UNIVERSITAT POLITÈCNICA DE VALÈNCIA

DEPARTAMENT DE MATEMÀTICA APLICADA

Atomic decompositions and frames in Fréchet spaces and their duals

TESI DOCTORAL REALITZADA PER:

Juan Miguel Ribera Puchades

DIRIGIDA PER:

José Bonet Solves

Carmen Fernández Rosell

Antonio Galbis Verdú

VALÈNCIA, FEBRER 2015

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

DEPARTAMENT DE MATEMÀTICA APLICADA

Atomic decompositions and frames in Fréchet spaces and their duals

TESI DOCTORAL REALITZADA PER:

Juan Miguel Ribera Puchades

DIRIGIDA PER:

José Bonet Solves

Carmen Fernández Rosell

Antonio Galbis Verdú

VALÈNCIA, FEBRER 2015

Don **José Bonet Solves**, Catedrático de Universidad de la Universitat Politècnica de València, Doña **Carmen Fernández Rosell**, Profesora Titular del Departament d'Anàlisi Matemàtica de la Universitat de València y Don **Antonio Galbis Verdú**, Catedrático de Universidad de la Universitat de València

CERTIFICAN:

que la presente memoria "Atomic decompositions and frames in Fréchet spaces and their duals" ha sido realizada bajo nuestra dirección por Juan Miguel Ribera Puchades y constituye su tesis para optar al grado de Doctor en Ciencias Matemáticas, con mención "Doctor Internacional".

Y para que así conste en cumplimiento de la legislación vigente presentamos y apadrinamos ante la Escuela de Doctorado de la Universitat Politècnica de València la referida tesis firmando el presente certificado.

Valencia, Febrero de 2015

Los directores:

José Bonet Solves, Carmen Fernández Rosell y Antonio Galbis Verdú

a la memòria de mon pare Juan Bautista
a la memòria dels meus iaios Miguel i Milieta
a la meua mare

Agraïments

Aquestes dues pàgines no són prou per incloure els agraïments que voldria donar a totes les persones que m'han acompanyat, en els últims quatre anys, mentre elaborava aquesta tesi.

Estic molt agraït als meus tres directors de tesi, els professors *José Bonet*, *Carmina Fernández* i *Antonio Galbis*, qui han sigut molt bons directors i grans companys de treball. D'ells sempre he rebut bons consells i indicacions per seguir endavant en la meua carrera i, sobretot, un exemple de dedicació i amor per les matemàtiques. Sense dubte, he sigut afortunat tenint la possibilitat de treballar amb ells.

I would also like to thank professors *Leonhard Frerick* and *Jochen Wergenroth* for being my advisors in my stays in Trier (Germany). Both of them teach me the importance of the perfection in my work as well as a related topic in my mathematical research.

Gràcies també, a tots els professors de matemàtiques que he tingut al llarg de la meua vida i que m'han dut fins a aquest moment. Entre ells es mereixen una menció especial (els ja amics) *Pep Borràs* i *Salvador (Voro) Messeguer* qui, des de que estudiava batxillerat fins a l'actualitat, sempre m'han donat ànims i vitalitat per tal de donar el màxim de mi; així com grans esmorzars on ells em donaven consells i jo els apropava el món universitari de les matemàtiques de nou.

Cal recordar també, a tots els professors de la Facultat de Matemàtiques de la Universitat de València que m'han format com a matemàtic. Destacar a un professor de la Facultat, que no fou professor meu, però que m'ha ensenyat moltes coses, no sols de matemàtiques, sinó de la vida en general amb els seus consells, *Rafael Crespo*. A ell, donar-li les gràcies per haver comptat amb mi com a professor de matemàtiques al programa EstalmatCV i a altres 'bolos' que tant m'han aportat i que m'han servit per a formar-me com a professor de matemàtiques.

Pero igual que Rafa ha sido como un padrino en el mundo de las matemáticas, mi madrina es *Clara Jiménez*, profesora en la Universidad de la Rioja. Desde que la

conocí en el año 2007 hasta la actualidad, me une a ella una gran amistad, que siempre ha venido acompañada de sabios consejos en mi trabajo por teléfono.

Tot i que no tinc germans de sang, tinc amics al voltant del món que són com germans. En primer lloc *Jaime Sánchez* i *Eva Primo*, un parell de germans inseparables de mi (i entre ells), que m'han fet sentir a València com si fos ma casa, i que han sigut també confidents i bons consellers. *Maria José Beltran*, o més bé, la germaneta, fou qui va estar al inici d'aquest camí acompanyat de partides de ping-pong que sempre guanyava ella. També està *Ronald Jiménez*, "mi hermano de otra madre", com ell deia, que va nàixer a Mèxic però que ha viscut una vida paral·lela a la meua; les nostres vides van interseccionar a València durant la seua estada investigadora on em va ensenyar a valorar tot el que tinc i a disfrutar de les coses bones que ens dona la vida. Als meus *norma-bro, pepper-bro* i *Chuck-bro*, que estan per tot arreu de Espanya i que han sigut companys de congressos durant la meua carrera investigadora. I en un lloc especial, una gran familia que es reuneix cada dimecres, des de fa més de 3 anys, quan encara sols érem *Víctor*, *Adolfo* i jo; i que ara compta amb una vintena de persones que m'han mostrat que hi ha que donar-ho tot, en tot moment. En un lloc *honorable* guarde a *Marina Murillo*, la meua companya de viatge(s) en la carrera investigadora i amb la que guarde records meravellosos.

Also, special mention to *Céline Anne Marie Esser*, who was in Valencia during some months but that always have been with me although I tell her bad jokes that make us happy. Oh men, this thesis, written by the happy men is "for you and for me and the entire human race". Of course, I want to thank also all the PhD students in Trier Universität that take care of me during my stay there. Special mention to *Bernhard Dierolf* and *Andreas Jung* who where also like brothers and that make my stay in Trier more familiar.

Recordar també a tots els membres del IUMPA que m'han acompanyat en el meu dia a dia els últims anys. Començant per *Alberto Conejero*, *Enrique Jordá* i *David Jornet* que m'han aconsellat i motivat en tot moment. Altres estudiants de doctorat, com jo, que han passat per el IUMPA com *Javi Falcó*, *Javier Aroza*, *Carme Zaragoza*, *Xavi Barrachina*, etc. companys de dinars i converses al començar el dia. I com no, a tota la resta dels membres del IUMPA i del Departament d'Anàlisi Matemàtica de la Universitat de València, sempre disposats a ajudar. Especialment, a *Félix* i *Minerva* per facilitar la part administrativa. I en general, vull agrair als integrants del grup predoc de la Facultat de Matemàtiques la creació d'un grup molt gran i motivador durant els anys de la tesi.

Per últim, volia deixar el final per a dos persones que són molt importants en la meua vida. La primera d'elles és una bonica al·lota mallorquina anomenada *Lucia Rotger*, que em fa feliç i de la qual estic enamorat; qui m'acompanya en tot moment i em fa despertar i anar a dormir tots els dies amb un somriure en la cara.

I l'altra es ma mare, *Mari Sales Puchades*, a qui li dec tot el que soc ara mateix; sempre m'ha aconsellat correctament i m'ha mostrat que calia que la bondat fos el més important. Inclòs en els mals moments que ella ha passat últimament, sempre ha estat al meu costat donant-me ànims. Per a ella, i per al meu difunt pare va dedicada aquesta tesi, perquè gràcies al seu constant esforç he arribat fins a ací. Gràcies.

Sueca, a 14 de Març de 2015 a les 9:26:35
Juan Miguel Ribera Puchades

Aquesta memòria ha estat elaborada a l'Institut Universitari de Matemàtica Pura i Aplicada (IUMPA) de la Universitat Politècnica de València, durant el període de gaudi d'una beca del programa VALi+d para investigadores en formación de la Generalitat Valenciana ACIF/2011/059 i d'una beca del Progama de Formació de Professorat Universitari F.P.U. AP2010-4400 (Ministeri d'Educació).

La beca Erasmus SST i els projectes d'investigació "Métodos del análisis funcional para el análisis matemático" MEC and FEDER Project MTM2010-15200 "Métodos del análisis funcional y teoría de operadores" MEC and FEDER Project MTM2013-43540-P i Programa de Apoyo a la Investigación y Desarrollo de la UPV PAID-06-12 SP20120807 ha permés a l'autor realitzar una estada d'investigació de 3 mesos a Trier, Alemania (abril-maig de 2013 i maig de 2014) dirigit per el professor Leonhard Frerick.

Aquesta investigació també ha estat finançada parcialment pels projectes: "Métodos del análisis funcional para el análisis matemático" MEC and FEDER Project MTM2010-15200 "Métodos del análisis funcional y teoría de operadores" MEC and FEDER Project MTM2013-43540-P, "Operadores de extensión de Whitney, curvas ultradiferenciables, iterados de operadores diferenciales y frente de ondas" (Project GV/2010/040, Conselleria d'Educació de la GVA) i Programa de Apoyo a la Investigación y Desarrollo de la UPV PAID-06-12 SP20120807.

Resumen

La presente memoria "Descomposiciones atómicas y frames en espacios de Fréchet y sus duales" trata diferentes áreas del análisis funcional con aplicaciones.

Los frames de Schauder se utilizan para representar un elemento arbitrario x de un espacio de funciones E mediante una serie a partir de un conjunto numerable fijado $\{x_j\}_j$ de elementos de este espacio de manera que los coeficientes de la reconstrucción de x dependen de forma lineal y continua de x. A diferencia de las bases de Schauder, la expresión de un elemento x en términos de la sucesión $\{x_j\}_j$, i.e. la fórmula de reconstrucción para x, no es necesariamente única. Las descomposiciones atómicas o los frames de Schauder son un estructura menos restrictiva que las bases, porque un subespacio complementado de un espacio de Banach con base tiene siempre un frame de Schauder natural, que se obtiene a partir de una base del superespacio. Incluso cuando el subespacio complementado tiene una base, no hay una forma sistemática de encontrarla. Las descomposiciones atómicas aparecen en aplicaciones al procesamiento de señales y la teoría de muestreo, entre otras áreas. Feichtinger caracterizó las descomposiciones atómicas de Gabor para espacios de modulación [24] que más tarde desarrolló en la teoría general presentada en el trabajo conjunto con Gröchenig [25] y [26]. Recientemente, Pilipovic y Stoeva [55] (véase también [54]) han estudiado el desarrollo en serie en límites inductivos y proyectivos (numerables) de espacios de Banach. En esta tesis empezamos un estudio sistemático de los frames de Schauder en espacios localmente convexos aunque nuestro interés principal son los espacios de Fréchet y sus duales. La diferencia principal respecto del concepto considerado en [55] es que nuestra aproximación no depende de una representación fijada del espacio de Fréchet como límite proyectivo de espacios de Banach.

El texto queda dividido en dos partes y un apéndice que incluye la notación, las definiciones y los resultados básicos que usaremos a lo largo de la tesis. La primera parte se centra en la relación entre las propiedades de un frame de Schauder en un espacio de Fréchet E y la estructura del espacio. En el segundo capítulo se definen y estudian los frames y las sucesiones de Bessel en espacios de Fréchet y sus duales. A continuación, presentamos una breve descripción de los capítulos:

En el Capítulo 1, estudiamos los frames de Schauder en los espacios de Fréchet y sus duales así como los resultados de perturbación. Definimos los frames de Schauder contractivos y acotadamente completos en espacios localmente convexos, estudiamos la dualidad de estos dos conceptos y su relación con la reflexividad del espacio. Caracterizamos cuándo un frame de Schauder incondicional es contractivo o acotadamente completo en términos de las propiedades del espacio. También se presentan varios ejemplos de frames de Schauder en espacios de funciones concretos. Nuestro interés principal en este capítulo es investigar la relación entre las propiedades de un frame de Schauder en un espacio de Fréchet E y la estructura del espacio, por ejemplo, si E es reflexivo o si contiene copias de c_0 o ℓ_1. La mayoría de los resultados incluidos en este capítulo están publicados por Bonet, Fernández, Galbis y Ribera en [13].

El segundo capítulo de la tesis está centrado en el estudio de las sucesiones de Λ-Bessel ($\{g_i\}_i \subset E'$), Λ-frames y frames respecto de Λ en el dual de un espacio localmente convexo de Hausdorff E, en particular, para espacios de Fréchet y espacios (LB) completos E, con Λ un espacio de sucesiones. Investigamos la relación de estos dos conceptos con los sistemas representantes en el sentido de Kadets y Korobeinik [34] y con los frames de Schauder, considerados en el Capítulo 1. Los resultados abstractos presentados aquí, cuando los aplicamos a espacios de funciones analíticas concretos, nos dan muchos ejemplos y consecuencias sobre los conjuntos de muestreo y los desarrollos en serie de Dirichlet. Presentamos varios resultados abstractos sobre Λ-frames en espacios (LB) completos. Finalmente, recogemos muchas aplicaciones, resultados y ejemplos alrededor de los conjuntos suficientes para espacios de Fréchet de funciones holomorfas y conjuntos débilmente suficientes para espacios pesados (LB) de funciones holomorfas. La mayoría de los resultados incluidos en este capítulo están enviados para publicar en un trabajo de Bonet, Fernández, Galbis y Ribera en [12].

En el apéndice introducimos algunos conceptos sobre espacios localmente convexos y sus duales con especial atención a los límites inductivos. Además, también introducimos algunos resultados relacionados con las bases topológicas. Establecemos las definiciones y las propiedades fundamentales que se pueden necesitar a lo largo de la tesis.

Resum

La tesi “Descomposicions atòmiques i frames en espais de Fréchet i els seus duals” presentada ací tracta diferents àrees de l’anàlisi funcional amb aplicacions.

Els frames de Schauder s’utilitzen per tal de representar un element arbitrari x d’un espai de funcions E com una reconstrucció en sèrie a partir d’un conjunt numerable fixat $\{x_j\}_j$ d’elements en aquest espai tal que els coeficients de la reconstrucció de x depenen de forma lineal i continua de x. A diferència de les bases de Schauder, l’expressió d’un element x en termes d’una successió $\{x_j\}_j$, i.e. la fórmula de reconstrucció per a x, no és necessàriament única. Les descomposicions atòmiques o els frames de Schauder són una estructura menys restrictiva que les bases, donat que un subespai complementat d’un espai de Banach amb base sempre té un frame de Schauder natural, el qual és obtingut a partir d’una base del superespai. Inclòs quan el subespai complementat disposa de una base, no hi ha una forma sistemàtica per tal de trobar-la. Les descomposicions atòmiques apareixen en aplicacions a processat de senyals i teoria de mostreig entre altres àrees. Feichtinger va caracteritzar les descomposicions atòmiques per a espais de modulació [24] que més tart va desenvolupar en el seu treball conjunt amb Gröchenig en [25] i [26]. Recentment, Pilipovic i Stoeva [55] (veure també [54]) han estudiat els desenvolupaments en sèrie en límits inductius o projectius (numerables) en espais de Banach. En aquesta tesi comencem un estudi sistemàtic dels frames de Schauder en espais localment convexos, tot i que el nostre interès està en els espais de Fréchet i els seus duals. La diferència més important amb el concepte estudiat en [55] és que el nostre estudi no depèn de una representació fixada del espai de Fréchet com a límit projectiu de espais de Banach.

El text està dividit en dos capítols i un apèndix que ens aporta la notació, definicions i els resultats bàsics que utilitzarem al llarg de la tesi. El primer dels capítols està centrat en la relació entre les propietats de un frame de Schauder en un espai de Fréchet E i la estructura del espai. En el segon capítol es defineixen i estudien els frames i les successions de Bessel en espais de Fréchet i els seus duals. En el que segueix, donem una breu descripció dels diferents capítols:

En el Capítol 1, estudiem els frames de Schauder en els espais de Fréchet i els seus duals, així com els resultats de pertorbació. Definim els frames de Schauder contractius i fitadament complets en espais localment convexos, estudiem la dualitat d'aquests dos conceptes i la seua relació amb la reflexivitat del espai. Caracteritzem, en quines situacions, un frame de Schauder incondicional és contractiu o fitadament complet en termes de les propietats del espai. També presentem alguns exemples de frames de Schauder concrets en espais de funcions. El nostre principal interès en aquest capítol és investigar la relació entre les propietats d'un frame de Schauder d'un espai de Fréchet E i la estructura del espai, per exemple, si E és reflexiu o si E conté còpies dels espais c_0 o ℓ_1. La majoria dels resultats inclosos en aquest capítol estan publicats per Bonet, Fernández, Galbis i Ribera en [13].

El segon capítol de la tesi està centrat en el estudi de les successions Λ-Bessel ($\{g_i\}_i \subset E'$), Λ-frames i frames respecte de Λ en el dual d'un espai localment convex de Hausdorff E, en particular, per a espais de Fréchet i espais (LB) complets E, amb Λ un espai de successions. Investiguem la relació d'aquests dos conceptes amb sistemes representants en el sentit de Kadets i Korobeinik [34] i amb els frames de Schauder, que han sigut investigats en el Capítol 1. Els resultats abstractes presentatsací, quan els apliquem a espais de funcions analítiques concrets, ens donen molts exemples i conseqüències sobre els conjunts de mostreig i els desenvolupaments en sèrie de Dirichlet. Presentem diversos resultats abstractes sobre Λ-frames en espais (LB) complets. Finalment, recollim moltes aplicacions, resultats i exemples al voltant dels conjunts suficients per a espais de Fréchet de funcions holomorfes i conjunts dèbilment suficients per a espais pesats (LB) de funcions holomorfes. La majoria dels resultats inclosos en aquest capítol estan sotmesos a publicació per Bonet, Fernández, Galbis i Ribera en [12].

En l'apèndix són introduïts alguns conceptes sobre espais localment convexos i els seus duals, amb especial atenció als límits inductius. A més a més, també introduïm alguns resultats sobre bases topològiques. Establim les definicions i les propietats fonamentals que necessitarem al llarg de la tesi.

Summary

The Ph.D. Thesis “Atomic decompositions and frames in Fréchet spaces and their duals” presented here treats different areas of functional analysis with applications.

Schauder frames are used to represent an arbitrary element x of a function space E as a series expansion involving a fixed countable set $\{x_j\}_j$ of elements in that space such that the coefficients of the expansion of x depend in a linear and continuous way on x. Unlike Schauder bases, the expression of an element x in terms of the sequence $\{x_j\}_j$, i.e. the reconstruction formula for x, is not necessarily unique. Atomic decompositions or Schauder frames are a less restrictive structure than bases, because a complemented subspace of a Banach space with basis has always a natural Schauder frame, that is obtained from the basis of the superspace. Even when the complemented subspace has a basis, there is not a systematic way to find it. Atomic decompositions appeared in applications to signal processing and sampling theory among other areas. Feichtinger characterized Gabor atomic decompositions for modulation spaces [24] and the general theory was developed in his joint work with Gröchenig [25] and [26]. Very recently, Pilipovic and Stoeva [55] (see also [54]) studied series expansions in (countable) projective or inductive limits of Banach spaces. In this thesis we begin a systematic study of Schauder frames in locally convex spaces, but our main interest lies in Fréchet spaces and their duals. The main difference with respect to the concept considered in [55] is that our approach does not depend on a fixed representation of the Fréchet space as a projective limit of Banach spaces.

The text is divided into two chapters and appendix that gives the notation, definitions and the basic results we will use throughout the thesis. The first one focuses on the relation between the properties of an existing Schauder frame in a Fréchet space E and the structure of the space. In the second chapter frames and Bessel sequences in Fréchet spaces and their duals are defined and studied. In what follows, we give a brief description of the different chapters:

In Chapter 1, we study Schauder frames in Fréchet spaces and their duals, as well as perturbation results. We define shrinking and boundedly complete Schauder

frames on a locally convex space, study the duality of these two concepts and their relation with the reflexivity of the space. We characterize when an unconditional Schauder frame is shrinking or boundedly complete in terms of properties of the space. Several examples of concrete Schauder frames in function spaces are also presented. Our main purpose in this chapter is to investigate the relation between the properties of an existing Schauder frame in a Fréchet space E and the structure of the space, for example if E is reflexive or if it contains copies of c_0 or ℓ_1. Most of the results included in this chapter are published by Bonet, Fernández, Galbis and Ribera in [13].

The second chapter of the thesis is devoted to study Λ-Bessel sequences $\{g_i\}_i \subset E'$, Λ-frames and frames with respect to Λ in the dual of a Hausdorff locally convex space E, in particular for Fréchet spaces and complete (LB)-spaces E, with Λ a sequence space. We investigate the relation of these concepts with representing systems in the sense of Kadets and Korobeinik [34] and with the Schauder frames, that were investigated in Chapter 1. The abstract results presented here, when applied to concrete spaces of analytic functions, give many examples and consequences about sampling sets and Dirichlet series expansions. We present several abstract results about Λ-frames in complete (LB)-spaces. Finally, many applications, results and examples concerning sufficient sets for weighted Fréchet spaces of holomorphic functions and weakly sufficient sets for weighted (LB)-spaces of holomorphic functions are collected. Most of the results are submitted for publication in a preprint of Bonet, Fernández, Galbis and Ribera in [12].

In the appendix, some concepts about locally convex spaces and their duals are introduced with special attention to inductive limits. In addition, we also introduce some results concerning topological bases. We establish the definitions and the fundamental properties that we shall need through the thesis.

Contents

Introduction

Schauder frames are used to represent an arbitrary element x of a function space E as a series expansion involving a fixed countable set $\{x_j\}_j$ of elements in that space such that the coefficients of the expansion of x depend in a linear and continuous way on x; that is

$$x = \sum_{j=1}^{\infty} x'_j(x)\, x_j, \qquad \text{for all } x \in E.$$

Unlike Schauder bases, the expression of an element x in terms of the sequence $\{x_j\}_j$, i.e. the reconstruction formula for x, is not necessarily unique. In the classical literature of function spaces the Schauder frames are usually referred to as atomic decompositions. In abstract theory of Banach spaces the concept of atomic decomposition is often associated with a certain sequence space selected a priori while the notion of Schauder frame makes no reference to any sequence space. However, the two concepts are closely related and some papers in the area ([20], [16], [17]) are written in terms of atomic decompositions whereas others ([19], [3], [43]) are stated in terms of Schauder frames. Atomic decompositions appeared in applications to signal processing and sampling theory among other areas. Feichtinger characterized Gabor atomic decompositions for modulation spaces [24] and the general theory was developed in his joint work with Gröchenig [25] and [26]. In these papers, the authors show that reconstruction through atomic decompositions is not limited to Hilbert spaces. Indeed, they obtain atomic decompositions for a large class of Banach spaces, namely the coorbit spaces. Atomic decompositions or Schauder frames are a less restrictive structure than bases, because a complemented subspace of a Banach space with basis has always a natural Schauder frame, that is obtained from the basis of the superspace. Even when the complemented subspace has a basis, there is not a systematic way to find it. There is a vast literature dedicated to the subject. The related topic of frame expansions in Banach spaces was considered for example in [18] and [20].

Carando and Lasalle [16] and [17] studied atomic decompositions and their relationship with duality and reflexivity of Banach spaces. They extended the concepts of shrinking and boundedly complete Schauder bases to the atomic decomposition framework. They considered when an atomic decomposition for a Banach space generates, by duality, an atomic decomposition for its dual space and characterized the reflexivity of a Banach space in terms of properties of its atomic decompositions. Unconditional atomic decompositions allowed them to prove James-type results characterizing shrinking and boundedly complete unconditional atomic decompositions in terms of the containment in the Banach space of copies of ℓ_1 and c_0 respectively.

Very recently, Pilipovic and Stoeva [55] (see also [54]) studied series expansions in (countable) projective or inductive limits of Banach spaces. In this thesis we begin a systematic study of Schauder frames in locally convex spaces, but our main interest lies in Fréchet spaces and their duals. The main difference with respect to the concept considered in [55] is that our approach does not depend on a fixed representation of the Fréchet space as a projective limit of Banach spaces. We mention the following preliminary example as a motivation for our work: Leontiev proved that for each bounded convex domain G of the complex plane $\mathbb{C}$ there is a sequence of complex numbers $\{\lambda_j\}_j$ such that every holomorphic function $f \in H(G)$ can be expanded as a series of the form $f(z) = \sum_{j=1}^{\infty} a_j e^{\lambda_j z}$, converging absolutely and uniformly on the compact subsets of G. It is well-known that this expansion is not unique. We refer the reader e.g. to Korobeinik's survey [37]. A priori it is not clear whether the coefficients a_j in the expansion can be selected depending continuously on the function f. However, Korobeinik and Melikhov [40, Th. 4.3 and Remark 4.4(b)] showed that this is the case when the boundary of the open set G is of class C^2; thus obtaining what we call below an unconditional Schauder frame for the Fréchet space $H(G)$. These are the type of phenomena and reproducing formulas that we try to understand in our thesis.

The starting point of the first chapter has been the article [17] by Carando, Lasalle and Schmidberg about the reconstruction formula for Banach frames and duality. We show that a similar situation holds in our context of locally convex spaces. Our main purpose in this chapter is to investigate the relation between the properties of an existing Schauder frame in a Fréchet space E and the structure of the space, for example if E is reflexive or if it contains copies of c_0 or ℓ_1. Other precise references to work in this direction in the Banach space setting can be seen in [18].

In Section 1.1 we introduce Schauder frames and for complete barrelled spaces we show in 1.1.4 that having a Schauder frame is equivalent to being complemented in a locally convex space with a Schauder basis.

In Section 1.2 we start by giving some perturbation results for Schauder frames as in Theorem 1.2.1. We also show an equivalence between a Schauder frame being bounded below and equicontinuity of the coefficient functionals.

In Section 1.3 we introduce shrinking and boundedly complete Schauder frames on a locally convex space, study the duality of these two concepts and their relation with the reflexivity of the space; see Theorem 1.3.14.

Unconditional Schauder frames are studied in Section 1.4. We completely characterize, for a given unconditional Schauder frame, when it is shrinking or boundedly complete in terms of properties of the space in Theorems 1.4.11 and 1.4.14. As a tool, that could be of independent interest, we show Rosenthal's ℓ_1 theorem for boundedly retractive inductive limits of Fréchet spaces.

