Relations between $\{K, s+1\}$-Potent Matrices and Different Classes of Complex Matrices

Leila Lebtahi* Óscar Romero† Néstor Thome*

Abstract

In this paper, $\{K, s+1\}$-potent matrices are considered. A matrix $A \in \mathbb{C}^{n\times n}$ is called $\{K, s+1\}$-potent when $KA^{s+1}K = A$ where K is an involutory matrix and $s \in \{1, 2, 3, \dots\}$. Specifically, $\{K, s+1\}$-potent matrices are analyzed considering their relations to different classes of complex matrices. These classes of matrices are: $\{s+1\}$-generalized projectors, $\{K\}$-Hermitian matrices, normal matrices, and matrices $B \in \mathbb{C}^{n\times n}$ (anti-)commuting with K or such that KB is involutory, Hermitian or normal. In addition, some new relations for K-generalized centrosymmetric matrices have been derived.

Keywords: Involutory matrix; idempotent matrix; $\{K, s+1\}$-potent matrix; normal matrix, centrosymmetric matrix; $\{s+1\}$-potent matrix.
AMS subject classification: Primary: 15A24; Secondary: 15B57

1 Introduction and background

A matrix $A \in \mathbb{C}^{n\times n}$ is said to be (skew-)centrosymmetric if $JAJ = A$ ($JAJ = -A$), where $J \in \mathbb{C}^{n\times n}$ is the exchange matrix with ones on the anti-diagonal (lower left to upper right) and zeros elsewhere. These matrices have been widely studied by several authors and have applications in differential equations, in signal processing, in Markov processes, in engineering problems, etc. (see, for example, [1, 2, 8, 14, 21]).

In [17], Stuart gave a generalization of a centrosymmetric matrix called P-commutative matrices where P is a permutation. In [13], Li and Feng analyzed mirror matrices and exchange matrices, and that work is a special case of the generalizations that we consider in this paper. They have applications on multi-conductor transmission lines. Further results related to this class of matrices can be found, for instance, in [7, 9, 10, 15, 16, 18, 19, 20].

*Instituto Universitario de Matemática Multidisciplinar. Universitat Politècnica de València, 46022. Valencia, Spain. E-mail: {leilebep,njthome}@mat.upv.es. This work has been partially supported by grant DGI MTM2010-18228 and grant Universitat Politècnica de València, PAID-06-09. Ref.: 2659.

†Departamento de Comunicaciones. Universitat Politècnica de València, 46022. Valencia, Spain. E-mail: oromero@dcom.upv.es.

The common property that these classes of matrices (exchange, permutation, etc.) share is that they are all involutory ones. They allow the introduction of the so-called K-generalized centrosymmetric matrices which are the matrices $A \in \mathbb{C}^{n\times n}$ satisfying $A = KAK$ where $K \in \mathbb{C}^{n\times n}$ is an involutory matrix [22].

In recent years, matrices A such that $A^{s+1} = A$ for some positive integer s have been studied as an extension of the idempotent matrices; such matrices are called $\{s+1\}$-potent. In addition, the relation between the $\{s+1\}$-potent matrix A and its group inverse has been given [5, 6].

All these ideas motivate the following definition. For a given positive integer s and a given involutory matrix $K \in \mathbb{C}^{n\times n}$, a matrix $A \in \mathbb{C}^{n\times n}$ is called $\{K, s+1\}$-potent when $KA^{s+1}K = A$. In this paper the relation among different classes of matrices is analyzed.

Let Ω_k be the set of all k^{th} roots of unity with k a positive integer. If we define $\omega_k = e^{2\pi i/k}$ then $\Omega_k = \{\omega_k^1, \omega_k^2, \ldots, \omega_k^k\}$. Moreover, the following set will be used:

$$\mathbf{\Omega}^{(s)} = \{A \in \mathbb{C}^{n\times n} :\ \sigma(A) \subseteq \{0\} \cup \Omega_{s+1}\}, \tag{1}$$

where $\sigma(A)$ denotes the set of all eigenvalues of A, that is, the spectrum of A.

Throughout this paper, we will assume that the matrix $K \in \mathbb{C}^{n\times n}$ is involutory and that $s \in \{1, 2, 3 \ldots\}$. Moreover,

$$\mathbf{P}^{(K,s)} = \{A \in \mathbb{C}^{n\times n} :\ KA^{s+1}K = A\} \tag{2}$$

In addition, the notations $K\mathbf{S} = \{KB :\ B \in \mathbf{S}\}$ and $\mathbf{S}K = \{BK :\ B \in \mathbf{S}\}$ will be useful where K is the above fixed matrix and $\mathbf{S}$ is a prescribed subset of $\mathbb{C}^{n\times n}$.

A matrix $A \in \mathbb{C}^{n\times n}$ with the property $KA^*K = A$ is called $\{K\}$-Hermitian. In [11], Hill and Waters called κ-Hermitian matrices to $\{K\}$-Hermitian matrices. They emphasize the permutation κ and we stress on the matrix K for coherence with the remaining definitions. The equality $KA^*K = A$ is equivalent to $KAK = A^*$ because $K^2 = I_n$, where I_n denotes the identity matrix.

Some properties concerning the $\{K, s+1\}$-potent matrices were established in [12]. In particular, a matrix $A \in \mathbb{C}^{n\times n}$ is $\{K, s+1\}$-potent if and only if any of the following equivalent conditions holds: $KAK = A^{s+1}$, $KA = A^{s+1}K$, or $AK = KA^{s+1}$. Moreover, the following result holds:

Lemma 1 *If $A \in \mathbf{P}^{(K,s)}$ then the following properties hold:*

(a) $KA^jK = A^{j(s+1)}$ *for all integer* $j \geq 1$.

(b) $(KA)^2 = A^{s+2}$.

(c) $(KA)^{2s+1} = KA$ *and* $(AK)^{2s+1} = AK$.

(d) If K is Hermitian (that is, $K^* = K$) then $A^* \in \mathbf{P}^{(K,s)}$.

(e) If A is a nonsingular matrix then $A^{-1} \in \mathbf{P}^{(K,s)}$.

Proof. By induction, let us suppose that $KAK = A^{s+1}$ and $KA^jK = A^{j(s+1)}$ hold for some $j > 1$. Since $K^2 = I_n$,

$$KA^{j+1}K = (KA^jK)(KAK) = A^{j(s+1)}A^{s+1} = A^{(j+1)(s+1)}.$$

Then, property (a) is shown. Property (b) follows directly by definition. The first part of property (c) can be demonstrated as follows

$$\begin{aligned}(KA)^{2s+1} &= (KA)((KA)^2)^s = (KA)(A^{s+2})^s = KA^{(s+1)^2} \\ &= K(A^{s+1})^{s+1} = KKAK^{s+1} = K(KA^{s+1}K) = KA.\end{aligned}$$

The sencond part is similar. Properties (d) and (e) can be directly obtained by definition. This finishes the proof. ■

In what follows we will need the following spectral theorem.

