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# A generalized version of the rings $C_K(X)$ and $C_\infty(X)$– an enquery about when they become Noetheri

Sudip Kumar Acharyya [a], Kshitish Chandra Chattopadhyay [b] and Pritam Rooj [a,*]

[a] Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata 700019, West Bengal, India (sdpacharyya@gmail.com, prooj10@gmail.com)
[b] Department of Mathematics, University of Burdwan, Golapbag, Burdwan 713104, West Bengal, India (kcchattopadhyay2009@gmail.com)

Abstract

*Suppose $F$ is a totally ordered field equipped with its order topology and $X$ a completely $F$-regular topological space. Suppose $\mathcal{P}$ is an ideal of closed sets in $X$ and $X$ is locally-$\mathcal{P}$. Let $C_\mathcal{P}(X,F) = \{f\colon X \to F \mid f$ is continuous on $X$ and its support belongs to $\mathcal{P}\}$ and $C_\infty^\mathcal{P}(X,F) = \{f \in C_\mathcal{P}(X,F) \mid \forall\varepsilon > 0$ in $F$, $cl_X\{x \in X : |f(x)| > \varepsilon\} \in \mathcal{P}\}$. Then $C_\mathcal{P}(X,F)$ is a Noetherian ring if and only if $C_\infty^\mathcal{P}(X,F)$ is a Noetherian ring if and only if $X$ is a finite set. The fact that a locally compact Hausdorff space $X$ is finite if and only if the ring $C_K(X)$ is Noetherian if and only if the ring $C_\infty(X)$ is Noetherian, follows as a particular case on choosing $F = \mathbb{R}$ and $\mathcal{P} =$ the ideal of all compact sets in $X$. On the other hand if one takes $F = \mathbb{R}$ and $\mathcal{P} =$ the ideal of closed relatively pseudocompact subsets of $X$, then it follows that a locally pseudocompact space $X$ is finite if and only if the ring $C_\psi(X)$ of all real valued continuous functions on $X$ with pseudocompact support is Noetherian if and only if the ring $C_\infty^\psi(X) = \{f \in C(X) \mid \forall\varepsilon > 0, cl_X\{x \in X : |f(x)| > \varepsilon\}$ is pseudocompact $\}$ is Noetherian. Finally on choosing $F = \mathbb{R}$ and $\mathcal{P} =$ the ideal of all closed sets in $X$, it follows that: $X$ is finite if and only if the ring $C(X)$ is Noetherian if and only if the ring $C^*(X)$ is Noetherian.*

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## 1. Introduction

A commutative ring $R$ with or with out identity, is called Noetherian/Artinian if any ascending/descending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots / I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ in it is stationary in the sense that there is an $m \in \mathbb{N}$ with $I_j = I_m$ for all $j > m$. Noetherian rings play an important role in Commutative Algebra and also in Algebraic Geometry. A principal result for these rings is that if $R$ is Noetherian, then the polynomial ring $R[x_1, x_2, \ldots, x_n]$ in finitely many indeterminates is also Noetherian. Thus quite a large number of "good" rings appear to be Noetherian. The theory of rings $C(X)$ of all real valued continuous functions over topological spaces $X$ became an important area of research with the pioneering work of M. H. Stone [11], Gelfand and Kolmogorov [5] and Hewitt [6]. Two important subrings of the last mentioned ring viz. the ring $C_K(X)$ of all real valued continuous functions over $X$, which have compact support and the ring $C_\infty(X)$ of all those functions in $C_K(X)$ which vanish at infinity have also received the attention of some Mathematicians, mentioned may be made of C. W. Kohl ([8] and [9]). A large number of well known Mathematicians subsequently got attracted to this area, particularly after the classic text book of Gillman and Jerison [4] came into being in the year 1960. Many properties by and large common to most of the "good" rings have also been shared by the rings $C(X)$ and some of these by the rings $C_K(X)$ and $C_\infty(X)$ also. Since most of the so-called "good" rings are Noetherian one may ask, if so are also the rings $C(X)$, $C_K(X)$ and $C_\infty(X)$. In this article we establish a general result from which it follows that for a Tychonoff space $X$, $C(X)$ is Noetherian if and only if $X$ is a finite set and if $X$ is locally compact and Hausdorff, then $C_K(X)$ is Noetherian if and only if $C_\infty(X)$ is Noetherian when and only when $X$ is a finite set. It also follows from the same general result that a locally pseudocompact Tychonoff space $X$ is finite if and only if the ring $C_\psi(X)$ of all functions in $C(X)$ with pseudocompact support is Noetherian if and only if the ring $C_\infty^\psi(X)$ of all functions $f$ in $C(X)$ for which for each $\varepsilon > 0$, the set $cl_X\{x \in X : |f(x)| > \varepsilon\}$ is pseudocompact is Noetherian. The last ring may be thought of as the pseudocompact analogue of the ring $C_\infty(X)$.

