APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 16, no. 1(2015), 37-44
doi:10.4995/agt.2015.2305

Nonself KKM maps and corresponding theorems in Hadamard manifolds

Parin Chaipunya and Poom Kumam*

Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.
(parin.cha@mail.kmutt.ac.th, poom.kum@kmutt.ac.th)

Abstract

In this paper, we consider the KKM maps defined for a nonself map and the correlated intersection theorems in Hadamard manifolds. We also study some applications of the intersection results. Our outputs improved the results of Raj and Somasundaram [17, V. Sankar Raj and S. Somasundaram, KKM-type theorems for best proximity points, *Appl. Math. Lett.*, 25(3):496–499, 2012.].

2010 MSC: *53C22; 53C25; 47H04.*

Keywords: *KKM Maps; Hadamard manifolds; generalized equilibrium problems; best proximity points.*

1. Introduction

The KKM theory, as the term coined by Park [15], is the study of the equivalent formulations, variants, and extensions to the 1929 geometric result due to Knaster, Kuratowski, and Mazurkiewicz. This result is known nowadays as the KKM lemma, and it provides a firm foundation to many different areas of mathematics, *e.g.,* fixed point theory, minimax theory, game theory, variational inequality, equilibrium theory, and henceforth. This lemma is also known for being equivalent to both the Brouwer's fixed point theorem and the Sperner's lemma (see [16] for further discussions). One of the most important

*Corresponding author

Received 19 April 2014 – Accepted 3 October 2014

enhancement of the KKM lemma is due to Fan [7], whose result is obtained in a topological vector space.

In [17], the nonself KKM maps have been introduced and studied under the framework of a normed linear space. As naturally occurs, the best proximity point theorem is deduced in relation to the nonself KKM lemma.

On the other hand, Colao *et al.* [6] proved the KKM lemma in a Hadamard manifold, as an auxiliary tool for proving several results on the existence of solutions to equilibrium problems. Also, the fixed point, variational inequality and Nash equilibrium are investigated by the authors.

In this paper, we occupy the nonself KKM lemma in Hadamard manifolds. The nonself version of the Browder's fixed point theorem as well as the solvability of a generalized equilibrium problem are studied, as applications of our KKM lemma.

2. Preliminaries

Recall first that a Hadamard manifold M is a complete simply-connected smooth Riemannian manifold whose sectional curvature is non-positive. At each point $x \in M$, we write T_xM to represent the tangent plane at x, which is at the same time a manifold.

With this structure, we can define an exponential map $\exp_p : T_pM \to M$ by $\exp_p(\nu) := \gamma_\nu(1)$, where γ_ν is a geodesic defined by its position p and velocity ν at p. Recall that exponential maps are diffeomorphisms.

The exponential maps allow us to characterize the minimal geodesic joining a point p to another point q by the function $t \mapsto \exp_p(t \exp_p^{-1}(q))$, with $t \in [0,1]$. Naturally, a subset $K \subset M$ is said to be geodesically convex if minimal geodesics correspond to each of its elements are contained in K. For any nonempty subset $A \subset M$, denoted by $\mathrm{co}(A)$ the geodesically convex hull of A, *i.e.,* the smallest geodesically convex set containing A. Note that the geodesically convex hull of any finite subset is compact. Moreover, the geodesic distance $d(p,q)$ between two points $p, q \in M$ defined the length of its minimal geodesic induces the original topology of M.

A real function $f : M \to \mathbb{R}$ is said to be geodesically convex if the composition $f \circ \gamma$ is convex (in ordinary sense), provided that γ is the minimal geodesic joining two arbitrary points in M. In particular, the geodesic distance is geodesically convex in both of its arguments.

