) The element a_{n1} of A is at position $(n, l+1)$. If $a_{ii} \neq 0$ and $a_{i+1i+1} \neq 0$, then $a_{ii+1} \neq 0$ and, after the permutation, it will be in submatrix B_{22}. If $i_j + 1 \neq i_{j+1}$ then position (i_j, i_{j+1}) of the permutated matrix B will be unspecified. Now we partially complete B_{22} as follows: $a_{1n} = 1/a_{n1}$ and $a_{i_j i_j+1} = 1$ for all $j \in \{1, 2, \ldots, p\}$, and by Theorem 5 we get that there exists an N_0-completion of B_{22} named B_{22_c}.

(b) The element a_{n1} of A is not at position $(n, l+1)$. We complete B_{22} with 1's and −1's in order to obtain a matrix with all their diagonals formed alternatively by 1's and −1's. All the principal minors of this new matrix will be zero.

The completion of B, B_c, is an N_0-matrix since

(a) the principal minors lying rows and columns with indices in N_l are zero, as we can prove by developing by the nonzero elements (there is as maximum one nonzero entry by line);

(b) the principal minors lying rows and columns with indices in N_p are less than or equal to zero, since B_{22_c} is an N_0-matrix completion either a_{n1} of A is at position $(n, l+1)$ or not;

Table 1

(a)

$\mathbf{a_{ii+1}}$	$a_{i+1i+1}=0$	$a_{i+1i+1}\neq 0$
$a_{ii}=0$	B_{11}	B_{12}
$a_{ii}\neq 0$	B_{21}	B_{22}

(b)

$\mathbf{a_{n1}}$	$a_{11}=0$	$a_{11}\neq 0$
$a_{nn}=0$	$(l,1)$ (in B_{11})	$(l,l+1)$ (in B_{12})
$a_{nn}\neq 0$	$(n,1)$ (in B_{21})	$(n,l+1)$ (in B_{22})

(c) the principal minors with rows and columns with indices in N_l and N_p are zero, because of they have a row of zeros or, as before, by developing by the only nonzero element of each row, the minor is reduced to one of submatrix B_{21}.

So, matrix A has an N_0-matrix completion. □

Theorem 7. *Let A be an $n\times n$, n even (resp., odd), $n\geq 3$, partial N_0-matrix with some null main diagonal entries, whose associated graph is a directed cycle. Let one suppose that if $a_{ii}a_{i+1i+1}\neq 0$ for all $i=1,2,\ldots,n$ where the indices are expressed module n, then $a_{ii+1}\neq 0$. If $a_{ii+1}\neq 0$ for all $i\in\{1,2,\ldots,n\}$, where the indices are expressed module n, then the condition $a_{12}a_{23}\cdots a_{n1}>0$ for n even (resp., $a_{12}a_{23}\ldots a_{n1}<0$ for n odd) is necessary and sufficient for the existence of an N_0-matrix completion.*

Proof. Let n be even. If $a_{12}a_{23}\ldots a_{n1}>0$ some changes in the proof of the above theorem gives the desired completion. Specifically, choose the zero diagonal entry a_{kk} with less index. If $1\leq k\leq n-1$ since $a_{12}a_{23}\cdots a_{n1}>0$ matrix A can be transformed by diagonal similarity in a matrix such that if p is even, $a_{n1}=1$ and a_{kk+1} are equal to a positive value and if p is odd, $a_{n1}=-1$ and a_{kk+1} are equal to a negative value. By permutation similarity we get a block-matrix analogous to the previous result and by a similar reasoning we get that there exists an N_0-matrix completion of A. If $k=n$ we transform A by the permutation $P=[e_n,e_1,\ldots,e_{n-2},e_{n-1}]$, where e_k is the canonical vector for all $k\in\{1,2,\ldots,n\}$ and we proceed in a similar way to the previous result.

