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THE POINCARÉ PROBLEM, ALGEBRAIC INTEGRABILITY AND DICRITICAL DIVISORS

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ABSTRACT. We solve the Poincaré problem for plane foliations with only one dicritical divisor. Moreover, in this case, we give an algorithm that decides whether a foliation has a rational first integral and computes it in the affirmative case. We also provide an algorithm to compute a rational first integral of prefixed genus $g \neq 1$ of any type of plane foliation \mathcal{F} . When the number of dicritical divisors $\text{dic}(\mathcal{F})$ is larger than two, this algorithm depends on suitable families of invariant curves. When $\text{dic}(\mathcal{F}) = 2$, it proves that the degree of the rational first integral can be bounded only in terms of g , the degree of \mathcal{F} and the local analytic type of the dicritical singularities of \mathcal{F} .

1. INTRODUCTION AND RESULTS

Denote by \mathbb{P}^2 the projective plane over the field of complex numbers. Poincaré, in [34], observed that “to find out whether a differential equation of the first order and of the first degree is algebraically integrable, it is enough to get an upper bound on the degree of the integral. Afterwards, one only needs to perform purely algebraic computations.” The motivation for this observation, expressed in modern terminology, was the problem of deciding whether a singular algebraic foliation \mathcal{F} on \mathbb{P}^2 (plane foliation) has a rational first integral and, when the answer is positive, to compute it. The so-called Poincaré problem consists of obtaining an upper bound of the degree of the first integral depending only on the degree of the foliation. Although it is well-known that such a bound does not exist in general, in the forthcoming Theorem 1 (our main goal) we shall give a bound of this type under the assumption that the minimal resolution of the singularities of \mathcal{F} (which exists by a result of Seidenberg [36]) has only one dicritical (i.e., non-invariant by \mathcal{F}) exceptional divisor. Also, under this assumption, we shall give an algorithm that solves the above mentioned decision problem whose inputs are a differential 1-form Ω defining the foliation and the part of the minimal resolution of \mathcal{F} corresponding to dicritical singularities (see Definition 1), which can be obtained from Ω . This algorithm only involves simple integer arithmetics and resolution of systems of linear equations.

The natural extended version of the Poincaré problem consists of bounding the degree of the algebraic integral (reduced and irreducible) invariant curves of a foliation \mathcal{F} (without assuming algebraic integrability) in terms of data obtained from the foliation and/or invariants related with the invariant curves themselves. There has been (and there is) a lot of activity concerning this or related problems, some of the main results (including higher dimension) being [9, 7, 5, 40, 38, 39, 41, 32, 15, 16, 8, 17].

The above mentioned problem was stated at the end of the 19th century as the problem of deciding whether a complex polynomial differential equation on the complex plane is algebraically integrable. The usefulness of nonlinear ordinary differential equations in practically any science turns this problem into a very attractive one, especially because

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when a differential equation admits a first integral, its study can be reduced in one dimension and because it is related with other interesting challenges. For example, it is related with the second part of the XVI Hilbert problem which tries to bound the number of limit cycles for a real polynomial vector field [28, 29], with the solutions of Einstein's field equations in general relativity [21] and with the center problem for vector fields [35, 14].

Algebraic integrability problem has a long history. In the 19th century, the main contributors were Darboux [12], Poincaré [33, 34], Painlevé [30] and Autonne [1]. They laid the foundations of a theory that has inspired a large quantity of papers, many of them published in the last twenty years. It was Darboux who gave a bound on the number of invariant integral algebraic curves of a polynomial differential equation that, when it is exceeded, implies the existence of a first integral. A close result was proved by Jouanolou [23] to guarantee that a foliation \mathcal{F} as above has a rational first integral and that if one has enough reduced invariant curves, then the rational first integral can be computed. The existence of a first integral of that type is also equivalent to the fact that every invariant curve by \mathcal{F} is algebraic and to the fact that there exist infinitely many invariant integral curves. These results have been adapted and extended to foliations on other varieties [22, 23, 4, 18, 11]. In [16], the authors gave an algorithm to decide about the existence of a rational first integral (and to compute it in the affirmative case) assuming that one has a well-suited set of $\text{dic}(\mathcal{F})$ reduced invariant curves, where we stand $\text{dic}(\mathcal{F})$ for the number of dicritical divisors appearing in the resolution of \mathcal{F} . In the same paper, it was also shown how to get sets of invariant curves as above for foliations such that the cone of curves of the surface obtained by the resolution of the dicritical singularities is polyhedral.