Referring to [18], Hadamard manifolds behave nicely with probability measures defined on them. Let $\mathscr{P}(M)$ be the collection of probability measures μ on M whose supports are separable and $\int_M d(x,y)\mathrm{d}\mu(y) < \infty$ for every[1] $x \in M$. Then, to each $\mu \in \mathscr{P}(M)$ and $y \in M$, we associate a point $z_* \in M$ that minimizes the (uniformly) geodesically convex function $z \mapsto \int_M [d^2(z,x) - d^2(y,x)]\mathrm{d}\mu(x)$. Such point z_* is independent of $y \in M$, so we prefer writing

[1] The quantifier 'for some' is used in some texts. They are however identical as one can deduce from the triangle inequality.

$b(\mu)$ in place of z_*. Moreover, we say that it is the barycenter of μ. If $\mathrm{supp}(\mu)$ is contained in some closed geodesically convex set K, it is the case that $b(\mu) \in K$.

We can make $\mathscr{P}(M)$ into a metric space by endowing it with the Wasserstein metric given by:

$$d^W(\mu,\nu) := \inf \iint_{M\times M} d(x,y)\mathrm{d}\lambda(x,y), \quad \forall \mu,\nu \in \mathscr{P}(M),$$

where the infimum is taken over $\lambda \in \mathscr{P}(M\times M)$ whose marginals are μ and ν. With respect to this metric, the map $\mu \mapsto b(\mu)$ is nonexpansive.

3. Nonself KKM Maps

The pair (A,B) set up by two given nonempty subsets A and B of a metric space (S,d) is called a *proximal pair* if to each point $(x,y) \in A\times B$, there corresponds a point $(\bar{x},\bar{y}) \in A\times B$ such that

$$d(x,\bar{y}) = d(\bar{x},y) = \mathrm{dist}(A,B),$$

where $\mathrm{dist}(A,B) := \inf\{d(x,y), x\in A, y\in B\}$. In addition, if both A and B are convex, we say that (A,B) is a *convex proximal pair.*

In the future contents, we assume that M is a Hadamard manifold with the geodesic distance d. Given a point $x \in M$ and two nonempty subsets $A,B \subset M$, we write $d(x,A) := \inf_{z\in A} d(x,z)$.

Definition 3.1. Let (A,B) be a proximal pair in a Hadamard manifold M. A nonself map $T : A \rightrightarrows B$ is said to be KKM if for each finite subset $D := \{x_1,x_2,\cdots,x_m\} \subset A$, there is a subset $E := \{y_1,y_2,\cdots,y_m\} \subset B$ such that $d(x_i,y_i) = \mathrm{dist}(A,B)$, $\forall i \in \{1,2,\cdots,m\}$, and

$$\mathrm{co}(\{y_i, i\in I\}) \subset T(\{x_i, i\in I\})$$

for every $\varnothing \neq I \subset \{1,2,\cdots,m\}$.

Theorem 3.2. *Suppose that (A,B) is a proximal pair in a Hadamard manifold M and $T : A \rightrightarrows B$ is a KKM map with nonempty closed values. Then, the family $\{T(x), x\in A\}$ has the finite intersection property.*

Proof. Assume to the contrary that there is a finite subset $D := \{x_1,x_2,\cdots,x_m\} \subset A$ such that $\bigcap_{x\in D} T(x) = \varnothing$. Since T is KKM, we can find a subset $E := \{y_1,y_2,\cdots,y_m\} \subset B$ so that

$$\mathrm{co}(\{x_j, j\in J\}) \subset T(\{y_j, j\in J\}),$$

for every $\varnothing \neq J \subset \{1,2,\cdots,m\}$. Set $K := \mathrm{co}(E)$. Define a function $\lambda : K \to \mathbb{R}$ by

$$\lambda(y) := \sum_{i=1}^{m} d(y, K\cap T(x_i)), \quad \forall y \in K.$$

At each $y \in K$, we have $\lambda(y) > 0$ since $\bigcap_{x\in D} T(x) = \varnothing$. Then, the map

$$y \in K \mapsto \mu_y := \sum_{i=1}^{m} \left[\frac{d(y, K\cap T(x_i))\delta_{y_i}}{\lambda(y)}\right],$$

where δ_u is the Dirac probability measure corresponding to $u \in K$, is continuous (from K into $\mathscr{P}(K)$). Thus, the composition $y \mapsto \mu_y \mapsto b(\mu_y)$ is continuous from K into itself, and it therefore has a fixed point $y_0 \in K$ (see [11]).