Now we state our principal result. Let $X$ be a Hausdorff topological space and $F$, a totally ordered field equipped with the order topology. Then

$$C(X, F) = \{f \colon X \to F \mid f \text{ is continuous on } X\}$$

makes a commutative lattice ordered ring with identity if the compositions are defined pointwise on $X$. We get the familiar ring $C(X)$ on choosing $F = \mathbb{R}$. $X$ is called completely $F$-regular (CFR in short) if given a closed set $K$ in $X$ and a point $x \in X - K$, there exists an $f \in C(X, F)$ such that $f(x) = 0$ and $f(K) = 1$. The ring $C(X, F)$ together with a few of its subrings were investigated by Acharyya, Chattopadhyaya and Ghosh [2]. Their purpose is to look into a few aspects on the possible interplay between the topological structure on $X$ and the algebraic structure of $C(X, F)$ and their subrings mentioned in the last sentence. It was observed in the same paper [2] that a CFR-space with $F$ not isomorphic to $\mathbb{R}$ is zero-dimensional, in particular Tychonoff. Conversely each zero-dimensional Hausdorff topological space becomes CFR-space for any ordered field $F$. Thus zero-dimensionality of a Hausdorff space $X$ is realised as a kind of separation axiom effected by $F$-valued continuous functions on $X$.

A family $\mathcal{P}$ of closed sets in $X$ is called an ideal of closed sets if:

(1) $A \in \mathcal{P}$, $B \in \mathcal{P} \Rightarrow A \cup B \in \mathcal{P}$ and
(2) $A \in \mathcal{P}$ and $C \subseteq A$ with $C$ closed in $X \Rightarrow C \in \mathcal{P}$.

With any such ideal $\mathcal{P}$, we associate the following two subsets of $C(X,F)$, each of which is a subring of $C(X,F)$ (possibly without identity):

$$\begin{aligned} C_{\mathcal{P}}(X,F) &= \{f \in C(X,F) : cl_X(X - Z(f)) \in \mathcal{P}\} \text{ and} \\ C_\infty^{\mathcal{P}}(X,F) &= \{f \in C(X,F) : \forall \varepsilon > 0 \text{ in } F, \{x \in X : |f(x)| \geq \varepsilon\} \in \mathcal{P}\} \\ &= \{f \in C(X,F) : \forall \varepsilon > 0 \text{ in } F, cl_X\{x \in X : |f(x)| > \varepsilon\} \in \mathcal{P}\} \end{aligned}$$

here $Z(f) = \{x \in X : f(x) = 0\}$ is the zero set of $f$. It is easy to check that $C_{\mathcal{P}}(X,F) \subseteq C_\infty^{\mathcal{P}}(X,F)$. Given an ideal $\mathcal{P}$ of closed sets in $X$, $X$ is called locally-$\mathcal{P}$ if each point $x \in X$ has an open neighbourhood $U$ such that $cl_X U \in \mathcal{P}$. Thus locally compact spaces $X$ are locally-$\mathcal{P}$ if $\mathcal{P}$ is the ideal of compact sets in $X$. The principal result of this paper is stated as follows:

**Theorem 1.1** (Main Theorem). *Given an ideal $\mathcal{P}$ of closed sets in $X$ and a totally ordered field $F$, the following statements are equivalent for a locally-$\mathcal{P}$, CFR-space $X$:*

(1) *$C_{\mathcal{P}}(X,F)$ is a Noetherian ring.*
(2) *$C_{\mathcal{P}}(X,F)$ is an Artinian ring.*
(3) *$C_\infty^{\mathcal{P}}(X,F)$ is a Noetherian ring.*
(4) *$C_\infty^{\mathcal{P}}(X,F)$ is an Artinian ring.*
(5) *$X$ is a finite set.*

## 2. Two subsidiary results and the proof of the main result

**Lemma 2.1.** *Let $\{R_1, R_2, \ldots, R_n\}$ be a finite family of commutative rings with identity. The ideals of the direct product $R_1 \times R_2 \times \cdots \times R_n$ are exactly of the form $I_1 \times I_2 \times \cdots \times I_n$, where for $k = 1, 2, \ldots, n$, $I_k$ is an ideal of $R_k$.*

*Proof.* If $I_k$ is an ideal of $R_k$ for $k = 1, 2, \ldots, n$ then it follows trivially that $I_1 \times I_2 \times \cdots \times I_n$ is an ideal of $R_1 \times R_2 \times \cdots \times R_n$.