Painlevé in [30] posed the problem of recognizing the genus of the general invariant algebraic curve of a foliation admitting a rational first integral. Mixing the ideas of Painlevé and Poincaré, one can try to bound the degree of the rational first integral using also its genus. When \mathcal{F} is non-degenerated, Poincaré himself provided a bound proving that $d(r - 4) \leq 4(g - 1)$, where d (respectively, r) is the degree of the first integral (respectively, \mathcal{F}) and g the mentioned genus. In the same sense, for foliations \mathcal{F} as above with Kodaira dimension equal to 2, there exists a bound on the degree of the rational first integral which only depends on its genus, the degree of \mathcal{F} and the sequence $\{h^0(\mathbb{P}^2, \mathcal{K}_{\mathcal{F}}^{\otimes m})\}_{m>0}$, $\mathcal{K}_{\mathcal{F}}$ being the canonical sheaf of the foliation \mathcal{F} (see [32] for a proof). With this philosophy, we shall show in clause a) of Theorem 2 that, for a foliation \mathcal{F} on \mathbb{P}^2 having a rational first integral of genus $g \neq 1$ and such that $\text{dic}(\mathcal{F}) \leq 2$, there exists a bound on the degree of the first integral which only depends on g , the degree of \mathcal{F} and the local analytic type of the dicritical singularities of \mathcal{F} . It is worthwhile to add that, in [27], Lins Neto showed that, in general, such a bound does not exist. Clause b) of Theorem 2 states that, for foliations \mathcal{F} of \mathbb{P}^2 satisfying also $\text{dic}(\mathcal{F}) \leq 2$, there exists an algorithm of the same type as the one of clause b) of Theorem 1 that decides whether \mathcal{F} has a rational first integral of fixed genus $g \neq 1$ (and computes it in the affirmative case).

Theorem 3 extends the results of Theorem 2 to the case when $\text{dic}(\mathcal{F}) \geq 3$. Here it is required, as an additional hypothesis, the existence (and the knowledge) of a set of $\text{dic}(\mathcal{F}) - 2$ independent algebraic solutions of \mathcal{F} (see Definition 3). In a sense, this theorem is related with the above mentioned Darboux and Jouanolou's results because the knowledge of enough invariant curves allows us to obtain information concerning the rational first integral.

We finish this introduction stating the main results and summarizing briefly the aim of each section of the paper.

Theorem 1. *Let \mathcal{F} be a singular algebraic foliation on \mathbb{P}^2 of degree r such that $\text{dic}(\mathcal{F}) = 1$.*

a): If \mathcal{F} is algebraically integrable and d is the degree of a general integral invariant curve, then

$$d \leq \frac{(r+2)^2}{4}.$$

Therefore, the Poincaré problem is solved in this case.

b): There exists an algorithm to decide whether \mathcal{F} has a rational first integral (and to compute it, in the affirmative case) whose inputs are: an homogeneous 1-form defining \mathcal{F} and the minimal resolution of the dicritical singularities of \mathcal{F} .

Theorem 2. Let \mathcal{F} be a singular algebraic foliation on \mathbb{P}^2 such that $\text{dic}(\mathcal{F}) \leq 2$. Let $g \neq 1$ be a non-negative integer.

a): Assume that \mathcal{F} has a rational first integral of genus g . Then, there exists a bound on the degree of the first integral depending only on the degree of \mathcal{F} , a bound on g and the local analytic type of the dicritical singularities of \mathcal{F} .

b): There exists an algorithm to decide whether \mathcal{F} has a rational first integral of genus g (and to compute it, in the affirmative case) whose inputs are: a homogeneous 1-form defining \mathcal{F} and the minimal resolution of the dicritical singularities of \mathcal{F} .