Take $J := \{j \in \{1, 2, \cdots, m\}, d(y_0, K \cap T(x_j)) > 0\}$. It is immediate that $y_0 \notin \bigcup_{j\in J} T(x_j)$. As a matter of fact, we have $\operatorname{supp}(\mu_{y_0}) \subset \operatorname{co}(\{x_j, j \in J\})$ which implies that $y_0 = b(\mu_{y_0}) \in \operatorname{co}(\{x_j, j \in J\}) \subset \bigcup_{j\in J} T(x_j)$, a contradiction. Therefore, the family $\{T(x), x \in A\}$ must possess the finite intersection property. □

Theorem 3.3. *Suppose that (A, B) is a proximal pair in a Hadamard manifold M and $T : A \rightrightarrows B$ is a KKM map with nonempty closed values. If $T(x_0)$ is compact at some $x_0 \in A$, then the intersection $\bigcap\{T(x), x \in A\}$ is nonempty.*

Proof. By Theorem 3.2, we know that $T(x_0) \cap T(x)$ is nonempty and closed for all $x \in A$. Moreover, the family $\{T(x_0) \cap T(x), x \in A\}$ has the f.i.p. The conclusion follows as $T(x_0)$ is compact. □

Remark 3.4. With the same proofs, theorems presented above can also be extended to CAT(0) spaces, but with an additional assumption that every continuous map from a compact convex subset of M into itself has a fixed point. In particular, if A and B are identical, we can obtain KKM results as of [6, 12]

4. Some Applications

We observe here some applications of our results in the previous section. Before we go into the main subjects, let us observe the following fact about convex hulls of finitely many points between a convex proximal pair.

Lemma 4.1. *Let (A, B) be a convex proximal pair of a Hadamard manifold M. Assume that $x_1, x_2, \cdots, x_m \in A$ and $y_1, y_2, \cdots, y_m \in B$ are points such that*

$$d(x_i, y_i) = \operatorname{dist}(A, B), \quad \forall i \in \{1, 2, \cdots, m\}.$$

Then, $(\operatorname{co}(\{x_1, x_2, \cdots, x_m\}), \operatorname{co}(\{y_1, y_2, \cdots, y_m\}))$ is a proximal pair with

$$\operatorname{dist}(\operatorname{co}(\{x_1, x_2, \cdots, x_m\}), \operatorname{co}(\{y_1, y_2, \cdots, y_m\})) = \operatorname{dist}(A, B). \tag{4.1}$$

Proof. The equity (4.1) is obvious, so let us prove the former part. Let us write $C_1 := \{x_1, x_2, \cdots, y_m\}$, and for $j \geq 2$, let C_j be the union of minimal geodesics that join pairs of points in C_{j-1}. In the same way, we let $D_1 := \{y_1, y_2, \cdots, y_m\}$, and for $j \geq 2$, let D_j be the union of minimal geodesics that join pairs of points in D_{j-1}. One can simply show, by using mathematical induction, that (C_j, D_j) is a proximal pair for all $j \geq 1$. Now, apply [14, Proposition 2.5.5] to complete the proof. □

4.1. **Generalized Equilibrium Problems.** Given a nonempty set Q and a bifunction $\psi : Q \times Q \to \mathbb{R}$, the equilibrium problem concerns the existence (and the determination) of a point $\bar{x} \in Q$ that makes $\psi(\bar{x}, \cdot)$ into a non-negative function.

This equilibrium problem is first considered by Fan [7, 8] under Euclidean spaces. It is then improved and enriched in [3]. As it unifies many problems in optimization, for examples, minimization problem, variational inequality, minimax inequality, and Nash equilibrium problem, the equilibrium theory gained its fame very quickly. Consult [10, 13, 2, 5, 9, 1] for richer details.