In the last section of Chapter 1 we include some examples of concrete Schauder frames in function spaces. Our Theorem 1.5.2 shows a remarkable relation between the existence of a continuous linear extension operator for C^∞ functions defined on a compact subset K of $\mathbb{R}^n$ and the existence of an unconditional Schauder frame in $C^\infty(K)$ using exponentials.

Most of the results included in this chapter are published by Bonet, Fernández, Galbis and Ribera in [13].

The purpose of chapter 2 is twofold. On the one hand we study Λ-Bessel sequences $\{g_i\}_i \subset E'$, Λ-frames and frames with respect to Λ in the dual of a Hausdorff locally convex space E, in particular for Fréchet spaces and complete (LB)-spaces E, with Λ a sequence space. We investigate the relation of these concepts with representing systems in the sense of Kadets and Korobeinik [34] and with the Schauder frames, that were investigated in Chapter 1. On the other hand Chapter 2 emphasizes the deep connection of frames for Fréchet and (LB)-spaces with the sufficient and weakly sufficient sets for weighted Fréchet and (LB)-spaces of holomorphic functions. These concepts correspond to sampling sets in the case of Banach spaces of holomorphic functions. Our general results in Sections 2.2 and 2.3 permit us to obtain as a consequence many examples and results in the literature in a unified way in Section 2.4.

Section 2.2 is inspired by the work of Casazza, Christensen and Stoeva [18] in the context of Banach spaces. Their characterizations of Banach frames and frames with respect to a BK-sequence space gave us the proper hint to present here the right definitions in our more general setting. A point of view different from ours concerning frames in Fréchet spaces was presented by Pilipovic and Stoeva [55] and [56].

Motivated by the applications to weakly sufficient sets for weighted (LB)-spaces of holomorphic functions we present several abstract results about Λ-frames in complete (LB)-spaces, that require a delicate analysis, in Section 2.3. Our main result is Theorem 2.3.4.

Finally, many applications, results and examples are collected in Section 2.4 concerning sufficient sets for weighted Fréchet spaces of holomorphic functions and weakly sufficient sets for weighted (LB)-spaces of holomorphic functions. We include here consequences related to the work of many authors; see [1], [2], [11], [36], [38], [49], [50], [60] and [65].

The results in Chapter 2 are included in a preprint of Bonet, Fernández, Galbis and Ribera in [12].

Our notation for functional analysis and operator theory is standard. We refer the reader e.g. to [23], [58], [48], [53], [9] and [59]. More details are presented in the appendix.

Chapter 1

Schauder frames in locally convex spaces

In this chapter we study Schauder frames in Fréchet spaces and their duals, as well as perturbation results. We define shrinking and boundedly complete Schauder frames on a locally convex space, study the duality of these two concepts and their relation with the reflexivity of the space. We characterize when an unconditional Schauder frame is shrinking or boundedly complete in terms of properties of the space. Several examples of concrete Schauder frames in function spaces are also presented. Most of the results included in this chapter are published by Bonet, Fernández, Galbis and Ribera in [13].

Throughout this chapter, E denotes a locally convex Hausdorff linear topological space (briefly, a lcs) with additional hypotheses added as needed and $cs(E)$ is the system of continuous seminorms describing the topology of E.

1.1 Schauder frames

In this section, we introduce Schauder frames and we show that having a Schauder frame is equivalent to being complemented in a locally convex space with a Schauder basis.

Definition 1.1.1 Let E be a lcs, $\{x_j\}_{j=1}^{\infty} \subset E$ and $\{x_j'\}_{j=1}^{\infty} \subset E'$. We say that $\left(\{x_j'\}, \{x_j\}\right)$ is a *Schauder frame of E* if

$$x = \sum_{j=1}^{\infty} x_j'(x)\, x_j, \qquad \text{for all } x \in E,$$

the series converging in E.

A lcs E which admits a Schauder frame is separable. Let E be a lcs with a Schauder basis $\{e_j\}_{j=1}^{\infty} \subset E$ and let $\{e_j'\}_{j=1}^{\infty} \subset E'$ denote the coefficient functionals (more precisely defined in Appendix 3). Clearly, $\left(\{e_j'\}, \{e_j\}\right)$ is a Schauder frame for E. The main difference with Schauder basis is that, in general, one may have a sequence $\{x_j\}_{j=1}^{\infty} \subset E$ and two different sequences $\{x_j'\}_{j=1}^{\infty} \subset E'$ and $\{y_j'\}_{j=1}^{\infty} \subset E'$ so that both $\left(\{x_j'\}, \{x_j\}\right)$ and $\left(\{y_j'\}, \{x_j\}\right)$ are Schauder frames. See the comments after Corollary 1.2.4.

Proposition 1.1.2 *Let E be a lcs and let $P : E \to E$ be a continuous linear projection. If $\left(\{x_j'\}, \{x_j\}\right)$ is a Schauder frame for E, then $\left(\{P'(x_j')\}, \{P(x_j)\}\right)$ is a Schauder frame for $P(E)$.*

In particular, if E is isomorphic to a complemented subspace of a lcs with a Schauder basis, then E admits a Schauder frame.

Proof. Since $P'(x_j')(y) = x_j'(P(y)) = x_j'(y)$ for all $y \in P(E)$ and $j \in \mathbb{N}$, we obtain a Schauder frame:

$$y = P(y) = P\left(\sum_{j=1}^{\infty} x_j'(y)\, x_j\right) = \sum_{j=1}^{\infty} P'\left(x_j'\right)(y) P(x_j).$$

□

Lemma 1.1.3 *Let $\{x_j\}$ be a fixed sequence of non-zero elements in a lcs E and let us denote by $\bigwedge$ the vector space*

$$\bigwedge := \left\{\alpha = \{\alpha_j\}_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j \textit{ is convergent in } E\right\}. \tag{1.1.1}$$

Endowed with the system of seminorms

$$\mathcal{Q} := \left\{q_p\left(\{\alpha_j\}_j\right) := \sup_n p\left(\sum_{j=1}^{n} \alpha_j x_j\right), \textit{ for all } p \in cs(E)\right\}, \tag{1.1.2}$$

$\bigwedge$ is a sequence space and the canonical unit vectors form a Schauder basis. If E is complete, then $\bigwedge$ is complete. In particular, if E is a Fréchet (resp. Banach) space, so is $\bigwedge$.

Proof. The fact that $\mathcal{Q}$ is a fundamental system of seminorms follows easily from the fact that the topology of E is determined by the elements of $cs(E)$.

To see that $\bigwedge \subset \omega$ is continuous, for every $j \in \mathbb{N}$, choose $p_j \leq p_{j+1} \in cs(E)$ such that $p_j(x_j) > 0$. For $\alpha \in \bigwedge$ and $l \in \mathbb{N}$:

$$\begin{aligned} |\alpha_l| &= \frac{1}{p_l(x_l)} p_l(\alpha_l x_l) \leq \frac{1}{p_l(x_l)} \left(p_l\left(\sum_{j=1}^{l} \alpha_j x_j\right) + p_l\left(\sum_{j=1}^{l-1} \alpha_j x_j\right)\right) \\ &\leq \frac{2}{p_l(x_l)} q_{p_l}(\alpha). \end{aligned}$$

To show that the canonical unit vectors are a Schauder basis of $\bigwedge$ observe that, by definition of the space and of its topology, $\bigwedge = \overline{\operatorname{span}\{e_j : j \in \mathbb{N}\}}$ and clearly

$$q\left(\sum_{i=1}^{n} \alpha_i e_i\right) \leq q\left(\sum_{i=1}^{n+m} \alpha_i e_i\right)$$

for every $q \in \mathcal{Q}$ and for all $m, n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_{n+m} \in \mathbb{K}$ we can apply [33, 14.3.6] to conclude that the unit vectors are also a Schauder basis.

Assume that E is complete. To prove that $\bigwedge$ is complete, let $\{\{\alpha_j^\gamma\}_j\}_{\gamma\in\Gamma}$ be a Cauchy net in $\bigwedge$. Denote by $y_\gamma := \sum_{j=1}^{\infty} \alpha_j^\gamma x_j$. As $p(y_\gamma) \leq q_p(\alpha^\gamma)$, $\{y_\gamma\}_{\gamma\in\Gamma}$ is a Cauchy net in E, hence convergent to some $y \in E$. As $\bigwedge$ is continuously included in ω, $\{\{\alpha_j^\gamma\}_j\}_{\gamma\in\Gamma}$ converges to $\alpha \in \omega$ which implies that for every n, the net $\{\sum_{j=1}^{n} \alpha_j^\gamma x_j\}_{\gamma\in\Gamma}$ converges $\sum_{j=1}^{n} \alpha_j x_j$. Given $\varepsilon > 0$ and $p \in cs(E)$ we can take $\gamma_0 \in \Gamma$ such that for $\gamma, \gamma' \geq \gamma_0$ we have $p(y - y_\gamma) < \varepsilon$ and $\sup_n p(\sum_{j=1}^{n} \alpha_j^\gamma x_j - \sum_{j=1}^{n} \alpha_j^{\gamma'} x_j) < \varepsilon$, and therefore taking limits we have $\sup_n p(\sum_{j=1}^{n} \alpha_j^\gamma x_j - \sum_{j=1}^{n} \alpha_j x_j) \leq \varepsilon$. From here it is immediate that the series $\sum_{j=1}^{\infty} \alpha_j x_j$ converges to y in E, thus $\alpha \in \bigwedge$ and $q_p(\alpha^\gamma - \alpha) \leq \varepsilon$ whenever $\gamma \geq \gamma_0$.
□

Theorem 1.1.4 *Let E be a complete barrelled lcs. The following conditions are equivalent:*

(1) *E admits a Schauder frame.*

(2) *E is isomorphic to a complemented subspace of a complete sequence space with the canonical unit vectors as Schauder basis.*

(3) *E is isomorphic to a complemented subspace of a complete lcs with a Schauder basis.*

In particular, a Fréchet space E admits a Schauder frame if and only if it is isomorphic to a complemented subspace of a Fréchet space with a Schauder basis.

Proof. (1) $\Rightarrow$ (2) Let $\left(\{x_j'\},\{x_j\}\right)$ be a Schauder frame of E. We may assume that $x_j \neq 0$ for all $j \in \mathbb{N}$. Let $\bigwedge$ be the complete lcs of sequences defined as in Lemma 1.1.3. We define $F_n : E \longrightarrow E$ as $F_n\left(x\right) := \sum_{j=1}^{n} x_j'\left(x\right) x_j$. Since E is barrelled the sequence $\{F_n\}_n$ is equicontinuous, that is, for every $p \in cs(E)$ there exists $p' \in cs(E)$ such that $p\left(F_n\left(x\right)\right) \leq p'\left(x\right)$ for every $x \in E$ and for every $n \in \mathbb{N}$. Consequently the map

$$\begin{array}{rcl} U : E & \longrightarrow & \bigwedge \\ x & \longrightarrow & U\left(x\right) := \left\{x_j'\left(x\right)\right\}_j \end{array}$$

is injective and continuous. Moreover, the map

$$\begin{array}{rcl} S : \bigwedge & \longrightarrow & E \\ \{\alpha_j\}_j & \longrightarrow & S\left(\{\alpha_j\}_j\right) := \sum_{j=1}^{\infty} \alpha_j x_j \in E. \end{array}$$

is linear and continuous, since

$$p\left(S\left(\{\alpha_j\}_j\right)\right) = p\left(\sum_{j=1}^{\infty} \alpha_j x_j\right) \leq \sup_n p\left(\sum_{j=1}^{n} \alpha_j x_j\right) = q_p\left(\{\alpha_j\}_j\right).$$

As $S \circ U = I_E$ we conclude that U is an isomorphism into its range $U\left(E\right)$ and $U \circ S$ is a projection of $\bigwedge$ onto $U\left(E\right)$.

The proof ends if we show that the canonical unit vectors form a Schauder basis. We give now an argument different from Lemma 1.1.3. Note that the canonical unit vectors form a topological basis in $\bigwedge$ since there exists, for each $\{\alpha_j\}_j \in \bigwedge$, a unique sequence $\{y_n\}_n \in \omega$ such that there is a unique $\{\alpha_j\}_j = \sum_{n=1}^{\infty} y_n e_n$ due to:

$$\lim_{n\to\infty} q_p\left(\alpha - \sum_{j=1}^{n} \alpha_j e_j\right) = \lim_{n\to\infty} \sup_{m>n} p\left(\sum_{j=n+1}^{m} \alpha_j x_j\right) = 0,$$

since $\sum_{j=1}^{\infty}\alpha_j e_j$ is convergent in E. Now, since $e_j \notin \overline{span\{e_i : i \neq j\}}$, by Proposition 3 in [33, p. 292] we obtain that $\{e_j\}$ form a Schauder basis.

(2) $\Rightarrow$ (3) is trivial, while (3) $\Rightarrow$ (1) is consequence of Proposition 1.1.2. $\square$

The following Corollary is a consequence of an important result of Pełczyński. A locally convex space is said to satisfy the bounded approximation property if the identity of E is the pointwise limit of an equicontinuous net of finite rank operators. If the locally convex space is separable, then the net can be replaced by a sequence. Pełczyński [52] (see also [44, Theorem 2.11]) proved that a separable Fréchet space has the bounded approximation property if and only if it is isomorphic to a complemented subspace of a Fréchet space with a Schauder basis. For more information about the bounded approximation property for Fréchet spaces see J. Bonet's seminar [10].

Corollary 1.1.5 *A separable Fréchet space E admits a Schauder frame if and only if E has the bounded approximation property.*

Proof. It follows from Theorem 1.1.4 and the aforementioned result of Pełczyński [52]. $\square$

Taskinen [62] gave examples of a complemented subspace F of a Fréchet Schwartz space E with a Schauder basis, such that F is nuclear and does not have a basis. By Theorem 1.1.4, F has a Schauder frame. Vogt [66] gave examples of nuclear (hence separable) Fréchet spaces E which do not have the bounded approximation property. These separable Fréchet spaces E do not admit a Schauder frame, although by Kōmura-Kōmura's Theorem [48, Theorem 29.8] they are isomorphic to a subspace of the countable product $s^{\mathbb{N}}$ of copies of the space of rapidly decreasing sequence, that has a Schauder basis.

Remark 1.1.6 Let $\left(\{x_j'\},\{x_j\}\right)$ be a Schauder frame of a sequentially retractive (LF)-space $E = \mathrm{ind}_{n\rightarrow} E_n$. Let $F_n := \sum_{j=1}^{n} x_j'(\cdot)x_j$. We know that the sequence $\{F_n\}_n$ is equicontinuous, hence if B is a bounded set,

$$\tilde{B} := \overline{\cup_n F_n(B)}$$

is bounded, therefore there is k such that $\tilde{B}$ is contained and bounded in E_k. Moreover, as the sequence $\{F_n(x)\}_n$ converges to x in E, there is m such that $F_n(x) \in E_m$ and $x = \sum_{j=1}^{\infty} x_j'(x)x_j$ with convergence in E_m.

Define

$$\Lambda_k := \left\{ \alpha \in \omega : \sum_{j=1}^{n} \alpha_j x_j \in E_k \text{ for each } n \text{ and } \sum_{j=1}^{\infty} \alpha_j x_j \text{ converges in } E_k \right\}$$

with the seminorms defined as usual. Then Λ_k is a Fréchet sequence space. Let $\Lambda = \cup_k \Lambda_k$ with the topology of the inductive limit of the Λ_k's. The map

$$\begin{array}{rcl} U : E & \longrightarrow & \Lambda \\ x & \longrightarrow & U(x) := \left\{ x_j'(x) \right\}_j \end{array}$$

is continuous as it maps bounded sets into bounded sets and E, being an (LF)-space, is bornological. The map $\Phi : \Lambda \to E$, $\alpha \to \sum_j \alpha_j x_j$ is continuous, since by the definition of the space, Λ_k is sent continuously into E_k. Clearly $\Phi \circ U = I_E$. Moreover given $\alpha \in \Lambda$ take k such that $\alpha \in \Lambda_k$, the $\alpha - \sum_{j=1}^{n} \alpha_j e_j$ converges to zero in Λ_k which implies that $\alpha = \sum_{j=1}^{\infty} \alpha_j e_j$ in Λ, the representation is unique and the coefficients depend continuously on α as Λ is a sequence space, then $\{e_j\}_j$ is a Schauder basis. This implies that E is complemented in a sequence (LF)-space with a Schauder basis.

Proposition 1.1.7 *Let E and F be Hausdorff locally convex spaces such that $\left(\left\{x_j'\right\}, \{x_j\}\right)$ is a Schauder frame for E and $\left(\left\{y_j'\right\}, \{y_j\}\right)$ is a Schauder frame for F. Then, there exists $\left\{z_j'\right\} \subset (E \times F)'$ and $\{z_j\} \subset (E \times F)$ such that $\left(\left\{z_j'\right\}, \{z_j\}\right)$ is a Schauder frame for $E \times F$ where $E \times F$ is the product space.*

Proof. Define $\{z_j\} \subset (E \times F)$ by $z_{2j-1} = (x_j, 0)$ and $z_{2j} = (0, y_j)$ for every $n \in \mathbb{N}$. Define now $\left\{z_j'\right\} \subset (E \times F)'$ by $z_{2j-1}'(x, y) = x_j'(x)$ and $z_{2j}' = y_j'(y)$ for every $n \in \mathbb{N}$ and for every $(x, y) \in E \times F$. Thus, for every $(x, y) \in E \times F$

$$\begin{aligned} \sum_{n=1}^{\infty} z_j'(x, y)\, z_j &= \sum_{n=1}^{\infty} z_{2j}'(x, y)\, z_{2j} + \sum_{n=1}^{\infty} z_{2j-1}'(x, y)\, z_{2j-1} = \\ &= \left(\sum_{n=1}^{\infty} x_j'(x)\, x_j, \sum_{n=1}^{\infty} y_j'(y)\, y_j \right) = (x, y)\,. \end{aligned}$$

□

1.2 Perturbation results

The following result, that is needed below, can be found in [29, page 436]: *Let E be a complete lcs and let $T : E \to E$ be an operator with the property that there exists $p_0 \in cs(E)$ such that for all $p \in cs(E)$ there is $C_p > 0$ such that $p(Tx) \le C_p p_0(x)$ for all $x \in E$ (that is, T maps a neighborhood into a bounded set) and moreover C_{p_0} can be chosen strictly smaller than 1. Then $I - T$ is invertible (with continuous inverse on E).*

Theorem 1.2.1 *Let $(\{x'_j\}, \{x_j\})$ be a Schauder frame of a complete lcs E.*

(1) *If $\{y_j\}_j$ is a sequence in E satisfying that there is $p_0 \in cs(E)$ such that for all $p \in cs(E)$ there is $C_p > 0$ with*

(i) $\sum_{j=1}^{\infty} |x'_j(x)| p(x_j - y_j) \le p_0(x) C_p$ *for each $x \in E$ and*

(ii) C_{p_0} *can be chosen strictly smaller than* 1

then, there exists a sequence $\{y'_j\}_j$ in E' such that $(\{y'_j\}, \{y_j\})$ is a Schauder frame for E.

(2) *If $\{y'_j\}_j$ is a sequence in E' satisfying that there is $p_0 \in cs(E)$ such that for all $p \in cs(E)$ there is $C_p > 0$ with*

(i) $\sum_{j=1}^{\infty} |(x'_j - y'_j)(x)| p(x_j) \le p_0(x) C_p$ *for each $x \in E$ and*

(ii) C_{p_0} *can be chosen strictly smaller than* 1

then, there exists $\{y_j\}_j$ a sequence in E such that $(\{y'_j\}, \{y_j\})$ is a Schauder frame for E.

Proof. In case (1) we consider the operator $T(x) = \sum_{j=1}^{\infty} x'_j(x)(x_j - y_j)$. It is well defined as the series is absolutely convergent in E, hence convergent, and T is continuous as

$$p\,(Tx) \le \sum_{j=1}^{\infty} \left| x'_j(x) \right| p\,(x_j - y_j) \le p_0\,(x)\, C_p$$

and the hypothesis imply the invertibility of $I + T$. Now, as $(I + T)(x_j) = y_j$ we have that

$$x = (I+T)(I+T)^{-1}(x) = (I+T)\left(\sum x'_j\left((I+T)^{-1}(x)\right)(x_j)\right) = \sum y'_j(x) y_j$$

where $y'_j = x'_j \circ (I + T)^{-1}$.

In case (2) we argue in the same way with the operator $T(x) = \sum_{j=1}^{\infty}(x'_j - y'_j)(x)x_j$, and the sequence $\{y_j\}_j$ is given by $S^{-1}(x_j)$, $j \in \mathbb{N}$.

$\square$

Our next result should be compared with [21, Proposition 2].

Corollary 1.2.2 *Let $(\{x'_j\}, \{x_j\})$ be a Schauder frame of a complete lcs E. Suppose that there exists $p_0 \in cs(E)$ such that $|x'_j(x)| \leq p_0(x)$ for every $x \in E, j \in \mathbb{N}$. Let $\{y_j\}_j \subset E$ such that $\sum_{j=1}^{\infty} p(y_j - x_j) < \infty$ for every $p \in cs(E)$ and $\sum_{j=1}^{\infty} p_0(y_j - x_j) < 1$. Then there exists $\{y'_j\}_j \subset E'$ such that $(\{y'_j\}, \{y_j\})$ is a Schauder frame for E.*

Proof. It is enough to check that the hypothesis of Theorem 1.2.1 hold. In fact, given $p \in \mathrm{cs}(E)$,

$$\sum_j \left|x'_j(x)\right| p(x_j - y_j) \leq p_0(x) C_p$$

with $C_p = \sum_j p(x_j - y_j)$ for each $x \in E$. Hence, $C_{p_0} < 1$. $\square$

We present now an equivalence between a Schauder frame being *bounded below* (i.e. there exists $p \in cs\,(E)$ such that $p\,(x_j) \geq 1$ for every $j \in \mathbb{N}$) and equicontinuity of the coefficient functionals.

Proposition 1.2.3 *Let $\left(\{x'_j\}, \{x_j\}\right)$ be a Schauder frame for a barrelled lcs E.*

(i) Assume that $\{x_j\}_j$ is bounded below. Then, $\{x'_j\}_j$ is equicontinuous in E'.

(ii) Assume now $\{x'_j\}_j \subset E'$ is equicontinuous in E' and $\lambda := \inf_{j \in \mathbb{N}} \left|x'_j\,(x_j)\right| > 0$. Then $\{x_j\}_j$ is bounded below.

Proof. In case (i), given $p \in cs\,(E)$ there is $p' \in cs\,(E)$ such that $p\left(x'_j\,(x)\,x_j\right) \leq p'\,(x)$ for every $x \in E$ and for every $j \in \mathbb{N}$. Then, $p\left(x'_j\,(x)\,x_j\right) = \left|x'_j\,(x)\right| p\,(x_j) \leq p'\,(x)$ for every $x \in E$ and for every $j \in \mathbb{N}$. Since $\{x_j\}_j$ is bounded below, this implies that $\left|x'_j\,(x)\right| \leq p'\,(x)$ for every $x \in E$ and for every $j \in \mathbb{N}$. We conclude that $\{x'_j\}_j \subset E'$ is equicontinuous.

By assumption, in case (ii), there exists $p \in cs\,(E)$ such that $\left|x'_j\,(x)\right| \leq p\,(x)$ for every $j \in \mathbb{N}$ and for every $x \in E$. Then, for every $j \in \mathbb{N}$, $\lambda \leq \left|x'_j\,(x_j)\right| \leq p\,(x_j)$. Therefore, the seminorm $\frac{1}{\lambda} p \in cs\,(E)$ satisfies $\frac{1}{\lambda} p\,(x_j) \geq 1$ for every $j \in \mathbb{N}$. $\square$

Observe that, in case (ii), $\{x'_j\}_j \subset E'$ being equicontinuous in E' and $\lambda := \inf_{j\in\mathbb{N}} \left|x'_j(x_j)\right| > 0$ happens if $\{x_j\}_j$ is a basis and $\{x'_j\}_j$ are the coefficient functionals, since $x'_j(x_j) = 1$ for every $j \in \mathbb{N}$.

Corollary 1.2.4 *Let E be a Fréchet space with fundamental system of seminorms $\{p_k\}_k$ and let $(\{x'_j\},\{x_j\})$ be a Schauder frame of E. Suppose that $\{y'_j\}_j \subset E'$ satisfies*

$$p_1^*(x'_j - y'_j) < \frac{1}{1 + j^2 p_j(x_j) + 3^j p_1(x_j)} \quad \text{where } p_1^*(x') = \sup\{|x'(x)| : p_1(x) \le 1\}.$$

Then there exists $\{y_j\}_j \subset E$ such that $(\{y'_j\},\{y_j\})$ is a Schauder frame for E.

Given a Schauder frame $(\{x'_j\},\{x_j\})$ on a complete lcs E, if $x'_1(x_1) \neq 1$ the map $x \mapsto \sum_{j=2}^{\infty} x'_j(x)x_j$ is invertible as 1 is not an eigenvalue of the rank one operator $x \mapsto x'_1(x)x_1$; see [35, p. 207]. Hence there exists $\{y'_j\}_j \subset E'$ such that $(\{y'_j\}_j,\{x_{j+1}\}_j)$ is a Schauder frame and similarly there exists $\{y_j\}_j \subset E$ such that $(\{x'_{j+1}\}_j,\{y_j\}_j)$ is a Schauder frame. That is, we can remove an element and still obtain Schauder frames. We recall that for a Schauder basis $\{x_j\}_j$ with coefficient functionals $\{x'_j\}_j$ one has $x'_1(x_1) = 1$.

1.3 Duality of Schauder frames

Given a Schauder frame $\left(\{x'_j\},\{x_j\}\right)$ of E it is rather natural to ask whether $\left(\{x_j\},\{x'_j\}\right)$ is a Schauder frame of E'. This is always the case when E' is endowed with the weak* topology $\sigma(E',E)$.