Theorem 1 ([3]) *Let $A \in \mathbb{C}^{n\times n}$ with k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then A is diagonalizable if and only if there exist disjoint projectors $P_1, P_2, \ldots, P_k$, (i.e., $P_iP_j = \delta_{ij}P_i$ for $i, j \in \{1, 2, \ldots, k\}$) such that $A = \sum_{j=1}^k \lambda_j P_j$ and $I_n = \sum_{j=1}^k P_j$. Moreover, when A is a normal matrix, the projectors $P_1, P_2, \ldots, P_k$ are orthogonal (i.e., $P_i^* = P_i$ for $i \in \{1, 2, \ldots, k\}$).*

In this paper, some properties of $\{K, s+1\}$-potent matrices are studied. Specifically, in section 2, we derive relations between $\{K, s+1\}$-potent matrices and $\{K\}$-Hermitian matrices, $\{s+1\}$-generalized projectors, unitary matrices, and matrices $B \in \mathbb{C}^{n\times n}$ such that KB is involutory. In section 3, relations with normal matrices and matrices $B \in \mathbb{C}^{n\times n}$ such that KB is Hermitian or normal have been analyzed. In section 4, Hermitian and skew-Hermitian matrices, and matrices $B \in \mathbb{C}^{n\times n}$ (skew-)commuting with K are related to $\{K, s+1\}$-potent matrices. In all the cases, the class equivalences have been established, or in case of obtaining only one inclusion, the necessary examples are presented showing that they are proper subsets. In addition, a set of new properties has been derived for centrosymmetric matrices.

2 Analysis of $\mathbf{P}^{(K,s)}$ through $\mathbf{H}^{(K)}$, $\mathbf{GP}^{(s)}$, $K\mathbf{I}^{(s)}$, and $\mathbf{U}$

Let $s \in \{1, 2, 3, \ldots\}$, and let $\varphi : \{0, 1, 2, \ldots, (s+1)^2 - 2\} \to \{0, 1, 2, \ldots, (s+1)^2 - 2\}$ be the function given by $\varphi(j) = b_j$ where b_j is the smallest nonnegative integer such that $b_j \equiv j(s+1)$ [mod $((s+1)^2 - 1)$]. In [12] it was proved that this function φ is bijective. Moreover, in that paper the authors have shown that the eigenvalues of a $\{K, s+1\}$-potent matrix are included in the set constituted by $0, w^1_{(s+1)^2-1}, \ldots, w^{(s+1)^2-2}_{(s+1)^2-1}, 1$ and such a matrix A has associated certain projectors. Specifically, we will consider matrices P_j's satisfying the relations

$$KP_jK = P_{\varphi(j)} \quad \text{and} \quad KP_{(s+1)^2-1}K = P_{(s+1)^2-1} \tag{3}$$

for $j \in \{0, 1, \ldots, (s+1)^2 - 2\}$ where $P_0, P_1, \ldots, P_{(s+1)^2-1}$ are the projectors appearing in the spectral decomposition of A given in Theorem 1 associated to the eigenvalues

$$0, w^1_{(s+1)^2-1}, \ldots, w^{(s+1)^2-2}_{(s+1)^2-1}, 1,$$

respectively. From now on, we will say that projectors P_j's satisfying (3) fulfil condition $(\mathcal{P})$.

Characterizations of $\{K, s+1\}$-potent matrices are presented in the next theorem established in [12].

Theorem 2 *Let $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

(a) $A \in \mathbf{P}^{(K,s)}$.

(b) A is diagonalizable, $\sigma(A) \subseteq \{0\} \cup \Omega_{(s+1)^2-1}$, and P_j's satisfy condition $(\mathcal{P})$.

(c) $A^{(s+1)^2} = A$, and P_j's satisfy condition $(\mathcal{P})$.

(d) $A^{\#} = A^{(s+1)^2-2}$, and P_j's satisfy condition $(\mathcal{P})$.

Considering the sets

$$\mathbf{D} = \{A \in \mathbb{C}^{n\times n} : \ A \text{ is diagonalizable}\},$$

$$\mathbf{\Phi} = \{A \in \mathbb{C}^{n\times n} : \ \text{ projectors } P_j \text{ 's satisfy condition } (\mathcal{P})\},$$

$$\mathbf{P}^{(s)} = \{A \in \mathbb{C}^{n\times n} : \ A^{s+1} = A\},$$

$$\mathbf{G}^{(s)} = \{A \in \mathbb{C}^{n\times n} : \ A^{\#} = A^{s+1}\},$$

we can rewrite Theorem 2 as follows:

$$\mathbf{P}^{(K,s)} = \mathbf{D} \cap \mathbf{\Omega}^{(s+1)^2-2} \cap \mathbf{\Phi} = \mathbf{P}^{((s+1)^2-1)} \cap \mathbf{\Phi} = \mathbf{G}^{((s+1)^2-3)} \cap \mathbf{\Phi}.$$

According to the above notation, we have $\mathbf{P}^{(I_n,s)} = \mathbf{P}^{(s)}$. Moreover, $\mathbf{P}^{(1)}$ is the set of all $n \times n$ idempotent matrices. In addition, $\mathbf{P}^{(s+2)} = \mathbf{G}^{(s)}$.

We recall that a matrix $A \in \mathbb{C}^{n\times n}$ with the property $A^{s+1} = A^*$ for some positive integer s is called an $\{s+1\}$-generalized projector [4, 9], and also that A is called $\{K\}$-Hermitian when $KA^*K = A$. The following sets will be useful later:

$$\mathbf{GP}^{(s)} = \{A \in \mathbb{C}^{n\times n} : \ A^{s+1} = A^*\},$$

$$\mathbf{H}^{(K)} = \{A \in \mathbb{C}^{n\times n} : \ KA^*K = A\}.$$

In the following result we derive the relationship between the known concepts of $\{s+1\}$-generalized projector and of a $\{K\}$-Hermitian matrix and the newer concept of a $\{K, s+1\}$-potent matrix, which was introduced in [12].

The matrix

$$A = \frac{1}{2}\begin{bmatrix} 1+i & -1-i \\ 1+i & 1+i \end{bmatrix} \tag{4}$$

is a $\{3\}$-generalized projector for $K = I_2$. However, A is neither $\{K, 3\}$-potent nor $\{K\}$-Hermitian. On the other hand, the matrix

$$A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{5}$$

is $\{K\}$-Hermitian for $K = I_2$. Nevertheless, A is neither $\{K, s+1\}$-potent nor an $\{s+1\}$-generalized projector for any odd positive integer s. Finally, the matrix

$$A = \begin{bmatrix} -3 & -2\sqrt{7} \\ \frac{\sqrt{7}}{2} & 2 \end{bmatrix} \tag{6}$$

is $\{K, 2\}$-potent matrix for

$$K = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

However, A is neither $\{K\}$-Hermitian nor a $\{2\}$-generalized projector.