In this section, we shall consider the case where the bifunction ψ is defined on the product $P \times Q$, with P, Q being nonempty and possibly distinct sets. This leads to a more general aspect of equilibrium problems.

Theorem 4.2. *Suppose that (P, Q) is a geodesically convex proximal pair in a Hadamard manifold M, and $\psi : P \times Q \to \mathbb{R}$ is a bifunction. Assume that the following conditions hold:*

(i) $\psi(x, y) \geq 0$ *provided* $x \in P$, $y \in Q$, *and* $d(x, y) = \operatorname{dist}(P, Q)$,
(ii) $\forall x \in P$, *the set* $\{y \in Q, \psi(x, y) < 0\}$ *is geodesically convex,*
(iii) $\forall y \in Q$, *the function* $\psi(\cdot, y)$ *is u.s.c.,*
(iv) there exists a nonempty compact set $L \subset M$ *such that both* $L \cap P$ *and* $L \cap Q$ *are nonempty and*

$$\psi(x, \bar{y}) < 0, \quad \forall x \in P \setminus L,$$

for some point $\bar{y} \in L \cap Q$.

Then, there exists a point $\bar{x} \in L \cap P$ *such that*

$$\psi(\bar{x}, y) \geq 0, \quad \forall y \in Q.$$

Proof. Define a map $G : Q \rightrightarrows P$ by

$$G(y) := \{x \in P, \psi(x, y) \geq 0\}, \quad \forall y \in Q.$$

Since $\psi(\cdot, y)$ is u.s.c., $G(y)$ is closed for each $y \in Q$. From (iv), we have $G(\bar{y}) \subset L$ and so $G(\bar{y})$ is compact.

We shall prove next that G is a KKM map. Suppose that $\{y_1, y_2, \cdots, y_m\} \subset Q$ and $\{x_1, x_2, \cdots, x_m\} \subset P$ such that $d(x_i, y_i) = \operatorname{dist}(P, Q)$, $\forall i \in \{1, 2, \cdots, m\}$. Let us assume to the contrary that there exists a subset $\varnothing \neq J \subset \{1, 2, \cdots, m\}$ and a point $x_0 \in \operatorname{co}(\{x_j, j \in J\})$ such that $x_0 \notin G(\{y_j, j \in J\})$. Equivalently, $\psi(x_0, y_j) < 0$, $\forall j \in J$. By Lemma 4.1, we can choose a point $y_0 \in \operatorname{co}(\{y_j, j \in J\})$ with $d(x_0, y_0) = \operatorname{dist}(P, Q)$. Note that $y_j \in \{y \in Q, \psi(x_0, y) < 0\}$ for each $j \in J$ and $\{y \in Q, \psi(x_0, y) < 0\}$ is geodesically convex. Therefore, we have $y_0 \in \operatorname{co}(\{y_j, j \in J\}) \subset \{y \in Q, \psi(x_0, y) < 0\}$, which contradicts the hypothesis (i). Hence, G is a KKM map, and the desired result follows immediately from the construction of G. □

Remark 4.3. If $P = Q$, then the condition (i) reads as follows:

(i') $\psi(x, x) \geq 0, \quad \forall x \in P$,

where it is always assumed in classical equilibrium theory.

4.2. **Best Proximity Points.** Suppose that (S, d) is a metric space and $A \subset S$ is nonempty. Given a map $T : A \rightrightarrows S$, a point $x_0 \in A$ is a *fixed point* of F if $x_0 \in T(x_0)$. In particular, if T is closed valued, a fixed point is expressed metrically by $d(x_0, T(x_0)) = 0$.

Suppose that $B \subset S$ is nonempty. Then, it may be the case that the map $T : A \rightrightarrows B$ does not have a fixed point. In fact, it is evident that $d(x, T(x)) \geq \operatorname{dist}(A, B)$ for all $x \in A$. In this case, instead of fixed points, we can consider the best proximity point $x_0 \in A$, *i.e.,* the point such that $d(x_0, T(x_0)) = \operatorname{dist}(A, B)$.