Lemma 1.3.1 *If $\left(\{x'_j\},\{x_j\}\right)$ is a Schauder frame of E, then $\left(\{x_j\},\{x'_j\}\right)$ is a Schauder frame of $(E',\sigma(E',E))$.*

Proof. For every $x' \in E'$ and $x \in E$ we have

$$x'(x) = x'\left(\sum_{j=1}^{\infty} x'_j(x)\,x_j\right) = \sum_{j=1}^{\infty} x'_j(x)\,x'(x_j) = \left(\sum_{j=1}^{\infty} x'(x_j)\,x'_j\right)(x),$$

and $x' = \sum_{j=1}^{\infty} x'(x_j)\,x'_j$ with convergence in $(E',\sigma(E',E))$. □

We study conditions to ensure that $\left(\{x_j\},\{x'_j\}\right)$ is a Schauder frame of the strong dual $(E',\beta(E',E))$ of E. Moreover we investigate the relation between the existence of certain Schauder frames and reflexivity. We recall that in the case of bases this questions lead to the concept of shrinking basis and boundedly complete basis defined in the appendix in 3 ; see [33].

Definition 1.3.2 Let $\left(\{x_j'\},\{x_j\}\right)$ be a Schauder frame of a lcs E, we consider the finite rank continuous linear operator defined as

$$\begin{array}{rcl} F_n : E & \longrightarrow & E \\ x & \longrightarrow & F_n\left(x\right) := \sum_{j=1}^{n} x_j'\left(x\right) x_j . \end{array}$$

Also, we consider the linear operator defined as

$$\begin{array}{rcl} T_n : E & \longrightarrow & E \\ x & \longrightarrow & T_n\left(x\right) := \sum_{j=n+1}^{\infty} x_j'\left(x\right) x_j . \end{array}$$

Observe that $T_n := I - F_n \in L\left(E\right), n \in \mathbb{N}$. If we assume that E is a barrelled lcs, then $\{F_n\}_n$ is an equicontinuous subset of $L\left(E\right)$ as we proved in the proof of Theorem 1.1.4.

Definition 1.3.3

1. A Schauder frame $\left(\{x_j'\},\{x_j\}\right)$ of a lcs E is said to be *shrinking* if, for all $x' \in E'$,

   $$\lim_{n\to\infty} x' \circ T_n = 0$$

   uniformly on the bounded subsets of E.

2. A Schauder frame $\left(\{x_j'\},\{x_j\}\right)$ of a lcs E is said to be *boundedly complete* if the series $\sum_{j=1}^{\infty} x_j'\left(x''\right) x_j$ converges in E for every $x'' \in E''$.

Proposition 1.3.4 *Let E be a lcs and let $\left(\{x_j'\},\{x_j\}\right)$ be a Schauder frame of E. The following are equivalent:*

(1) $\left(\{x_j\},\{x_j'\}\right)$ *is a Schauder frame for* E_β'.

(2) *For all* $x' \in E'$, $\sum_{j=1}^{\infty} x'\left(x_j\right) x_j'$ *is convergent in* E_β'.

(3) $\left(\{x_j'\},\{x_j\}\right)$ *is shrinking.*

Proof. (1) $\Rightarrow$ (2) is clear by the definition of Schauder frame.

(2) $\Rightarrow$ (3) if we suppose that for all $x' \in E'$, $\sum_{j=1}^{\infty} x'\left(x_j\right) x_j'$ is convergent in E_β', then there exists $\lim_{n\to\infty} \sum_{j=1}^{n} x'\left(x_j\right) x_j'$ in E_β'. This means that

$$\lim_{n\to\infty}\left[\sum_{j=1}^{\infty} x'\left(x_j\right) x_j' - \sum_{j=1}^{n} x'\left(x_j\right) x_j'\right] = \lim_{n\to\infty} \sum_{j=n+1}^{\infty} x'\left(x_j\right) x_j' = 0, \text{ in } E_\beta'.$$

Fixing $x' \in E'$ we want to see that $\lim_{n\to\infty}(x' \circ T_n) = 0$ in E'_β. Note that

$$\begin{aligned}(x' \circ T_n)(x) &= x'\left(\sum_{j=n+1}^{\infty} x'_j(x)\, x_j\right) = \sum_{j=n+1}^{\infty} x'_j(x)\, x'(x_j) \\ &= \left(\sum_{j=n+1}^{\infty} x'(x_j)\, x'_j\right)(x)\,.\end{aligned}$$

Then

$$x' \circ T_n = \sum_{j=n+1}^{\infty} x'(x_j)\, x'_j \longrightarrow 0, \text{ in } E'_\beta.$$

Finally, we prove (3) $\Rightarrow$ (1). Every $x' \in E'$ can be written as $x' = \sum_{j=1}^{\infty} x'(x_j)\, x'_j$ with convergence in the weak* topology $\sigma(E', E)$. Given a bounded set B in E,

$$\sup_{x\in B}\left|\left(x' - \sum_{j=1}^{n} x'(x_j)\, x'_j\right)(x)\right| = \sup_{x\in B}|x' \circ T_n(x)|$$

which tends to zero, hence $x' = \sum_{j=1}^{\infty} x'(x_j)\, x'_j$ in the topology $\beta(E', E)$.

□

A space E is called Montel if it is barrelled and every bounded subset of E is relatively compact. Since the pointwise convergence of an equicontinuous sequence of operators implies the uniform convergence on the compact sets, every Schauder frame of a Montel space E is shrinking. Beanland, Freeman and Liu [3] have shown that every infinite dimensional Banach space which admits a Schauder frame has also a Schauder frame which is not shrinking. The main tool in their proof is the existence of weak* null sequences in the unit sphere of E'. This result inspired the following characterization of Fréchet spaces with a Schauder frame that are Montel. In fact, since a Fréchet space E is Montel if and only if every weak* null sequence in E' is also strongly convergent [15], an adaptation of the proof of [3, Theorem 2.3] gives the following result.

Theorem 1.3.5 *Let E be a separable Fréchet space with the bounded approximation property. Then E is Montel if and only if every Schauder frame of E is shrinking.*

Proof. If E is Montel, then every Schauder frame is shrinking, according to our comments above. We prove the converse. By assumption E has a Schauder frame

$\big(\{x'_j\},\{x_j\}\big)$. If it is not shrinking, we are done. Thus, we assume that $\big(\{x'_j\},\{x_j\}\big)$ is shrinking. Fix $x_0 \in E$, $x_0 \neq 0$, and let $x'_0 \in E'$ such that $x'_0(x_0) = 1$. Since E is not Montel, we can apply [15] to find $(u'_j)_j \subset E'$ such that $u'_j(x) \to 0$ as $j \to \infty$ for each $x \in E$, but there are a bounded set B in E and $\varepsilon_0 > 0$ such that $\sup_{b\in B} |u'_j(b)| > \varepsilon_0$ for each $j \in \mathbb{N}$ and each $b \in B$. We now define

$$y_{3j-2} := x_j, \qquad y_{3j-1} := x_0, \qquad y_{3j} := x_0$$

$$y'_{3j-2} := x'_j, \qquad y'_{3j-1} := -u'_j, \qquad y'_{3j} := u_j.$$

We first show that $\big(\{y'_j\},\{y_j\}\big)$ is a frame in E. To do this, fix $x \in E$, a continuous seminorm p on E and $\delta > 0$. There is $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, we have $p(x - \sum_{j=1}^n x'_j(x)x_j) < \frac{\delta}{2}$ and $|u'_n(x)| < \frac{\delta}{2(p(x_0)+1)}$. For $m > 3n_0$, $x - \sum_{j=1}^m y'_j(x)y_j$ coincides with $x - \sum_{j=1}^n x'_j(x)x_j$ (if $m = 3n$ or $m = 3n-2$) and with $x - \sum_{j=1}^n x'_j(x)x_j + u'_j(x)x_0$ (if $m = 3n-1$). In the first case

$$p(x - \sum_{j=1}^m y'_j(x)y_j) = p(x - \sum_{j=1}^n x'_j(x)x_j) < \frac{\delta}{2} < \delta,$$

and in the second case

$$p(x - \sum_{j=1}^m y'_j(x)y_j) \leq p(x - \sum_{j=1}^n x'_j(x)x_j) + |u'_j(x)|p(x_0) < \delta.$$

This shows that $x = \sum_{j=1}^\infty y'_j(x)y_j$ in E for each $x \in E$. It remains to prove that $\big(\{y'_j\},\{y_j\}\big)$ is not shrinking. As $\big(\{x'_j\},\{x_j\}\big)$ is shrinking and $B \subset E$ is bounded, there is $N_0 \in \mathbb{N}$ such that, for each $b \in B$ and each $M \geq N_0$, $|x'_0(\sum_{j=M+1}^\infty x'_j(b)x_j)| < \frac{\varepsilon_0}{2}$. For each $M \geq N_0$ there is $b_M \in B$ with $|u'_M(b_M)| > \varepsilon_0$. Then

$$\begin{aligned}
\sup_{b\in B}\left|x'_0(\sum_{i=3M}^\infty y'_i(b)y_i)\right| &\geq \left|x'_0(\sum_{i=3M}^\infty y'_i(b_M)y_i)\right| \\
&\geq |x'_0(u'_M(b_M)x_0)| - \left|x'_0(\sum_{j=M+1}^\infty x'_j(b_M)x_j)\right| \\
&> |u'_M(b_M)| - \left|x'_0(\sum_{j=M+1}^\infty x'_j(b_M)x_j)\right| > \varepsilon_0 - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2}.
\end{aligned}$$

Hence we have that $(x'_0 \circ T_{3M})_{M=1}^\infty$ does not converge to 0 uniformly on the bounded subsets of E. Thus $\big(\{y'_j\},\{y_j\}\big)$ is not shrinking. $\square$

Proposition 1.3.6 *Let E be a lcs and let $\left(\{x_j'\}, \{x_j\}\right)$ be a shrinking Schauder frame of E. Then $\left(\{x_j\}, \{x_j'\}\right)$ is a boundedly complete Schauder frame of E_β'.*

Proof. Since $\left(\{x_j'\}, \{x_j\}\right)$ is a shrinking Schauder frame of E, then $\left(\{x_j\}, \{x_j'\}\right)$ is a Schauder frame of E_β'. Moreover, given $x''' \in E'''$ set $x' := x'''|_E$ to obtain

$$\sum_{j=1}^{\infty} x'''\left(x_j\right) x_j' = \sum_{j=1}^{\infty} \left(x'''|_E\right)\left(x_j\right) x_j' = \sum_{j=1}^{\infty} x'\left(x_j\right) x_j' = x'.$$

□

Recall that a boundedly complete Schauder basis $\{e_j\}_j$ in a lcs E is a basis such that if $\{\alpha_j\}_j \in \omega$ and $\{\sum_{j=1}^k \alpha_j e_j\}_k$ is bounded, then $\sum_{j=1}^\infty \alpha_j e_j$ is convergent. We refer to the appendix in Chapter 3 for the definition.

In [16] it is shown that a basis $\{e_j\}_j$ in a Banach space X is boundedly complete if and only if the Schauder frame $\left(\{e_j'\}, \{e_j\}\right)$ is boundedly complete. This extends to arbitrary barrelled spaces.

Proposition 1.3.7 *Let E be a barrelled lcs with a Schauder basis $\{e_j\}_j$. Then the following are equivalent:*

(1) *The basis is boundedly complete.*

(2) *The Schauder frame $\left(\{e_j'\}, \{e_j\}\right)$ is boundedly complete.*

Proof. To prove (1) $\Rightarrow$ (2) we fix $x'' \in E''$ and we prove that $\sum_{j=1}^\infty e_j'\left(x''\right) e_j$ converges in E. For every $x' \in E'$ and $x \in E$ we have

$$\lim_{k\to\infty} \left(\sum_{j=1}^{k} x'\left(e_j\right) e_j'\right)(x) = \lim_{k\to\infty} x'\left(\sum_{j=1}^{k} e_j'\left(x\right) e_j\right) = x'(x).$$

Since E is barrelled we conclude that $\left\{\sum_{j=1}^k x'\left(e_j\right) e_j', k \in \mathbb{N}\right\}$ is $\beta(E', E)$-bounded. Consequently $\left\{\sum_{j=1}^k x''\left(e_j'\right) x'(e_j), k \in \mathbb{N}\right\}$ is a bounded set of scalars for every $x' \in E'$, which means that $\left\{\sum_{j=1}^k x''\left(e_j'\right) e_j, k \in \mathbb{N}\right\}$ is $\sigma(E, E')$-bounded. As all topologies of the same dual pair have the same bounded sets ([33, 8.3.4]) we finally obtain that $\left\{\sum_{j=1}^k x''\left(e_j'\right) e_j, k \in \mathbb{N}\right\}$ is a bounded subset of E and the conclusion follows.

To prove (2) $\Rightarrow$ (1) we fix $\{\alpha_j\}_j \subset \mathbb{K}$ such that $\{\sum_{j=1}^{k} \alpha_j e_j\}_k$ is bounded and we show that $\sum_{j=1}^{\infty} \alpha_j e_j$ is convergent in E. Since $\{\sum_{j=1}^{k} \alpha_j e_j\}_k$ is $\sigma(E'', E')$-relatively compact it has a $\sigma(E'', E')$-cluster point $x'' \in E''$. By hypothesis, $\sum_{j=1}^{\infty} x''\left(e_j'\right) e_j$ is convergent in E, so to conclude it suffices to check that $x''\left(e_j'\right) = \alpha_j$. To this end we fix $j \in \mathbb{N}$ and $k > j$ and observe that

$$e_j'\left(\sum_{i=1}^{k} \alpha_i e_i\right) = \sum_{i=1}^{k} \alpha_i e_j'(e_i) = \alpha_j.$$

As $x''(e_j')$ is a cluster point of $\left\{e_j'\left(\sum_{i=1}^{k} \alpha_i e_i\right)\right\}_{k=1}^{\infty}$ we finally deduce $x''\left(e_j'\right) = \alpha_j$. $\square$

Proposition 1.3.8 *Let E be a lcs. The following holds:*

(1) *Let $P : E \to E$ be a continuous linear projection. If $\left(\{x_j'\}, \{x_j\}\right)$ is a shrinking Schauder frame for E, then $\left(\{P'\left(x_j'\right)\}, \{P(x_j)\}\right)$ is a shrinking Schauder frame for $P(E)$.*

(2) *Let $P : E \to E$ be a continuous linear projection. If $\left(\{x_j'\}, \{x_j\}\right)$ is a boundedly complete Schauder frame for E, then $\left(\{P'\left(x_j'\right)\}, \{P(x_j)\}\right)$ is a boundedly complete Schauder frame for $P(E)$.*

Theorem 1.3.9 *Let $\left(\{x_j'\}, \{x_j\}\right)$ be a Schauder frame of a lcs E. If $\left(\{x_j'\}, \{x_j\}\right)$ is boundedly complete Schauder frame, E is a barrelled and complete lcs E with E_β'' barrelled, then E is complemented in its bidual E_β''.*

Proof. We define the canonical inclusion

$$\begin{array}{rcl} j : E & \longrightarrow & E_\beta'' \\ x & \longrightarrow & j(x)(x') := x'(x), \text{ for every } x' \in E'; \end{array}$$

since E is barrelled, j is continuous. On the other hand, we define

$$\begin{array}{rcl} g : E_\beta'' & \longrightarrow & E \\ x'' & \longrightarrow & g(x'') := \sum_{j=1}^{\infty} x''\left(x_j'\right) x_j; \end{array}$$

a well-defined map since $\left(\{x_j'\}, \{x_j\}\right)$ is a boundedly complete Schauder frame. Note that g is linear. Since $g(x'') = \lim_{k\to\infty} \sum_{j=1}^{k} x''\left(x_j'\right) x_j$ with $\sum_{j=1}^{k} x''\left(x_j'\right) x_j$ continuous for every $k \in \mathbb{N}$ and since E_β'' is barrelled, by Banach-Steinhaus theorem, g is continuous. Now, by [32, Proposition 2.7.3], we only have to show that $g \circ j = I_E$. Let $x \in E$, then:

$$g \circ j(x) = \sum_{j=1}^{\infty} j(x)\left(x_j'\right) x_j = \sum_{j=1}^{\infty} x_j'(x)\, x_j = x.$$

□

For a Fréchet space E, the bidual E''_β is again a Fréchet space, therefore barrelled. For (LB)-spaces, this is not always the case. In fact, if $\lambda_1(A)$ is the Grothendieck example of a non-distinguished Fréchet space, $\lambda_1(A)$ is the strong dual of an (LB)-space E. The bidual of E, being the strong dual of $\lambda_1(A)$, is not barrelled. See [41, Chapter 31, Sections 6 and 7] and [48, Example 27.19].

Theorem 1.3.10 *Let $(\{x'_j\},\{x_j\})$ be a Schauder frame of a lcs E. If $(\{x'_j\},\{x_j\})$ is shrinking and boundedly complete, then E is semi-reflexive. If, in addition, E is barrelled then it is reflexive.*

Proof. Fix $x'' \in E''$. Since the Schauder frame is boundedly complete then $\sum_{j=1}^{\infty} x'_j\left(x''\right) x_j$ converges to an element $x \in E$. We claim that $x'' = x$. In fact, since the Schauder frame is shrinking, for every $x' \in E'$ we have $x' = \sum_{j=1}^{\infty} x'\left(x_j\right) x'_j$ with convergence in E'_β. Thus

$$\begin{aligned} \langle x'', x' \rangle &= \langle x'', \sum_{j=1}^{\infty} x'\left(x_j\right) x'_j \rangle = \sum_{j=1}^{\infty} x'\left(x_j\right) x''\left(x'_j\right) \\ &= \left(\sum_{j=1}^{\infty} x''\left(x'_j\right) x_j \right) \left(x'\right) = \langle x, x' \rangle . \end{aligned}$$

It follows $x'' = x$. □

We recall that if E is barrelled, then $\{F_n\}_n \subset L\left(E\right)$ is equicontinuous.

Lemma 1.3.11 *Let E be a barrelled lcs and let F_n be as Definition 1.3.2, then $\{F'_n\}_n \subset L\left(E'_\beta\right)$ is equicontinuous.*

Proof. Given $B \in \mathcal{B}(E)$, we show that $C := \bigcup_{n\in\mathbb{N}} F_n\left(B\right)$ is bounded in E. Indeed, fix $U \in \mathcal{U}_0\left(E\right)$ there is $V \in \mathcal{U}_0\left(E\right)$ such that $F_n\left(V\right) \subset U$ for each $n \in \mathbb{N}$. Select $\lambda > 0$ such that $B \subset \lambda V$. This implies $F_n\left(B\right) \subset \lambda F_n\left(V\right) \subset \lambda U$ for each $n \in \mathbb{N}$. Hence $C \subset \lambda U$ and C is bounded. We conclude that for every $B \in \mathcal{B}\left(E\right)$, there exists $C \in \mathcal{B}\left(E\right)$ such that $F'_n\left(C^\circ\right) \subset B^\circ$. □

Lemma 1.3.12 *Suppose that $(\{x'_j\},\{x_j\})$ is a Schauder frame of a barrelled lcs E such that for all $k \in \mathbb{N}$*

$$\lim_{n\to\infty} \left(x'_k - \sum_{j=1}^{n} x'_k\left(x_j\right) x'_j \right) = 0 \text{ in } E'_\beta. \tag{1.3.1}$$

Then $\left(\{x_j\},\{x'_j\}\right)$ *is a Schauder frame of the closed linear span* $H=\overline{\operatorname{span}\left\{x'_j\right\}}^{E'_\beta}$.

Proof. We fix $x'\in H$ and show that $x'=\sum_{j=1}^{\infty}x'(x_j)\,x'_j$ with convergence in E'_β. To this end we fix U a neighborhood of zero in E'_β and consider $F_n(x)=\sum_{j=1}^{n}x'_j(x)x_j$, $n\in\mathbb{N},x\in E$. Since $\{F'_n\}_n\subset L(E')$ is equicontinuous, there is another $\beta(E',E)$-neighborhood V, $V\subset U$, such that $F'_n(V)\subset\frac{1}{3}U$ for each $n\in\mathbb{N}$. Find $u=\sum_{k=1}^{s}\alpha_k x'_k,\alpha_k\in\mathbb{K},s\in\mathbb{N}$, with $x'-u\in\frac{1}{3}V$. By condition (1.3.1) we can find $n_0\in\mathbb{N}$ such that $u-F'_n(u)\in\frac{1}{3}V$ for each $n\geq n_0$. Finally,

$$x'-F'_n(x')=x'-u-F'_n(x'-u)+u-F'_n(u)\in\frac{1}{3}V+\frac{1}{3}U+\frac{1}{3}V\subset U \text{ if } n\geq n_0.$$

Thus $E'_\beta\text{-}\lim_{n\to\infty}F'_n(x')=x'$ and the conclusion follows. □

Remark 1.3.13 (a) Observe that if $\left(\{x_j\},\{x'_j\}\right)$ is a Schauder frame of the closed linear span $H=\overline{\operatorname{span}\left\{x'_j\right\}}^{E'_\beta}$ then (1.3.1) holds since $x'_k\in H$ for each $k\in\mathbb{N}$.

(b) If $\{x_j\}$ is a Schauder basis in E with coefficient functionals $\{x'_j\}$ then (1.3.1) also holds, since $x'_k-\sum_{j=1}^{n}x'_k(x_j)\,x'_j=0$ for every $n\geq k$.

(c) If E is a Montel space, (1.3.1) holds since every weakly convergent sequence in a Montel space is also strongly convergent to the same limit, by [33, 11.6.2].

Theorem 1.3.14 *Let* $\left(\{x'_j\},\{x_j\}\right)$ *be a Schauder frame of a lcs* E*. If* E *is reflexive and* (1.3.1) *in Lemma 1.3.12 holds, then* $\left(\{x'_j\},\{x_j\}\right)$ *is shrinking.*

Proof. As E is reflexive then it is barrelled and Lemmas 1.3.1 and 1.3.12 hold. In particular, for each $x'\in H=\overline{\operatorname{span}\{x'_j\}}^{E'_\beta}$ we have $x'=\sum_{j=1}^{\infty}x'(x_j)\,x'_j$ with convergence in E'_β. Since E is semi-reflexive, $\beta(E',E)$ and $\sigma(E',E)$ are topologies of the same dual pair. Hence, by Lemma 1.3.1 we obtain $H=\overline{\operatorname{span}\{x'_j\}}^{E'_\beta}=\overline{\operatorname{span}\{x'_j\}}^{\left(E',\sigma\left(E',E\right)\right)}=E'$. The result follows by Proposition 1.3.4. □

1.4 Unconditional Schauder frames

In this section we assume that E is a complete lcs and we denote by $\mathcal{U}_0(E)$ the set of absolutely convex and closed 0-neighborhoods. We refer the reader to [33] for unconditional convergence of series in locally convex spaces.

Definition 1.4.1 A Schauder frame $\left(\{x_j'\},\{x_j\}\right)$ for a lcs E is said to be *unconditional* if for every $x \in E$ we have $x = \sum_{j=1}^{\infty} x_j'(x)\, x_j$ with unconditional convergence.

Remark 1.4.2 By [45, p.116] a series $\sum_{j=1}^{\infty} x_j$ in a (sequentially) complete lcs converges unconditionally if and only if the limits $\lim\limits_{n\to\infty}\sum\limits_{j=1}^{n} a_j x_j$ exist uniformly for $\{a_j\}_j$ in the unit ball of ℓ_∞, and the operator

$$\begin{array}{rcl} \ell_\infty & \longrightarrow & E \\ \{a_j\}_j & \longrightarrow & \sum_{j=1}^{\infty} a_j x_j; \end{array}$$

is continuous.

Lemma 1.4.3 *Let X be a normed space, E a barrelled space and G any lcs. Then every separately continuous bilinear map $B : X \times E \to G$ is continuous.*

Proof. Let $W \in \mathcal{U}_0(G)$ and let U_X be the closed unit ball of X. We take $T := \{y \in E : \ B(x,y) \in W \ \text{ for every } \ x \in U_X\} = \bigcap_{x\in U_X} B_x^{-1}(W)$, where $B_x = B(x,\cdot)$. Note that T is an absolutely convex closed subset since each $B_x : E \to G$ is continuous. Fixing $y \in E$, since $B_y : X \to G$ is continuous then $B_y^{-1}(W) \in \mathcal{U}_0(X)$, which means that there exists $\lambda > 0$ such that $\lambda U_X \subset B_y^{-1}(W)$. Therefore $B(x,\lambda y) \in W$ for every $x \in U_X$ and thus $\lambda y \in T$, that is, T is absorbent. Since E is barrelled, $T \in \mathcal{U}_0(E)$ and from $B(U_X \times T) \subset W$ we conclude that B is continuous. $\square$

Corollary 1.4.4 *Let $\left(\{x_j'\},\{x_j\}\right)$ be an unconditional Schauder frame for a complete barrelled lcs E. Then, the bilinear map*

$$\begin{array}{rcl} B : E \times \ell_\infty & \longrightarrow & E \\ (x,a) & \longrightarrow & B(x,a) := \sum_{j=1}^{\infty} a_j x_j'(x)\, x_j; \end{array}$$

is continuous.

The property of having an unconditional Schauder frame is also inherited by complemented subspaces.

Proposition 1.4.5 *Let E be a lcs and let $P : E \to E$ be a continuous linear projection. If $\left(\{x_j'\}, \{x_j\}\right)$ is an unconditional Schauder frame for E, then $\left(\{P'(x_j')\}, \{P(x_j)\}\right)$ is an unconditional Schauder frame for $P(E)$.*

Proof. By Proposition 1.1.2 we obtain that it is a Schauder frame and it is an unconditional Schauder frame since P is linear; then, for every $b \in B_{\ell^\infty}$, we obtain:

$$P\left(\sum_{j=1}^{\infty} b_j x_j'(y)\, x_j\right) = \sum_{j=1}^{\infty} b_j \langle P'\left(x_j'\right), y\rangle P(x_j).$$

□

Corollary 1.4.6 *If E is isomorphic to a complemented subspace of a lcs with an unconditional Schauder basis, then E admits an unconditional Schauder frame.*

Similarly to Lemma 1.1.3 we have the following.

Lemma 1.4.7 *Let $\{x_j\}_j$ be a fixed sequence of non-zero elements in a lcs E and let us denote by $\widetilde{\bigwedge}$ the space*

$$\widetilde{\bigwedge} := \left\{\alpha = \{\alpha_j\}_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j \text{ is unconditionally convergent in } E\right\}. \tag{1.4.1}$$

Endowed with the system of seminorms

$$\widetilde{\mathcal{Q}} := \left\{\widetilde{q}_p\left(\{\alpha_j\}_j\right) := \sup_{b \in B_{\ell^\infty}} p\left(\sum_{j=1}^{\infty} b_j \alpha_j x_j\right), \text{ for all } p \in cs(E)\right\}, \tag{1.4.2}$$

$\widetilde{\bigwedge}$ is a complete lcs of sequences and the canonical unit vectors are an unconditional basis.