With these three examples, we tried to find some relations between $\{s+1\}$-generalized projectors, $\{K, s+1\}$-potent matrices and $\{K\}$-Hermitian matrices. In order to satisfy one of them, it is easy to see that the other two are required.

Theorem 3 *The following inclusions hold:*

(a) $\mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} \subseteq \mathbf{GP}^{(s)} \cap \mathbf{P}^{(s+2)} = \mathbf{GP}^{(s)}$.

(b) $\mathbf{P}^{(K,s)} \cap \mathbf{GP}^{(s)} \subseteq \mathbf{H}^{(K)} \cap \mathbf{P}^{(s+2)}$.

(c) $\mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)} \subseteq \mathbf{P}^{(K,s)} \cap \mathbf{P}^{(s+2)}$.

Proof. Let $A \in \mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)}$. Then, $KAK = A^{s+1}$ and $KAK = A^*$. So, $A \in \mathbf{GP}^{(s)}$. From Theorem 2.1 in [4] we have that $A^{s+3} = A$, thus $A \in \mathbf{P}^{(s+2)}$. In addition, we have obtained that $\mathbf{GP}^{(s)} \subseteq \mathbf{P}^{(s+2)}$, then $\mathbf{GP}^{(s)} \cap \mathbf{P}^{(s+2)} = \mathbf{GP}^{(s)}$. Thus, (a) is shown.

By using the definitions, we can easily see that both intersections $\mathbf{P}^{(K,s)} \cap \mathbf{GP}^{(s)}$ and $\mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)}$ are included in $\mathbf{H}^{(K)}$ and $\mathbf{P}^{(K,s)}$, respectively. Again, $\mathbf{GP}^{(s)} \subseteq \mathbf{P}^{(s+2)}$ leads to properties (b) and (c). ■

The examples preceding Theorem 3 show that none of the conditions imply the other two, and that, in general, all three inclusions are proper.

Note that Theorem 3 links one set that depends only on K to two others that depend only on s, and to another set that depends on both K and s. Furthermore, the three smaller sets in the inclusions are non-empty because all of them contain the identity matrix.

A particularly important case is when the matrices A are required to be nonsingular. In this case, the relations with unitary matrices appear. Let

$$\mathbf{GL} = \{A \in \mathbb{C}^{n \times n} : \ A \text{ is nonsingular}\},$$

$$\mathbf{U} = \{A \in \mathbb{C}^{n \times n} : \ AA^* = A^*A = I_n\}.$$

Theorem 4 *The following inclusion holds:*

$$\mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} \cap \mathbf{GL} \subseteq \mathbf{U}.$$

Proof. Let A be both $\{K, s{+}1\}$-potent and $\{K\}$-Hermitian. Item (a) of Theorem 3 implies that $A^{s+3} = A$. Since A is nonsingular, we get $A^{s+2} = I_n$ and then $AA^* = AA^{s+1} = I_n$, which shows that A is unitary. The proof is then completed. ■

We can say even more. In general,

$$\mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} \cap \mathbf{GL} \subsetneq \mathbf{U} \tag{7}$$

because the matrix given in (4) is unitary and belongs to $\mathbf{P}^{(K,4)}$ but it does not belong to $\mathbf{H}^{(K)}$ for $K = I_2$. Also, the matrix given in (5) is unitary and $\{K\}$-Hermitian but it does not belong to $\mathbf{P}^{(K,s)}$ when s is odd for $K = I_2$.

Several interesting observations are obtained as direct consequences of Theorem 3 and Theorem 4. The first one is:

$$\mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)} = \mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} = \mathbf{P}^{(K,s)} \cap \mathbf{GP}^{(s)} = \mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)}, \tag{8}$$

that is, if an element belongs to two of the three sets considered, then it belongs to the remaining one.

Another general observation based on (7) and (8) is that the following proper inclusions hold:

(a) $\mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)} \cap \mathbf{GL} \subsetneq \mathbf{U}$.

(b) $\mathbf{P}^{(K,s)} \cap \mathbf{GP}^{(s)} \cap \mathbf{GL} \subsetneq \mathbf{U}$.

(c) $\mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)} \cap \mathbf{GL} \subsetneq \mathbf{U}$.

Even more, from Theorem 2.1 in [4] it can be shown a more general inclusion holds.

Theorem 5 *The following inclusion holds:*

$$\mathbf{GP}^{(s)} \cap \mathbf{GL} \subseteq \mathbf{U}.$$

Example (5) shows that the last inclusion is strict: s must be odd.

In (7) we have presented an inclusion involving the set of unitary matrices. What happens with $\supseteq$ if we intersect $\mathbf{U}$ with another set?

Let

$$\mathbf{I}^{(s)} = \{A \in \mathbb{C}^{n\times n} : \ A^{s+1} = I_n\}.$$

Note that, when $s = 1$ the set $\mathbf{I}^{(1)}$ corresponds to the involutory matrices, and moreover, it is clear that $\mathbf{I}^{(s)} \subsetneq \mathbf{P}^{(s+1)}$ and $\mathbf{I}^{(s)} = \mathbf{P}^{(s+1)} \cap \mathbf{GL}$.

Theorem 6 *The following inclusions hold:*

(a) $\mathbf{U} \cap K\mathbf{I}^{(1)} \subseteq \mathbf{H}^{(K)}$.

(b) $\mathbf{U}\cap\mathbf{H}^{(K)}\subseteq K\mathbf{I}^{(1)}$.

(c) $\mathbf{U}\cap\mathbf{P}^{(s+2)}\subseteq\mathbf{GP}^{(s)}$.

(d) $K\mathbf{I}^{(1)}\cap\mathbf{H}^{(K)}\subseteq\mathbf{U}$.

Proof. Suppose that A is unitary and there is $B\in\mathbf{I}^{(1)}$ such that $A=KB$. Then $KAKA=(KA)^2=B^2=I_n$ and so, $KAK=A^{-1}=A^*$. Thus, (a) is proved.

In order to prove item (b), post-multiplying by A equality $KAK=A^*$ we obtain $(KA)^2=I_n$ because $A\in\mathbf{U}$. Then, $KA\in\mathbf{I}^{(1)}$ and so $A\in K\mathbf{I}^{(1)}$.

Let $A\in\mathbf{U}\cap\mathbf{P}^{(s+2)}$. Since A is nonsingular, $A^{s+3}=A$ implies that $A^{s+2}=I_n$ which is equivalent to $A^{s+1}=A^{-1}=A^*$. Thus, $A\in\mathbf{GP}^{(s)}$.