Theorem 3. Let \mathcal{F} be a singular algebraic foliation on \mathbb{P}^2 such that $\text{dic}(\mathcal{F}) \geq 3$ and assume the existence (and the knowledge) of a $[\text{dic}(\mathcal{F}) - 2]$ -set S of independent algebraic solutions of \mathcal{F} (see Definition 3). Let $g \neq 1$ be a non-negative integer.

a): Assume that \mathcal{F} has a rational first integral of genus g . Then there exists a bound on the degree of the first integral which depends on the degree of \mathcal{F} , a bound on g , the local analytic type of the dicritical singularities of \mathcal{F} and the degrees of the curves in S and their multiplicities at the centers of the sequence of blow-ups $\pi_{\mathcal{F}}$ giving rise to the minimal resolution of the dicritical singularities of \mathcal{F} .

b): There exists an algorithm to decide whether \mathcal{F} has a rational first integral of genus g (and to compute it, in the affirmative case). Its inputs are: a homogeneous 1-form defining \mathcal{F} , $\pi_{\mathcal{F}}$ and the degrees of the curves in S and their above mentioned multiplicities.

Section 2 provides the notations and preliminary facts devoted to make easier the reading of the paper. Section 3 contains the mentioned study of rational first integrals with fixed genus; we describe the algorithm announced in clause b) of Theorem 3 (Algorithm 1), which is supported mainly in Lemma 1; the algorithm of clause b) of Theorem 2 is nothing but a particular case. Clause a) in both theorems is deduced as a consequence of the obtained algorithm. Section 4 is devoted to prove Theorem 1. We describe first the algorithm of clause b), which is divided in two parts (Algorithms 2 and 3). The bound on the degree of the first integral is deduced from the auxiliary results supporting the algorithm. Finally, in Section 5, we give several examples that show how our algorithms work.

2. PRELIMINARIES

2.1. Basic definitions. Let Z be an algebraic smooth projective complex surface. A *singular algebraic foliation* \mathcal{F} (or simply a *foliation* in the sequel) on Z is given by a set of pairs $\{(U_i, v_i)\}_{i \in I}$, where $\{U_i\}_{i \in I}$ is an open covering of Z , $v_i \in \mathcal{T}_Z(U_i)$ (where \mathcal{T}_Z denotes the tangent sheaf of Z) and, if $i, j, k \in I$, there exist functions $g_{ij} \in \mathcal{O}_Z^*(U_i \cap U_j)$ such that $v_i = g_{ij}v_j$ on $U_i \cap U_j$ and $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. If \mathcal{L} denotes the invertible sheaf defined by the multiplicative cocycle given by $\{g_{ij}\}$, we can regard \mathcal{F} as a global section of the sheaf $\mathcal{L} \otimes \mathcal{T}_Z$ and, therefore, there is an induced morphism $\mathcal{L}^{-1} \rightarrow \mathcal{T}_Z$. The sheaf \mathcal{L}

is called the *canonical sheaf* of the foliation \mathcal{F} , and it will be denoted by $\mathcal{K}_{\mathcal{F}}$. Conversely, given an invertible sheaf \mathcal{J} on Z and a morphism $\mathcal{J} \rightarrow \mathcal{T}_Z$, we can canonically associate with \mathcal{J} a foliation such that \mathcal{J}^{-1} is its canonical sheaf.

From the dual point of view, the natural product map $\Omega_Z^1 \otimes \Omega_Z^1 \rightarrow \Omega_Z^2$ gives rise to an isomorphism $\Omega_Z^1 \rightarrow \mathcal{T}_Z \otimes \Omega_Z^2$. Under this isomorphism the map $\mathcal{K}_{\mathcal{F}}^{-1} \rightarrow \mathcal{T}_Z$ corresponds to a global section of $\Omega_Z^1 \otimes \mathcal{K}_{\mathcal{F}} \otimes \mathcal{K}_Z^{-1}$, where \mathcal{K}_Z denotes the canonical sheaf of Z .