Proof. The fact that $\widetilde{\mathcal{Q}}$ is a fundamental system of seminorms for the space $\widetilde{\bigwedge}$ is clear as $cs(E)$ is a fundamental system of seminorms of E. Moreover, $\mathbb{K}^{(\mathbb{N})} \subset \widetilde{\bigwedge}$ and $\widetilde{\bigwedge} \subset \bigwedge$ ($\bigwedge$ as in Lemma 1.1.3) continuously, hence $\widetilde{\bigwedge}$ is a lcs of sequences.

From the definition of $\widetilde{\bigwedge}$ and 1.4.2, we have $\widetilde{\bigwedge} = \overline{\operatorname{span}\{e_i : i \in \mathbb{N}\}}$, to conclude that the canonical unit vectors are a basis we argue as in Lemma 1.1.3.

To see the completeness, let $\{\{\alpha_j^\gamma\}_j\}_{\gamma\in\Gamma}$ be a Cauchy net in $\widetilde{\bigwedge}$. As $\widetilde{\bigwedge} \subset \bigwedge$ continuously and $\bigwedge$ is complete, we have that the net converges to $\alpha \in \bigwedge$, even more, for each $b \in B_{\ell^\infty}$, $\{\{b_j\alpha_j^\gamma\}_j\}_{\gamma\in\Gamma}$ converges to $\{b_j\alpha_j\}_j \in \bigwedge$, which shows that $\alpha \in \widetilde{\bigwedge}$. Given $\varepsilon > 0$ and $p \in cs(E)$ we find γ_0 such that for $\gamma, \gamma' \geq \gamma_0$

$$p\left(\sum_{j=1}^{n} b_j\alpha_j^\gamma x_j - \sum_{j=1}^{n} b_j\alpha_j^{\gamma'} x_j\right) < \varepsilon$$

for all $n \in \mathbb{N}$ and all $b \in B_{\ell^\infty}$, hence taking limits $p(\sum_{j=1}^n b_j\alpha_j^\gamma x_j - \sum_{j=1}^n b_j\alpha_j x_j) \leq \varepsilon$ for all $n \in \mathbb{N}$ and all $b \in B_{\ell^\infty}$, which implies that

$$p\left(\sum_{j=1}^{\infty} b_j\alpha_j^\gamma x_j - \sum_{j=1}^{\infty} b_j\alpha_j x_j\right) \leq \varepsilon$$

for all $n \in \mathbb{N}$ and all $b \in B_{\ell^\infty}$, thus the net converges to α in $\widetilde{\bigwedge}$. □

Theorem 1.4.8 *Let E be a complete, barrelled lcs. The following conditions are equivalent:*

(1) *E admits an unconditional Schauder frame.*

(2) *E is isomorphic to a complemented subspace of a complete sequence space with the canonical unit vectors as unconditional Schauder basis.*

(3) *E is isomorphic to a complemented subspace of a complete sequence space with unconditional Schauder basis.*

Proof. The proof follows the steps of Theorem 1.1.4 but the continuity of the map

$$U : E \longrightarrow \widetilde{\bigwedge},\ x \to \{x_j'(x)\}_j,$$

now follows from Corollary 1.4.4. □

In our next two results, bipolars are taken in E'' that is $U^{\circ\circ} = \{x'' \in E'' : |x''(x')| \leq 1 \text{ for all } x' \in U^\circ\}$.

Lemma 1.4.9 *Let E be a lcs and let U be an absolutely convex and closed 0-neighborhood. For every $z \in E''$ such that $p_{U^{\circ\circ}}(z) > 0$ there exists $\{x_\alpha\}_\alpha \subset E$ with $p_U(x_\alpha) \leq p_{U^{\circ\circ}}(z)$ and $x_\alpha \to z$ in $\sigma(E'', E')$.*

Proof. First, we take $x := \frac{z}{p_{U^{\circ\circ}}(z)}$; note that $p_{U^{\circ\circ}}(x) = 1$. Since $U^{\circ\circ}$ is $\sigma(E'', E')$-closed, then is closed for the natural topology in E'' [41, p. 300] with $U^{\circ\circ} = \{y \in E'' : p_{U^{\circ\circ}}(y) \leq 1\}$. Thus $x \in U^{\circ\circ} = \overline{U}^{\sigma(E'',E')}$ (by Theorem 4 in [58, p. 35]); this means that there exists $\{y_\alpha\}_\alpha \subset U$ such that $y_\alpha \to x$ in $\sigma(E'', E')$. Now, $p_{U^{\circ\circ}}(z)\, y_\alpha \to z$ and $x_\alpha := p_{U^{\circ\circ}}(z)\, y_\alpha$ satisfy $p_U(x_\alpha) = p_{U^{\circ\circ}}(z)\, p_U(y_\alpha) \leq p_{U^{\circ\circ}}(z)$ as $p_U(y_\alpha) \leq 1$. □

The following result is well-known. A proof for Banach spaces can be seen e.g. in [33, 8.5.9]. We give an idea of the proof in the general case for the convenience of the reader.

Lemma 1.4.10 (*Sobzcyk's theorem for lcs*). *Let E be a separable lcs. Let H be a subspace isomorphic to c_0. Then H is complemented in E.*

Proof. Let $T : H \to c_0$ be a topological isomorphism. Denote by $e'_j \in (c_0)'$ the continuous linear form mapping each $x \in c_0$ to its n-th coordinate. Define $v_j : H \to \mathbb{K}$ by $v_j(x) := e'_j(T(x)), x \in H$. The continuity of T implies that the sequence $\{v_j\}_j \subset H'$ is equicontinuous. Moreover, since $T(x) \in c_0$ for each $x \in H$, it follows that $\{v_j\}_j$ converges to 0 in $\sigma(H', H)$. We apply [42, 33.5 (2)] to find an equicontinuous sequence $u_j : E \to \mathbb{K}, j \in \mathbb{N}$, such that the restriction of u_j to H coincides with v_j for each j and $\{u_j\}_j$ converges to 0 for the topology $\sigma(E', E)$. Define $S : E \to c_0$ by $S(z) := \{u_j(z)\}_j, z \in E$. The map S is well defined, linear, continuous and $S(x) = T(x)$ for each $x \in H$. Now it is easy to show that $P := T^{-1} \circ S$ is a continuous projection from E to H. □

Theorem 1.4.11 *Let E be a complete, barrelled lcs which admits an unconditional Schauder frame $\left(\{x'_j\}, \{x_j\}\right)$. Then, $\left(\{x'_j\}, \{x_j\}\right)$ is boundedly complete if and only if E does not contain a copy of c_0.*

Proof. Suppose that E contains a copy of c_0. Since E is separable, there exists a projection $P : E \to E$ such that $P(E)$ is isomorphic to c_0 ([33, 8.5.9]). If $\left(\{x'_j\}, \{x_j\}\right)$ is boundedly complete, then $\left(\{P'(x'_j)\}, \{P(x_j)\}\right)$ is a boundedly complete Schauder frame in $P(E) \simeq c_0$. By Proposition 1.3.14, c_0 is complemented in its bidual, a contradiction.

In order to show the converse, suppose that E does not contain a copy of c_0 and $\left(\{x'_j\}, \{x_j\}\right)$ is not boundedly complete. Then there exists $x'' \in E''$, $x'' \neq 0$, such that $\sum_{j=1}^{\infty} x''\left(x'_j\right) x_j$ is not convergent in E. We can find an absolutely convex 0-neighborhood U_1 and two sequences (p_i), (q_i) of natural numbers such that $p_1 < q_1 < p_2 < q_2 < \ldots$ and $p_{U_1}\left(\sum_{i=p_j}^{q_j} x''(x'_i)\, x_i\right) \geq 1$ for each $j \in \mathbb{N}$. We set $y_j := \sum_{i=p_j}^{q_j} x''(x'_i)\, x_i$ and define $T : \varphi \to E$ by $T(\{a_j\}_j) := \sum_{j=1}^{\infty} a_j y_j$.

We first prove that T is continuous when φ is endowed with the $\|\cdot\|_\infty$- norm. To this end, take U an absolutely convex neighborhood of the origin in E. Since $x'' \neq 0$, $x'' \in E''$, there is an absolutely convex 0-neighborhood U_2 in E such that $p_{U_2^{\circ\circ}}(x'') > 0$. Put $V := U_1 \cap U_2 \cap U$. Clearly $p_{V^{\circ\circ}}(x'') \geq p_{U_2^{\circ\circ}}(x'') > 0$. We can apply Corollary 1.4.4 to find an absolutely convex closed 0-neighborhood W in E such that $W \subset V$ and

$$p_V\left(\sum_{i=1}^{\infty} d_i x_i'(z)\, x_i\right) \leq p_W(z)\, \|d\|_\infty \tag{1.4.3}$$

for each $n \in \mathbb{N}$, each $d \in \ell^\infty$ and $z \in E$. For $a = \{a_j\}_j \in \varphi$, $a \neq 0$, and $s := \max(\operatorname{supp} a)$, the support of a being the set of non-zero coordinates of a, we define $b_i = a_j$ for $p_j \leq i \leq q_j$, and $b_i = 0$ otherwise. We have

$$\sum_{j=1}^{\infty} a_j y_j = \sum_{j=1}^{s} a_j y_j = \sum_{i=p_1}^{q_s} b_i x''(x_i')\, x_i.$$

Given $\varepsilon > 0$, we can apply Lemma 1.4.9 to find $y \in E$, $p_W(y) \leq p_{W^{\circ\circ}}(x'')$ and

$$\max_{p_1 \leq i \leq q_s} |(x'' - y)(x_i')| \leq \frac{\varepsilon}{2q_s \|a\|_\infty \max(p_V(x_i), 1)}.$$

This implies

$$\begin{aligned} p_V\left(\sum_{i=p_1}^{q_s} b_i x''(x_i')\, x_i\right) &\leq p_V\left(\sum_{i=p_1}^{q_s} b_i x_i'(y)\, x_i\right) + \sum_{i=p_1}^{q_s} |b_i|\, |(x''-y)(x_i')|\, p_V(x_i) \\ &\leq p_V\left(\sum_{i=p_1}^{q_s} b_i x_i'(y)\, x_i\right) + \frac{\varepsilon}{2}. \end{aligned}$$

Now, by the estimate (1.4.9), we obtain

$$p_V\left(\sum_{i=p_1}^{q_s} b_i x_i'(y)\, x_i\right) \leq \left(\max_{p_1 \leq i \leq q_s} |b_i|\right) p_W(y) \leq \left(\max_{1 \leq j \leq s} |a_j|\right) p_{W^{\circ\circ}}(x'').$$

Then,

$$p_V\left(\sum_{j=1}^{s} a_j y_j\right) \leq \|a\|_\infty\, p_{W^{\circ\circ}}(x'') + \frac{\varepsilon}{2}.$$

Since this holds for each $\varepsilon > 0$, we get

$$p_U\left(\sum_{j=1}^{\infty} a_j y_j\right) \leq p_V\left(\sum_{j=1}^{\infty} a_j y_j\right) \leq \|a\|_\infty \, p_{W^{\circ\circ}}\left(x''\right).$$

Thus the operator $T : (\varphi, \|\cdot\|_\infty) \to E$ is continuous. Since E is complete, T admits a unique continuous extension $\widetilde{T} : c_0 \to E$. As by assumption E does not contain c_0, we can apply Theorem 4 in [57, p.208] to conclude that $\{\widetilde{T}(e_j)\}_j$ has a convergent subsequence $\{\widetilde{T}(e_{j_k})\}_k$. That is, $\{y_j\}_j$ admits a convergent subsequence $\{y_{j_k}\}_k$. Moreover, since $\widetilde{T} : (c_0, \sigma(c_0, l_1)) \to (E, \sigma(E, E'))$ is also continuous then $\{\widetilde{T}(e_j)\}_j = \{y_j\}_j$ is $\sigma(E, E')$-convergent to 0, hence $\{y_{j_k}\}_k$ must converge to 0 in E. This is a contradiction, since $p_{U_1}(y_j) \geq 1$ for each $j \in \mathbb{N}$. $\square$

Now, we give the definition of boundedly retractive inductive limits in the case of (LF)-spaces. We define in the appendix in 3 the general case.

Definition 1.4.12 [53] An (LF)-space $E = \mathrm{ind}_{n\to} E_n$ is called *boundedly retractive* if for every bounded set B in E there exists $m = m(B)$ such that B is contained and bounded in E_m and E_m and E induce the same topology on B.

By [27] an (LF)-space E is boundedly retractive if and only if each bounded subset in E is in fact bounded in some step E_n and for each n there is $m > n$ such that E_m and E induce the same topology on the bounded sets of E_n.

For (LB)-spaces, this is equivalent to the a priori weaker condition that for all $n \in \mathbb{N}$, there exists $m > n$ such that for all $k > m$, E_m and E_k induce the same topology in the unit ball B_n of E_n ([51]). In particular (LB)-spaces with compact linking maps $E_n \hookrightarrow E_{n+1}$ are boundedly retractive. More information about these and related concepts can be seen in [67].

Obviously, each Fréchet space F can be seen as a boundedly retractive (LF)-space, just take $F_n = F$ for all $n \in \mathbb{N}$. In particular Theorem 1.4.14 below holds for Fréchet spaces. Every strict (LF)-space is boundedly retractive. In particular, for an open subset Ω in $\mathbb{R}^d$, the space $\mathcal{D}(\Omega)$ is a boundedly retractive (LF)-space. The space $\mathcal{E}'(\Omega)$ and the space VH in Example 1 of Section 1.5 are boundedly retractive (LB)-spaces.

Rosenthal's ℓ_1 theorem was extended to Fréchet spaces by Díaz in [21], showing that every bounded sequence in a Fréchet space has a subsequence that is either weakly Cauchy or equivalent to the unit vectors in ℓ_1.

Proposition 1.4.13 (*Rosenthal's ℓ_1 theorem for (LF)-spaces*). *Let $E = \mathrm{ind}_{n\rightarrow} E_n$ be a boundedly retractive (LF)-space. Every bounded sequence in E has a subsequence which is $\sigma(E, E')$-Cauchy or equivalent to the unit vector basis of ℓ_1. In particular, E does not contain a copy of ℓ_1 if and only if every bounded sequence in E has a $\sigma(E, E')$-Cauchy subsequence.*

Proof. Let $\{x_j\}_j$ be a bounded sequence in E and assume that it has no $\sigma(E, E')$-Cauchy subsequence. There is $n_0 \in \mathbb{N}$ such that $\{x_j\}_j$ is a bounded sequence in E_{n_0}. Now select $m \geq n_0$ such that E_m and E induce the same topology on the bounded sets of E_{n_0}. Since $\{x_j\}_j$ is bounded in E_m and it has no $\sigma(E_m, E'_m)$-Cauchy subsequence, we can apply Rosenthal's ℓ_1 theorem in the Fréchet space E_m to conclude that there is a subsequence $\{x_{j_k}\}_k$ which is equivalent to the unit vector basis of ℓ_1. That is, there exist c_1 and a continuous seminorm p in E_m such that

$$c_1 \sum_{k=1}^{\infty} |\alpha_k| \leq p\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq \sup_k p(x_{j_k}) \sum_{k=1}^{\infty} |\alpha_k|,$$

for every $\alpha = \{\alpha_k\}_k \in \ell_1$.

As the inclusion $E_{n_0} \hookrightarrow E_m$ is continuous, we find a continuous seminorm q in E_{n_0} such that for $x \in E_{n_0}$ one has $p(x) \leq q(x)$. Then, for each $\alpha = \{\alpha_k\}_k \in \ell_1$,

$$c_1 \sum_{k=1}^{\infty} |\alpha_k| \leq p\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq q\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq \sup_k q(x_{j_k}) \sum_{k=1}^{\infty} |\alpha_k|.$$

Set $F := \{\sum_{k=1}^{\infty} \alpha_k x_{j_k} : \alpha = \{\alpha_k\}_k \in \ell_1\} \subset E_{n_0}$. Then p and q restricted to F are equivalent norms, and F endowed with any of them is a Banach space isomorphic to ℓ_1. The spaces E_{n_0} and E_m induce on F the same (Banach) topology. Denote by U_F the closed unit ball of F and by τ_m and τ the topologies of E_m and E, respectively. Then τ and τ_m coincide on U_F, which is an absolutely convex 0-neighbourhood for $\tau_m|_F$. Applying a result of Roelcke [53, 8.1.27] we conclude that τ_m and τ coincide in F; hence, there is a continuous seminorm r on E such that $p(z) \leq r(z)$ for every $z \in F$. This implies, for each $\alpha = \{\alpha_k\}_k \in \ell_1$,

$$c_1 \sum_{k=1}^{\infty} |\alpha_k| \leq p\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq r\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq \left(\sup_k r(x_{j_k})\right) \sum_{k=1}^{\infty} |\alpha_k|.$$

Thus, $\{x_{j_k}\}_k$ is equivalent to the unit vectors of ℓ_1 in E and the inclusion $F \hookrightarrow E$ is a topological isomorphism into. Then, E contains an isomorphic copy of ℓ_1. $\square$

In the proof of the next result we utilize the fact that a boundedly retractive (LF)-space E does not contain ℓ_1 if and only if every $\mu(E', E)$-null sequence in E' is $\beta(E', E)$-convergent to 0. This was proved by Domański and Drewnowski and by Valdivia independently for Fréchet spaces. The proof can be seen in [14] and the proof for arbitrary boundedly retractive (LF)-spaces follows the same steps as in [14, Theorem 10] but using Proposition 1.4.13 instead of Rosenthal ℓ_1-theorem for Fréchet spaces.

Theorem 1.4.14 *Let E be a boundedly retractive (LF)-space. Assume that E admits an unconditional Schauder frame $\left(\{x'_j\}, \{x_j\}\right)$. Then, $\left(\{x'_j\}, \{x_j\}\right)$ is shrinking if and only if E does not contain a copy of ℓ_1.*

Proof. We first assume that $\left(\{x'_j\}, \{x_j\}\right)$ is shrinking. Then, by Proposition 1.3.4, $\left(\{x'_j\}, \{x_j\}\right)$ is a Schauder frame for E'_β and, in particular, E'_β is separable. Consequently E contains no subspace isomorphic to ℓ_1.

Conversely, assume that E does not contain a copy of ℓ_1. By Lemma 1.3.1, $\left(\{x_j\}, \{x'_j\}\right)$ is a Schauder frame of $(E', \sigma(E', E))$. We check that, for all $x' \in E'$,

$$\sum_{j=1}^{\infty} x'(x_j)x'_j \tag{1.4.4}$$

is subseries summable to x' in E'_β. Since for each $x \in E$ the convergence of

$$\sum_{j=1}^{\infty} x'_j(x)x_j \tag{1.4.5}$$

is unconditional and E is sequentially complete, then (1.4.5) is subseries summable and we conclude that (1.4.4) is also $\sigma(E', E)$-subseries summable. We can apply Orlicz-Pettis' Theorem ([33, p. 308]) to obtain that (1.4.4) is $\mu(E', E)$-unconditionally convergent to x'. Therefore it is $\beta(E', E)$-convergent to x', as E does not contain a copy of ℓ_1. Consequently $\left(\{x'_j\}, \{x_j\}\right)$ is shrinking. $\square$

1.5 Examples

In this section we present some examples of Schauder frames on locally convex spaces. These Schauder frames are shrinking and boundedly complete since all the spaces involved are Montel spaces.

Example 1. This example was obtained by Taskinen in [63]. Denote by $\mathbb{D}$ the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and for each n let v_n be the weight

$v_n(z) := \min\left\{1, |\log(1-|z|)|^{-n}\right\}$. We consider the weighted Banach space of holomorphic functions

$$H_{v_n}^{\infty} := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic } : \|f\|_{v_n} = \sup_{z\in\mathbb{D}} |f(z)|\, v_n(z) < \infty \right\}.$$

Since $v_{n+1} \leq v_n$ then $H_{v_n}^{\infty} \subset H_{v_{n+1}}^{\infty}$ continuously and we consider the inductive limit

$$VH = \mathrm{ind}_{n\to\infty} H_{v_n}^{\infty}.$$

The unit disc $\mathbb{D}$ is decomposed as $\mathbb{D} := \bigcup_j D_j$ with $\overset{\circ}{D}_j \neq \emptyset$ for all $j \in \mathbb{N}$ in such a way that the set of elements of $\mathbb{D}$ belonging to more that one of the D_j's has Lebesgue measure 0. Let us fix, for all $j \in \mathbb{N}$, $\lambda_j \in \overset{\circ}{D}_j$. As proved in [63], we can obtain such a decomposition with the property that

$$S : VH \to VH,\ f \mapsto (Sf)(z) := \sum_{j=1}^{\infty} \frac{m(D_j)\, f(\lambda_j)}{\left(1 - \overline{\lambda_j} z\right)^2},$$

is an isomorphism.

Theorem 1.5.1 [63, Theorem 1] *Under the conditions above, let* $f_j(z) := \frac{m(D_j)}{(1-\overline{\lambda_j}z)^2}$ *and* $u_j(f) := \left(S^{-1}f\right)(\lambda_j)$ *be given. Then* $(\{u_j\}, \{f_j\})$ *is a shrinking and boundedly complete Schauder frame for* VH.

Proof. Each $f \in VH$ can be written as

$$f = S\left(S^{-1}(f)\right) = \sum_{j=1}^{\infty} \left(S^{-1}f\right)(\lambda_j)\, f_j,$$

hence $(\{u_j\}, \{f_j\})$ is a Schauder frame in VH. Since VH is a Montel space we can apply Theorem 1.3.14 to conclude that the Schauder frame is shrinking. □

As pointed out in [63, p. 330], the coefficients in the series expansion above are not unique.

Example 2. Let K be a compact subset of $\mathbb{R}^p$ that coincides with the closure of its interior, i.e. $K = \overline{\overset{\circ}{K}}$. Let $C^{\infty}(K)$ be the space of all complex-valued functions

$f \in C^{\infty}(\overset{\circ}{K})$ uniformly continuous in $\overset{\circ}{K}$ together with all partial derivatives. The Fréchet space topology in $C^{\infty}(K)$ is defined by the norms:

$$q_n(f) := \sup\left\{\left|f^{(\alpha)}(x)\right| : x \in K,\ |\alpha| \leq n\right\}, n \in \mathbb{N}_0.$$

A continuous and linear extension operator is a continuous and linear operator $T : C^{\infty}(K) \to C^{\infty}(\mathbb{R}^p)$ such that $T(f)|_K = f$. Not every compact set admits a continuous and linear extension operator but every convex compact set does. Further information can be found in [28].

Theorem 1.5.2 *Let $K \subset \mathbb{R}^p$ be a compact set which is the closure of its interior. The following conditions are equivalent:*

(1) *There exists a continuous and linear extension operator $T : C^{\infty}(K) \to C^{\infty}(\mathbb{R}^p)$.*

(2) *There are $\{\lambda_j\}_j \subset \mathbb{R}^p$ and $\{u_j\}_j \in C^{\infty}(K)'$ such that $\left(\{u_j\}, \left\{e^{2\pi i x\cdot\lambda_j}\right\}\right)$ is an unconditional Schauder frame for $C^{\infty}(K)$.*

Proof. (1) $\Rightarrow$ (2). We consider $M > 0$ such that $K \subset [-M, M]^p$ and choose $\phi \in \mathcal{D}((-2M, 2M)^p)$ such that $\phi(x) = 1$ for all x in a neighborhood of $[-M, M]^p$. For every $f \in C^{\infty}(K)$ we define $Hf := \phi \cdot T(f) \in \mathcal{D}((-2M, 2M)^p)$. Then $H : C^{\infty}(K) \to \mathcal{D}((-2M, 2M)^p)$ is a continuous and linear map and $Hf|_K = f$. After extending Hf as a periodic C^{∞} function in $\mathbb{R}^p$ we get

$$Hf(x) := \sum_{j \in \mathbb{Z}^p} a_j e^{2\pi i x\cdot\lambda_j}, \text{ where } \lambda_j = \frac{1}{4M}(j_1, \ldots, j_p)$$

and $a_k = a_k(Hf)$ are the Fourier coefficients of Hf. By [39], $\sup_{j \in \mathbb{Z}^p} |a_j| \, |j|^m < \infty$ for every m, which implies that the series $f = \sum_{j \in \mathbb{Z}^p} a_j e^{2\pi i x\cdot\lambda_j}$ converges absolutely in $C^{\infty}(K)$. Each a_k, being a Fourier coefficient of Hf, depends linearly and continuously on f. Then $\left(\{u_j(\cdot)\}, \{e^{2\pi i x\cdot\lambda_j}\}\right)_{j \in \mathbb{Z}^p}$ is a Schauder frame for $C^{\infty}(K)$, with $u_j \in C^{\infty}(K)'$ defined by $u_j(f) = a_j(Hf)$.

(2) $\Rightarrow$ (1). For every $f \in C^{\infty}(K)$ we have

$$f(x) = \sum_{j=1}^{\infty} u_j(f) e^{2\pi i x\cdot\lambda_j} \text{ in } C^{\infty}(K)$$

and

$$\sum_{j=1}^{\infty} u_j(f) b_j e^{2\pi i x\cdot\lambda_j}$$

converges in $C^{\infty}(K)$ for every $\{b_j\} \in \ell_{\infty}$. After differentiation, we obtain that the series

$$\sum_{j=1}^{\infty} u_j(f)(2\pi)^{\alpha} b_j \lambda_j^{\alpha} e^{2\pi i x\cdot\lambda_j}$$

converges in $C^{\infty}(K)$ for every $\alpha \in \mathbb{N}_0^p$ and $\{b_j\} \in \ell_{\infty}$. In particular, this series converges for a fixed x_0 in the interior of K, from where it follows

$$\sum_{j=1}^{\infty} \left| u_j(f)(2\pi)^{\alpha} \lambda_j^{\alpha} \right| < +\infty$$

for every $\alpha \in \mathbb{N}_0^p$. Consequently $T(f)(x) := \sum_{j=1}^{\infty} u_j(f) e^{2\pi i x\cdot\lambda_j}$ defines a C^{∞} function in $\mathbb{R}^p$ and we obtain that $T : C^{\infty}(K) \to C^{\infty}(\mathbb{R}^p)$ is a linear extension operator. The continuity of T follows from the Banach-Steinhaus theorem, as $T(f)$ is the pointwise limit of $T_n(f) := \sum_{j=1}^{n} u_j(f) f_j$, $f_j(x) := e^{2\pi i x\cdot\lambda_j}$. □

Assume that condition (1) in the previous theorem holds. Then, for a fixed $j_0 \in \mathbb{Z}^p$ we can choose ϕ such that the j_0-th Fourier coefficient of $\phi T(e^{2\pi i \lambda^{j_0}\cdot})$ is not equal to 1. According to the comment after Corollary 1.2.4, we may remove one of the exponentials in the Schauder frame above and still obtain a Schauder frame.