Now, we are going to prove the necessity of the condition by induction on n; that is, if there exists an N_0-completion of A, $A_c=(c_{ij})$, then $c_{12}c_{23}\cdots c_{n1}>0$ if n is even and $c_{12}c_{23}\cdots c_{n1}<0$ if n is odd. We denote $\alpha=c_{12}c_{23}\cdots c_{n1}$.

For $n=3$, if we suppose that $\alpha>0$ by analyzing the seven different cases that arise depending on the number of zeros, one, two, or three in the main diagonal, we get that $\det A_c>0$, that is a contradiction. Then $\alpha<0$.

Let $n>3$ be. Suppose n is even (if n is odd the process is analogous). We are showing that in the mentioned conditions if there exists an N_0-completion of A, then $\alpha>0$. Let A_c be an N_0-completion of A.

Let us suppose by hypothesis of induction (HI) that the statement is true for all $3\leq k\leq n-1$. Since A_c is an N_0-matrix, we obtain, from the 2×2 principal minors, that all the entries of the first under diagonal are greater than or equal to zero; that is, $c_{i-1i}\geq 0$ for all $i\in\{1,2,\ldots,n\}$.

In addition, if we consider $\det A[\{i,i+1,\ldots,i+p\}]$ with $i\in\{1,2,\ldots,n-p\}$, $p\in\{2,\ldots,n-2\}$, we get that $c_{ij}>0$ if $i-j$ is odd and $c_{ij}<0$ if $i-j$ is even for $i\in\{1,2,\ldots,n-1\}$, $j\in\{1,2,\ldots,n-2\}$, $i>j$.

From the nonpositivity of $\det A[\{i,j\}]$ for all $i\in\{1,2,\ldots,n-2\}$, $j\in\{3,\ldots,n\}$, we obtain that the upper diagonals follow the same rule of signs of the under diagonals, alternatively positive and negative, but with the option of zero; that is, $c_{ij}\geq 0$ if $j-i$ is odd and $c_{ij}\leq 0$ if $j-i$ is even for $i\in\{1,2,\ldots,n-2\}$, $j\in\{3,\ldots,n\}$, $i<j$.

Now we study the case in which there exists $i\in\{3,\ldots,n\}$ such that $a_{1i}\neq 0$.

(a) If $i\in\{3,\ldots,n-1\}$ we consider $\det A[\{1,i,i+1,\ldots,n\}]$. The $(n-i+2)\times(n-i+2)$ submatrix of A_c, $A'=A_c[\{1,i,i+1,\ldots,n\}]$ can be considered as a completion of a partial N_0 matrix of size strictly smaller than n with all the first upper diagonal formed by 1's. If A' has at least a zero diagonal entry, taking into account that $c_{1i}\leq 0$ if i is odd and $c_{1i}\geq 0$ if i is even, the hypothesis of induction allows us to assure that the entry in position $(n-i+2,1)$ of A'; that is, α is positive. In the other case, if all the diagonal entries are nonzero we get the same conclusion by Theorem 5. This ends the proof in this case.

(b) If $i=n$ and $c_{1i}=0$ with $i\in\{1,\ldots,n-1\}$, we analyze two cases: (b.1) $c_{11}=0$ or $c_{nn}=0$ and (b.2) $c_{11}\neq 0$ and $c_{nn}\neq 0$. In the first one, we get $\alpha>0$ from $\det A_c[\{1,3,n\}]$. If $c_{11}\neq 0$, $c_{nn}\neq 0$ and $c_{22}=0$ and $c_{ii}\neq 0$ for all $i\in\{3,\ldots,n-1\}$ we get $\alpha>0$ from $\det A_c[\{1,n-1,n\}]\leq 0$. If $c_{11}\neq 0$, $c_{nn}\neq 0$ and there exists $i\in\{3,\ldots,n-1\}$ such that $c_{ii}=0$ we get also the same result from $\det A_c[\{1,i,n\}]\leq 0$.