Given a point $p \in Z$, take an open set U_i such that $p \in U_i$. The *algebraic multiplicity* of \mathcal{F} at p , $\nu_p(\mathcal{F})$, is the order of v_i at p , that is, $\nu_p(\mathcal{F}) = s$ if and only if $(v_i)_p \in \mathfrak{m}_p^s \mathcal{T}_{Z,p}$ and $(v_i)_p \notin \mathfrak{m}_p^{s+1} \mathcal{T}_{Z,p}$, where \mathfrak{m}_p denotes the maximal ideal of the local ring $\mathcal{O}_{Z,p}$. The *singularities* of \mathcal{F} are those points p in Z such that $\nu_p(\mathcal{F}) \geq 1$. We shall assume that all considered foliations are saturated, that is they have finitely many singularities. Notice that if $\mathcal{F} \in H^0(Z, \mathcal{K}_{\mathcal{F}} \otimes \mathcal{T}_Z)$ vanishes on a divisor H of Z , one can regard \mathcal{F} as a global section of $\mathcal{K}_{\mathcal{F}} \otimes \mathcal{T}_Z \otimes \mathcal{O}_Z(-H)$ which defines a foliation \mathcal{F}^s , called saturation of \mathcal{F} , with isolated singularities such that $\mathcal{K}_{\mathcal{F}^s} = \mathcal{K}_{\mathcal{F}} \otimes \mathcal{O}_Z(-H)$.

Recall that an integral (i.e., reduced and irreducible) projective curve $C \subseteq Z$ is called to be invariant by \mathcal{F} if the restriction map $\mathcal{K}_{\mathcal{F}}^{-1}|_C \rightarrow \mathcal{T}_Z|_C$ factorizes through the natural inclusion $\mathcal{T}_C \rightarrow \mathcal{T}_Z|_C$ and that a projective curve $C \subseteq Z$ is named *invariant* by \mathcal{F} if all its integral components are invariant. Integral invariant curves of a foliation \mathcal{F} are usually called *algebraic solutions* of \mathcal{F} . Locally, it means that for all closed point $p \in Z$, $v_p(f) \in I_{C,p}$, whenever $f \in I_{C,p}$, $I_{C,p}$ being the ideal of C and v_p a generator of \mathcal{F} both at p ; or, dually, that the local differential 2-form $\omega_p \wedge df$ is a multiple of f , ω_p being a local equation as a form of \mathcal{F} and $f = 0$ a local equation of C at p .

Assume now that \mathcal{F} is a foliation on \mathbb{P}^2 (the projective plane over the complex field) and let r be the non-negative integer such that $\mathcal{K}_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^2}(r-1)$; r is named the *degree* of the foliation. The Euler sequence $0 \rightarrow \Omega_{\mathbb{P}^2}^1 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^3 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow 0$, in fact the dual sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^3 \rightarrow \mathcal{T}_{\mathbb{P}^2} \rightarrow 0$, allows us to regard \mathcal{F} as induced by a homogeneous vector field

$$\mathbf{X} = U\partial/\partial X_0 + V\partial/\partial X_1 + W\partial/\partial X_2,$$

where U, V, W are homogeneous polynomials of degree r in homogeneous coordinates $(X_0 : X_1 : X_2)$ on \mathbb{P}^2 ; two vector fields define the same foliation if, and only if, they differ by a multiple of the radial vector field of the form $H(X_0, X_1, X_2)(X_0\partial/\partial X_0 + X_1\partial/\partial X_1 + X_2\partial/\partial X_2)$, where H is a homogeneous polynomial of degree $r-1$. A detailed description of this fact, using coordinates, can be seen in [20, Capítulo 1.3].

Returning to the dual point of view, the foliation \mathcal{F} corresponds to a global section of the sheaf $\Omega_{\mathbb{P}^2}^1 \otimes \mathcal{O}_{\mathbb{P}^2}(r+2)$. Taking into account the Euler sequence, this section corresponds to three homogeneous polynomials A, B and C of degree $r+1$, without common factors, such that $X_0A + X_1B + X_2C = 0$ (Euler condition); equivalently, the section can be seen as the homogeneous differential 1-form on \mathbb{A}^3 :

$$\mathbf{\Omega} := AdX_0 + BdX_1 + CdX_2.$$

Notice that the equality

$$\det \begin{pmatrix} dX_0 & dX_1 & dX_2 \\ X_0 & X_1 & X_2 \\ U & V & W \end{pmatrix} = \mathbf{\Omega}$$

allows us to compute $\mathbf{\Omega}$ from \mathbf{X} and that a curve on \mathbb{P}^2 defined by a homogeneous equation $F = 0$ is invariant by \mathcal{F} if, and only if, the polynomial F divides the projective 2-form $\mathbf{\Omega} \wedge dF$.

