APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 16, no. 1(2015), 45-52
doi:10.4995/agt.2015.3006

On faint continuity

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Abstract

Recently the class of strongly faintly α-continuous functions between topological spaces has been defined and studied in some detail. We consider this class of functions from the perspective of change(s) of topology. In particular, we conclude that each member of this class of functions belongs the usual class of continuous functions between topological spaces when the domain and codomain of the function in question have been retopologized appropriately. Some consequences of this fact are considered in this paper.

2010 MSC: *06A06; 54H10.*

Keywords: *Change of topology; strongly faint α-continuity; faint α-continuity.*

1. Introduction

There can be no argument that the concept of continuity is one of the most important ideas in the whole of mathematics. So much so that it has become fashionable to speak of ‘continuous mathematics’ and ‘discrete mathematics’ as a fundamental division of mathematics, rather than the more traditional ‘pure mathematics’ and ‘applied mathematics’. Over recent decades, many generalizations and variants of continuous functions between topological spaces have been introduced. In 1982, Long and Herrington [7] considered the class of *faintly continuous* functions. Three weak forms of faint continuity were introduced by Noiri and Popa [15]. More recently, Nasef and Noiri [12] have defined and studied three strong forms of faint continuity under the names of *strongly faint semi-continuity, strongly faint precontinuity* and *strongly faint*

Received 26 May 2014 – Accepted 4 December 2014

β*-continuity*. Very recently, Nasef [11] has introduced two more strong forms of faint continuity using the terms *strongly faint* α*-continuity* and *strongly faint* γ*-continuity*, and has provided arguments for their significance outside of mathematics.

Nasef [11] takes care to distinguish between strongly faint α-continuity and other variants of continuity, especially in the diagram of Remark 4.1 and the set of Examples 4.1 to 4.5. Our primary purpose is to argue that strongly faint α-continuity is merely continuity in disguise. Indeed, if the domain and codomain spaces of a strongly faintly α-continuous function f are each re-topologized in a suitable fashion (see Proposition 3.1), then f is simply a continuous function. This observation puts the notion of strongly faint α-continuity in a more natural context, and it permits alternative proofs of most of the results of Nasef [11]. In the language of category theory, we are claiming that a strongly faintly α-continuous function f arises because the wrong source and target have been chosen for the morphism f in the category Top of topological spaces and continuous functions.

In Section 2, we provide the relevant definitions of the classes of functions we consider in this paper. In particular, we examine the basic properties of α-topologies and θ-topologies. Section 3 investigates the class of strongly faintly α-continuous functions introduced by Nasef [11], especially from the perspective of change of topology. Section 4 provides a decomposition of faint α-continuity.

Our notation and terminology are standard; see for example Dugundji [3]. No separation properties are assumed for topological spaces unless explicitly stated. We denote the interior of a subset A of the topological space (X, τ) by $\mathsf{int}A$, and the closure by $\mathsf{cl}A$.

2. Definitions and Preliminaries

Definition 2.1. A subset A of a topological space (X, τ) is said to be

(i) α-open if $A \subseteq \mathsf{int}(\mathsf{cl}(\mathsf{int}A))$
(ii) semi-open if $A \subseteq \mathsf{cl}(\mathsf{int}A)$
(iii) preopen if $A \subseteq \mathsf{int}(\mathsf{cl}A)$
(iv) γ-open if $A \subseteq \mathsf{cl}(\mathsf{int}A) \cup \mathsf{int}(\mathsf{cl}A)$
(v) θ-open if for each $x \in A$ there exists an open set U such that $x \in U \subseteq \mathsf{cl}U \subseteq A$.

The family of all α-open (respectively semi-open, pre-open, γ-open) subsets of X is denoted by τ^α (respectively $SO(X)$, $PO(X)$, $\gamma O(X)$). The complement of a θ-open (respectively γ-open, α-open) set is said to be θ-closed (respectively γ-closed, α-closed).