Conversely let $I$ be an ideal of $R_1 \times R_2 \times \cdots \times R_n$. Suppose $\pi_k \colon R_1 \times R_2 \times \cdots \times R_n \to R_k$ is the $k$-th projection map for $k = 1, 2, \ldots, n$ defined in the usual manner. Let $I_k = \{\pi_k(x) : x \in I\}$. Then $I_k$ is an ideal of $R_k$ for $k = 1, 2, \ldots, n$ and $I \subseteq I_1 \times I_2 \times \cdots \times I_n$. Now we choose $(x_1, x_2, \ldots, x_n) \in I_1 \times I_2 \times \cdots \times I_n$. Then for $k = 1, 2, \ldots, n$, there exists $y_k \in I$ such that $\pi_k(y_k) = x_k$. This implies that $(0, 0, \ldots, x_k, \ldots, 0)$, with its $k$-th co-ordinate $x_k$ belongs to $I$, consequently $(x_1, x_2, \ldots, x_n) \in I$. Hence $I = I_1 \times I_2 \times \cdots \times I_n$. □

**Lemma 2.2.** *Given a totally ordered field $F$ and a CFR-space $X$, the following statements are equivalent:*

(1) *$X$ is locally-$\mathcal{P}$.*
(2) *$\{Z(f) : f \in C_{\mathcal{P}}(X,F)\}$ is a closed base for $X$.*
(3) *$\{Z(f) : f \in C_{\infty}^{\mathcal{P}}(X,F)\}$ is a closed base for $X$.*

(A special case of this result with $F = \mathbb{R}$ appeared in a paper in 2010 (see [1, Theorem 4.3]))

*Proof.* The proof is just a simple adaptation of the proof of Theorem 4.3 in the paper [1], mentioned above. We have only to take care of the fact that a completely $F$-regular space is regular. □

*Proof of Theorem 1.1.* First assume that $X$ is a finite set with '$n$' elements. Since $X$ is Hausdorff it becomes a discrete space and therefore $\mathcal{P} = p(X) \equiv$ the power set of $X$. Consequently $C_{\mathcal{P}}(X,F) = C_{\infty}^{\mathcal{P}}(X,F) = C(X,F) = F^n \equiv F \times F \times \cdots \times F$ ($n$ times). Since $\{0\}$ and $F$ are the only ideals of the field $F$, it follows from Lemma 2.1 that, there are exactly $2^n$ many ideals of the product ring $F^n$. Hence $C_{\mathcal{P}}(X,F)$ and $C_{\infty}^{\mathcal{P}}(X,F)$ are both Noetherian rings and Artinian rings, trivially.

Conversely let $X$ be an infinite set. As, a commutative ring is Noetherian if and only if each ideal in this ring is finitely generated, therefore to show that $C_{\mathcal{P}}(X,F)$ is not a Noetherian ring, we shall construct an ideal in this ring which is not finitely generated. Now like any infinite Hausdorff space, $X$ contains a copy of $\mathbb{N} = \{1,2,3,\dots\}$. We take $I = \{f \in C_{\mathcal{P}}(X,F) : f(1) = 0$ and $f(k) = 0$ for all but possibly finitely many $k$'s from $\mathbb{N}\}$. It is easy to check that $I$ is an ideal of $C_{\mathcal{P}}(X,F)$. We assert that $I$ is not finitely generated. For that purpose we select any finite number of elements $f_1, f_2, \dots, f_n$ from $I$, $n \in \mathbb{N}$. Then the set $\bigcap_{i=1}^{n} Z(f_i)$ contains the point 1 and also all but finitely many points from the set $\mathbb{N}$. So we can choose $m \neq 1$ from $\mathbb{N}$ such that $m \in \bigcap_{i=1}^{n} Z(f_i)$. Since $m$ is isolated in the space $\mathbb{N}$, there exists an open neighbourhood $U$ of $m$ in $X$ such that $U \cap \mathbb{N} = \{m\}$. As $X$ is locally-$\mathcal{P}$, it follows from Lemma 2.2 that there is an $f \in C_{\mathcal{P}}(X,F)$ such that $f(m) \neq 0$ and $f(X - U) = 0$. Since $X - U$ contains all the points of $\mathbb{N} - \{m\}$ and $m \neq 1$, it is clear that $f \in I$. But the choice that $f(m) \neq 0$, while $f_1(m) = f_2(m) = \cdots = f_n(m) = 0$ tells us that, there does not exists $\alpha_1, \alpha_2, \dots, \alpha_n \in C_{\mathcal{P}}(X,F)$, for which we can write $f = \sum_{i=1}^{n} \alpha_i f_i$. This shows that $I$ is not finitely generated. Altogether $C_{\mathcal{P}}(X,F)$ is not a Noetherian ring. Analogous arguments can be made to prove that $C_{\infty}^{\mathcal{P}}(X,F)$ is not a Noetherian ring.