Finally, if we suppose that $KAK=A^*$ and $A=KB$ with $B^2=I_n$, then item (d) follows from $A^*A=(KAK)A=(KA)^2=B^2=I_n$. Thus, the proof is completed. ∎

In order to see that, in general, the inclusions in Theorem 6 are proper we can consider the following examples:

$$\begin{bmatrix}1&0\\0&0\end{bmatrix},\ \begin{bmatrix}0&2i\\-\frac{1}{2}i&0\end{bmatrix},\ \begin{bmatrix}1&0\\0&0\end{bmatrix}\ \text{with } s\in\{1,2,3,\dots\},\text{ and the matrix given by (4),}$$

all for $K=I_2$, corresponding respectively to items $(a)-(d)$.

For the set $K\mathbf{I}^{(1)}$ we can also obtain the following inclusions:

(a) $K\mathbf{I}^{(1)}\cap\mathbf{P}^{(K,s)}\subseteq\mathbf{I}^{(s+1)}$.

(b) $\mathbf{GP}^{(s)}\cap K\mathbf{I}^{(1)}\cap\mathbf{GL}\subseteq\mathbf{P}^{(K,s)}$.

(c) $\mathbf{P}^{(s+2)}\cap K\mathbf{I}^{(1)}\cap\mathbf{H}^{(K)}\subseteq\mathbf{P}^{(K,s)}$.

The proof requires property (b) in Lemma 1 and Theorem 2.1 in [4]. In general, all the inclusions are proper as the following respective examples show: the same last matrix A for (a), and the matrix given in (6) for (b) and (c).

Again, what happens in (7) with $\supseteq$ if we intersect $\mathbf{U}$ with $\mathbf{P}^{(K,s)}$? The following result can be given.

Proposition 1 *Let* $(\mathbf{P}^{(K,s)})^*=\{A^*:\ A\in\mathbf{P}^{(K,s)}\}$. *Then the following property holds:*

$$\mathbf{U}\cap\mathbf{P}^{(K,s)}=\mathbf{U}\cap(\mathbf{P}^{(K,s)})^*.$$

Proof. Let $A\in\mathbf{U}$. We have to prove that $A\in\mathbf{P}^{(K,s)}$ if and only if $A^*\in\mathbf{P}^{(K,s)}$. In fact, if $A\in\mathbf{P}^{(K,s)}$ by using property (e) from Lemma 1 we have

$$KA^*K=KA^{-1}K=(A^{-1})^{s+1}=(A^*)^{s+1}.$$

Thus, $A^*\in\mathbf{P}^{(K,s)}$ and so $A\in(\mathbf{P}^{(K,s)})^*$. The converse can be obtained in a similar way. This ends the proof. ∎

What happens with the equality $\mathbf{P}^{(K,s)} = (\mathbf{P}^{(K,s)})^*$ when $A \notin \mathbf{U}$? In general, it is not satisfied. In fact, the matrix given in (6) belongs to $\mathbf{P}^{(K,1)}$ and $A \notin \mathbf{U}$, but A does not belong to $(\mathbf{P}^{(K,1)})^*$.

However, the equality remains valid when matrix K is Hermitian. The following result extends the properties (d) and (e) presented in Lemma 1.

Lemma 2 *Let* $(\mathbf{P}^{(K,s)})^{-1} = \{A^{-1} : \ A \in \mathbf{P}^{(K,s)} \cap \mathbf{GL}\}$*. Then*

(a) $\mathbf{P}^{(K^*,s)} = (\mathbf{P}^{(K,s)})^*$*. In particular,* $\mathbf{P}^{(K,s)} = (\mathbf{P}^{(K,s)})^*$ *when* K *is Hermitian.*

(b) $\mathbf{P}^{(K,s)} \cap \mathbf{GL} = (\mathbf{P}^{(K,s)})^{-1}$.

In particular, $A \in (\mathbf{P}^{(K,s)})^{-1}$ if and only if $A^{-1} \in (\mathbf{P}^{(K,s)})^{-1}$.

Furthermore, in general, the equality $\mathbf{P}^{(K,s)} = \mathbf{P}^{(K^*,s)}$ does not hold when K is not Hermitian. This can be checked using the matrix of example (6).

In addition, we can assure that in general

$$\mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)} \nsubseteq \mathbf{U},$$

as the $\{K, 5\}$-potent matrix

$$A = \begin{bmatrix} i & 0 & 0 \\ 0 & 5 & -2 \\ 0 & 15 & -6 \end{bmatrix}$$

shows for

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Note that $\mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)} \neq \emptyset$ because it contains the zero matrix.

In order to see that in general $\mathbf{U} \nsubseteq \mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)}$, we can consider $A = -I_n$, any positive odd number s, and any involutory matrix $K \in \mathbb{C}^{n\times n}$.

We close this section by analyzing the last considered set $\mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)}$. We recall that K^*K is a Hermitian matrix. Hence, there is a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D \in \mathbb{C}^{n\times n}$ such that

$$K^*K = UDU^* \tag{9}$$

Theorem 7 *Let us consider a unitary matrix* U *and a diagonal matrix*

$$D = \mathrm{diag}(\lambda_1 I_{r_1}, \lambda_2 I_{r_2}, \dots, \lambda_t I_{r_t})$$

as in (9) with $r_1 + r_2 + \cdots + r_t = n$ *and* $\lambda_i \neq \lambda_j$ *for* $i \neq j$*. Then the set* $\mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)}$ *is given by*

$$\left\{ U\tilde{D}U^* : \ \tilde{D} = \mathrm{diag}(A_{11}, A_{22}, \dots, A_{tt}) \in \mathbf{P}^{(U^*KU,s)} \cap \mathbf{P}^{(U^*K^*U,s)}, A_{ii} \in \mathbb{C}^{r_i\times r_i}, i = 1, \dots, t \right\}.$$

Proof. Let $A \in \mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)}$. Then $A^{s+1} = KAK$ and $(A^*)^{s+1} = KA^*K$. By making some algebraic manipulations we get $K^*KA = AK^*K$. Since K^*K is Hermitian, we can suppose that $K^*K = UDU^*$ with U and D as in the statement. Then $U^*AUD = DU^*AU$. Now, we partition matrix U^*AU as follows

$$\tilde{A} = U^*AU = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1t} \\ A_{21} & A_{22} & \dots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ A_{t1} & A_{t2} & \dots & A_{tt} \end{bmatrix}$$

according to the sizes of the blocks of D. Thus, $\tilde{A}D = D\tilde{A}$ yields to $A_{ij} = O$ for all $i, j \in \{1, 2, \dots, t\}$ with $i \neq j$, and consequently

$$A = U\mathrm{diag}(A_{11}, A_{22}, \dots, A_{tt})U^*.$$

Since $A \in \mathbf{P}^{(K,s)}$, we have that

$$U^*KU \begin{bmatrix} A_{11}^{s+1} & O & \dots & O \\ O & A_{22}^{s+1} & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A_{tt}^{s+1} \end{bmatrix} U^*KU = \begin{bmatrix} A_{11} & O & \dots & O \\ O & A_{22} & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A_{tt} \end{bmatrix}.$$

Then $\tilde{D} = \mathrm{diag}(A_{11}, A_{22}, \dots, A_{tt})$ is $\{U^*KU, s+1\}$-potent because U^*KU is involutory. A similar reasoning is valid for $A \in \mathbf{P}^{(K^*,s)}$. Then, $\tilde{D} \in \mathbf{P}^{(U^*KU,s)} \cap \mathbf{P}^{(U^*K^*U,s)}$. The other inclusion is obvious. $\blacksquare$

The previous theorem allows us to obtain another representation for the set $\mathbf{P}^{(K,s)}$ (see Theorem 2). Its advantage is that the diagonalization is given by a unitary matrix U, however, unlike Theorem 2, the diagonal matrix $\tilde{D}$ is given by blocks.