Njastad [14] defined α-open subsets in 1965 and he showed that for any topological space (X, τ), the collection τ^α of all α-open subsets of (X, τ) is a topology on X larger than τ. In 1963, Levine [5] introduced semi-open sets. The term *preopen subset* was introduced by Mashhour, Abd El-Monsef and El-Deeb [10] but the concept had appeared much earlier. For example, Carson and Michael [2] used the term *locally dense* for preopen sets in 1964. The family of γ-open subsets of (X,τ) was studied by Andrijević [1] in 1996 under the name of *b-open* subsets. Velicko [17] defined θ-open subsets and showed that the collection of all θ-open subsets of (X,τ) forms a topology τ_θ on X which is smaller than τ. Thus for any topological space (X,τ), we have $\tau_\theta \subset \tau \subset \tau^\alpha$.

There is one very significant difference between the families of α-open subsets and γ-open subsets of a topological space (X,τ). This is that τ^α is always a topology on X, while $\gamma O(X)$ is not a topology on X in general. Consider the following example:

Example 2.2. Let X be the set of real numbers $\mathbb{R}$, and let τ be the usual (Euclidean) topology on X. If $\mathbb{Q}$ is the subset of all rational numbers, then $\mathbb{Q}$ is γ-open. Furthermore, if $A = [0,1)$, the half-open unit interval, then A is γ-open. However $B = A \cap \mathbb{Q}$ is not γ-open, since $0 \in B$ but $0 \notin \mathsf{cl}(\mathsf{int}B) \cup \mathsf{int}(\mathsf{cl}B)$. Thus $\gamma O(X,\tau)$ is not closed under finite intersection, and so $\gamma O(X,\tau)$ is not a topology on X.

Despite the comment of Nasef [11, page 540] immediately preceding his Theorem 4.4, the collection of all γ-open sets of (Y,σ) does not form a topology on Y. Andrijevic [1] showed no such result. He showed that there is a topology on Y generated in a natural way by the family $\gamma O(Y,\sigma)$. In fact, this topology is $\{V \subset Y : V \cap S \in \gamma O(Y,\sigma) \text{ whenever } S \in \gamma O(Y,\sigma)\}$.

Definition 2.3. A function $f : X \to Y$ is said to be *faintly continuous* (respectively faintly semi-continuous, faintly precontinuous, faintly α-continuous) if for each $x \in X$ and each θ-open set V containing $f(x)$ there exists an open (respectively semi-open, preopen, α-open) set U containing x such that $f(U) \subseteq V$.

Definition 2.4. A function $f : X \to Y$ is said to be *strongly faintly semi-continuous* (respectively strongly faintly precontinuous, strongly faintly α-continuous, strongly faintly γ-continuous) if for each $x \in X$ and each semi-open (respectively preopen, α-open, γ-open) set V containing $f(x)$, there exists a θ-open set U containing x such that $f(U) \subseteq V$.

3. Change of topology

The fundamental defining feature of the class of strongly faintly α-continuous functions between topological spaces is given by the following result, which is an immediate corollary of Theorem 3.1 of Nasef [11].

Proposition 3.1. *A function $f : (X,\tau) \to (Y,\sigma)$ is strongly faintly α-continuous if and only if $f : (X,\tau_\theta) \to (Y,\sigma_\alpha)$ is continuous.*

One immediate conclusion from Proposition 3.1 is that each strongly faintly α-continuous function is a morphism in the category Top whose objects are topological spaces and whose morphisms are continuous functions between topological spaces. If $f : (X,\tau) \to (Y,\sigma)$ is strongly faintly α-continuous, then f is a morphism in Top from (X,τ_θ) to (Y,σ_α). It is not the case that f lies outside of Top. It is the case that the wrong objects in Top have been chosen for the source and target of the morphism f in the category Top.