2.2. Resolution of singularities. Throughout this paper, we shall consider sequences of morphisms

$$(1) \quad X_{n+1} \xrightarrow{\pi_n} X_n \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 := \mathbb{P}^2,$$

such that each π_i is the blow-up of the variety X_i at a closed point $p_i \in X_i$, $1 \leq i \leq n$. The set of closed points $\mathfrak{K} = \{p_1, p_2, \dots, p_n\}$ given by a sequence as (1) will be called a *configuration* over \mathbb{P}^2 and the variety X_{n+1} the *sky* of \mathfrak{K} ; we shall identify two configurations with \mathbb{P}^2 -isomorphic skies. We shall denote by E_{p_i} (respectively, $\tilde{E}_{p_i}, E_{p_i}^*$) the exceptional divisor provided by the blow-up π_i (respectively, its strict transform, its total transform on X_{n+1}). Also, given two points p_i, p_j in \mathfrak{K} , we shall say that p_i is *infinitely near to* p_j (denoted $p_i \geq p_j$) if either $p_i = p_j$ or $i > j$ and $\pi_j \circ \pi_{j+1} \circ \cdots \circ \pi_{i-1}(p_i) = p_j$. The relation \geq is a partial ordering among the points of the configuration \mathfrak{K} . Furthermore, a point p_i will be called *proximate* to other one p_j whenever p_i is in the strict transform of the exceptional divisor E_{p_j} on the surface containing p_i . To represent sequences as (1), we shall use a combinatorial invariant named the *proximity graph*. It is a graph whose vertices correspond to the points p_i in \mathfrak{K} and the edges join points p_i and p_j whenever p_i is proximate to p_j . This edge is dotted excepting the case when $p_i \in E_{p_j}$.

If \mathcal{F} is a foliation on \mathbb{P}^2 , a sequence of morphisms (1) induces, for each $i = 2, 3, \dots, n+1$, a foliation \mathcal{F}_i on X_i given by the pull-back of \mathcal{F} (see [3], for instance). By a result of Seidenberg [36] there exists a *resolution of singularities* of \mathcal{F} , that is, a sequence of blow-ups as (1) such that the foliation \mathcal{F}_{n+1} on the last obtained surface X_{n+1} has only simple singularities. A singularity $p \in U_i$ is *simple* (or *reduced*) if at least one of the eigenvalues α and β of the linear part of the vector field v_i (that are well defined since $v_i(p) = 0$) does not vanish and, assuming $\beta \neq 0$, the quotient α/β is not an strictly positive rational number. These singularities have the property that they cannot be removed by blowing-up.

In the sequel, we shall denote by $\mathcal{S}_{\mathcal{F}}$ the configuration $\{p_i\}_{i=1}^n$ given by the centers of the blow-ups involved in a *minimal* (with respect to the number of blow-ups) resolution of singularities of \mathcal{F} , however in our development we shall not use the whole resolution of singularities, but only the sequence of blow-ups concerning the so-called configuration of *dicritical* points that we define next.

Definition 1. An exceptional divisor E_{p_i} (respectively, a point $p_i \in \mathcal{S}_{\mathcal{F}}$) of a minimal resolution of singularities of a foliation \mathcal{F} on \mathbb{P}^2 is called *non-dicritical* if it is invariant by the foliation \mathcal{F}_{i+1} (respectively, all the exceptional divisors E_{p_j} , with $p_j \geq p_i$, are non-dicritical). Otherwise, E_{p_i} (respectively, p_i) is said to be *dicritical*. We shall denote by $\mathcal{B}_{\mathcal{F}}$ the configuration of dicritical points in $\mathcal{S}_{\mathcal{F}}$ and by $Z_{\mathcal{F}}$ the sky of $\mathcal{B}_{\mathcal{F}}$.

2.3. Foliations having a rational first integral. This paper is devoted to study algebraic integrability of certain type of foliations on the projective plane \mathbb{P}^2 , so we start this brief section by defining this concept.

Definition 2. A *rational first integral* of a foliation \mathcal{F} on \mathbb{P}^2 is a rational map $f : \mathbb{P}^2 \cdots \rightarrow \mathbb{P}^1$ such that the closures of its fibers are invariant curves by \mathcal{F} . Equivalently, and from an algebraic point of view, if f is given by a rational function R , f is a rational first integral if, and only if, $\Omega \wedge dR = 0$. \mathcal{F} is called to be *algebraically integrable* (or that it has a *rational first integral*) whenever there exists such a rational map.