Note that, with respect to the spectrum of every A_{ii}, for $i = 1, 2, \dots, t$, we have

$$\sigma(A_{ii}) \subseteq \sigma(\tilde{D}) \subseteq \{0\} \cup \Omega_{(s+1)^2-1}.$$

3 Analysis of $\mathbf{P}^{(K,s)}$ through $\mathbf{GP}^{(s)}$, $K\mathbf{H}$, $\mathbf{N}$, and $K\mathbf{N}$

More relations can be obtained when Hermitian matrices are involved. Let

$$\mathbf{H} = \{A \in \mathbb{C}^{n\times n} : \ A^* = A\}. \tag{10}$$

Note that $\mathbf{H}^{(I_n)} = \mathbf{H}$. When $K \in \mathbf{H}$ it can be shown that $\mathbf{H} = K\mathbf{H}^{(K)} = \mathbf{H}^{(K)}K$ or equivalently $\mathbf{H}^{(K)} = K\mathbf{H} = \mathbf{H}K$.

Without assuming any hypothesis on matrix K we can give the following result.

Theorem 8 *The following inclusion holds:*

$$K\mathbf{H} \cap \mathbf{P}^{(K,s)} \subseteq K\mathbf{P}^{(2)}.$$

Proof. It is well-known that the spectrum of the Hermitian matrix KA is real and also that it is included in $\{0\} \cup \Omega_{2s}$ when $A \in \mathbf{P}^{(K,s)}$ by using property (c) of Lemma 1. Hence, KA has its spectrum included in $\{0, 1, -1\}$. Since KA is diagonalizable, it is tripotent which ends the proof. ■

In general, the last inclusion is strict as the following matrices show:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ -3 & -2 & 1 \end{bmatrix} = KB$$

where

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix}.$$

However, the additional condition on K to be Hermitian allows us to obtain a more interesting result: an equality for the intersection $K\mathbf{H} \cap \mathbf{P}^{(K,s)}$.

Theorem 9 *Let $K \in \mathbf{H}$. Then*

$$K\mathbf{H} \cap \mathbf{P}^{(K,s)} = \mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)}.$$

Proof. Let $A \in \mathbf{P}^{(K,s)}$. We have to prove that $A \in K\mathbf{H}$ if and only if $A \in \mathbf{GP}^{(s)}$. In fact, if there is $B \in \mathbf{H}$ such that $A = KB$ then it is obvious that $KA = B$ and so $KA \in \mathbf{H}$. Moreover, the equalities

$$KA = (KA)^* = A^*K$$

allow us to conclude that $A^{s+1} = KAK = A^*$. Thus $A \in \mathbf{GP}^{(s)}$.

Conversely, if $A \in \mathbf{GP}^{(s)}$, by definition $A^{s+1} = A^*$. Then, $KAK = A^*$ and $KA = A^*K$. So, we get $A = K(A^*K)$, that is, $A = KB$ where $B = A^*K$ satisfies

$$B^* = K^*A = KA = A^*K = B.$$

Thus, $B \in \mathbf{H}$ and finally $A \in K\mathbf{H}$. This concludes the proof. ■

We now consider the set of normal matrices:

$$\mathbf{N} = \{A \in \mathbb{C}^{n \times n} : \; AA^* = A^*A\}.$$

In [4], the equivalences between the following statements have been established:

$$A^{s+1} = A^* \Longleftrightarrow A \text{ is a normal matrix, } \sigma(A) \subseteq \Omega_{s+2} \Longleftrightarrow A \text{ is a normal matrix, } A^{s+3} = A. \tag{11}$$

From Theorem 9 and from the aforementioned result, we derive a relationship between $\{K, s+1\}$-potent matrices and normal matrices, $\{s+1\}$-generalized projectors, and $\{s+1\}$-potent matrices.

Proposition 2 *Let $K \in \mathbf{H}$. Then*

$$\mathbf{N} \cap \mathbf{\Omega}^{(s+1)} \cap \mathbf{P}^{(K,s)} = \mathbf{N} \cap \mathbf{P}^{(s+2)} \cap \mathbf{P}^{(K,s)} = K\mathbf{H} \cap \mathbf{P}^{(K,s)}. \tag{12}$$

Proof. It follows directly from (11) and Theorem 9. ■

In particular, in Proposition 2 we have established that

$$\mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)} \subseteq \mathbf{N} \cap \mathbf{P}^{(K,s)}$$

holds. However, in general

$$\mathbf{N} \cap \mathbf{P}^{(K,s)} \not\subseteq \mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)}$$

as the following example shows. In fact, it is clear that matrix $A = iI_2$ is normal and, considering $K = I_2$, we have that A is $\{K, 5\}$-potent and $A^5 \neq A^*$.

Furthermore, in general

(a) $\mathbf{N} \cap \mathbf{\Omega}^{(s+1)} \not\subseteq \mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)}$ (as matrix (4) shows by setting $s = 2$).

(b) $\mathbf{P}^{(s+2)} \cap \mathbf{P}^{(K,s)} \not\subseteq \mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)}$ (as the following matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tag{13}$$

shows by considering $K = I_2$ and $s = 1$).

(c) $\mathbf{\Omega}^{(s+1)} \cap \mathbf{P}^{(K,s)} \not\subseteq \mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)}$ (as the same example than in item (b) shows).

(d) $\mathbf{GP}^{(s)} \not\subseteq K\mathbf{H} \cap \mathbf{P}^{(K,s)}$ (as matrix (4) shows by setting $s = 2$).

Now, we can state that $K\mathbf{N}$ is closed under multiplication by K for the $\{K, s+1\}$-potent matrices.