Throughout his paper, Nasef [11] has presented his results in a way that strongly suggests that there is an exact parallel between strongly faint α-continuity and strongly faint γ-continuity. Example 2.2 shows that there is no analogue of Proposition 3.1 for strongly faint γ-continuity. Change of topology methods of proof cannot be used for strongly faint γ-continuity, as they are in the rest of this paper for strongly faint α-continuity.

The equivalence given in Proposition 3.1 shows that strongly faint α-continuity is a μ-continuity property in the sense of Gauld, Mrsevic, Reilly and Vamanamurthy [4]. The fact that strongly faint α-continuity is equivalent to continuity under these changes of topology can be exploited to yield alternative elegant proofs of existing results, and to suggest new results. Definitions 3.2 and 3.3 of Nasef [11] describe variations of the standard Hausdorff (or T_2) and compactness properties, denoted α-T_2 and θ-T_2, and α-compact and θ-compact, respectively. The following results are immediate from these definitions.

Lemma 3.2.

(i) *(X, τ) is α-T_2 if and only if (X, τ_α) is T_2.*
(ii) *(X, τ) is θ-T_2 if and only if (X, τ_θ) is T_2.*

Lemma 3.3.

(i) *(X, τ) is α-compact if and only if (X, τ_α) is compact.*
(ii) *(X, τ) is θ-compact if and only if (X, τ_θ) is compact.*

Using the preservation of compactness by continuous functions, we can obtain an elegant proof of a whole space version of Theorem 3.8 of Nasef [11], as follows:

Proposition 3.4. *If $f : (X, \tau) \to (Y, \sigma)$ is a strongly faintly α-continuous surjection and (X, τ) is θ-compact, then (Y, σ) is α-compact.*

Proof. We have that $f : (X, \tau_\theta) \to (Y, \sigma_\alpha)$ is continuous and (Y, σ_α) is the image of the compact space (X, τ_θ). Hence (Y, σ_α) is compact, and therefore (Y, σ) is α-compact. □

The topic considered in Theorem 4.7 of Nasef [11] yields to a similar approach. Note that (X, τ) is α-connected [θ-connected] if and only if (X, τ_α) [(X, τ_θ)] is connected. As expected, (X, τ) is defined to be θ-connected if X cannot be expressed as the union of two nonempty disjoint θ-open subsets of X.

Proposition 3.5. *If $f : (X, \tau) \to (Y, \sigma)$ is a strongly faintly α-continuous surjection and (X, τ) is θ-connected, then (Y, σ) is α-connected.*

Proof. We have that $f : (X, \tau_\theta) \to (Y, \sigma_\alpha)$ is a continuous surjection, and (X, τ_θ) is connected. Thus (Y, σ_α) is connected; that is, (Y, σ) is α-connected. □

Classical topological results can be used with this change of topology technique to obtain new results. We now provide some examples.

Proposition 3.6. *Let $f, g : (X, \tau) \to (Y, \sigma)$ be strongly faintly α-continuous, and let (Y, σ) be α-T_2. Then the equalizer $E = \{x \in X : f(x) = g(x)\}$ of f and g is θ-closed in (X, τ).*

Proof. We have that $f, g : (X, \tau_\theta) \to (Y, \sigma_\alpha)$ are continuous by Proposition 3.1, and that (Y, σ_α) is Hausdorff by Lemma 3.2(i). So the classical result of Dugundji [3, page 140, 1.5(1)] implies that E is closed in (X, τ_θ); that is, E is θ-closed in (X, τ). □

Proposition 3.7. *If $f : (X, \tau) \to (Y, \sigma)$ is strongly faintly α-continuous and (Y, σ) is α-T_2, then the graph $G(f)$ of f is closed in $(X \times Y, \tau_\theta \times \sigma_\alpha)$.*

Proof. We use the fact that (Y, σ_α) is Hausdorff and that $f : (X, \tau_\theta) \to (Y, \sigma_\alpha)$ is continuous and a classical result (see for example Dugundji [3, page 140, 1.5(3)]) to conclude that $G(f)$ is closed in $(X \times Y, \tau_\theta \times \sigma_\alpha)$. □

Our Proposition 3.7 is essentially a restatement of Theorem 3.9 of Nasef [11].