To complete this theorem we shall show that $C_{\mathcal{P}}(X,F)$ (respectively $C_{\infty}^{\mathcal{P}}(X,F)$) is not an Artinian ring (with the same hypothesis that $X$ is an infinite set). At this stage, we may be tempted to argue as follows: since a commutative Artinian ring is known to be Noetherian, therefore $C_{\mathcal{P}}(X,F)$ (respectively $C_{\infty}^{\mathcal{P}}(X,F)$) is not an Artinian ring, because we have just realised that (with $X$ infinite), $C_{\mathcal{P}}(X,F)$ is not a Noetherian ring (respectively $C_{\infty}^{\mathcal{P}}(X,F)$ is not a Noetherian ring). But a word of caution here, the proof of the fact that a commutative Artinian ring $R$ is Noetherian crucially uses the tacit assumption that $1 \in R$ (see [3, Theorem 3, Chapter 16]). But in our situation each of the rings $C_{\mathcal{P}}(X, F)$ and $C_{\infty}^{\mathcal{P}}(X, F)$ may well lack identity elements. Indeed if $X$ is a non compact locally compact Hausdorff space, then none of the rings $C_K(X)$ and $C_{\infty}(X)$ possesses identity. Thus we feel it necessary to prove independently that $C_{\mathcal{P}}(X, F)$ (respectively $C_{\infty}^{\mathcal{P}}(X, F)$) is not an Artinian ring. Indeed for each $k \in \mathbb{N}$, if we set $I_k = \{f \in C_{\mathcal{P}}(X, F) : f(1) = f(2) = \cdots = f(k) = 0\}$, then on using Lemma 2.2, it is not at all hard to check that $I_1 \supsetneqq I_2 \supsetneqq I_3 \supsetneqq \cdots$ is a strictly decreasing sequence of ideals in $C_{\mathcal{P}}(X, F)$ which never terminates at a finite stage. This proves that $C_{\mathcal{P}}(X, F)$ is not an Artinian ring. Analogously one can see that $C_{\infty}^{\mathcal{P}}(X, F)$ is also not an Artinian ring. The theorem is completely proved. □

*Remark* 2.3. A careful scrutiny into the proof of the above theorem yields that if $X$ is an infinite locally-$\mathcal{P}$, CFR-space, then none of the rings that lie between $C_{\mathcal{P}}(X, F)$ and $C_{\infty}^{\mathcal{P}}(X, F)$ is Noetherian (respectively Artinian). Therefore we can write the following improved version of our Main Theorem.

**Theorem 2.4.** *The following statements are equivalent for a locally-$\mathcal{P}$, CFR-space $X$:*

(1) *There exists a Noetherian ring lying between $C_{\mathcal{P}}(X, F)$ and $C_{\infty}^{\mathcal{P}}(X, F)$.*
(2) *There exists an Artinian ring lying between $C_{\mathcal{P}}(X, F)$ and $C_{\infty}^{\mathcal{P}}(X, F)$.*
(3) *$X$ is a finite set.*

## 3. A few interesting special cases of the Main Theorem

The choice $F = \mathbb{R}$ and $\mathcal{P} =$ the ideal of all closed sets in $X$ together with the fact that $C^*(X)$ is isomorphic to $C(\beta X)$, where $\beta X$ is the Stone-Čech compactification of $X$ yields the following special case:

**Theorem 3.1.** *The following statements are equivalent for a Tychonoff space $X$:*

(1) *$C(X)$ is a Noetherian ring.*
(2) *$C(X)$ is an Artinian ring.*
(3) *$C^*(X)$ is a Noetherian ring.*
(4) *$C^*(X)$ is an Artinian ring.*
(5) *$X$ is a finite set.*

Since a point and a closed subset of $X$ missing that point could always be separated by a function in $C^*(X)$, it follows by making a simple modification of the proof of the converse part of Theorem 1.1 that, for an infinite set $X$, no ring lying between $C^*(X)$ and $C(X)$ can ever be Noetherian (respectively Artinian).