Theorem 10 *Let $K \in \mathbf{H}$. Then $\mathbf{N} \cap \mathbf{P}^{(K,s)} \subseteq K\mathbf{N}$.*

Proof. Let $A \in \mathbf{N} \cap \mathbf{P}^{(K,s)}$. Theorems 1 and 2 assure that

$$A = \sum_{j=1}^{(s+1)^2-1} \omega_{(s+1)^2-1}^{j} P_j$$

where P_j's are orthogonal disjoint projectors satisfying condition $(\mathcal{P})$. Note that $P_{j_0} = O$ if there exists $j_0 \in \{1, 2, \ldots, (s+1)^2 - 1\}$ such that $\omega_{(s+1)^2-1}^{j_0} \notin \sigma(A)$ and moreover, $P_0 = O$ when $0 \notin \sigma(A)$. Then,

$$\begin{aligned} KA(KA)^* &= \left(\sum_{j=1}^{(s+1)^2-1} \omega_{(s+1)^2-1}^{j} KP_j \right) \left(\sum_{j=1}^{(s+1)^2-1} \omega_{(s+1)^2-1}^{-j} P_j K \right) \\ &= \sum_{j=1}^{(s+1)^2-1} KP_jK = \sum_{j=1}^{(s+1)^2-2} P_{\varphi(j)} + P_{(s+1)^2-1} = \sum_{i=1}^{(s+1)^2-1} P_i. \end{aligned}$$

In the last equality we have used relation (3). A similar computation yields to

$$(KA)^*KA = \sum_{j=1}^{(s+1)^2-1} P_j.$$

Thus, $KA \in \mathbf{N}$. This concludes the proof. $\blacksquare$

Nevertheless, we cannot establish that $\mathbf{N}$ is closed under multiplication by K. Even more, in general, the opposite inclusion in Theorem 10 is not true, that is $\mathbf{N} \cap \mathbf{P}^{(K,s)} \subsetneq K\mathbf{N}$. In fact, the following matrix

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$$

is not normal and $KA \in \mathbf{N}$ where

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In order to state a characterization of the $\{K, s+1\}$-potent matrices which are also normal matrices we give the following result.

Theorem 11 *The following equality holds:*

$$\mathbf{N} \cap \mathbf{P}^{(K,s)} = \mathbf{GP}^{((s+1)^2-3)} \cap \mathbf{P}^{(K,s)}.$$

Proof. Let $A \in \mathbf{P}^{(K,s)}$. We have to prove that $A \in \mathbf{N}$ if and only if $A \in \mathbf{GP}^{((s+1)^2-3)}$. In fact, if $A \in \mathbf{N}$, by Theorem 1 and Theorem 2, we have that all the distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of A belong to $\{0\} \cup \Omega_{(s+1)^2-1}$ and there exist disjoint orthogonal projectors $P_1, P_2, \ldots, P_k$ such that $A = \sum_{j=1}^{k} \lambda_j P_j$. Under these conditions, it is easy to see that

$$A^* = \sum_{j=1}^{k} \bar{\lambda}_j P_j \quad \text{and} \quad A^m = \sum_{j=1}^{k} \lambda_j^m P_j$$

for each $m \in \{1, 2, 3, \ldots\}$. For all $\lambda_j \in \Omega_{(s+1)^2-1}$, we get $\lambda_j \bar{\lambda}_j = 1 = \lambda_j \lambda_j^{(s+1)^2-2}$ and then

$$A^* = \sum_{j=1}^{k} \lambda_j^{-1} P_j = \sum_{j=1}^{k} \lambda_j^{(s+1)^2-2} P_j = A^{(s+1)^2-2}.$$

This proves that $A \in \mathbf{GP}^{((s+1)^2-3)}$.

The converse is obvious because A^* is a power of A. $\blacksquare$

If we use the set $\mathbf{GP}^{(s)}$ instead of $\mathbf{N}$ in the equality (12) then it looks like as follows:

$$\mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)} = \mathbf{\Omega}^{(s+1)} \cap \mathbf{GP}^{((s+1)^2-3)} \cap \mathbf{P}^{(K,s)} = \mathbf{P}^{(s+2)} \cap \mathbf{GP}^{((s+1)^2-3)} \cap \mathbf{P}^{(K,s)}$$

as Theorems 9 and 11 show. As a consequence, Theorems 10 and 11 derive the following result.

Corollary 1 *Let $K \in \mathbf{H}$. Then*

$$\mathbf{GP}^{((s+1)^2-3)} \cap \mathbf{P}^{(K,s)} \subseteq K\mathbf{N} \cap \mathbf{P}^{(K,s)}.$$

Observe that, in general, the opposite inclusion in Corollary 1 is not true. For that, we consider the matrices given in (13). We can check that $A \in \mathbf{P}^{(K,1)}$, $A \in K\mathbf{N}$, and it is easy to see that $A \notin \mathbf{GP}^{(1)}$. So, we can assure that, in general,

$$\mathbf{GP}^{((s+1)^2-3)} \cap \mathbf{P}^{(K,s)} \subsetneq K\mathbf{N} \cap \mathbf{P}^{(K,s)}$$

holds.

We now derive a relation between $K\mathbf{N}$ and $K\mathbf{GP}^{(m)}$ for some special $m \in \{1, 2, 3, \dots\}$.

Theorem 12 *The following equality holds:*

$$K\mathbf{N} \cap \mathbf{P}^{(K,s)} = K\mathbf{GP}^{(2s-2)} \cap \mathbf{P}^{(K,s)}.$$

Proof. Let $A \in \mathbf{P}^{(K,s)}$. We have to prove that $A \in K\mathbf{N}$ if and only if $A \in K\mathbf{GP}^{(2s-2)}$. In fact, if $KA \in \mathbf{N}$, by Theorems 1, 2 and Lemma 1 (c), we have that all the distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of KA belong to $\{0\} \cup \Omega_{2s}$ and there exist disjoint orthogonal projectors $P_1, P_2, \dots, P_k$ (i.e., $P_iP_j = \delta_{ij}P_i$ and $P_i^* = P_i$ for $i, j \in \{1, 2, \dots, k\}$) such that $KA = \sum_{j=1}^k \lambda_j P_j$. Under these conditions, it is easy to see that $(KA)^* = \sum_{j=1}^k \bar{\lambda}_j P_j$ and $(KA)^m = \sum_{j=1}^k {\lambda_j}^m P_j$ for any $m \in \{1, 2, 3, \dots\}$. For all $\lambda_j \in \Omega_{2s}$, we get $\lambda_j\bar{\lambda}_j = 1 = \lambda_j^{2s}$ and then

$$(KA)^* = \sum_{j=1}^k \bar{\lambda}_j P_j = \sum_{j=1}^k \lambda_j^{-1} P_j = \sum_{j=1}^k \lambda_j^{2s-1} P_j = (KA)^{2s-1}.$$

This proves that $KA \in \mathbf{GP}^{(2s-2)}$.