Definition 3.8. A function $f : (X, \tau) \to (Y, \sigma)$ is defined to be:

(i) strongly α-irresolute [9] if $f^{-1}(V)$ is open in (X, τ) for every α-open subset V of (Y, σ),
(ii) α-irresolute [8] if $f^{-1}(V)$ is α-open in (X, τ) for every α-open subset V of (Y, σ),
(iii) α-continuous [16] if $f^{-1}(V)$ is α-open in (X, τ) for every open subset V of (Y, σ),
(iv) strongly θ-continuous [14] if for each point x in X and each open set V containing $f(x)$, there is an open set U containing x such that $f(\mathsf{cl}U) \subset V$,
(v) quasi-θ-continuous [14] if for each point x in X and each θ-open set V containing $f(x)$, there is a θ-open set U containing x such that $f(U) \subset V$.

Each of these properties of functions reduces to continuity provided appropriate changes of topology are made on the domain and/or the codomain. In particular, we have

Proposition 3.9. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then*

(i) *f is strongly α-irresolute if and only if $f : (X, \tau) \to (Y, \sigma_\alpha)$ is continuous,*
(ii) *f is α-irresolute if and only if $f : (X, \tau_\alpha) \to (Y, \sigma_\alpha)$ is continuous [16],*
(iii) *f is α-continuous if and only if $f : (X, \tau_\alpha) \to (Y, \sigma)$ is continuous [16],*
(iv) *f is strongly θ-continuous if and only if $f : (X, \tau_\theta) \to (Y, \sigma)$ is continuous [6],*
(v) *f is quasi-θ-continuous if and only if $f : (X, \tau_\theta) \to (Y, \sigma_\theta)$ is continuous,*
(vi) *f is faintly continuous if and only if $f : (X, \tau) \to (Y, \sigma_\theta)$ is continuous [7],*

(vii) *f is faintly α-continuous if and only if $f : (X, \tau_\alpha) \to (Y, \sigma_\theta)$ is continuous.*

We observe that (vi) and (vii) follow from Definition 2.3.

Corollary 3.10. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:*

(1) $f : (X, \tau) \to (Y, \sigma)$ is strongly faintly α-continuous.
(2) $f : (X, \tau_\theta) \to (Y, \sigma_\alpha)$ is continuous.
(3) $f : (X, \tau_\theta) \to (Y, \sigma)$ is strongly α-irresolute.
(4) $f : (X, \tau) \to (Y, \sigma_\alpha)$ is strongly θ-continuous.

Change of topology allows us to prove the next set of results simply by observing that the composition of two continuous functions is continuous. There is no need to use first principles to prove such results, as Nasef [11, Theorem 4.5] has done. Note that (2), (3) and (4) of our Proposition 3.11 are Theorems 4.6 (iii), 4.5 and 4.6 (ii) of Nasef [11] respectively, for the α case.

Proposition 3.11. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$ be functions.*

(1) *If f is faintly continuous and g is strongly faintly α-continuous, then $g \circ f$ is strongly α-irresolute.*
(2) *If f is quasi-θ-continuous and g is strongly faintly α-continuous, then $g \circ f$ is strongly faintly α-continuous.*
(3) *If f is strongly faintly α-continuous and g is α-irresolute, then $g \circ f$ is strongly faintly α-continuous.*
(4) *If f is strongly faintly α-continuous and g is α-continuous, then $g \circ f$ is strongly θ-continuous.*
(5) *If f is strongly θ-continuous and g is strongly α-irresolute, then $g \circ f$ is strongly faintly α-continuous.*
(6) *If f is α-continuous and g is faintly continuous, then $g \circ f$ is faintly α-continuous.*
(7) *If f is faintly α-continuous and g is strongly faintly α-continuous, then $g \circ f$ is α-irresolute.*
(8) *If f is faintly α-continuous and g is strongly θ-continuous, then $g \circ f$ is α-continuous.*
(9) *If f is is strongly faintly α-continuous and g is faintly α-continuous, then $g \circ f$ is quasi-θ-continuous.*