The notion of best proximity point is stronger than the best approximation. In details, if T is single-valued and x_0 is a best proximity point of T, then $T(x_0)$ is a best approximant to x_0 for B.

Now, we state our nonself version of Browder's fixed point theorem [4] in the setting of Hadamard manifolds.

Theorem 4.4. *Let (A, B) be a proximal pair of a Hadamard manifold M, where A is assumed to be compact and geodesically convex, and $T : A \rightrightarrows B$ be a map such that*

(i) $\forall x \in A$, $T(x)$ is nonempty and geodesically convex,
(ii) T is an open fibre map, i.e., *$\forall y \in B$, the inverse image*

$$T^{-1}(y) := \{x \in A, y \in T(x)\}$$

is open.

Then, there exists a point $\bar{x} \in A$ such that $d(\bar{x}, T(\bar{x})) = \operatorname{dist}(A, B)$.

Proof. Consider the dual map $G : B \rightrightarrows A$ defined by

$$G(y) := A \setminus T^{-1}(y), \quad \forall y \in B.$$

If $G(y_0) = \varnothing$ for some $y_0 \in B$, *i.e.,* $T^{-1}(y_0) = A$. This means $y_0 \in T(x)$ for all $x \in A$. Now, since (A, B) is a proximal pair, we can find a point $\bar{x} \in A$ such that $d(\bar{x}, y_0) = \operatorname{dist}(A, B)$. In particular, $y_0 \in T(\bar{x})$. Thus, we have

$$d(\bar{x}, T(\bar{x})) \leq d(\bar{x}, y_0) = \operatorname{dist}(A, B),$$

yielding the desired result.

On the other hand, suppose that $G(y)$ is nonempty for every $y \in B$. Moreover, G is closed valued. Observe that

$$\bigcap_{y \in B} G(y) = \bigcap_{y \in B} (A \setminus T^{-1}(y)) = A \setminus \bigcup_{y \in B} T^{-1}(y).$$

Since $\{T^{-1}(y), y \in B\}$ is an open cover of A, we obtain from the above equality that $\bigcap_{y \in B} G(y)$ is empty. We conclude from Theorem 3.3 that G is not a KKM map. Thus, suppose that $\{y_1, y_2, \cdots, y_m\} \subset B$ and $\{x_1, x_2, \cdots, x_m\} \subset A$ are sets such that $d(x_i, y_i) = \operatorname{dist}(A, B)$, $\forall i \in \{1, 2, \cdots, m\}$ and $\operatorname{co}(\{x_1, x_2, \cdots, x_m\})$ is not contained in $G(\{y_1, y_2, \cdots, y_m\})$.

In particular, choose $\bar{x} \in \operatorname{co}(\{x_1, x_2, \cdots, x_m\})$ such that $\bar{x} \notin G(\{y_1, y_2, \cdots, y_m\})$. Hence, $\bar{x} \in T^{-1}(y_i)$ for all $i \in \{1, 2, \cdots, m\}$, or equivalently, $y_i \in T(\bar{x})$ for all $i \in \{1, 2, \cdots, m\}$. According to Lemma 4.1, we can choose a point

$z \in \text{co}(\{y_1, y_2, \cdots, y_m\})$ with $d(\bar{x}, z) = \text{dist}(A, B)$. Since $T(\bar{x})$ is convex, we get $z \in \text{co}(\{y_1, y_2, \cdots, y_m\}) \subset T(\bar{x})$. We have again

$$d(\bar{x}, T(\bar{x})) \leq d(\bar{x}, z) = \text{dist}(A, B),$$

which leads to the desired result. □

In case A and B are identical, we have the following variant of Browder's theorem in the setting of Hadamard manifolds.