In this case $c_{1i}=0$ for all $i\in\{3,\ldots,n\}$ if some entry c_{ij} of the upper triangular part of A_c is nonzero, we also obtain that $\alpha>0$ by using $\det A_c[\{1,2,\ldots,i-1,i,j,j+1,\ldots,n\}]\leq 0$ and HI.

Therefore, it remains to analyze the case in which all the upper triangular part except the first upper diagonal is zero. As we will see now, most of the cases can not be given.

Let c_{ii} be the nonzero diagonal entry with less index. If $i\geq 5$ from $\det A_c[\{1,2,3,i\}]\leq 0$ or if $i\in\{1,2,3,4\}$ and $n\geq i+4$ from $\det A_c[\{1,n-2,n-1,n\}]\leq 0$ we get a contradiction. Now we study the remaining cases depending on the values of n and i.

If $n=4$ from $\det A_c\leq 0$ we get $\alpha>0$ (if $i=4$ to show it we also use $\det A_c[\{1,2,4\}]\leq 0$).

If $n=5$ and $c_{i+1i+1}=0$ from $\det A_c\leq 0$ we obtain $\alpha<0$ (if $i=3$ to show it we also use $\det A_c[\{1,2,4,5\}]\leq 0$ and $\det A_c[\{1,2,4\}]\leq 0$ if $i=4$). If $n=5$, $c_{i+1i+1}\neq 0$ and $i\in\{2,3\}$ from $\det A_c\leq 0$ we get $\alpha<0$ (if $i=2$ to show it we also use $\det A_c[\{2,4,5\}]\leq 0$ and $\det A_c[\{1,2,4\}]\leq 0$ if $i=3$); if $i=4$ $\det A_c[\{1,2,4\}]\leq 0$ and $\det A_c[\{1,2,3,5\}]\leq 0$ leads a contradiction.

If $n=6$ and $i=3$ from $\det A_c\leq 0$ we get $\alpha>0$ (if $c_{i+1i+1}=0$ to show it we also use $\det A_c[\{1,2,5,6\}]\leq 0$ and

Table 2: *Under the assumption that $a_{ii}a_{i+1i+1} \neq 0$ implies $a_{ii+1} \neq 0$.

	$\forall i \quad a_{ii+1} \neq 0$	$\exists i \quad a_{ii+1} = 0^*$
$\forall i \quad a_{ii} \neq 0$	$a_{12}a_{23}\dots a_{n1} > 0$ iff $\exists N_0$-matrix completion (Theorem 5)	N_0-matrix completion has no sense (Proposition 1)
$\exists i \quad a_{ii} = 0$	$a_{12}a_{23}\dots a_{n1} > 0$ iff $\exists N_0$-matrix completion (Theorem 7)	$\exists N_0$-matrix completion (Theorem 6)

A is an $n \times n$, **n even**, partial N_0-matrix, whose associated graph is a directed cycle.

$\det A_c[\{1,2,4,5,6\}] \leq 0$ and if $c_{i+1i+1} \neq 0$, the nonpositivity of $\det A_c[\{1,2,4\}]$. If $n = 6$, $i = 4$ and $c_{i+1i+1} = 0$ from $\det A_c \leq 0$ and $\det A_c[\{1,2,4\}] \leq 0$ we get $\alpha > 0$. In this case if $c_{i+1i+1} \neq 0$ the nonpositivity of $\det A_c[\{1,2,4\}]$ and $\det A_c[\{1,2,3,5\}]$ leads to a contradiction.

Finally, if $n = 7$ and $i = 4$ we get a contradiction by using $\det A_c[\{1,2,4\}] \leq 0$ and also $\det A_c[\{1,2,3,5\}] \leq 0$ if $c_{i+1i+1} \neq 0$, or if $c_{i+1i+1} = 0$ the nonpositivity of $\det A_c[\{1,2,3,5,6\}]$ and $\det A_c[\{1,2,3,5,6,7\}]$. □

We sum up the results of Theorems 6 and 7 in Table 2. One can consider a similar one for n odd.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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