The converse is obvious because $(KA)^*$ is a power of KA. Then, the proof is finished. ■

Remark 1 *We note that $\mathbf{GP}^{(0)} = \mathbf{H}$. From Theorem 12 we can deduce that*

$$K\mathbf{N} \cap \mathbf{P}^{(K,1)} = K\mathbf{H} \cap \mathbf{P}^{(K,1)}.$$

4 Further results

In this section we present some properties similar to those studied so far, but we want to focus on the cases (anti-)commutative matrices, skew-Hermitian matrices, and matrices for which any of its powers coincides with its opposite.

4.1 Analysis of $\mathbf{P}^{(K,s)}$ through H, SH, $\mathbf{C}^{(K,m)}$, and $\mathbf{SP}^{(s)}$

We start this subsection by defining the set of skew-Hermitian matrices and the set of all matrices that commute (anti-commute) with the matrix K:

$$\mathbf{SH} = \{A \in \mathbb{C}^{n\times n} : \ A^* = -A\}$$

$$\mathbf{C}^{(K,m)} = \{A \in \mathbb{C}^{n\times n} : \ AK = mKA\} \text{ where } m \in \{+,-\}.$$

We also introduce the set given by:

$$\mathbf{SP}^{(s)} = \{A \in \mathbb{C}^{n\times n} : \ A^{s+1} = -A\}.$$

Now, we can state the following results.

Theorem 13 *The following equalities hold:*

(a) $\mathbf{H} \cap \mathbf{P}^{(K,s)} = \mathbf{C}^{(K,+)} \cap \mathbf{T}$ *where*

$$\mathbf{T} = \begin{cases} \mathbf{P}^{(1)} \cap \mathbf{GP}^{(1)} & \textit{when } s \textit{ is odd,} \\ \mathbf{P}^{(2)} \cap \mathbf{GP}^{(2)} & \textit{when } s \textit{ is even.} \end{cases}$$

(b) *(i)* $\mathbf{SH} \cap \mathbf{P}^{(K,s)} = \{O\}$ *when* s *is odd.*

(ii) $\mathbf{SH} \cap \mathbf{P}^{(K,s)} = \mathbf{SP}^{(2)} \cap \mathbf{GP}^{(2)} \cap \mathbf{T}$ *where*

$$\mathbf{T} = \begin{cases} \mathbf{C}^{(K,+)} & \textit{when } s = 4t, \\ \mathbf{C}^{(K,-)} & \textit{when } s = 4t+2, \end{cases}$$

for $t \in \{1, 2, 3, \dots\}$.

Proof. In order to show (a) we suppose that $A^* = A$ and $KA^{s+1}K = A$. Then

$$\sigma(A) \subseteq \mathbb{R} \cap (\{0\} \cup \Omega_{(s+1)^2-1}) = \begin{cases} \{0,1\} & \text{when } s \text{ is odd,} \\ \{0,1,-1\} & \text{when } s \text{ is even,} \end{cases}$$

so,

$$A = \begin{cases} A^2 & \text{when } s \text{ is odd,} \\ A^3 & \text{when } s \text{ is even,} \end{cases}$$

and thus, we get $KAK = A^{s+1} = A$. The converse can be easily checked.

Item (b) follows trivially when s is odd. If s is even and $A \in \mathbf{SH} \cap \mathbf{P}^{(K,s)}$ then $\sigma(A) \subseteq i\mathbb{R} \cap (\{0\} \cup \Omega_{s(s+2)}) = \{0, i, -i\}$. Hence $A^3 = -A$, and so, when $s = 4t$ then

$$KAK = A^{s+1} = A^{4t+1} = -A^3 = A,$$

for every $t \in \{1, 2, 3, \dots\}$. In a similar way, the case $s = 4t+2$ follows from $A^{4t} = -A^2$. As before, the converse can be easily checked. ∎

Furthermore, the $\{K, s+1\}$-potent matrices which are also in $K\mathbf{SH}$ satisfy the following relations.

Theorem 14 *The following statements hold:*

(a) $K\mathbf{SH} \cap \mathbf{P}^{(K,s)} \subseteq K\mathbf{SP}^{(2)}$ *when* s *is even.*

(b) $K\mathbf{SH} \cap \mathbf{P}^{(K,s)} = \{O\}$ *when* s *is odd.*

Proof. Let $A \in K\mathbf{SH}$. We first observe that $KA \in \mathbf{SH}$ and thus $\sigma(KA) \subseteq i\mathbb{R}$. Now, for $A \in \mathbf{P}^{(K,s)}$ we get $(KA)^{2s+1} = KA$ and so, $\sigma(KA) \subseteq \{0\} \cup \Omega_{2s}$. Hence, the case for s odd follows trivially. When s is even, we can see that $\sigma(KA) \subseteq \{0, i, -i\}$. Since KA is diagonalizable, we obtain that $(KA)^3 = -KA$, which means that $A \in K\mathbf{SP}^{(2)}$. ■

Note that, in general, the inclusion in (a) is strict as can be checked by using the example given in (13).

We can also observe that $K\mathbf{SP}^{(2)} \subseteq \mathbf{SP}^{(s+2)}$ because $(KA)^3 = -KA$ implies that $A^{s+3} = -A$. In general, in this case the inclusion is strict because the 1×1 matrix defined by $A = \begin{bmatrix} \omega \end{bmatrix}$ satisfies the required conditions assuming that $\omega^3 = -1 \neq \omega^2$ holds for $K = \begin{bmatrix} 1 \end{bmatrix}$.

Finally, when $K^* = K$ we can observe that the following conditions are equivalent:

(a) $KA \in \mathbf{SH}$.

(b) $KA^*K = -A$.

(c) $AK \in \mathbf{SH}$.

Also, the condition $A \in \mathbf{SH}$ is equivalent to $K(KA)^*K = -KA$.

In [12], the definition of $\{K, s+1\}$-potent matrices considered $s \in \{1, 2, 3, \dots\}$ excluding the case $s = 0$. The reason is that Theorem 2 does not give any information for the case $s = 0$. So, the second subsection provides with some results valid for $s = 0$.

4.2 New results on K-generalized centrosymmetric matrices

Firstly, we observe that $\mathbf{P}^{(K,0)} = \mathbf{C}^{(K,+)}$, which is the set of the K-generalized centrosymmetric matrices. By using the spectral decomposition, we can easily deduce the following result.

Theorem 15 *Let* $A \in \mathbb{C}^{n\times n}$ *be a diagonalizable matrix with spectral decomposition given by* $A = \sum_{i=1}^{k} \lambda_i P_i$ *as in Theorem 1 and* $K \in \mathbb{C}^{n\times n}$ *be an involutory matrix. Then* $A \in \mathbf{C}^{(K,+)}$ *if and only if* $KP_i = P_iK$*, for all* $i \in \{1, 2, \dots, k\}$.