Choosing $\psi \neq \phi$ in the proof above, we find a different sequence $(v_j) \in C^{\infty}(K)'$ such that $\left(\{v_j\}, \{e^{2\pi i x\cdot\lambda_j}\}\right)$ is an unconditional Schauder frame for $C^{\infty}(K)$. In fact, according to [39], no system of exponentials can be a basis in $C^{\infty}([0,1])$.

Example 3. We give a Schauder frame of the Schwartz space $\mathcal{S}(\mathbb{R}^p)$ of rapidly decreasing functions. It is inspired by the work of Pilipovic, Stoeva and Teofanov [54], although their Theorem 4.2 cannot be directly applied to conclude that one gets a Schauder frame. Let $a, b > 0$, and $\Lambda = a\mathbb{Z}^p \times b\mathbb{Z}^p$ be given. For $z = (x, \xi) \in \mathbb{R}^{2p}$ and $f \in L^2(\mathbb{R}^p)$ we put $\pi(z)f(t) = e^{2\pi i \xi t} f(t - x)$. Let us assume that $g \in \mathcal{S}(\mathbb{R}^p)$ and $\{\pi(\lambda)g : \ \lambda \in \Lambda\}$ is a Gabor frame in $L^2(\mathbb{R}^p)$. As proved by Janssen (see [30, Corollary 11.2.6]) the dual window is also a function $h \in \mathcal{S}(\mathbb{R}^p)$ and every $f \in L^2(\mathbb{R}^p)$ can be written as

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g \rangle \, \pi(\lambda)h. \tag{1.5.1}$$

For every $\lambda \in \Lambda$ we consider $u_\lambda \in \mathcal{S}'(\mathbb{R}^p)$ defined by $u_\lambda(f) = \langle f, \pi(\lambda)g \rangle$.

Proposition 1.5.3 *$(\{u_\lambda\}_{\lambda\in\Lambda}, \{\pi(\lambda)h\}_{\lambda\in\Lambda})$ is an unconditional Schauder frame for $\mathcal{S}(\mathbb{R}^p)$.*

Proof. According to [30, Corollary 11.2.6], the topology of $\mathcal{S}(\mathbb{R}^p)$ can be described by the sequence of seminorms

$$q_n(f) := \sup_{z\in\mathbb{R}^{2p}} |\langle f, \pi(z)g\rangle|\, v_n(z),\ \ n\in\mathbb{N},$$

where $v_n(z) = (1+|z|)^n$. So, we only need to check that, for every $n\in\mathbb{N}$,

$$\sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|\, q_n\left(\pi(\lambda)h\right) < \infty. \tag{1.5.2}$$

To this end, we fix $N > n$ large enough. Since

$$|\langle \pi(\lambda)h, \pi(z)g\rangle| \leq |\langle h, \pi(z-\lambda)g\rangle| \leq q_N(h)v_N(z-\lambda)^{-1}$$

and v_n is submultiplicative we obtain that (1.5.2) is dominated by

$$q_N(h)q_N(f)\sum_{\lambda\in\Lambda}\left(v_N(\lambda)\right)^{-1} v_n(\lambda) < \infty$$

and the proof is finished. □

This example is closely related to the fact that $\{\pi(\lambda)g :\ \lambda\in\Lambda\}$ is a Gabor frame for each modulation space defined in terms of a polynomially moderate weight; see for instance [30, Corollary 12.2.6].

Chapter 2

Frames in locally convex spaces

In this chapter frames and Bessel sequences in Fréchet spaces and their duals are defined and studied. Their relation with Schauder frames and representing systems is analyzed. The abstract results presented here, when applied to concrete spaces of analytic functions, give many examples and consequences about sampling sets and Dirichlet series expansions. Most of the results are submitted for publication in a preprint of Bonet, Fernández, Galbis and Ribera in [12].

2.1 Notation and preliminaries

Definition 2.1.1 Given a sequence space Λ its β-dual space is defined as

$$\Lambda^\beta := \left\{ \{y_j\}_j \in \omega : \sum_{j=1}^{\infty} x_j y_j \text{ converges for every } \{x_i\}_i \in \Lambda \right\}.$$

Clearly, (Λ, Λ^β) is a dual pair. Under additional assumptions we even have the relation given in next lemma.

Lemma 2.1.2 Let Λ be a barrelled sequence lcs for which the canonical unit vectors $\{e_j\}_j$ form a Schauder basis. Then, its topological dual Λ' can be algebraically identified with its β-dual Λ^β and the canonical unit vectors $\{e_j\}_j$ are a basis for $(\Lambda^\beta, \mu(\Lambda^\beta, \Lambda))$. Moreover if we consider on Λ^β the system of seminorms given by

$$p_B((y_i)_i) := \sup_{x \in B} \left| \sum_i x_i y_i \right|,$$

where B runs on the bounded subsets of Λ then $\left(\Lambda^{\beta}, (p_B)_B\right)$ is topologically isomorphic to $(\Lambda', \beta(\Lambda', \Lambda))$.

Proof. The map

$$\psi : \Lambda' \mapsto \Lambda^{\beta},\ h \to \{h(e_i)\}_i$$

is a linear bijection. In fact, for every $h \in \Lambda'$ and $x = \{x_i\}_i \in \Lambda$ we have $h(x) = \sum_i x_i h(e_i)$ which implies that ψ is well defined and obviously linear and injective. The barrelledness of Λ and the Banach-Steinhauss theorem give the surjectivity.

If $K \subset \Lambda$ is $\sigma(\Lambda, \Lambda^{\beta})$-compact, given $y \in \Lambda^{\beta}$ and $\varepsilon > 0$ there is n_0 such that for $n \geq n_0$,

$$\left| \sum_{i=n}^{\infty} y_i x_i \right| < \varepsilon$$

for all $x \in K$, from where $y = \sum_i y_i e_i$ in the Mackey topology $\mu(\Lambda^{\beta}, \Lambda)$.

As for each bounded subset B of Λ we have

$$p_B\left(\{h(e_i)\}_i\right) = \sup_{x \in B} |h(x)|,$$

the topological identity follows. $\square$

From now on, if the sequence space Λ satisfies the assumption in Lemma 2.1.2, we identify Λ' with Λ^{β} and use always Λ'.

2.2 General results

Definition 2.2.1 Let E be a lcs and Λ be a sequence space.

1. $\{g_i\}_i \subset E'$ is called a Λ-*Bessel sequence* in E' if the analysis operator

$$\begin{aligned} U = U_{\{g_i\}_i} : E &\longrightarrow \Lambda \\ x &\longmapsto \{g_i(x)\}_i \end{aligned}$$

   is continuous.

2. $\{g_i\}_i \subset E'$ is called a Λ-*frame* if the analysis operator U is an isomorphism into. If in addition the range of the analysis operator, $R\left(U_{\{g_i\}_i}\right)$, is complemented in Λ then $\{g_i\}_i$ is said to be a *frame for E with respect to* Λ. In this case there exists $S : \Lambda \to E$ such that $S \circ U = id|_E$.

Clearly, given a lcs E each sequence $\{g_i\}_i \subset E'$ is an ω-Bessel sequence. On the other hand, if Λ is a Hilbert space and E is complete, each Λ-frame is a frame with respect to Λ. Obviously a lcs space has a Λ-frame if and only if it is isomorphic to a subspace of Λ and it has a frame with respect to Λ if and only if it is isomorphic to a complemented subspace of Λ. Therefore, the property of having a Λ-frame is inherited by subspaces whereas having a frame with respect to Λ is inherited by complemented subspaces.

Remark 2.2.2 Let E be a lcs, Λ_1, Λ_2 sequence spaces, $\{g_i^1\}_i \subset E'$ a Λ_1-Bessel sequence and $\{g_i^2\}_i \subset E'$ a Λ_2-frame. We define $\{f_k\}_k \subset E'$ as $f_k = g_i^1$ for $k = 2i - 1$ and $f_k = g_i^2$ when $k = 2i$. Consider the sequence space

$$\Lambda := \{\{\alpha_k\}_k : \{\alpha_{2k-1}\}_k \in \Lambda_1, \text{ and } \{\alpha_{2k}\}_k \in \Lambda_2\},$$

with the topology given by the seminorms

$$||\alpha||_{p,q} := p\left(\{\alpha_{2k-1}\}_k\right) + q\left(\{\alpha_{2k}\}_k\right), \text{ where } p \in cs(\Lambda_1),\ q \in cs(\Lambda_2).$$

Then $\{f_k\}_k$ is a Λ-frame for E. In the case that $\Lambda_1 = \Lambda_2$ is one of the spaces c_0 or ℓ_p then $\Lambda = \Lambda_1 = \Lambda_2$.

Recall the definition of Schauder frames given in Definition 1.1.1. Note that, if $\left(\{x_i'\}_i, \{x_i\}_i\right)$ is a Schauder frame for a lcs E, the associated sequence space is $\Lambda := \{\alpha = \{\alpha_i\}_i \in \omega : \sum_{i=1}^{\infty} \alpha_i x_i \text{ is convergent in } E\}$. Endowed with the system of seminorms $\mathcal{Q}$ defined in Proposition 1.1.2, Λ is a sequence space and the canonical unit vectors form a Schauder basis by Lemma 1.1.3.

There is a close connection between Λ-frames and Schauder frames.

Proposition 2.2.3 *(a) Let $\left(\{x_i'\}_i, \{x_i\}_i\right)$ be a Schauder frame for a barrelled and complete lcs E and Λ the associated sequence space. Then $\{x_i'\}_i \subset E'$ is a frame for E with respect to Λ. If moreover Λ is barrelled then $\{x_i\}_i \subset E$ is a frame for E' with respect to Λ'.*

(b) If $\{x_i'\}_i \subset E'$ is a frame for E with respect to a sequence space Λ, and Λ has a Schauder frame, then E also admits a Schauder frame.

Proof. (a) According to the proof of Theorem 1.1.4 the operators $U : E \to \Lambda$ and $S : \Lambda \to E$ given by $U(x) := \{x_i'(x)\}_i$ and $S(\{\alpha_i\}_i) := \sum_{i=1}^{\infty} \alpha_i x_i$ respectively, are continuous and $S \circ U = id_E$. Consequently $\{x_i'\}_i$ is a frame for E with respect to Λ. Under the additional assumption that Λ is barrelled we have that $\Lambda' = \Lambda^\beta$ is a sequence space. Moreover $S'(x') := \{x'(x_i)\}_i$ for each $x' \in E'$ and from $U' \circ S' = id_{E'}$ we conclude that $\{x_i\}_i$ is frame for E' with respect to Λ'.

Statement (b) follows from the fact that having a Schauder frame is inherited by complemented subspaces. That is, letting $P : \Lambda \to R\left(U_{\{g_i\}_i}\right)$ the projection on the complemented subspace, the Schauder frame will be given by $x = \sum_{i=1}^{\infty} g_i\left(x\right) U^{-1}_{\{g_i\}_i}\left(P\left(e_i\right)\right)$ for every $x \in E$. □

The barrelledness of the sequence space Λ naturally associated to a Schauder frame follows for instance, if E is a Banach, Fréchet or a sequentially retractive (LF)-space as we mentioned in Remark 1.1.6. Observe that the dual space E' need not be separable, in which case neither need Λ' be.

Definition 2.2.4 ([34]) A *representing system* in a lcs E is a sequence $\{x_i\}_i$ in E such that each $x \in E$ admits a representation

$$x = \sum_i c_i x_i$$

the series converging in E.

The coefficients in the representation need not be unique, that is, one can have

$$0 = \sum_i d_i x_i$$

for a non-zero sequence $\{d_i\}_i$. Moreover, we do not assume that it is possible to find a representation of this type with coefficients depending continuously on the vectors.

Clearly each topological basis is a representing system. Given a Schauder frame $(\{x'_n\}_n, \{x_n\}_n)$, the sequence $\{x_n\}_n$ is a representing system. However, there are representing systems that are neither basis nor coming from a Schauder frames. In fact, each separable Fréchet space has a representing system [34, Theorem 1] but only those Fréchet spaces with the bounded approximation property admit a Schauder frame by 1.1.5

Definition 2.2.5 A Λ-*representing system* in a lcs E is a sequence $\{x_i\}_i$ in E such that each $x \in E$ admits a representation $x = \sum_i c_i x_i$ with $\{c_i\}_i \in \Lambda$.

Proposition 2.2.6 *Let E be a barrelled lcs and let Λ be a barrelled sequence lcs for which the canonical unit vectors $\{e_i\}_i$ form a Schauder basis. Then*

(i1) $\{g_i\}_i \subset E'$ is a Λ-Bessel sequence if and only if the operator

$$T : (\Lambda', \mu(\Lambda', \Lambda)) \to (E', \mu(E', E)),\ \{d_i\}_i \mapsto \sum_{i=1}^{\infty} d_i g_i$$

is well defined and continuous.

(i2) $\{g_i\}_i \subset E'$ *is* Λ'*-Bessel for* E *if and only if* $T : \Lambda \to (E', \beta(E', E))$ *given by* $T((d_i)_i) := \sum_{i=1}^{\infty} d_i g_i$ *is well defined and continuous.*

(ii) If $\{g_i\}_i \subset E'$ *is a* Λ*-frame in* E*, then* $\{g_i\}_i$ *is a* Λ'*-representing system for* $(E', \mu(E', E))$*. If moreover* E *is reflexive, then* $\{g_i\}_i$ *is a* Λ'*-representing system for* $(E', \beta(E', E))$*.*

(iii) If $\{g_i\}_i \subset E'$ *is a* Λ*-Bessel sequence which is also a* Λ'*-representing system for* $(E', \mu(E', E))$ *then* $\{g_i\}_i$ *is a* $(\Lambda, \sigma(\Lambda, \Lambda'))$*-frame for* $(E, \sigma(E, E'))$*.*

(iv) If in addition E *and* Λ *are Fréchet spaces, then* $\{g_i\}_i \subset E'$ *is a* Λ*-frame for* E *if, and only if,* $\{g_i\}_i$ *is* Λ*-Bessel and a* Λ'*-representing system for* $(E', \mu(E', E))$*.*

Proof. (i1) Let us assume that $\{g_i\}_i$ is a Λ-Bessel sequence and consider $T = U'$ the transposed map of the analysis operator $U : E \to \Lambda$, $U(x) = \{g_i(x)\}_i$. Then $T : (\Lambda', \mu(\Lambda', \Lambda)) \to (E', \mu(E', E))$ is continuous and $T(e_i) = g_i$. As the canonical unit vectors are a basis for $(\Lambda', \mu(\Lambda', \Lambda))$ we conclude $T(\{d_i\}_i) = \sum_{i=1}^{\infty} d_i g_i$. Conversely, if T is a well defined and continuous map, then its transposed $T' : E \to \Lambda$ is also continuous which means that $\{g_i\}_i$ is a Λ-Bessel sequence. (i2) is proved similarly considering that the dual $(\Lambda', \beta(\Lambda', \Lambda))$ is a sequence space.

(ii) If $\{g_i\}_i$ is a Λ-frame then U is a topological isomorphism into, hence $T = U'$ is surjective. In particular $\{g_i\}_i$ is a Λ'-representing system in $(E', \mu(E', E))$.

(iii) From (i), the map $T : (\Lambda', \mu(\Lambda', \Lambda)) \to (E', \mu(E', E))$, $\{d_i\}_i \mapsto \sum_{i=1}^{\infty} d_i g_i$, is well defined, continuous and surjective. Consequently $T' : (E, \sigma(E, E')) \to (\Lambda, \sigma(\Lambda, \Lambda'))$ is an isomorphism into [33, 9.6.1], hence $\{g_i\}_i$ is a $(\Lambda, \sigma(\Lambda, \Lambda'))$-frame for $(E, \sigma(E, E'))$.

(iv) Necessity follows from (ii) and sufficiency follows from the closed range theorem [33, 9.6.3] and (iii). □

Proposition 2.2.7 *Let* E *be a reflexive space and let* Λ *be a reflexive sequence space for which the canonical unit vectors* $\{e_i\}_i$ *form a Schauder basis. If either*

(i) E *and* Λ *are Fréchet spaces*

or

(ii) E *is the strong dual of a Fréchet-Montel space and* Λ *is an* (LB)*-space,*

then $\{g_i\}_i \subset E'$ is a Λ-frame for E if, and only if, $\{g_i\}_i$ is Λ-Bessel and a Λ'-representing system for $(E', \beta(E', E))$.

Proof. The case (i) is Proposition 2.2.6 (iv). Only the sufficiency in case (ii) has to be proved. Let us assume that $\{g_i\}_i$ is Λ-Bessel and a Λ'-representing system for $(E', \beta(E', E))$ and consider the continuous map $U : E \to \Lambda$, $U(x) = \{g_i(x)\}_i$. Then $T = U' : \Lambda' \to E'$ is a well-defined continuous and surjective map. Since E' is a Fréchet-Montel space, the map T lifts bounded sets, that is, for every bounded set B in E' we can find a bounded set C in Λ' such that $B \subset T(C)$. Hence $U : E \to \Lambda$ is a topological isomorphism into, which means that $\{g_i\}_i$ is a Λ-frame for E. $\Box$

Example 2.2.8 Let F be the strong dual of a Fréchet-Montel space E. Since E is separable it admits a representing system $\{g_i\}_i \subset F'$. Consider the Fréchet sequence space

$$\Lambda = \left\{ \{\alpha_i\}_i \in \omega : \sum_i \alpha_i g_i \text{ is convergent in } E \right\}$$

endowed with the system of seminorms as in (1.1.2). Then $\{g_i\}_i$ is a Λ-representing system for E and also a $(\Lambda', \beta(\Lambda', \Lambda))$-frame for F.

In fact, the continuous map $T : \Lambda \to E$, $\{\alpha_i\}_i \mapsto \sum_i \alpha_i g_i$, lifts bounded sets, which implies that $U = T' : E \to \Lambda'$ is an isomorphism into. $\Box$

Example 2.2.9 Let E be a Fréchet-Schwartz space. Then there are a Fréchet sequence space Λ and a sequence $\{g_j\}_j \subset E'$ which is a Λ-frame for E.

In fact, $F := E'_\beta = \mathrm{ind}_k F_k$ is a sequentially retractive (LB)-space, hence it is sequentially separable and admits a representing system $\{g_j\}_j$ ([34, Theorem 1]). Now, for each k we put

$$\Gamma_k := \left\{ \alpha \in \omega \ : \alpha_j g_j \in F_k \text{ for all } j \text{ and } \sum_{j=1}^{\infty} \alpha_j g_j \text{ converges in } F_k \right\}.$$

Without loss of generality we may assume that Γ_k is non-trivial for each k and we endow it with the norm

$$q_k(\alpha) = \ \sup_n || \sum_{j=1}^{n} \alpha_j g_j ||_k,$$

where $\|\cdot\|_k$ denotes the norm of the Banach space F_k. Then $\{\Gamma_k, q_k\}_k$, $k \in \mathbb{N}$, is an increasing sequence of Banach spaces with continuous inclusions and $\bigcup_k \Gamma_k$ coincides algebraically with

$$\Gamma := \left\{ \alpha \in \omega \,:\, \sum_{j=1}^{\infty} \alpha_j g_j \text{ converges in } F \right\}.$$

Hence Γ, endowed with its natural (LB)-topology, is a sequence space with the property that the canonical unit vectors are a basis. Moreover, the map

$$\Gamma \to F,\, \alpha \mapsto \sum_{j=1}^{\infty} \alpha_j g_j$$

is well defined, continuous and surjective. Therefore, $\Lambda = \Gamma'$ is a Fréchet sequence space and $\{g_j\}_j$ is a Λ-frame for E. $\square$

The following result relates Λ-Bessel sequences with frames with respect to Λ when Λ is a barrelled sequence space for which the canonical unit vectors $\{e_i\}_i$ form a Schauder basis. Note that, if $\{g_i\}_i \subset E'$ is a frame with respect to Λ, by definition, $R\left(U_{\{g_i\}_i}\right)$ is complemented in Λ; this means that the operator $U^{-1}_{\{g_i\}_i} : R(U) \to E$ can be extended to a continuous linear operator $S : \Lambda \to E$.

Proposition 2.2.10 *Let E be a barrelled and complete lcs and let Λ be a barrelled sequence space for which the canonical unit vectors $\{e_i\}_i$ form a Schauder basis. If $\{g_i\}_i \subset E'$ is Λ-Bessel for E then the following conditions are equivalent:*

(i) *$\{g_i\}_i \subset E'$ is a frame with respect to Λ.*

(ii) *There exists a family $\{f_i\}_i \subset E$, such that $\sum_{i=1}^{\infty} c_i f_i$ is convergent for every $\{c_i\}_i \in \Lambda$ and $x = \sum_{i=1}^{\infty} g_i(x) f_i$, for every $x \in E$.*

(iii) *There exists a Λ'-Bessel sequence $\{f_i\}_i \subset E \subseteq E''$ for E' such that $x = \sum_{i=1}^{\infty} g_i(x) f_i$ for every $x \in E$.*

If the canonical unit vectors form a basis for both Λ and Λ'_β, (i)- (iii) are also equivalent to

(iv) *There exists a Λ'-Bessel sequence $\{f_i\}_i \subset E \subseteq E''$ for E' such that $x' = \sum_{i=1}^{\infty} x'(f_i) g_i$ for every $x' \in E'$ with convergence in the strong topology.*

If each of the cases (iii) and (iv) hold then $\{f_i\}_i$ is actually a frame for E' with respect to Λ' . Moreover, $(\{g_i\}_i, \{f_i\}_i)$ is a shrinking Schauder frame.

Proof. We consider U as in Definition 2.2.1 which is an isomorphism into.

(i) → (ii) Let $S : \Lambda \to E$ be a continuous linear extension of $U^{-1}_{\{g_i\}_i}$ such that $S \circ U = I|_E$. Define $f_i := S(e_i)$ and observe that, for all $\{c_i\}_i \in \Lambda$,

$$\sum_{i=1}^{\infty} c_i f_i = \sum_{i=1}^{\infty} c_i S(e_i) = S\left(\sum_{i=1}^{\infty} c_i e_i\right) = S(\{c_i\}_i).$$

Moreover, for every $x \in E$, $x = (S \circ U)(x) = \sum_{i=1}^{\infty} g_i(x) f_i$.

(ii) → (i) Assume that (ii) is satisfied, we define $S : \Lambda \to E$ by $S(\{c_i\}_i) := \sum_{i=1}^{\infty} c_i f_i$ with $\{c_i\}_i$. Observe that, by Banach-Steinhaus theorem, S is a continuous operator. Taking $\{g_i(x)\}_i \in R(U)$ we obtain

$$S(\{g_i(x)\}_i) = \sum_{i=1}^{\infty} g_i(x) f_i = x.$$

We obtain that S is a continuous extension of U^{-1}, and (i) holds.

(ii) → (iii) Let $V : \Lambda \to E$ be a linear continuous extension of $U^{-1}_{\{g_i\}_i}$. Set $f_i := V(e_i)$. By Lemma 2.1.2, for every $x' \in E'$ we have $\{x'(f_i)\}_i = \{x'(V(e_i))\}_i \in \Lambda'$ and $\{f_i\}_i$, considered as a sequence in E'', is an Λ'-Bessel sequence for E'. Note that we can also prove the result using that $S' : E'_\beta \to \Lambda'$, given by $S'(x') := \{x'(f_i)\}_i$ is continuous since it is the transpose of S.

(iii) → (ii) By Proposition 2.2.6 (i2), if (iii) is valid then the operator $T : \Lambda' \to E \subset E''$ given by $T(\{c_i\}_i) := \sum_{i=1}^{\infty} c_i f_i$ is well defined and continuous, hence (ii) is satisfied.

(iii) → (iv) Denote the canonical basis of Λ by $\{e_i\}_i$ and the canonical basis of Λ'_β by $\{z_i\}_i$. If (iii) is valid, there exists $\{f_i\}_i \subset E \subseteq E''$ that is Λ'-Bessel for E' such that $x = \sum_{i=1}^{\infty} g_i(x) f_i$. Observe that, as $\{x'(f_i)\}_i$ belongs to Λ', then $\{x'(f_i)\}_i = \sum_{i=1}^{\infty} x'(f_i) z_i$ in $(\Lambda', \beta(\Lambda', \Lambda))$. Given a bounded set $B \in E$ then $C = \{\{g_i(x)\}_i : x \in B\}$ is a bounded set in Λ. If $p_B \in cs(E'_\beta)$ is the continuous seminorm defined by $p_B(u') := \sup_{x \in B} |u'(x)|$ then

$$\begin{aligned}
p_B\left(x' - \sum_{i=1}^{n} x'(f_i)\, g_i\right) &= \sup_{x\in B}\left|x'(x) - \sum_{i=1}^{n} x'(f_i)\, g_i(x)\right| \\
&= \sup_{x\in B}\left|x'\left(\sum_{i=1}^{\infty} g_i(x)\, f_i\right) - \sum_{i=1}^{n} x'(f_i)\, g_i(x)\right| \\
&= \sup_{x\in B}\left|\sum_{i=n+1}^{\infty} x'(f_i)\, g_i(x)\right| \\
&= \sup_{\phi\in C}\left|\phi\left(\sum_{i=n+1}^{\infty} x'(f_i)\, z_i\right)\right| \\
&= q_C\left(\sum_{i=n+1}^{\infty} x'(f_i)\, z_i\right)
\end{aligned}$$

where $q_C \in cs(\Lambda')$ is given by $q_C(\alpha) := \sup_{\Phi\in C}|\Phi(\alpha)|$ for every $\alpha \in \Lambda'_\beta$. Then, $q_C\left(\sum_{i=n+1}^{\infty} x'(f_i)\, z_i\right)$ converges to 0 as n converges to infinity since $\{x'(f_i)\}_i = \sum_{n=1}^{\infty} x'(f_i) z_i$ in Λ'_β.