Theorem 4.5. *Let K be a nonempty, compact, and geodesically convex subset of a Hadamard manifold M and $T : K \rightrightarrows K$ an open fibre map whose values are nonempty and geodesically convex. Then, T has a fixed point.*

Conclusion

We have proved the intersection theorem for nonself KKM maps in Hadamard manifolds, which extends the existed result of [17]. We have also provide some applications of our intersection result towards the existence of an equilibrium point and a best proximity point.

Acknowledgements. *The first author is supported by the Thailand Research Fund through the Royal Golden Jubilee Ph.D. Program (Grant No. PHD/0045/2555) and the King Mongkut's University of Technology Thonburi under the RGJ-Ph.D. scholarship. Moreover, we would like to gratefully thank the anonymous referees for their suggestions, which improve this paper significantly.*

References

[1] M. Bianchi and S. Schaible, *Generalized monotone bifunctions and equilibrium problems,* J. Optimization Theory Appl. 90, no. 1 (1996), 31–43.

[2] G. Bigi, A. Capătă and G. Kassay, *Existence results for strong vector equilibrium problems and their applications,* Optimization 61, no. 4-6 (2012), 567–583.

[3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, The Mathematics Student 63, no. 1-4 (1994), 123–145.

[4] F. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Mathematische Annalen 177, no. 4 (1968), 283–301.

[5] M. Castellani and M. Giuli, *On equivalent equilibrium problems,* J. Optim. Theory Appl. 147, no. 1 (2010), 157–168.

[6] V. Colao, G. López, G. Marino and V. Martín-Márquez, *Equilibrium problems in Hadamard manifolds,* J. Math. Anal. Appl. 388, no. 1 (2012), 61–77.

[7] K. Fan, *A generalization of Tychonoff's fixed point theorem,* Math. Ann. 142 (1961), 305–310.

[8] K. Fan, *A minimax inequality and applications*, Inequalities, III (Proceedings of the Third Symposium, University of California, Los Angeles, CA, 1969; dedicated to the memory of Theodore S. Motzkin), pages 103–113, 1972.

[9] A. N. Iusem and W. Sosa, *New existence results for equilibrium problems,* Nonlinear Anal. Theory Methods Appl. 52, no. 2 (2003), 621–635.

[10] L.-J. Lin and H. I. Chen, *Coincidence theorems for families of multimaps and their applications to equilibrium problems,* Abstr. Appl. Anal. 2003, no. 5 (2003), 295–309.

[11] S. Németh, *Variational inequalities on Hadamard manifolds*, Nonlinear Anal. Theory Methods Appl. 52, no. 5 (2003), 1491–1498.

[12] C. P. Niculescu and I. Rovenţa, *Fan's inequality in geodesic spaces*, Applied Mathematics Letters 22, no. 10 (2009), 1529–1533.

[13] W. Oettli, *A remark on vector-valued equilibria and generalized monotonicity,* Acta Math. Vietnam. 22, no. 1 (1997), 213–221.

[14] A. Papadopoulos, Metric Spaces, Convexity and Nonpositive Curvature, IRMA lectures in mathematics and theoretical physics, European Mathematical Society, 2005.

[15] S. Park, *Some coincidence theorems on acyclic multifunctions and applications to kkm theory*, Fixed Point Theory and Applications (1992), pp. 248–277.

[16] S. Park, *Ninety years of the brouwer fixed point theorem*, Vietnam J. Math. 27 (1997), 187–222.

[17] V. Sankar Raj and S. Somasundaram, *KKM-type theorems for best proximity points,* Appl. Math. Lett. 25, no. 3 (2012), 496–499.

[18] K.-T. Sturm, *Probability measures on metric spaces of nonpositive curvature,* In: Heat kernels and analysis on manifolds, graphs, and metric spaces. Lecture notes from a quarter program on heat kernels, random walks, and analysis on manifolds and graphs, April 16–July 13, 2002, Paris, France, pp. 357–390. Providence, RI: American Mathematical Society (AMS), 2003.