Consider an algebraically integrable foliation \mathcal{F} on \mathbb{P}^2 . The second theorem of Bertini [25] shows that \mathcal{F} admits a *primitive* rational first integral $f : \mathbb{P}^2 \cdots \rightarrow \mathbb{P}^1$ (that is, such that the closures of its general fibers are integral curves). Taking projective coordinates, if $F(X_0, X_1, X_2)$ and $G(X_0, X_1, X_2)$ are the two homogeneous polynomials of the same

degree d which are the components of f , then the closures of the fibers of f are the elements of the irreducible pencil $\mathcal{P}_{\mathcal{F}} := \langle F, G \rangle \subseteq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ and, moreover, any algebraic solution of \mathcal{F} is a component of a curve in $\mathcal{P}_{\mathcal{F}}$. The *degree* (respectively, *genus*) of the first integral f will be the degree (respectively, geometric genus) of a general fiber of $\mathcal{P}_{\mathcal{F}}$.

Let $Z_{\mathcal{F}}$ be the sky of the configuration $\mathcal{B}_{\mathcal{F}}$ (see Definition 1). By comparing both processes, that of elimination of indeterminacies of the rational map f [2, Theorem II.7] and the resolution of the dicritical singularities of \mathcal{F} through $\pi_{\mathcal{F}} : Z_{\mathcal{F}} \rightarrow \mathbb{P}^2$, it can be proved that $\pi_{\mathcal{F}}$ is also the minimal resolution of the indeterminacies of f . Indeed, if $p_i \in \mathfrak{K}$ and \mathfrak{f} and \mathfrak{g} are local equations at p_i of the strict transform of two general elements F and G generating the pencil $\mathcal{P}_{\mathcal{F}}$, then the local solutions of the foliations \mathcal{F}_i at p_i are the irreducible components of the local pencil in the completion with respect to the maximal ideal of \mathcal{O}_{X_i, p_i} generated by \mathfrak{f} and \mathfrak{g} (see [19] for complete details). As a consequence if f is a rational first integral of a foliation \mathcal{F} as above, then the map $\tilde{f} := f \circ \pi_{\mathcal{F}} : Z_{\mathcal{F}} \rightarrow \mathbb{P}^1$ is a morphism.

3. RATIONAL FIRST INTEGRAL WITH GIVEN GENUS

Along this section \mathcal{F} will be a foliation on \mathbb{P}^2 of degree r and $Z_{\mathcal{F}}$ the sky of its configuration $\mathcal{B}_{\mathcal{F}}$ of dicritical points. Denote by $A(Z_{\mathcal{F}})$ the vector space over \mathbb{Q} , $\text{Pic}(Z_{\mathcal{F}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\text{Pic}(Z_{\mathcal{F}})$ stands for the Picard group of the surface $Z_{\mathcal{F}}$. Intersection theory provides a \mathbb{Z} -bilinear form: $\text{Pic}(Z_{\mathcal{F}}) \times \text{Pic}(Z_{\mathcal{F}}) \rightarrow \mathbb{Z}$ which induces a non-degenerate bilinear form over \mathbb{Q} : $A(Z_{\mathcal{F}}) \times A(Z_{\mathcal{F}}) \rightarrow \mathbb{Q}$. The image by this form of a pair $(x, y) \in A(Z_{\mathcal{F}}) \times A(Z_{\mathcal{F}})$ will be denoted $x \cdot y$.

Given a divisor A on $Z_{\mathcal{F}}$, we shall denote by $[A]$ its class in the Picard group $\text{Pic}(Z_{\mathcal{F}})$ and also its image into $A(Z_{\mathcal{F}})$. If C is either a curve on \mathbb{P}^2 or an exceptional divisor, \tilde{C} (respectively, C^*) will denote its strict (respectively, total) transform on the surface $Z_{\mathcal{F}}$ via the composition of blow-ups $\pi_{\mathcal{F}}$. It is well known that the set $\mathbf{B} := \{[L^*]\} \cup \{[E_p^*]\}_{p \in \mathcal{B}_{\mathcal{F}}}$ is a \mathbb{Z} -basis (respectively, \mathbb{Q} -basis) of $\text{Pic}(Z_{\mathcal{F}})$ (respectively, $A(Z_{\mathcal{F}})$), where L denotes a general line on \mathbb{P}^2 .