Proof. (1) $f : (X, \tau) \to (Y, \sigma_\theta)$ and $g : (Y, \sigma_\theta) \to (Z, \mu_\alpha)$ are continuous, so that $g \circ f : (X, \tau) \to (Z, \mu_\alpha)$ is continuous.
(2) $f : (X, \tau_\theta) \to (Y, \sigma_\theta)$ and $g : (Y, \sigma_\theta) \to (Z, \mu_\alpha)$ are continuous, so that $g \circ f : (X, \tau_\theta) \to (Z, \mu_\alpha)$ is continuous.
(3) $f : (X, \tau_\theta) \to (Y, \sigma_\alpha)$ and $g : (Y, \sigma_\alpha) \to (Z, \mu_\alpha)$ are continuous, so that $g \circ f : (X, \tau_\theta) \to (Z, \mu_\alpha)$ is continuous.
(4) $f : (X, \tau_\theta) \to (Y, \sigma_\alpha)$ and $g : (Y, \sigma_\alpha) \to (Z, \mu)$ are continuous, so that $g \circ f : (X, \tau_\theta) \to (Z, \mu)$ is continuous.

(5) $f : (X, \tau_\theta) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu_\alpha)$ are continuous, so that $g \circ f : (X, \tau_\theta) \to (Z, \mu_\alpha)$ is continuous.
(6) $f : (X, \tau_\alpha) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu_\theta)$ are continuous, so that $g \circ f : (X, \tau_\alpha) \to (Z, \mu_\theta)$ is continuous.
(7) $f : (X, \tau_\alpha) \to (Y, \sigma_\theta)$ and $g : (Y, \sigma_\theta) \to (Z, \mu_\alpha)$ are continuous, so that $g \circ f : (X, \tau_\alpha) \to (Z, \mu_\alpha)$ is continuous.
(8) $f : (X, \tau_\alpha) \to (Y, \sigma_\theta)$ and $g : (Y, \sigma_\theta) \to (Z, \mu)$ are continuous, so that $g \circ f : (X, \tau_\alpha) \to (Z, \mu)$ is continuous.
(9) $f : (X, \tau_\theta) \to (Y, \sigma_\alpha)$ and $g : (Y, \sigma_\alpha) \to (Z, \mu_\theta)$ are continuous, so that $g \circ f : (X, \tau_\theta) \to (Z, \mu_\theta)$ is continuous.

□

4. A decomposition of faint α-continuity

The following fundamental relationship between classes of generalized open sets in a topological space was first proved in 1985 by Reilly and Vamanamurthy [16].

Lemma 4.1. *In any topological space (X, τ), $\tau^\alpha = PO(X) \cap SO(X)$.*

The definition of faint α-continuity, Definition 2.3, together with Lemma 4.1 immediately implies the following decomposition of faint α-continuity.

Proposition 4.2. *A function $f : (X, \tau) \to (Y, \sigma)$ is faintly α-continuous if and only if f is faintly precontinuous and faintly semi-continuous.*

Nasef [11] has established by his set of Examples 4.1 to 4.5 and the diagram of his Remark 4.1 that faint α-continuity is distinct and independent from existing classes of functions. His diagram indicates that faint α-continuity implies each of faint precontinuity and faint semi-continuity. Proposition 4.2 reveals that there is a 'joint' converse. Together, these two notions are equivalent to faint α-continuity. The equivalence does not extend however to a 'strong' analogue. While strongly faint precontinuity and strongly faint semi-continuity each clearly imply strongly faint α-continuity (by Definition 2.4 and Lemma 4.1), Nasef and Noiri [12, Example 3.1] establish that the converse does not hold.

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