We now derive some new results for the K-generalized centrosymmetric matrices from those established in section 2 and 3. Henceforth, we can get these new relations by setting $s = 0$ in all the corresponding results after doing simple verifications. In what follows, we present the results deduced from section 2.

Theorem 16 *The following statements hold:*

(a) $\mathbf{C}^{(K,+)} \cap \mathbf{H}^{(K)} \subsetneq \mathbf{H}$.

(b) $\mathbf{C}^{(K,+)} \cap \mathbf{H} \subsetneq \mathbf{H}^{(K)}$.

(c) $\mathbf{H}^{(K)} \cap \mathbf{H} \subsetneq \mathbf{C}^{(K,+)}$.

(d) $\mathbf{C}^{(K,+)} \cap \mathbf{H}^{(K)} \cap \mathbf{H} = \mathbf{C}^{(K,+)} \cap \mathbf{H}^{(K)} = \mathbf{C}^{(K,+)} \cap \mathbf{H} = \mathbf{H}^{(K)} \cap \mathbf{H}$.

(e) $\mathbf{C}^{(K,+)} \cap \mathbf{H}^{(K)} \cap \mathbf{I}^{(1)} \subsetneq \mathbf{U}$.

(f) $\mathbf{C}^{(K,+)} \cap \mathbf{H} \cap \mathbf{I}^{(1)} \subsetneq \mathbf{U}$.

(g) $\mathbf{C}^{(K,+)} \cap K\mathbf{I}^{(1)} \subsetneq \mathbf{I}^{(1)}$.

(h) $\mathbf{I}^{(1)} \cap K\mathbf{I}^{(1)} \subsetneq \mathbf{C}^{(K,+)}$.

(i) $\mathbf{P}^{(2)} \cap K\mathbf{I}^{(1)} \subsetneq \mathbf{C}^{(K,+)}$.

(j) $\mathbf{U} \cap \mathbf{C}^{(K,+)} = \mathbf{U} \cap (\mathbf{C}^{(K,+)})^{*}$.

The matrix

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

allows us to show that the inclusion in item (e) is strict for $K = I_2$.

The results deduced from section 3 are the following.

Theorem 17 *Let $K \in \mathbf{H}$. Then*

(a) $\mathbf{H} \cap \mathbf{C}^{(K,+)} = K\mathbf{H} \cap \mathbf{C}^{(K,+)}$.

(b) $\mathbf{N} \cap \mathbf{\Omega}^{(1)} \cap \mathbf{C}^{(K,+)} = \mathbf{N} \cap \mathbf{P}^{(2)} \cap \mathbf{C}^{(K,+)} = K\mathbf{H} \cap \mathbf{C}^{(K,+)}$.

(c) $\mathbf{N} \cap \mathbf{C}^{(K,+)} \subsetneq K\mathbf{N}$.

5 Acknowledgements

We thank the referees for their valuable comments and their constructive suggestions which have allowed us to improve the manuscript.

References

[1] A. Andrew, *Eigenvector of certain matrices*, Linear Algebra and its Applications, 157–162, 1973.

[2] A. Andrew, *Centrosymmetric matrices*, SIAM Review, 40, 697–698, 1998.

[3] A. Ben-Israel, T. Greville. Generalized Inverses: Theory and Applications, John Wiley & Sons, 2nd Edition, 2003.

[4] J. Benítez, N. Thome, *Characterizations and linear combinations of k-generalized projectors.* Linear Algebra and its Applications, 410, 150–159, 2005.

[5] J. Benítez, N. Thome, *$\{k\}$-group periodic matrices.* SIAM J. Matrix Analysis and Applications, 28 (1), 9–25, 2006.

[6] R. Bru, N. Thome. *Group inverse and group involutory matrices*, Linear and Multilinear Algebra, 45 (2-3), 207–218, 1998.

[7] C. Coll, L. Lebtahi, N. Thome. *$\{P, s+1\}$-Potent Matrices*, XXI Congreso de Ecuaciones Diferenciales y Aplicaciones (CEDYA09), Ciudad Real, 2009 (in Spanish).

[8] L. Datta, S. Morgera, *On the reducibility of centrosymmetric matrices-applications in engineering problems*, Circuits Systems Signal Process, 8, 71–96, 1989.

[9] Hong-Ke Du, Yuan Li, *The spectral characterization of generalized projections*, Linear Algebra and its Applications, 400, 313–318, 2005.

[10] A. Gersho, *Adaptive equalization of highly dispersive channels for data transmission*, BSTJ 48, 55–70, 1969.

[11] R. D. Hill, S. R. Waters. *On κ-Real and κ-Hermitian Matrices*, Linear Algebra and its Applications, 169, 17–29, 1992.

[12] L. Lebtahi, O. Romero, N. Thome. *Characterizations of $\{K, s+1\}$-Potent Matrices and Applications.* To appear in Linear Algebra and its Applications. DOI: 10.1016/j.laa.2010.11.034.

[13] Guo-Lin Li, Zheng-He Feng. *Mirrorsymmetric matrices, their basic properties, and an application on odd/even mode decomposition of symmetric multiconductor transmission lines*, SIAM J. Matrix Analysis and Applications, 24, 1, 78–90, 2002.

[14] Z. Liu. *Some properties of centrosymmetric matrices.* Applied Mathematics & Computation, 141 (2-3), 17–26, 2002.

[15] Z. Liu, H. Faßbender. *Some properties of generalized K-centrosymmetric H-matrices*, Journal of Computational and Applied Mathematics, 215, 38–48, 2008.

[16] F.R. Magee Jr., J.G. Prokis, *An estimate of an upper bound on error probability for maximum likelihood sequence estimation on channels having finite-duration pulse response*, IEEE Trans. Inf. Theory IT-19 22 (September) (1973) 699–702.

[17] J.L. Stuart, *Inflation matrices and ZME-Matrices that commute with a permutation matrix*, SIAM J. Matrix Analysis Applications, 9 (3), 408–418, 1988.

[18] J.L. Stuart, J.R. Weaver, *Matrices that commute with a permutation matrix*, Linear Algebra and its Applications, 150, 255–265, 1991.

[19] D. Tao, M. Yasuda, *A spectral characterization of generalized real symmetric centrosymmetric and generalized real symmetric skew-centrosymmetric matrices*, SIAM J. Matrix Analysis Applications, 23, 3, 885–895, 2002.

[20] W.F. Trench, *Characterization and properties of matrices with generalized symmetry or skew symmetry*, Linear Algebra and its Applications, 377, 207–218, 2004.

[21] J. Weaver, *Centrosymmetric (cross-symmetric) matrices, their basic properties, eigenvalues, and eigenvectors*, American Mathematical Monthly, 92, 711–717, 1985.

[22] M. Yasuda, *A spectral characterization for Hermitian centrosymmetric K matrices and Hermitian skew-centrosymmetric K matrices*, SIAM J. Matrix Analysis Applications, 25, 3, 601–605, 2003.