(iv) $\to$ **(iii)** If (iv) is valid, then there exists $\{f_i\}_i$ a Λ'-Bessel sequence for E' such that $x' = \sum_{i=1}^{\infty} x'(f_i)\, g_i$. Given a bounded subset $B' \subset E'$ then $C' = \{\{x'(f_i)\}_i : x' \in B'\}$ is a bounded set in Λ'_β. If $p_{B'} \in cs(E)$ is the continuous seminorm defined by $p_{B'}(x) := \sup_{x'\in B'}|x'(x)|$ then

$$\begin{aligned}
p_{B'}\left(x - \sum_{i=1}^{n} g_i(x)\, f_i\right) &= \sup_{x'\in B'}\left|x'(x) - \sum_{i=1}^{n} x'(f_i)\, g_i(x)\right| \\
&= \sup_{x'\in B'}\left|\sum_{i=n+1}^{\infty} x'(f_i)\, g_i(x)\right| \\
&= \sup_{\phi'\in C'}\left|\phi'\left(\sum_{i=n+1}^{\infty} g_i(x)\, e_i\right)\right| \\
&= q\left(\sum_{i=n+1}^{\infty} g_i(x)\, e_i\right)
\end{aligned}$$

where q is a continuous seminorm in $\bigwedge_\beta$. Then, $q\left(\sum_{i=n+1}^{\infty} g_i(x)\, e_i\right)$ converges to 0 as n converges to infinity due to the fact that $\{g_i(x)\}_i = \sum_{i=1}^{\infty} g_i(x) e_i$ in Λ.

To conclude, observe that, if (iii) and (iv) hold, then $(\{g_i\}_i, \{f_i\}_i)$ and $(\{f_i\}_i, \{g_i\}_i)$ are Schauder frames for E and E' respectively. By Proposition 1.3.4 we obtain that $(\{g_i\}_i, \{f_i\}_i)$ is a shrinking Schauder frame. $\square$

Let us introduce now the concept of algebra.

Definition 2.2.11 A vector space $\mathcal{A}$ over $\mathbb{K}$ is called an *algebra over* $\mathbb{K}$ if a product xy is defined for any two elements x and y of $\mathcal{A}$, which satisfies the rules:

1. $(xy)z = x(yz)$.
2. $x(y+z) = xy + xz, (x+y)z = xz + yz$,
3. $(xy)\alpha = x(y\alpha) = (x\alpha)y$ with $\alpha \in \mathbb{K}$.

A locally convex algebra $\mathcal{A}$ is a lcs which is an algebra with separately continuous multiplication.

The spectrum of the algebra is the set of all non-zero multiplicative linear functionals. The following remark will be useful in Section 2.4.2.

Remark 2.2.12 (i) In many cases E is continuously included in a locally convex algebra $\mathcal{A}$ with non-empty spectrum, Λ is a solid sequence space, $\{g_i\}_i$ is a Λ-frame and every g_i is the restriction to E of a continuous linear multiplicative functional on $\mathcal{A}$. Let us assume that for some $a \in \mathcal{A}$ the operator

$$T : E \to E(\subset \mathcal{A}), \quad x \mapsto ax$$

is well defined and it is a topological isomorphism into, and that $\alpha := \{g_i(a)\}_i$ defines by pointwise multiplication a continuous operator on Λ. Then, $\{h_i\}_i$, where

$$h_i := \begin{cases} g_i, & \text{if} \quad g_i(a) \neq 0 \\ 0 & \text{if} \quad g_i(a) = 0 \end{cases}$$

is a Λ-frame. In fact, since $U \circ T$ is a topological isomorphism into then for every continuous seminorm p on E there is a continuous seminorm q on Λ such that

$$p(x) \leq q\left(\{g_i(ax)\}_i\right) = q\left(\{g_i(a)g_i(x)\}_i\right) = q\left(\{g_i(a)h_i(x)\}_i\right).$$

Finally, since the pointwise multiplication with $\{g_i(a)\}_i$ is a continuous operator on Λ we find a continuous seminorm r on Λ with

$$p(x) \leq r\left(\{h_i(x)\}_i\right), \quad x \in E.$$

(ii) If E is a locally convex algebra with non-empty spectrum, Λ is a sequence space and $\{g_i\}_i$ is a Λ-frame consisting of continuous linear multiplicative functionals on E, then $U(E)$ is a locally convex algebra under pointwisse multiplication. Fix $\beta \in U(E)$ such that there exists $b \in E$ which $\beta = \{g_i(b)\}_i$. Let us consider now the operator

$$\widehat{T} : U(E) \to U(E), \quad \gamma \mapsto \beta\gamma;$$

since E is a locally convex algebra we obtain that if $\gamma = \{g_i(x)\}_i$ for a $x \in E$ then

$$\beta\gamma = \{g_i(b)\}_i\{g_i(x)\}_i = \{g_i(bx)\}_i.$$

Therefore $\widehat{T}$ is well defined. Now, we also check that it is continuous. Given $p \in cs(\Lambda)$ and using that $\{g_i\}_i$ is a Λ-frame consisting of continuous linear multiplicative functionals, there exists $q \in cs(E)$ such that

$$p(\{g_i(b)g_i(x)\}_i) = p(\{g_i(bx)\}_i) \leq q(bx).$$

Also, using that E is a locally convex algebra then there exists $q' \in cs(E)$ such that

$$q(bx) \leq q'(x).$$

Finally, using that $\{g_i\}_i$ is a Λ-frame we find a continuous seminorm r on Λ with

$$p(\{g_i(b)g_i(x)\}_i) \leq q'(x) \leq r\left(\{g_i(x)\}_i\right).$$

Hence, if E has no zero-divisors, the analysis map U cannot be surjective. In fact, if there are $x, y \in E$ such that $U(x) = e_1$ and $U(y) = e_2$ then $U(x \cdot y) = e_1 \cdot e_2 = 0$ and the injectivity of U implies $x \cdot y = 0$, which is a contradiction. Since the range of U is a topological subspace of Λ, the non-surjectivity of U implies the non-injectivity of the transposed map U'. Consequently the expression of any element in E' as a convergent series

$$\sum_i \alpha_i g_i$$

with $\alpha \in \Lambda'$ is never unique.

2.3 Λ-frames in (LB)-spaces

Let $E = \mathrm{ind}_n(E_n, \|\cdot\|_n)$ and $\Lambda = \mathrm{ind}_n(\Lambda_n, r_n)$ be complete (LB)-spaces and $\{g_i\}_i \subset E'$ a Λ-Bessel sequence. Let $U : E \to \Lambda$ be the continuous and linear map of Definition 2.2.1 and, for each $n \in \mathbb{N}$, consider the seminormed space (F_n, q_n) where

$$F_n = \{x \in E : U(x) \in \Lambda_n\}$$

and $q_n(x) := r_n(U(x))$. Let us consider the topologies on E

$$(E,\tau_1) = \mathrm{ind}_n(E_n, \|\cdot\|_n),\ (E,\tau_2) = \mathrm{ind}_n(F_n, q_n).$$

Finally, denote by τ_3 the topology on E given by the system of seminorms $x \mapsto p(U(x))$, when p runs in $cs(\Lambda)$. Then

$$\tau_1 \geq \tau_2 \geq \tau_3,$$

but observe that τ_2 and τ_3 need not be even Haussdorf.

We observe that $\{g_i\}_i \subset E'$ is a Λ-frame if, and only if, the former three topologies coincide.

The coincidence $\tau_1 = \tau_2$ is easily characterized under the mild additional assumption that the closed unit ball of Λ_n is also closed in ω. This is the case for all (weighted) ℓ_p spaces, $1 \leq p \leq \infty$, but not for c_0.

Proposition 2.3.1 *Assume that the closed unit ball of Λ_n is closed in ω. Then, $\tau_1 = \tau_2$ if and only if (F_n, q_n) is a Banach space for each n.*

Proof. Assume that $\tau_1 = \tau_2$, which in particular implies that τ_2 is Hausdorff. Since (F_n, q_n) is continuously injected in (E, τ_2), q_n is a norm. Moreover, if $x \in E$, $x \neq 0$, there is n such that $x \in F_n$, hence $q_n(x) > 0$. We have $0 < q_n(x) = r_n(U(x))$, which implies $U(x) \neq 0$. Thus U is injective. Let $\{x_j\}_j$ be a Cauchy sequence in (F_n, q_n). Then, it converges to a vector x in the complete (LB)-space E, and therefore its image under the analysis map $\{U(x_j)\}_j$ is convergent to $U(x)$ in Λ. Now, given $\varepsilon > 0$ we can find j_0 such that $r_n(U(x_j) - U(x_k)) \leq \varepsilon$ whenever j, $k \geq j_0$. That is, for $k \geq j_0$,

$$U(x_k) \in U(x_j) + (\alpha \in \Lambda_n : r_n(\alpha) \leq \varepsilon),$$

and then

$$U(x) \in \overline{U(x_j) + (\alpha \in \Lambda_n : r_n(\alpha) \leq \varepsilon)}^{\,\omega}$$

for all $j \geq j_0$. By hypothesis we get $U(x) \in \Lambda_n$ and $r_n(U(x - x_j)) \leq \varepsilon$ for all $j \geq k_0$. Hence (F_n, q_n) is a Banach space.

The converse holds since, by the open mapping theorem, two comparable (LB)-topologies must coincide. $\square$

The next result depends on Grothendieck's factorization theorem (see [48, 24.33]).

Corollary 2.3.2 *Assume that the closed unit ball of Λ_n is closed in ω. Then $\tau_1 = \tau_2$ if and only if for each n there are m and C such that $F_n \subset E_m$ and*

$$||x||_m \leq C q_n(x)$$

for each $x \in F_n$.

Proposition 2.3.3 *If E is Montel and $\tau_1 = \tau_2$, then $\{g_i\}_i$ is a Λ-frame for E.*

Proof. As in Proposition 2.3.1, U is injective and each (F_n, q_n) is a normed space. By Baernstein's lemma (see [53, 8.3.55]), as E is a Montel space and Λ is a complete (LB)-space, it suffices to show that for each bounded subset B of Λ, the pre-image $U^{-1}(B)$ is bounded in (E, τ_2). Since Λ is regular, because it is complete, there is n such that B is contained and bounded in Λ_n, hence $U^{-1}(B)$ is contained and bounded in F_n, therefore bounded in E. □

Proposition 2.3.4 *Let $E = \mathrm{ind}_n(E_n, \|\cdot\|_n)$ be a (DFS)-space and also let $\Lambda = \mathrm{ind}_n(\Lambda_n, r_n)$ be a complete (LB)-space. Assume that the closed unit ball of Λ_n is closed in ω. If $\{g_i\}_i \subset E'$ is a Λ-Bessel sequence then the following conditions are equivalent:*

(i) $\{g_i\}_i$ *is a Λ-frame,*

(ii) *The map $U : E \to \Lambda$, $U(x) = \{g_i(x)\}_i$, is injective and for every $n \in \mathbb{N}$ there exists $m > n$ such that $F_n \subset E_m$.*

Proof. If (i) is satisfied, $\tau_1 = \tau_2 = \tau_3$. The injectivity of U follows as in the proof of Proposition 2.3.1 and the rest of (ii) follows by Corollary 2.3.2.

We prove that (ii) implies (i). Without loss of generality we can assume that $E_n \subset E_{n+1}$ with compact inclusion, $E_n \subset F_n$ and $q_n(x) \leq \|x\|_n$ for every $x \in E_n$. It suffices to show that, under condition (ii), the inclusion $F_n \subset E_{m+2}$ is continuous. In fact, this implies the coincidence of the topologies $\tau_1 = \tau_2$, hence the Λ-frame property by Proposition 2.3.3.

We take

$$B = \left\{x \in F_n : \ q_n(x) \leq 1, \quad \|x\|_{m+1} > 1\right\}$$

and

$$A = \left\{ y = \frac{x}{\|x\|_{m+1}} : \ x \in B \right\}.$$

We can assume B is an infinite set since otherwise the inclusion $F_n \subset E_{m+1}$ is continuous and we are done. Let $\{p_j\}_j$ denote a fundamental system of seminorms for the Fréchet space ω. We *claim* that there are $j_0 \in \mathbb{N}$ and $C > 0$ such that

$$\|x\|_{m+1} \leq C p_{j_0}(U(x))$$

for every $x \in B$. On the contrary there is a sequence $\{y_j\}_j \subset A$ such that

$$p_j(U(y_j)) \leq \frac{1}{j!}. \tag{2.3.1}$$

Assume that $\{y_j\}_j$ is bounded in E_m. Then it would be compact in E_{m+1}. Therefore, there is a subsequence $\{y_s\}_s$ of $\{y_j\}_j$ that converges to y in E_{m+1}. Hence $U(y_s) \to U(y)$ in Λ, hence in ω. We can apply (2.3.1) to conclude that $U(y) = 0$, hence $y = 0$, since U is injective. This contradicts $\|y_s\|_{m+1} = 1$ for all s. Consequently, $\{y_j\}_j$ is unbounded in E_m. Hence, for $j_1 = 1$ there exists $j_2 > j_1$ such that

$$\frac{1}{6 \cdot 2^2}\|y_{j_2}\|_m > 3\|y_{j_1}\|_m.$$

Hence there is ψ in the unit ball $B_{E'_m}$ of E'_m such that

$$\frac{1}{6 \cdot 2^2}\left|\psi(y_{j_2})\right| > 3\|y_{j_1}\|_m > 2\left|\psi(y_{j_1})\right|.$$

Since $U : E_m \to \omega$ is a continuous and injective map then $U' : \omega' \to E'_m$ has $\sigma(E'_m, E_m)$-dense range and we can find $\varphi_2 \in \omega'$ such that

$$\max_{k=1,2}\left|(\varphi_2 \circ U - \psi)(y_{j_k})\right|$$

is so small that

$$\frac{1}{6 \cdot 2^2}\left|\varphi_2(U(y_{j_2}))\right| > 3\|y_{j_1}\|_m > 2\left|\varphi_2(U(y_{j_1}))\right|.$$

By condition (2.3.1) there is j'_2 such that

$$\left|\varphi_2(U(y_j))\right| < \left|\varphi_2(U(y_{j_2}))\right|, \quad j > j'_2.$$

Proceeding by induction it is possible to obtain a sequence $\{\varphi_\ell\}_\ell \subset \omega'$ and an increasing sequence $\{j_\ell\}_\ell$ of indices such that $\varphi_\ell \circ U \in B_{E'_m}$ and

$$\frac{1}{\ell(\ell+1)2^\ell}\left|\varphi_\ell(U(y_{j_\ell}))\right| > 3\sum_{k=1}^{\ell-1}\|y_{j_k}\|_m > 2\sum_{k=1}^{\ell-1}\left|\varphi_\ell(U(y_{j_k}))\right|$$

while

$$\left|\varphi_\ell(U(y_{j_k}))\right| < \left|\varphi_\ell(U(y_{j_\ell}))\right| \quad \forall k > \ell.$$

We now consider

$$y = \sum_{k=1}^{\infty} \frac{1}{k2^k} y_{j_k} \in E_{m+1}.$$

Then

$$\varphi_\ell(U(y)) = \sum_{k=1}^{\infty} \frac{1}{k2^k} \varphi_\ell(U(y_{j_k})),$$

hence

$$\begin{aligned}
|\varphi_\ell(U(y))| &\geq \frac{1}{\ell 2^\ell} |\varphi_\ell(U(y_{j_\ell}))| - \sum_{k<\ell} \frac{1}{k2^k} |\varphi_\ell(U(y_{j_k}))| - \sum_{k>\ell}^{\infty} \frac{1}{k2^k} |\varphi_\ell(U(y_{j_k}))| \\
&\geq \left(\frac{1}{\ell 2^\ell} - \sum_{k>\ell} \frac{1}{k2^k} \right) |\varphi_\ell(U(y_{j_\ell}))| - \sum_{k<\ell} \frac{1}{k2^k} |\varphi_\ell(U(y_{j_k}))| \\
&\geq \frac{1}{\ell(\ell+1)2^\ell} |\varphi_\ell(U(y_{j_\ell}))| - \frac{3}{2} \sum_{k<\ell} \|y_{j_k}\|_m \\
&\geq \sum_{k<\ell} \|y_{j_k}\|_m \geq \sum_{k<\ell} \|y_{j_k}\|_{m+1} = \ell - 1.
\end{aligned}$$

On the other hand $r_n(U(y_{j_k})) \leq 1$ for every $k \in \mathbb{N}$, which implies that the series

$$\sum_{k=1}^{\infty} \frac{1}{k2^k} U(y_{j_k})$$

converges in the Banach space Λ_n. Hence $y \in F_n \subset E_m$. Since $\varphi_\ell \circ U \in B'_{E_m}$ then $|\varphi_\ell(U(y))| \leq 1$, which is a contradiction. Consequently the claim is proved and there are $j_0 \in \mathbb{N}$ and $C > 0$ such that

$$\|x\|_{m+1} \leq C p_{j_0}(U(x))$$

for every $x \in B$. In order to conclude that the inclusion $F_n \subset E_{m+2}$ is continuous, it suffices to check that B is bounded in E_{m+2}. To this end we first observe that

$$1 \leq \|x\|_{m+1} \leq C p_{j_0}(U(x)) \leq C' \|x\|_{m+2}$$

for some $C' > 0$ and for all $x \in B$. Then

$$\left\{ \frac{x}{\|x\|_{m+2}} : \ x \in B \right\} \subset E_m$$

is a bounded set in E_{m+1}, hence relatively compact in E_{m+2}. We now proceed by contradiction and assume that B is unbounded in E_{m+2}. Then there exists a sequence $\{x_j\}_j \subset B$ with $\|x_j\|_{m+2} \geq j$. Passing to a subsequence if necessary we can assume that

$$y_j := \frac{x_j}{\|x_j\|_{m+2}}$$

converges to some element $z \in E_{m+2}$ such that $\|z\|_{m+2} = 1$. Since the inclusion $E_{m+2} \subset F_{m+2}$ is continuous we get

$$\lim_{j\to\infty} q_{m+2}(y_j - z) = 0.$$

From the injectivity of U we get $q_{m+2}(z) = r_{m+2}(U(z)) = a > 0$, and there is $j_0 \in \mathbb{N}$ such that $q_{m+2}(y_j) \geq \frac{a}{2}$ whenever $j \geq j_0$, which implies $q_{m+2}(x_j) \geq \frac{a}{2} j$ for all $j \geq j_0$. This is a contradiction, since $(m > n)$

$$q_{m+2}(x_j) \leq q_n(x_j) \leq 1.$$

The proof is complete.

$\square$

2.4 Examples

2.4.1 Weighted spaces of holomorphic functions

Let G be either an open disc centered at the origin or $\mathbb{C}$. A *radial weight* on G is a strictly positive continuous function v on G such that $v(z) = v(|z|),\ z \in G$. Then, the *weighted Banach space of holomorphic functions* is defined by

$$Hv(G) := \{f \in \mathcal{H}(G)\ :\ ||f||_v := \sup_{z\in G} v(|z|)|f(z)| < +\infty\}.$$

Let $V = \{v_n\}_n$ be a decreasing sequence of weights on G. Then the weighted inductive limit of spaces of holomorphic functions is defined by

$$VH := \mathrm{ind_n}\ Hv_n(G),$$

that is, $VH(G)$ is the increasing union of the Banach spaces $Hv_n(G)$ with the strongest locally convex topology for which all the injections $Hv_n(G) \to VH(G)$ become continuous.

Similarly, given an increasing sequence of weights $W = \{w_n\}_n$ on G, the weighted projective limit of spaces of entire functions is defined by

$$HW(G) := \mathrm{proj_n} Hw_n(G),$$

that is, $HW(G)$ is the decreasing intersection of the Banach spaces $Hw_n(G)$ whose topology is defined by the sequence of norms $||\cdot||_{w_n}$. It is a Fréchet space.

In both cases, when $G = \mathbb{C}$ we will simply write VH and HW.

Given any sequence $S := \{z_i\}_i \subset G$ and a decreasing sequence of weights V on G let us put
$$\nu_n(i) = v_n(z_i)$$
and
$$V\ell_\infty(S) = \mathrm{ind}_n \ell_\infty(\nu_n).$$
For an increasing sequence of weights $W = \{w_n\}_n$ on G, we put
$$\omega_n(i) := w_n(z_i)$$
and
$$\ell_\infty W(S) = \bigcap_n \ell_\infty(\omega_n).$$
Obviously, the restriction maps
$$R : VH(G) \to V\ell_\infty(S),\ f \mapsto \{f(z_i)\}_i$$
and
$$R : HW(G) \to \ell_\infty W(S),\ f \mapsto \{f(z_i)\}_i$$
are well defined and continuous, that is, $\{\delta_{z_i}\}_i$ is a $V\ell_\infty(S)$-Bessel sequence for $VH(G)$ and a $\ell_\infty W(S)$-Bessel sequence for $HW(G)$. We want to analyze when these Bessel sequences are in fact frames, that is, when the restriction map is an isomorphism into.

Let us first concentrate on the Fréchet case. In this case, $\{\delta_{z_i}\}_i$ is a $\ell_\infty W(S)$-frame if and only if for every n there are m and C such that
$$\sup_{z\in G} |f(z)|w_n(z) \le C \sup_i |f(z_i)|w_m(z_i)$$
for every $f \in HW(G)$. This is the same as saying that S is a *sufficient set* for $HW(G)$. The concept of sufficient set was introduced by Ehrenpreis in [22].

The (LB)-case is more delicate. Following the notation of section 2.3, if $E_n := Hv_n(G)$, the space $F_n := \{f \in VH(G) : R(f) \in \ell_\infty(\nu_n)\}$ is usually denoted by $A(S, v_n)$ and the corresponding seminorm q_n is denoted $\|\cdot\|_{n,S}$, that is,
$$\|f\|_{n,S} = \sup_{i\in\mathbb{N}} |f(z_i)|\, \nu_n(i), \quad f \in A(S, v_n).$$

Then τ_1 is the topology of the inductive limit $VH(G)$ and τ_2 is the topology of $\mathrm{ind}_n A(S, v_n)$. We recall that S is said to be *weakly sufficient* for $VH(G)$ when $VH(G) = \mathrm{ind}_n A(S, v_n)$ topologically. It should be mentioned that this definition a priori is not restricted to discrete sets, but this is the most interesting case. According to Proposition 2.3.1 and Corollary 2.3.2 we recover the following well-known result.

Theorem 2.4.1 *The following statements are equivalent:*

(i) $S := \{z_i\}_i$ *is weakly sufficient.*

(ii) $A(v_n, S)$ *is a Banach space for every* $n \in \mathbb{N}$.

(iii) For each n *there are* $m \geq n$ *and* $C > 0$ *such that for every* $f \in VH(G)$ *one has*

$$||f||_m \leq C||f||_{n,S}.$$

Also from Proposition 2.3.4 we get

Theorem 2.4.2 *Let us assume that* $\frac{v_{n+1}}{v_n}$ *vanishes at infinity on* G *for every* $n \in \mathbb{N}$. *Then, the following conditions are equivalent:*

(i) $S := \{z_i\}_i$ *is weakly sufficient.*

(ii) The restriction map $VH(G) \to V\ell_\infty(S)$ *is injective and for each* n *there are* $m \geq n$ *and* $C > 0$ *such that* $A(v_n, S) \subset Hv_m(G)$.

The injectivity of the restriction map means that S is a *uniqueness set* for $VH(G)$. As a consequence of Proposition 2.3.3 we obtain

Theorem 2.4.3 *If* $VH(G)$ *is Montel,* S *is weakly sufficient if and only if the restriction map*

$$R : VH(G) \to V\ell_\infty(S), \ f \mapsto f|_S,$$

is a topological isomorphism into.

If the sequence $V = \{v\}$ reduces to one weight, $\{\delta_{z_i}\}_i$ is a $\ell_\infty(\nu)$-frame for $Hv(G)$ if and only if S is a *sampling set* for $Hv(G)$. If $\{v_n\}_n$ is a decreasing sequence of weights on G and S is a sampling set for $Hv_n(G)$ for each n, then S is a weakly sufficient set for $VH(G)$. However, Khoi and Thomas [36] gave examples of countable weakly sufficient sets $S = \{z_i\}_i$ in the space

$$A^{-\infty}(\mathbb{D}) := \mathrm{ind}_n Hv_n(\mathbb{D}), \text{ with } v_n(z) = (1 - |z|)^n,$$

which are not sampling sets for any $Hv_n(\mathbb{D})$, $n \in \mathbb{N}$. As $A^{-\infty}(\mathbb{D})$ is Montel, $\{\delta_{z_i}\}_i$ is a $V\ell_\infty(S)$-frame for $A^{-\infty}(\mathbb{D})$ which is not a $\ell_\infty(\nu_n)$-frame for $Hv_n(\mathbb{D})$ for any n. Bonet and Domanski [11] studied weakly sufficient sets in $A^{-\infty}(\mathbb{D})$ and their relation to what they called (p, q)-sampling sets.

The dual of the space $A^{-\infty}(\mathbb{D})$ can be identified via the Laplace transform with the space of entire functions $A_{\mathbb{D}}^{-\infty} := HW(\mathbb{C})$ for the sequence of weights $W = \{w_n\}_n$,

$$w_n(z) = (1 + |z|)^n e^{-|z|},$$

(see [49] and also [1] for the several variables case). In [1] explicit constructions of sufficient sets for this space are given. For instance, for each k take $\ell_k \in \mathbb{N}$, $\ell_k > 2\pi k^2$, and let $z_{k,j} := kr_{k,j}$, $1 \le j \le \ell_k$, where $r_{k,j}$ are the ℓ_k-roots of the unity, then, with an appropriate order, $\{\delta_{z_{k,j}} : k \in \mathbb{N}, 1 \le j \le \ell_k\}$ is a $\ell_\infty W(S)$-frame in $A_{\mathbb{D}}^{-\infty}$. More examples for non-radial weights can be found in [2].