Now, let us suppose that \mathcal{F} admits a rational first integral f (which we assume to be primitive) and set $\tilde{\mathcal{F}}$ the foliation on $Z_{\mathcal{F}}$ given by the pull-back of \mathcal{F} by $\pi_{\mathcal{F}}$. $\tilde{f} := f \circ \pi_{\mathcal{F}}$ is a first integral of $\tilde{\mathcal{F}}$ and the integral invariant curves of $\tilde{\mathcal{F}}$ (which coincide with the integral components of the fibers of \tilde{f}) are, on the one hand, the strict transforms on $Z_{\mathcal{F}}$ of the integral invariant curves of \mathcal{F} and, on the other hand, the strict transforms of the exceptional divisors E_{p_i} (with $p_i \in \mathcal{B}_{\mathcal{F}}$) which are non-dicritical. Denote by $D_{\tilde{f}}$ a general fiber of \tilde{f} .

It is clear that every curve defined by an element of the pencil $\mathcal{P}_{\mathcal{F}}$ is the push-forward by $\pi_{\mathcal{F}}$ of some curve in $Z_{\mathcal{F}}$ that is linearly equivalent to $D_{\tilde{f}}$; so there is an inclusion of $\mathcal{P}_{\mathcal{F}}$ into the space of global sections $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*} \mathcal{O}_{Z_{\mathcal{F}}}(D_{\tilde{f}}))$. This inclusion is, in fact, an equality. Indeed, reasoning by contradiction, if we assume that the inclusion is strict, it follows that there exist infinitely many linearly equivalent to $D_{\tilde{f}}$ curves C on $Z_{\mathcal{F}}$ whose push-forwards to \mathbb{P}^2 are not defined by elements in the pencil $\mathcal{P}_{\mathcal{F}}$. But, taking into account that $D_{\tilde{f}}^2 = 0$, all these curves C satisfy $D_{\tilde{f}} \cdot C = 0$ and, therefore, they must be contracted by \tilde{f} ; so their push-forwards to \mathbb{P}^2 are reducible curves whose integral components are also components of curves defined by elements in $\mathcal{P}_{\mathcal{F}}$. This is a contradiction because the pencil $\mathcal{P}_{\mathcal{F}}$ is irreducible.

Notice that the invariant by $\tilde{\mathcal{F}}$ curves C are those that satisfy $D_{\tilde{f}} \cdot C = 0$ (because their irreducible components must be contracted by \tilde{f}). This condition and [26, Corollary 1.21] imply that the class $[C]$ of C in $A(Z_{\mathcal{F}})$ belongs to the boundary of the cone of curves of the surface $Z_{\mathcal{F}}$ and so $C^2 \leq 0$.

We summarize the above ideas in the following result:

Proposition 1. *Let \mathcal{F} and f be as above and let $D_{\tilde{f}}$ be a general fiber of \tilde{f} . Then,*

- (a) $h^0(Z_{\mathcal{F}}, \mathcal{O}_{Z_{\mathcal{F}}}(D_{\tilde{f}})) := \dim \left(H^0(Z_{\mathcal{F}}, \mathcal{O}_{Z_{\mathcal{F}}}(D_{\tilde{f}})) \right) = 2$.
- (b) *A curve C on $Z_{\mathcal{F}}$ is invariant by $\tilde{\mathcal{F}}$ if and only if $D_{\tilde{f}} \cdot C = 0$.*
- (c) *If C is a curve on $Z_{\mathcal{F}}$ which is invariant by $\tilde{\mathcal{F}}$ then $C^2 \leq 0$.*

Remark 1. The equality $h^0(Z_{\mathcal{F}}, \mathcal{O}_{Z_{\mathcal{F}}}(D_{\tilde{f}})) = 2$ proves that, to compute a primitive rational first integral f of \mathcal{F} , it is enough to know a divisor T linearly equivalent to the strict transform on $Z_{\mathcal{F}}$ of a general fiber of the pencil $\mathcal{P}_{\mathcal{F}}$ and two linearly independent global sections of $\pi_{\mathcal{F}*} \mathcal{O}_{Z_{\mathcal{F}}}(T)$, which will be the components of f .