If we choose $F = \mathbb{R}$ and $\mathcal{P} =$ the ideal of all compact sets in $X$, then the following special case of our Main Theorem emerges:

**Theorem 3.2.** *The following statements are equivalent for a locally compact Hausdorff space $X$:*

(1) $C_K(X)$ *is a Noetherian ring.*
(2) $C_K(X)$ *is an Artinian ring.*
(3) $C_\infty(X)$ *is a Noetherian ring.*
(4) $C_\infty(X)$ *is an Artinian ring.*
(5) $X$ *is a finite set.*

It follows from Remark 2.3 that for an infinite set $X$ none of the rings lying between $C_K(X)$ and $C_\infty(X)$ is Noetherian (respectively Artinian). It may be mentioned that in case $C_K(X) \neq C_\infty(X)$, there lies at least $2^{\aleph_1}$ many rings between $C_K(X)$ and $C_\infty(X)$ (see [4, 7G(1) and 14.13]).

Before mentioning the last important special case of the Main Theorem in this article, we recall that a subset $A$ of $X$ is called relatively pseudocompact if each $f \in C(X)$ is bounded on $A$. We write below the following result, proved in 1971 by Mark Mandelkar, which we will need for our present purpose.

**Theorem 3.3** (Mandelkar's theorem [10])**.** *A support in $X$ i.e. a subset of the form $cl_X(X - Z(f))$, $f \in C(X)$ is relatively pseudocompact if and only if it is pseudocompact.*

The closed pseudocompact subsets of a space $X$ may not form an ideal of closed sets. Indeed the right edge $\{\omega_1\} \times \omega_0$ of the Tychonoff plank $T \equiv (\omega_1+1)\times(\omega_0+1)-\{(\omega_1, \omega_0)\}$ is a closed subset of $T$ and $T$ is pseudocompact (see [4, 8.20]). Nevertheless, the set $C_\psi(X)$ of all real valued continuous functions on $X$ with pseudocompact support makes a subring, indeed an ideal of $C(X)$. This follows from Mandelkar's theorem and also the fact that the closed relatively pseudocompact subsets of a space $X$, do constitute an ideal of closed sets in $X$ (see [10, Corollary 2]). Mandelkar's theorem further implies that the set $C_\infty^\psi(X) = \{f \in C(X) : \forall\varepsilon > 0, cl_X\{x \in X : |f(x)| > \varepsilon\}$ is pseudocompact $\}$ which we may call the pseudocompact analogue of the ring $C_\infty(X)$, forms an ideal of $C(X)$ with $C_\psi(X) \subseteq C_\infty^\psi(X)$. Since the cozero sets make an open base for the topology of a Tychonoff space $X$, it is also a straight forward consequence of Mandelkar's theorem that $X$ is locally relatively pseudocompact if and only if it is locally pseudocompact, in the sense that each point $x$ of $X$ has an open neighbourhood whose closure is pseudocompact.

We now organize the above findings to conclude the following proposition, which we feel an interesting particular case of our Main Theorem 1.1.

**Theorem 3.4.** *The statements written below are equivalent for a locally pseudocompact (Tychonoff) space $X$:*

(1) $C_\psi(X)$ *is a Noetherian ring.*
(2) $C_\psi(X)$ *is an Artinian ring.*
(3) $C_\infty^\psi(X)$ *is a Noetherian ring.*
(4) $C_\infty^\psi(X)$ *is an Artinian ring.*
(5) $X$ *is a finite set.*

If $X$ is an infinite set, then no ring lying between $C_\psi(X)$ and $C_\infty^\psi(X)$ is Noetherian (respectively Artinian). This last assertion follows from Remark 2.3.

*Remark* 3.5 (Concluding remark). Rings of real valued continuous functions defined over Tychonoff spaces and many of their well known subrings are in general far from being Noetherian, yet all are important in their own right.

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