Finally we recover the following result. This should be compared with Corollary 2.4.18 below.

Theorem 2.4.4 ([1])

(i) $\{\lambda_k\}_k \subset \mathbb{C}$ *is sufficient for* $A_{\mathbb{D}}^{-\infty}$ *if and only if every function* $f \in A^{-\infty}(\mathbb{D})$ *can be represented as*

$$f(z) = \sum_k \alpha_k e^{\lambda_k z}$$

where

$$\sum_k |\alpha_k| (1 + |\lambda_k|)^{-n} e^{|\lambda_k|} < \infty \text{ for some } n \in \mathbb{N}.$$

(ii) $\{\lambda_k\}_k \subset \mathbb{D}$ *is weakly sufficient in* $A^{-\infty}(\mathbb{D})$ *if and only if each function* $f \in A_{\mathbb{D}}^{-\infty}$ *can be represented as*

$$f(z) = \sum_k \alpha_k e^{\lambda_k z}$$

where

$$\sum_k |\alpha_k| (1 - |\lambda_k|)^{-n} < \infty \text{ for every } n \in \mathbb{N}.$$

Remark 2.4.5 (a) Observe that the sequence $\{\lambda_k\}_k$ in Theorem 2.4.4(i) cannot be bounded. In fact, if we assume that $\{\lambda_k\}_k$ is bounded, condition

$$\sum_k |\alpha_k| (1 + |\lambda_k|)^{-n} e^{|\lambda_k|} < \infty \text{ for some } n \in \mathbb{N}.$$

simply means that

$$\sum_k |\alpha_k| < \infty,$$

and hence each $f \in A^{-\infty}(\mathbb{D})$ would be bounded, that is we would have the identity $A^{-\infty}(\mathbb{D}) = H^{\infty}(\mathbb{D})$.

In the same way, the sequence $\{\lambda_k\}_k$ in Theorem 2.4.4(ii) cannot be contained in a compact subset of $\mathbb{D}$. If this were the case,

$$\sum_k |\alpha_k| \, (1 - |\lambda_k|)^{-n} < \infty \text{ for every } n \in \mathbb{N}.$$

would be equivalent to $\{\alpha_k\}_k \in \ell_1$. Consequently, we could find $0 < a < 1$ such that each for each $f \in A_{\mathbb{D}}^{-\infty}$ there is $C > 0$ with

$$|f(z)| \leq Ce^{a|z|}.$$

(b) The convergence of the representations

$$f(z) = \sum_k \alpha_k e^{\lambda_k z}$$

in Theorem 2.4.4 is absolute. To this end, for $\lambda \in \mathbb{C}$, let us denote by e_λ the exponential $e_\lambda(z) := e^{\lambda z}$. Then, it is easy to see that

$$||e_\lambda||_{v_n} = \begin{cases} 1, & \text{if} \quad |\lambda| \leq n \\ (\frac{n}{e})^n \frac{e^{|\lambda|}}{|\lambda|^n}, & \text{if} \quad |\lambda| > n \end{cases}$$

Hence if $f \in A^{-\infty}(\mathbb{D})$ can be represented as

$$f(z) = \sum_k \alpha_k e^{\lambda_k z}$$

where

$$\sum_k |\alpha_k| \, (1 + |\lambda_k|)^{-n} e^{|\lambda_k|} < \infty \text{ for some } n \in \mathbb{N},$$

we have that

$$\sum_k |\alpha_k| \, ||e_\lambda||_{v_n} \leq$$

$$\sum_{|\lambda_k| \leq n} |\alpha_k| + (\tfrac{n}{e})^n \sum_{|\lambda_k| > n} |\alpha_k| \, \tfrac{e^{|\lambda|}}{|\lambda|^n} < \infty.$$

The other case is similar.

2.4.2 The Hörmander algebras

We use here and in what follows Landau's notation of little o-growth and capital O-growth. A function $p : \mathbb{C} \to [0, \infty[$ is called a *growth condition* if it is continuous, subharmonic, radial, increases with $|z|$ and satisfies

(α) $\log(1 + |z|^2) = o(p(|z|))$ as $|z| \to \infty$,

(β) $p(2|z|) = O(p(|z|))$ as $|z| \to \infty$.

Given a growth condition p, consider the weight $v(z) = e^{-p(|z|)}$, $z \in \mathbb{C}$, and the decreasing sequence of weights $V = \{v_n\}_n$, $v_n = v^n$. We define the following weighted spaces of entire functions (see e.g. [8], [5]):

$$A_p := \left\{ f \in \mathcal{H}(\mathbb{C}) : \text{ there is } A > 0 : \sup_{z \in \mathbb{C}} |f(z)| \exp(-Ap(z)) < \infty \right\},$$

that is, $A_p = VH$, endowed with the inductive limit topology, for which it is a (DFN)-algebra (see e.g. [46]). Given any sequence $S = \{z_i\}_i$ we will denote $A_p(S) = V\ell_\infty(S)$, that is,

$$A_p(S) = \bigcup_n \ell_\infty(\nu_n), \quad \nu_n(i) = e^{-np(|z_i|)}.$$

If we consider the increasing sequence of weights $W = \{w_n\}_n$, $w_n = v^{1/n}$, we define

$$A_p^0 := \left\{ f \in \mathcal{H}(\mathbb{C}) : \text{ for all } \varepsilon > 0 : \sup_{z \in \mathbb{C}} |f(z)| \exp(-\varepsilon p(z)) < \infty \right\},$$

that is, $A_p^0 = HW$, endowed with the projective limit topology, for which it is a nuclear Fréchet algebra (see e.g. [47]). Clearly $A_p^0 \subset A_p$. As before, given a sequence $S = \{z_i\}_i$ we will denote $A_p^0(S) = \ell_\infty W(S)$, that is,

$$A_p^0(S) = \bigcap_n \ell_\infty(\omega_n), \quad \omega_n(i) = e^{-\frac{1}{n}p(|z_i|)}.$$

Condition (α) implies that, for each $a > 0$, the weight $v_a(z) := e^{-ap(|z|)}$ is rapidly decreasing, consequently, the polynomials are contained and dense in $H^0_{v_a}$, and that for $a < b$ the inclusion $H_{v_a} \subset H^0_{v_b}$ is compact. Therefore the polynomials are dense in A_p and in A_p^0. Condition (β) implies that both spaces are stable under differentiation. By the closed graph theorem, the differentiation operator D is continuous on A_p and on A_p^0.

Weighted algebras of entire functions of this type, usually known as Hörmander algebras, have been considered since the work of Berenstein and Taylor [8] by many authors; see e.g. [5] and the references therein.

In order to present some examples we recall the definition of order and type of subharmonic functions in $\mathbb{C}$ which allows us to compare the order of growth of a function with different functions of $\log(|z|)$.

Definition 2.4.6 For a subharmonic function f in $\mathbb{C}$, we consider $M(f,r) = \sup_{|z|=r} f(z)$. We say that:

- It is of *class zero* if
$$\limsup_{r\to\infty} \frac{M(f,r)}{\log(r)} < \infty.$$
- It is of *finite class* $p \geq 1$ if p is the smallest integer $q \geq 1$ such that
$$\limsup_{r\to\infty} \frac{\log_q M(f,r)}{\log(r)} < \infty.$$
- It is of *infinite class* if no such integer exists.

If $p = 1$ one says also that f is of *finite order*. In this case, the number $a \in [0,\infty[$ defined by
$$a = \limsup_{r\to\infty} \frac{\log(M(f,r))}{\log(r)}$$
is called *the order of f*.

For functions of positive order $a > 0$, we have also the concept of the *type* τ:
$$\tau = \limsup_{r\to\infty} \frac{M(f,r)}{r^a}.$$

We say that f is of *minimal* (resp. *normal*, *maximal*) type if $\tau = 0$ (resp. it is finite and nonzero, infinite). The function f is said to be of *finite type* if it is of either minimal or normal type.

The entire functions most often considered are those of finite order and quite often they are of *exponential type*. These are the entire functions for which there are constants $A, B > 0$ such that $|f(z)| \leq A\exp(B|z|) \quad z \in \mathbb{C}$. In other words, these functions are precisely the entire functions of order < 1 together with the functions of order 1 and finite type.

As an example, when $p_a(z) = |z|^a$, then A_{p_a} consists of all entire functions of order a and finite type or order less than a, and $A^0_{p_a}$ is the space of all entire functions of order at most a and type 0. For $a = 1$, A_{p_1} is the space of all entire functions of exponential type, also denoted $Exp(\mathbb{C})$ and $A^0_{p_1}$ is the space of entire functions of infraexponential type.

As we recall in Theorem 2.4.15 and Corollary 2.4.16, the Fourier-Borel transform $\mathcal{F} : H(\mathbb{C})' \to Exp(\mathbb{C})$ defined by $\mathcal{F}(\mu) := \widehat{\mu}$, where $\widehat{\mu}(z) := \mu_\omega(e^{z\omega})$, is a topological isomorphism. As a consequence, the dual space of $Exp(\mathbb{C})$ can be identified with the space of entire functions, $H(\mathbb{C})$. In the same way, for $a > 1$ and b its conjugate exponent ($a^{-1} + b^{-1} = 1$) via the Fourier-Borel transform $\mathcal{F}$ we have the following identifications [64]

$$(A_{p_a})' = A^0_{p_b}, \text{ and } (A^0_{p_a})' = A_{p_b}.$$

Proposition 2.4.7 *Let $\{z_j\}_{j=1}^{\infty}$ be a (weakly) sufficient set for (A_p) A^0_p then we can remove finitely many points $\{z_j\}_{j=1}^{N}$ and still we have a (weakly) sufficient set.*

Proof. In fact, take Q a non constant polynomial which vanishes precisely at points $\{z_j\}_{j=1}^{N}$. The multiplication operator

$$T_Q(f)(z) = Q(z)f(z)$$

is a topological isomorphism from A_p (resp. A^0_p) into itself. Using that the zeros of the function are isolated, we obtain that, if $T_Q(f)(z) = 0$ therefore $f(z) = 0$ and we obtain that the multiplication operator is linear and injective. By [5, Lemma 2.5.9] we conclude that the range of T_Q is closed, therfore T_Q is an isomorphism into. Also, using that pointwisse multiplication by $\{Q(z_j)\}_j$ is continuous on $A_p(S)$ (resp. $A^0_p(S)$) it suffices to apply 2.2.12(i). $\square$

Definition 2.4.8 A multiplicity variety V is a sequence of pairs (z_k, m_k), z_k distinct points of $\mathbb{C}$, with $\lim_k |z_k| = \infty$ and the m'_ks are positive integers, called the multiplicities of the points z_k.

Given two multiplicity varieties, $V = \{(z_k, m_k) : k \in \mathbb{N}\}$ and $V' = \{(z'_k, m'_k) : k \in \mathbb{N}\}$, we say that $V' \subset V$ when $\{z'_k\}_k$ is a subsequence of $\{z_k\}_k$ and, for the corresponding indices, we have $m'_j \leq m_k$.

For an entire function $f \not\equiv 0$ its *multiplicity variety* $V = V(f)$ is the set of pairs (z_k, m_k), where z_k runs over all the zeros of f and m_k denotes the multiplicity of that zero.

By Weierstrass interpolation theorem (see e.g. [4]), the *restriction map*

$$R_V : \mathcal{H}(\mathbb{C}) \to \prod_{k\in\mathbb{N}} \mathbb{C}^{m_k}, \quad R_V(g) := \left\{ \left\{ \frac{g^{(l)}(z_k)}{l!} \right\}_{0\le l<m_k} \right\}_k,$$

is surjective.

We associate with a multiplicity variety $V = \{(z_k, m_k) : k \in \mathbb{N}\}$ and a growth condition p the following sequence spaces

$$A_p(V) := \left\{ a = (a_{k,l}) \in \prod_{k\in\mathbb{N}} \mathbb{C}^{m_k} \mid \text{there is } B > 0 : \sup_{k\in\mathbb{N}} \sum_{l=0}^{m_k-1} |a_{k,l}| \exp(-Bp(z_k)) < \infty \right\}$$

endowed with the inductive limit topology; and

$$A_p^0(V) := \left\{ a = (a_{k,l}) \in \prod_{k\in\mathbb{N}} \mathbb{C}^{m_k} \mid \text{ for all } \varepsilon > 0 : \sup_{k\in\mathbb{N}} \sum_{l=0}^{m_k-1} |a_{k,l}| \exp(-\varepsilon p(z_k)) < \infty \right\}$$

endowed with the projective topology, for which it is a Fréchet space.

It is well-known that $R_V(A_p) \subset A_p(V)$ and $R_V(A_p^0) \subset A_p^0(\mathbb{C})$; see [5], [6], [7], [8]. A multiplicity variety is called *interpolating* for A_p (resp. for A_p^0) if $R_V(A_p) = A_p(V)$ (resp. $R_V(A_p^0) = A_p^0(\mathbb{C})$). In case $m_k = 1$ for all k, we will simply say that the sequence $\{z_k\}_k$ is interpolating (for the corresponding spaces).

Given two multiplicity varieties $V' \subset V$ it is easy to see that V' has to be interpolating for A_p (resp. for A_p^0) whenever V is interpolating for A_p (resp. for A_p^0). In particular, if $V = \{(z_k, m_k) : k \in \mathbb{N}\}$ is interpolating for A_p (resp. for A_p^0), then $\{z_k\}_k$ is interpolating for A_p (resp. for A_p^0).

Since A_p^0 and A_p are algebras without zero divisors, a sequence $\{z_k\}_k$ cannot be simultaneously interpolating for A_p^0 (A_p) and (weakly) sufficient by 2.2.12(ii).

Also, from 2.2.12 we recover the following well-known result:

Lemma 2.4.9 *Let $\{z_k\}_k$ be interpolating for A_p (resp. A_p^0). Then there is $f \in A_p$, (resp. $f \in A_p^0$), $f \neq 0$ such that $f(z_k) = 0$ for all k.*

Proposition 2.4.10 *Let $S := \{z_k\}_k$ be a weakly sufficient set for A_p and assume that some subsequence $\{z_{k_j}\}_j$ is interpolating for A_p. Then $S \setminus \{z_{k_j} : j \in \mathbb{N}\}$ is a weakly sufficient set.*

Proof. By the previous lemma, there exists $f \in A_p$, $f \not\equiv 0$ such that $f(z_{k_j}) = 0$ for every $j \in \mathbb{N}$. Consider the multiplication operator

$$T_f(g)(z) = g(z)f(z).$$

By [5, 2.2.14, 2.2.15] is a topological isomorphism into of A_p. We conclude applying 2.2.12(i) as in the former proposition. $\square$

Now, we give examples of frames of type $\{\delta_{z_i}\}_i$ in these algebras. We deal first with the Fréchet case.

Theorem 2.4.11 *Given a growth condition q let $S := \{z_n\}_n$ be a sequence in $\mathbb{C}$ with $\lim_j |z_j| = \infty$ and assume that there is $C > 0$ such that the distance $d(z, S)$ satisfies $d(z, S) \le C|z|/\sqrt{q(|z|)}$ for all $z \in \mathbb{C}$. Then, the sequence $\{\delta_{z_j}\}_j$ is a $A_p^0(S)$-frame for A_p^0 whenever $p(r) = o(q(r))$ as $r \to \infty$.*

Proof. We take $V(r) = q(r)$. The family $\{ap,\, a > 0\}$ satisfies (i), (ii) and (iii) in [60, p.178] and the conclusion follows after applying [60, Theorem 5.1]. $\square$

In particular, if $p(r) = o(r^2)$ as $r \to \infty$ we may take $q(r) = r^2$.

Corollary 2.4.12 *If $p(r) = o(r^2)$, then for arbitrary α, $\beta > 0$ the regular lattice $\{\alpha m + i\beta m :\ n, m \in \mathbb{Z}\}$ is a sufficient set for $A_p^0(\mathbb{C})$. In other words, if $S = \{z_{n,m}\}$ where $z_{n,m} := \alpha n + i\beta m$ then the sequence $\{\delta_{z_{n,m}}\}$ is a $A_p^0(S)$-frame for $A_p^0(\mathbb{C})$.*

The former result is also true in the limit case $p(r) = r^2$. In fact,

Proposition 2.4.13 *If $p(r) = r^2$, then for arbitrary α, $\beta > 0$ the regular lattice $\{\alpha m + i\beta m :\ n, m \in \mathbb{Z}\}$ is a sufficient set for $A_p^0(\mathbb{C})$. In other words, if $S = \{z_{n,m}\}$ where $z_{n,m} := \alpha n + i\beta m$ then the sequence $\{\delta_{z_{n,m}}\}$ is a $A_p^0(S)$-frame for $A_p^0(\mathbb{C})$.*

Proof. First, we observe that in this case, $A_p^0(\mathbb{C})$ coincides algebraically and topologically with the intersection

$$\bigcap_{\gamma > 0} \mathcal{F}_\gamma^2$$

of the Bargmann-Fock spaces

$$\mathcal{F}_\gamma^2 := \left\{ h \in H(\mathbb{C}) : ||f||_\gamma := \int_{\mathbb{C}} |f(z)|^2 e^{-\gamma |z|^2} dz < \infty \right\}.$$

Then, by [61], there is γ_0 such that for $\gamma \geq \gamma_0$ we find constants A_γ, B_γ such that

$$A_\gamma ||f||_\gamma^2 \leq \sum_{n,m} |f(z_{n,m})|^2 e^{-\gamma |z_{n,m}|^2} \leq B_\gamma ||f||_\gamma^2.$$

To finish, it is enough to observe that in the definition of $A_p^0(S)$ one can replace the ℓ_∞ norms by ℓ_2-norms. □

According to [60] (see the comments after Corollary 4.9) there is an entire function of order 2 and finite type which vanishes at the lattice points $S = \{n+im : n, m \in \mathbb{Z}\}$. In the case $r^2 = o(p(r))$ we have $f \in A_p^0$, and the restriction map defined on A_p^0 by $f \mapsto f|_S$ is not injective. Consequently, the lattice points are not a sufficient set for A_p^0. Similarly, the lattice points are not a weakly sufficient set for A_p in the case $r^2 = O(p(r))$.

From [60, Proposition 8.1] and 2.4.2 we have that

Proposition 2.4.14 *If $p(r) = o(r^2)$, then for arbitrary α, $\beta > 0$ the regular lattice $\{\alpha m + i\beta m : n, m \in \mathbb{Z}\}$ is a sufficient set for $A_p(\mathbb{C})$. In other words, if $S = \{z_{n,m}\}$ where $z_{n,m} := \alpha n + i\beta m$ then the sequence $\{\delta_{z_{n,m}}\}$ is a $A_p(S)$-frame for $A_p(\mathbb{C})$.*

In particular, for the space $Exp(\mathbb{C})$, the sequence $\{\delta_{n+im}\}_{n,m\in\mathbb{Z}}$ is a $A_p(S)$-frame [65, Theorem 1], where $p(z) = |z|$ and $S = \{n + im\}_{n,m\in\mathbb{Z}}$.

By Proposition 2.2.6 if $S = \{z_i\}_i \subset G$ is a discrete (weakly) sufficient set in $HW(G)$ (resp. in $VH(G)$) each element in the dual space can be represented as a convergent series of type

$$\sum_i \alpha_i \delta_{z_i}$$

with coefficients in a given sequence space. Since the spaces under consideration are algebras, this representation is not unique by Remark 2.2.12(ii). As in many cases the dual space can be identified with a weighted space of holomorphic functions (via the Laplace or the Fourier-Borel transform). These vector spaces are algebraically isomorphic due to the following theorem and its corollary.

Theorem 2.4.15 *Fourier-Borel transform $\mathfrak{F} : \mathcal{H}(\mathbb{C})' \to Exp(\mathbb{C})$ defined by $\mathfrak{F}(\mu) := \widehat{\mu}$ where $\widehat{\mu} := \mu_\omega(e^{z\omega})$ is an algebraic isomorphism. Moreover, $B \in \mathcal{H}(\mathbb{C})'$ is equicontinuous if and only if there exists $k > 0$ such that $\mathfrak{F}(B)$ is bounded in $Exp(\mathbb{C})$.*

Corollary 2.4.16 *Fourier-Borel transform $\mathfrak{F} : \mathcal{H}(\mathbb{C})'_\beta \to Exp(\mathbb{C})$ defined by $\mathfrak{F}(\mu) := \widehat{\mu}$ where $\widehat{\mu} := \mu_w(e^{zw})$ is a topological isomorphism.*

Proof. $\mathcal{H}(\mathbb{C})$ is a Fréchet-Schwartz space (in fact, is a Fréchet nuclear space), therefore $\mathcal{H}(\mathbb{C})'_\beta$ is an inductive limit of Banach spaces which bounded sets are the equicontinuous sets. Therefore, $\mathfrak{F}$ is continuous since $\mathfrak{F}$ applies the bounded sets of $\mathcal{H}(\mathbb{C})'_\beta$ to the bounded sets of $Exp(\mathbb{C})$. The result holds applying the open mapping theorem since $Exp(\mathbb{C})$ is also an inductive limit of Banach spaces. □

Due to this relation, the point evaluations δ_{z_i} are identified with the exponentials $e^{z_i z}$, we get a representation of the elements in the dual space as Dirichlet series, thus recovering known results, for instance:

Corollary 2.4.17 *([65]) Every entire function $f(z)$ can be represented in the form*

$$f(z) = \sum_{n,m=-\infty}^{\infty} a_{n,m} e^{(n+im)z}$$

where $|a_{n,m}| e^{k(n^2+m^2)^{1/2}} \to 0$ as $n^2+m^2 \to +\infty$ for every $k>0$. Such expansion of f is never unique.

Corollary 2.4.18 *For $a \geq 2$ every function $f \in A_{p_a}$ and can be represented in the form*

$$f(z) = \sum_{n,m=-\infty}^{\infty} a_{n,m} e^{(n+im)z}$$

with coefficients $\{a_{n,m}\}$ satisfying

$$|a_{n,m}| \leq C\exp(-\varepsilon(n^2+m^2)^{b/2})$$

(b the conjugate of a) for some constants ε, $C>0$.

Proof. According to Corollary 2.4.12 and Proposition 2.4.13 the sequence $S = \{e^{(n+im)z} : \ n,m \in \mathbb{Z}\} \subset A_{p_a}$ is a $A^0_{p_b}(S)$-frame for $A^0_{p_b}$. Since the dual space of $\Lambda = A^0_{p_b}(S)$ is

$$\Lambda' = \Big\{ \{a_{n,m}\} : \ |a_{n,m}| \exp(\varepsilon(n^2+m^2)^{b/2}) < \infty \text{ for some } \varepsilon > 0 \Big\}$$

it suffices to apply Proposition 2.2.6 to conclude. □

Chapter 3

Appendix

In this appendix, some concepts about locally convex spaces and their duals are introduced with special attention to inductive limits. In addition, we also introduce some results concerning topological bases. We establish the definitions and the fundamental properties that we use through the thesis. We follow [9], [53],[33], [41], [48] and [58].

A subset M of a lcs E is called *bounded* if it is absorbed by every neighborhood (of the origin). If $\mathcal{U}$ is a basis of absolutely convex neighborhoods, the set M is bounded if and only if to each $U \in \mathcal{U}$ corresponds a positive λ with $M \subseteq \lambda U$. A subset M of a lcs is called *precompact* if, for every (absolutely convex) neighborhood U, there are $x_1, \ldots, x_n \in M$ such that $M \subseteq \cup_{i=1}^{n} (x_i + U)$.

Given a locally convex space E, we denote by E^* the algebraic dual of E, that is, the space of all linear forms $T : E \to \mathbb{C}$, and by E' its topological dual, i.e., the space of all continuous linear forms on E. Given a vector space E and F a vector subspace of E^* (the algebraic dual) we say that (E, F) is a *dual pair* if for each $x \in E$, $x \neq 0$, there is $y^* \in F$ such that $y^*(x) \neq 0$. The *weak topology* of E, denoted by $\sigma(E, E')$, is defined as follows: a net $\{x_\alpha\}_\alpha$ converges to x_0 in $(E, \sigma(E, E'))$ if and only if for all x' in E', $\{x'(x_\alpha)\}_\alpha$ converges to $x'(x_0)$ in $\mathbb{C}$. A net $\{x'_\alpha\}_\alpha$ converges to x'_0 in the *weak-star topology* on E', usually known as weak* or w^* topology, and denoted by $\sigma(E', E)$, if and only if for all x in E, $\{x'_\alpha(x)\}_\alpha$ converges to $x'_0(x)$ in $\mathbb{C}$. Let (E, E') be a dual pair and $\mathcal{A}$ a set of weakly bounded subsets of E. For each $M \in \mathcal{A}$, put $p'_M(x') = \sup\{|\langle x, x'\rangle| : x \in M\}$; then the set $\{p'_M : M \in \mathcal{A}\}$ determines the topology of $\mathcal{A}$-convergence. Taking $\mathcal{A}$ to be the set of all weakly bounded subsets of E, this topology is denoted by $\beta(E, E')$ and is sometimes called the *strong topology*. There is a finest topology of the dual pair (E, E'), namely the topology of uniform convergence on the set of all absolutely convex $\sigma(E', E)$-compact subsets of E'. This topology is denoted by $\mu(E, E')$ and sometimes called the *Mackey topology*. It is coarser than $\beta(E, E')$. For a locally convex space E, $cs(E)$ denotes a system of continuous seminorms determining the topology of E, and for two locally convex spaces E and F, the set of all continuous linear maps from E to F is denoted by $L(E, F)$. Each element $T \in L(E, F)$ is called an operator, and it defines another operator $T' : F' \to E'$, $T'(\lambda)(x) = \lambda(T(x))$, $\lambda \in F'$, $x \in E$, called its *transpose*.

A set M is *absorbing* if $\cup_n nM = E$. We say that a Hausdorff locally convex space E is *barrelled* if every *barrel*, that is, if every closed, absolutely convex (i.e., convex and balanced) and absorbing set in the space is a zero-neighbourhood. In most of the results we need the assumption that the lcs is barrelled. The reason is that Banach-Steinhaus theorem holds for barrelled lcs. Every Fréchet space is barrelled. We refer the reader to [33] and [53] for more information about barrelled spaces.

Theorem 3.0.19 (Banach - Steinhaus theorem) *Let E be a barrelled lcs space and F be a lcs. Further, let $M \subset L(E, F)$ be a pointwise bounded set (i.e., for each $x \in E$, $\{A(x) : A \in M\}$ is bounded in F). Then, for each zero neighborhood V in F there exists a zero neighborhood U in E with $A(U) \subset V$ for all $A \in M$ (i.e. M is an equicontinuous set).*

Corollary 3.0.20 *If E is a barrelled lcs and F is a Hausdorff lcs, if $\{A_n\}_n$ is a sequence of continuous linear mappings of E into F which is pointwise convergent to A_0, then A_0 is a continuous linear mapping and the convergence is uniform on every precompact subset of E.*

As usual ω denotes the countable product $\mathbb{K}^{\mathbb{N}}$ of copies of the scalar field, endowed with the product topology, and φ stands for the space of sequences with finite support. A *sequence locally convex space* $\bigwedge$ is a lcs which contains φ and is continuously included in ω.

Let (E, E') be a dual pair. If M is a subset of E, the subset of E' consisting of those x' for which $\sup\{|\langle x, x'\rangle| : x \in M\} \leq 1$ is called the *polar* of A in E' and denoted by M°. The following theorem implies that if E is a lcs and M is an absolutely convex subset of E. Then, $\overline{M} = (M^\circ)^\circ =: M^{\circ\circ}$.

Theorem 3.0.21 (Bipolar theorem) *Let (E, E') be a dual pair and F a vector subspace of E'' containing E. Then the bipolar $M^{\circ\circ}$ in F of a subset M of E is the $\sigma(F, E')$-closed absolutely convex envelope of M.*

In what follows we give an introduction to countable inductive of locally convex spaces. For the definitions, the proofs and more background, see e.g. [9] [48], [53].

Let E be a linear space, $\{E_n : n = 1, 2, \dots\}$ an increasing sequence of subspaces of E and $J_n : E_n \to E$, $J_{n,n+1} : E_n \to E_{n+1}$, the canonical injections. Suppose that each E_n is endowed with a Hausdorff locally convex topology τ_n such that each $J_{n,n+1} : (E_n, \tau_n) \to (E_{n+1}, \tau_{n+1})$ is continuous. Then $\mathcal{E} := \{(E_n, \tau_n) : n = 1, 2, \dots\}$ is called an *inductive sequence* with respect to the mappings $\{J_n : n = 1, 2, \dots\}$. An inductive sequence is *strict* if each $J_{n,n+1}$ is an isomorphism onto its image and *hyperstrict* if it is strict and each E_n is closed in (E_{n+1}, τ_{n+1}). Each (E_n, τ_n) is called a *step* of $\mathcal{E}$.

Let $\mathcal{E}$ be an inductive sequence and let τ be the finest locally convex topology on E such that each $J_n : (E_n, \tau_n) \to (E, \tau)$ is continuous. Then (E, τ) is called the *inductive limit* of the defining sequence $\mathcal{E}$ and we write $(E, \tau) = ind\ \mathcal{E} = ind\{(E_n, \tau_n) : n = 1, 2, \dots\}$. If $\mathcal{E}$ is strict (resp., hyperstrict), (E, τ) is said to be the strict (resp., hyperstrict) inductive limit of $\mathcal{E}$. If each (E_n, τ_n) of an inductive sequence is a Banach (resp., Fréchet) space, then (E, τ) is said to be an (LB)*-space* (resp., (LF)*-space*).

Proposition 3.0.22 ([53, 0.3.2]) *If* $(E, \tau) = ind\{(E_n, \tau_n) : n = 1, 2, \dots\}$ *and if*

(i) $\{n(k) : k = 1, 2, \dots\}$ *is a strictly increasing sequence of positive integers, then* $\mathcal{F} := \{(E_{n(k)}, \tau_{n(k)}) : k = 1, 2, \dots\}$ *is also a defining sequence for* (E, τ).

(ii) $T : (E, \tau) \to F$, *F being a Hausdorff locally convex space, is a linear mapping, then T is continuous if and only if each* $T \circ J_n : (E_n, \tau_n) \to F$ *is continuous.*

(iii) *U is an absolutely convex subset of E, then U is a* 0*-neighbourhood in* (E, τ) *if and only if each* $U \cap E_n$ *is a 0-neighbourhood in* (E_n, τ_n). *Thus a basis of* 0*-neighbourhoods in* (E, τ) *can be given by the sets* $\Gamma(\bigcup_{n \in \mathbb{N}} U_n)$, *where each* U_n *is a* 0*-neighborhood in* (E_n, τ_n) *and* Γ *denotes the absolutely convex hull.*

Theorem 3.0.23 (Grothendieck's Factorization Theorem [53, 1.2.20]) *Let F be a Baire space,* $E = ind_n E_n$ *a countable inductive limit of Fréchet spaces and* $T : F \to E$ *a linear mapping with closed graph in* $F \times E$. *Then, there exists a positive integer k such that $T(F)$ is contained in E_k and* $T : F \to (E_k, \tau_k)$ *is continuous.*

We say that an inductive limit $E = ind_n E_n$ is *regular* if, for every bounded set $B \subseteq E$, there exists $n \in \mathbb{N}$ such that B is a bounded subset of E_n. E is *boundedly retractive* if, for every bounded subset B of E, there exists $n \in \mathbb{N}$ such that B

is bounded in E_n and the topologies of E and E_n coincide on B, and E is said *strongly boundedly retractive* if E is regular and, for every $n \in \mathbb{N}$, there exists some $m \in \mathbb{N}$, $m \geq n$ such that the topology τ of the inductive limit E and the topology of E_m coincide on each bounded subset B of E_n. It is clear that a strongly boundedly retractive inductive limit is boundedly retractive.

Certain properties of locally convex spaces are preserved by the operation of taking inductive limits. In fact, an inductive limit of barrelled spaces is barrelled. However, even though we suppose that each of the locally convex spaces E_n in an inductive limit has a Hausdorff topology, it is possible that the inductive limit topology τ of $E = ind_n E_n$ is not Hausdorff. Regular inductive limits always carry a Hausdorff topology.

Now, we recall the concept of basis and of related objects in topological vector spaces and, in particular, in locally convex spaces. If $\{e_n\}_n$ is a basis in the Hausdorff lcs E, then e'_n will always be used without further explanation to denote the coefficient functionals associated with $\{e_n\}_n$.

Let E be a Haussdorf tvs. A infinite sequence $\{e_n\}_n$ in E is called a Schauder basis, if every $x \in E$ determines a unique sequence $\{\alpha_n\}_n$ in $\mathbb{K}$ such that $x = \sum_{n=1}^{\infty} \alpha_n e_n$ and its coefficient functionals $e'_n \in E'$, defined by $e'_n(\alpha) = \alpha_n$ for every $n \in \mathbb{N}$, are continuous for every $n \in \mathbb{N}$. It is clear that $\{e'_n\}_n$ is a Schauder basis in $(E', \sigma(E', E))$ provided $\{e_n\}_n$ is a Schauder basis in E. By a *shrinking basis* in E we mean a Schauder basis $\{e_n\}_n$ such that $\{e'_n\}_n$ is a Schauder basis even in $(E', \beta(E', E))$. By a boundedly complete basis in E one understands a Schauder basis $\{e_n\}_n$ in E with the following property: If $\{\alpha_n\}_n \subset \mathbb{K}$ is such that $\{\sum_{n=1}^{k} \alpha_n x_n\}_{k \in \mathbb{N}}$ is bounded in E, then $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in E.

Let E be a Hausdorff lcs and $\mathcal{U}$ a zero neighborhood basis in E consisting of absolutely convex sets. A sequence $\{x_n\}_n$ is called *unconditionally summable* if $\{\sum_{n=1}^{k} x_{\gamma(n)}\}_{k \in \mathbb{N}}$ converges in E, for each permutation γ of $\mathbb{N}$. All these sequences have the same limit, regardless of the permutation γ. If $\{x_n\}_n$ is a sequence in E which is unconditionally summable in the completion of E, then we call $\{x_n\}_n$ *unconditionally Cauchy.* A sequence $\{x_n\}_n$ in E is said to be *absolutely summable* if $\sum_{n=1}^{\infty} p_U(x_n)$ converges in $\mathbb{R}$, for every $U \in \mathcal{U}$, and if $\sum_{n=1}^{\infty} x_n$ exists in E. If the latter condition is omitted, the former still implies convergence of $\sum_{n=1}^{\infty} x_n$ in the completion of E. In that case we shall say that $\{x_n\}_n$ is *absolutely Cauchy.* A sequence $\{x_n\}_n$ in E is said to be *subseries summable* if $\sum_{k=1}^{\infty} x_{n_k}$ converges, for every strictly increasing sequence $\{n_k\}_k$ in $\mathbb{N}$. The corresponding Cauchy concept will not be needed here.

Proposition 3.0.24 ([33, Proposition 14.6.2]) *Every subseries summable sequence $\{x_n\}_n$ in E is unconditionally summable.*

Let E be a Hausdorff lcs. By an *unconditional basis* in E we mean a Schauder basis $\{e_n\}_n$ in E such that all sequences $\{\langle e'_n, x\rangle \cdot e_n\}_n$, $x \in E$, are unconditionally summable. Now, we introduce some characterizations of unconditional convergence.

Theorem 3.0.25 (McArthur - Retherford, [45, p. 116]) *Let E be a sequentially complete lcs and let ℓ_∞ the Banach space of bounded sequences of scalars $b = \{b_j\}_j$ with $\|b\| = \sup_j |b_j|$. For a series $\sum_{j=1}^{\infty} x_j$ in E the following are equivalent:*

(a) $\sum_{j=1}^{\infty} b_j x_j$ *converges for all* $b = \{b_j\}_j \in \ell_\infty$*;*

(b) $\sum_{j=1}^{\infty} b_j x_j$ *converges for all* $b = \{b_j\}_j$ *with* b_j *either 0 or 1 for each* j*;*

(c) $\sum_{j=1}^{\infty} x_j$ *is unconditionally convergent;*

(d) $\lim_n \sum_{j=1}^{n} b_j x_j$ *exists uniformly for* $b = \{b_j\}_j \in \ell_\infty$ *with* $\|b\| \le 1$*;*

(e) $\lim_n \sum_{j=1}^{n} b_j x_j$ *exists uniformly for* $b = \{b_j\}_j \in \ell_\infty$ *with* b_j *either 0 or 1 for each* j*.*

Since ℓ_∞ is non-separable, a Hausdorff lcs E with a shrinking basis cannot contain a subspace which is linearly homeomorphic to ℓ_1. In fact, otherwise the adjoint of $\ell_1 \hookrightarrow E$ would be a continuous surjection of the separable lcs $(E', \beta(E', E))$ onto ℓ_∞, and this is impossible.

Theorem 3.0.26 ([33, Theorem 14.7.3]) *Let E be a barrelled, sequentially complete lcs and $\{x_n\}_n$ an unconditional basis in E. $\{x_n\}_n$ is shrinking if and only if E does not contain a copy of ℓ_1 (i.e. no subspace of E is linearly homeomorphic to ℓ_1).*

Corollary 3.0.27 ([33, Corollary 14.7.4]) *If E is a sequentially complete, barrelled lcs with unconditionally basis $\{x_n\}_n$, then $\{x_n\}_n$ is shrinking if and only if $(E', \beta(E', E))$ is separable.*

Proceeding to the dual situation and consider unconditional basis which are boundedly complete.

Theorem 3.0.28 ([33, Theorem 14.7.5]) *Let E be a barrelled lcs with an unconditional basis $\{x_n\}_n$. The following are equivalent:*

1. $\{x_n\}_n$ *is boundedly complete.*

2. *$(E, \sigma(E, E'))$ is sequentially complete.*

3. *E contains no copy of c_0 and is sequentially complete.*

From the previous theorems 3.0.28 and 3.0.26, we may now conclude:

Corollary 3.0.29 ([33, Corollary 14.7.6]) *If E is a sequentially complete barrelled lcs with unconditional basis which contains neither a copy of c_0 nor a copy of ℓ_1, then E is reflexive.*

Finally, we introduce the concept of frames, which provide basis-like but usually redundant series representations of vectors in a Hilbert space. A sequence $\{x_n\}_n$ in a Hilbert space H is a *frame* for H if there exist constants $A, B > 0$ such that the following *pseudo-Plancherel formula* holds:

$$\text{For every } x \in H, \qquad A\|x\|^2 \leq \sum_n |\langle x, x_n\rangle|^2 \leq B\|x\|^2.$$

The constants A, B are called *frame bounds.* We refer to A as a *lower frame bound*, and to B as an *upper frame bound.* The largest possible lower frame bound is called the *optimal lower frame bound*, and the smallest possible upper frame bound is the *optimal upper frame bound.* $\{x_n\}_n$ is a frame if $\|x\| = \||\{\langle x, x_n\rangle\}_n\||_{\ell^2}$ is an equivalent norm for H, and if it is possible to take $A = B = 1$ then we actually have $\|x\| = \|\{\langle x, x_n\rangle\}_n\|_{\ell^2}$ and in this case we call $\{x_n\}_n$ a *Parseval frame.* Every orthonormal basis is a Parseval frame. A frame is a sequence, not a set, and hence repetitions of elements are allowed. Also, the zero vector is allowed to be an element of a frame. This gives us more trivial examples of frames that are not bases. The *analysis operator* $C : H \to \ell^2$ takes an element x to the sequence of coefficients $Cx = \{\langle x, x_n\rangle\}_n$ and its adjoint $D : \ell^2 \to H$ is the *synthesis operator.* The *frame operator* for $\{x_n\}_n$ is $S = D \circ C : H \to H$.

Frames yield unconditionally convergent, basis-like representations of vectors in a Hilbert space. In the statement of the next result, we use the operator notation $U \leq V$ if and only if $\langle Ux, x\rangle \leq \langle Vx, x\rangle$ for every $x \in H$.

Theorem 3.0.30 ([31, Theorem 8.13]) *Let $\{x_n\}_n$ be a frame for a Hilbert space H with frame bounds A, B. Then the following statements hold.*

1. *The frame operator S is a topological isomorphism of H onto itself, and $AI \leq S \leq BI$.*

2. *S^{-1} is a topological isomorphism, and $B^{-1}I \leq S^{-1} \leq A^{-1}I$.*

3. *$\{S^{-1}x_n\}_n$ is a frame for H with frame bounds B^{-1}, A^{-1}.*

4. *For each $x \in H$,*

$$x = \sum_n \langle x, S^{-1}x_n\rangle x_n = \sum_n \langle x, x_n\rangle S^{-1}x_n,$$

and these series converge unconditionally in the norm of H.

For more details about frames in Hilbert spaces we refer to [31].

Bibliography

[1] Abanin, A. V., and Khoi, L. H. Dual of the function algebra $A^{-\infty}(D)$ and representation of functions in Dirichlet series. *Proc. Amer. Math. Soc. 138*, 10 (2010), 3623–3635.

[2] Abanin, A. V., and Varziev, V. A. Sufficient sets in weighted Fréchet spaces of entire functions. *Sibirsk. Mat. Zh. 54*, 4 (2013), 725–741.

[3] Beanland, K., Freeman, D., and Liu, R. Upper and lower estimates for schauder frames and atomic decompositions. *arXiv:1202.2492v1*.

[4] Berenstein, C. A., and Gay, R. *Complex variables*, vol. 125 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. An introduction.

[5] Berenstein, C. A., and Gay, R. *Complex analysis and special topics in harmonic analysis*. Springer-Verlag, New York, 1995.

[6] Berenstein, C. A., and Li, B. Q. Interpolating varieties for spaces of meromorphic functions. *J. Geom. Anal. 5*, 1 (1995), 1–48.

[7] Berenstein, C. A., Li, B. Q., and Vidras, A. Geometric characterization of interpolating varieties for the (FN)-space A_p^0 of entire functions. *Canad. J. Math. 47*, 1 (1995), 28–43.

[8] Berenstein, C. A., and Taylor, B. A. A new look at interpolation theory for entire functions of one variable. *Adv. in Math. 33*, 2 (1979), 109–143.

[9] Bierstedt, K. D. An introduction to locally convex inductive limits. In *Functional analysis and its applications (Nice, 1986)*, ICPAM Lecture Notes. World Sci. Publishing, Singapore, 1988, pp. 35–133.

[10] Bonet, J. The bounded approximation property in fréchet spaces. Seminar delivered at the IUMPA, UPV (Valencia), 2012, pp. 35–133.

[11] Bonet, J., and Domański, P. Sampling sets and sufficient sets for $A^{-\infty}$. *J. Math. Anal. Appl. 277*, 2 (2003), 651–669.

[12] Bonet, J., Fernández, C., Galbis, A., and Ribera, J. M. Frames and representing systems in Fréchet spaces and their duals. Preprint (2014), available at `http://arxiv.org/abs/1411.3500`.

[13] Bonet, J., Fernández, C., Galbis, A., and Ribera, J. M. Shrinking and boundedly complete Schauder frames in Fréchet spaces. *J. Math. Anal. Appl. 410*, 2 (2014), 953–966.

[14] Bonet, J., and Lindström, M. Convergent sequences in duals of Fréchet spaces. In *Functional analysis (Essen, 1991)*, vol. 150 of *Lecture Notes in Pure and Appl. Math.* Dekker, New York, 1994, pp. 391–404.

[15] Bonet, J., Lindström, M., and Valdivia, M. Two theorems of Josefson-Nissenzweig type for Fréchet spaces. *Proc. Amer. Math. Soc. 117*, 2 (1993), 363–364.

[16] Carando, D., and Lassalle, S. Duality, reflexivity and atomic decompositions in Banach spaces. *Studia Math. 191*, 1 (2009), 67–80.

[17] Carando, D., Lassalle, S., and Schmidberg, P. The reconstruction formula for Banach frames and duality. *J. Approx. Theory 163*, 5 (2011), 640–651.

[18] Casazza, P. G., Christensen, O., and Stoeva, D. T. Frame expansions in separable Banach spaces. *J. Math. Anal. Appl. 307*, 2 (2005), 710–723.

[19] Casazza, P. G., Dilworth, S. J., Odell, E., Schlumprecht, T., and Zsák, A. Coefficient quantization for frames in Banach spaces. *J. Math. Anal. Appl. 348*, 1 (2008), 66–86.

[20] Casazza, P. G., Han, D., and Larson, D. R. Frames for Banach spaces. In *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)*, vol. 247 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 149–182.

[21] Díaz, J. C. Montel subspaces in the countable projective limits of $L^P(\mu)$-spaces. *Canad. Math. Bull. 32*, 2 (1989), 169–176.

[22] Ehrenpreis, L. *Fourier analysis in several complex variables.* Pure and Applied Mathematics, Vol. XVII. Wiley-Interscience Publishers A Division of John Wiley & Sons, New York-London-Sydney, 1970.

[23] Fabian, M., Habala, P., Hájek, P., Montesinos, V., and Zizler, V. *Banach space theory.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011. The basis for linear and nonlinear analysis.

[24] Feichtinger, H. G. Atomic characterizations of modulation spaces through Gabor-type representations. *Rocky Mountain J. Math. 19*, 1 (1989), 113–125. Constructive Function Theory—86 Conference (Edmonton, AB, 1986).

[25] Feichtinger, H. G., and Gröchenig, K. H. Banach spaces related to integrable group representations and their atomic decompositions i. *J. Funct. Anal. 86*, 2 (1989), 307–340.

[26] Feichtinger, H. G., and Gröchenig, K. H. Banach spaces related to integrable group representations and their atomic decompositions ii. *Monatsh. Math. 108*, 2-3 (1989), 129–148.

[27] Fernández, C. Regularity conditions on (LF)-spaces. *Arch. Math. (Basel) 54*, 4 (1990), 380–383.

[28] Frerick, L. Extension operators for spaces of infinite differentiable Whitney jets. *J. Reine Angew. Math. 602* (2007), 123–154.

[29] Garnir, H. G., De Wilde, M., and Schmets, J. *Analyse fonctionnelle. Théorie constructive des espaces linéaires à semi-normes. Tome I: Théorie générale.* Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, Band 36. Birkhäuser Verlag, Basel, 1968.

[30] Gröchenig, K. *Foundations of time-frequency analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2001.

[31] Heil, C. *A basis theory primer*, expanded ed. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2011.

[32] Horváth, J. *Topological vector spaces and distributions. Vol. I.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[33] Jarchow, H. *Locally convex spaces.* B. G. Teubner, Stuttgart, 1981. Mathematische Leitfäden. [Mathematical Textbooks].

[34] Kadets, V. M., and Korobeĭnik, Y. F. Representing and absolutely representing systems. *Studia Math. 102*, 3 (1992), 217–223.

[35] Kelley, J. L., and Namioka, I. *Linear topological spaces.* With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, Kennan T.

Smith. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J., 1963.

[36] Khôi, L. H., and Thomas, P. J. Weakly sufficient sets for $A^{-\infty}(\mathbf{D})$. *Publ. Mat. 42*, 2 (1998), 435–448.

[37] Korobeĭnik, Y. F. Representative systems. *Uspekhi Mat. Nauk 36*, 1(217) (1981), 73–126, 248.

[38] Korobeĭnik, Y. F. Inductive and projective topologies. Sufficient sets and representative systems. *Izv. Akad. Nauk SSSR Ser. Mat. 50*, 3 (1986), 539–565, 638.

[39] Korobeĭnik, Y. F. On absolutely representing systems in spaces of infinitely differentiable functions. *Studia Math. 139*, 2 (2000), 175–188.

[40] Korobeĭnik, Y. F., and Melikhov, S. N. A linear continuous right inverse operator for the representation operator, and applications to convolution operators. *Siberian Math. J. 34*, 1 (1993), 59–72.

[41] Köthe, G. *Topological vector spaces. I.* Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159. Springer-Verlag New York Inc., New York, 1969.

[42] Köthe, G. *Topological vector spaces. II*, vol. 237 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York, 1979.

[43] Liu, R. On shrinking and boundedly complete Schauder frames of Banach spaces. *J. Math. Anal. Appl. 365*, 1 (2010), 385–398.

[44] Matyszczyk, C. Approximation of analytic and continuous mappings by polynomials in Fréchet spaces. *Studia Math. 60*, 3 (1977), 223–238.

[45] McArthur, C. W., and Retherford, J. R. Some applications of an inequality in locally convex spaces. *Trans. Amer. Math. Soc. 137* (1969), 115–123.

[46] Meise, R. Sequence space representations for (DFN)-algebras of entire functions modulo closed ideals. *J. Reine Angew. Math. 363* (1985), 59–95.

[47] Meise, R., and Taylor, B. A. Sequence space representations for (FN)-algebras of entire functions modulo closed ideals. *Studia Math. 85*, 3 (1987), 203–227.

[48] MEISE, R., AND VOGT, D. *Introduction to functional analysis*, vol. 2 of *Oxford Graduate Texts in Mathematics*. The Clarendon Press Oxford University Press, New York, 1997. Translated from the German by M. S. Ramanujan and revised by the authors.

[49] MELIKHOV, S. N. (DFS)-spaces of holomorphic functions invariant under differentiation. *J. Math. Anal. Appl. 297*, 2 (2004), 577–586. Special issue dedicated to John Horváth.

[50] NAPALKOV, V. V. Comparison of topologies in some spaces of entire functions. *Dokl. Akad. Nauk SSSR 264*, 4 (1982), 827–830.

[51] NEUS, H. Über die Regularitätsbegriffe induktiver lokalkonvexer Sequenzen. *Manuscripta Math. 25*, 2 (1978), 135–145.

[52] PEŁCZYŃSKI, A. Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis. *Studia Math. 40* (1971), 239–243.

[53] PÉREZ, P., AND BONET, J. *Barrelled locally convex spaces*, vol. 131 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1987. Notas de Matemática [Mathematical Notes], 113.

[54] PILIPOVIĆ, S., STOEVA, D., AND TEOFANOV, N. Frames for Fréchet spaces. *Bull. Cl. Sci. Math. Nat. Sci. Math.*, 32 (2007), 69–84.

[55] PILIPOVIĆ, S., AND STOEVA, D. T. Series expansions in Fréchet spaces and their duals, construction of Fréchet frames. *J. Approx. Theory 163*, 11 (2011), 1729–1747.

[56] PILIPOVIĆ, S., AND STOEVA, D. T. Fréchet frames, general definition and expansions. *Anal. Appl. (Singap.) 12*, 2 (2014), 195–208.

[57] QINGYING, B., AND RONGLU, L. Locally convex spaces containing no copy of c_0. *J. Math. Anal. Appl. 172*, 2 (1993), 205–211.

[58] ROBERTSON, A. P., AND ROBERTSON, W. *Topological vector spaces*, second ed. Cambridge University Press, London, 1973. Cambridge Tracts in Mathematics and Mathematical Physics, No. 53.

[59] RUDIN, W. *Functional analysis*. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973. McGraw-Hill Series in Higher Mathematics.

[60] SCHNEIDER, D. M. Sufficient sets for some spaces of entire functions. *Trans. Amer. Math. Soc. 197* (1974), 161–180.

[61] Seip, K., and Wallstén, R. Density theorems for sampling and interpolation in the Bargmann-Fock space. II. *J. Reine Angew. Math. 429* (1992), 107–113.

[62] Taskinen, J. A Fréchet-Schwartz space with basis having a complemented subspace without basis. *Proc. Amer. Math. Soc. 113*, 1 (1991), 151–155.

[63] Taskinen, J. Atomic decomposition of a weighted inductive limit. *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 97*, 2 (2003), 325–337.

[64] Taylor, B. A. Atomic decomposition of a weighted inductive limit. *Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966)* (1968), 431–467.

[65] Taylor, B. A. Discrete sufficient sets for some spaces of entire functions. *Trans. Amer. Math. Soc. 163* (1972), 207–214.

[66] Vogt, D. An example of a nuclear Fréchet space without the bounded approximation property. *Math. Z. 182*, 2 (1983), 265–267.

[67] Wengenroth, J. *Derived functors in functional analysis*, vol. 1810 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2003.