The morphism $\tilde{f} : Z_{\mathcal{F}} \rightarrow \mathbb{P}^1$ is a fibration of the surface $Z_{\mathcal{F}}$ by the curve \mathbb{P}^1 in the sense that \tilde{f} is surjective and with connected fibers. Taking duals as $\mathcal{O}_{Z_{\mathcal{F}}}$ -modules in the corresponding to \tilde{f} sequence of differentials

$$0 \rightarrow \tilde{f}^* \Omega_{\mathbb{P}^1}^1 \rightarrow \Omega_{Z_{\mathcal{F}}}^1 \rightarrow \Omega_{Z_{\mathcal{F}}/\mathbb{P}^1} \rightarrow 0,$$

one gets

$$0 \rightarrow \mathcal{T}_{Z_{\mathcal{F}}/\mathbb{P}^1} \rightarrow \mathcal{T}_{Z_{\mathcal{F}}} \rightarrow \tilde{f}^* \mathcal{T}_{\mathbb{P}^1},$$

where $\mathcal{T}_{\mathbb{P}^1}$ and $\mathcal{T}_{Z_{\mathcal{F}}}$ denote the tangent sheaves of \mathbb{P}^1 and $Z_{\mathcal{F}}$, and $\mathcal{T}_{Z_{\mathcal{F}}/\mathbb{P}^1}$ the relative tangent sheaf of the fibration, which is an invertible sheaf [37, Section 1]. The morphism $\mathcal{T}_{Z_{\mathcal{F}}/\mathbb{P}^1} \rightarrow \mathcal{T}_{Z_{\mathcal{F}}}$ is given by the differential of \tilde{f} and it defines the foliation $\tilde{\mathcal{F}}$, therefore we obtain the equality $\mathcal{K}_{\tilde{\mathcal{F}}} = \mathcal{T}_{Z_{\mathcal{F}}/\mathbb{P}^1}^{-1}$. From [37, Lemma 1.1], it follows that

$$\mathcal{K}_{\tilde{\mathcal{F}}} = \mathcal{T}_{Z_{\mathcal{F}}/\mathbb{P}^1}^{-1} = \mathcal{K}_{Z_{\mathcal{F}}} \otimes \tilde{f}^* \mathcal{K}_{\mathbb{P}^1}^{-1} \otimes \mathcal{O}_{Z_{\mathcal{F}}} \left(- \sum_{i \in I} (n_i - 1) G_i \right),$$

where $\mathcal{K}_{Z_{\mathcal{F}}}$ and $\mathcal{K}_{\mathbb{P}^1}$ are the canonical sheaves of $Z_{\mathcal{F}}$ and \mathbb{P}^1 , respectively, and $\{G_i\}_{i \in I}$ the set of integral components of the singular fibers of \tilde{f} , n_i being the multiplicity of G_i in the fiber to which belongs to. If we take divisors $K_{\tilde{\mathcal{F}}}$ and $K_{Z_{\mathcal{F}}}$ such that $\mathcal{K}_{\tilde{\mathcal{F}}} = \mathcal{O}_{Z_{\mathcal{F}}}(K_{\tilde{\mathcal{F}}})$ and $\mathcal{K}_{Z_{\mathcal{F}}} = \mathcal{O}_{Z_{\mathcal{F}}}(K_{Z_{\mathcal{F}}})$, the above equality may be rewritten in the following form

$$(2) \quad K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}} \sim 2D_{\tilde{f}} - \sum_{i \in I} (n_i - 1) G_i,$$

where \sim means linear equivalence.

The linear equivalence class of the divisor $K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}$ can also be expressed in terms of the above basis \mathbf{B} of $\text{Pic}(Z_{\mathcal{F}})$. Indeed, denote, as above, the degree of \mathcal{F} by r and by $\nu_p(\mathcal{F})$ the algebraic multiplicity at p of the foliation given by the pull-back of \mathcal{F} on the surface to which p belongs. Set $\epsilon_p(\mathcal{F})$ the value 0 (respectively, 1) whenever the exceptional divisor E_p is non-dicritical (respectively, dicritical). Then, by [6, Proposition 1.1], it happens that

$$\pi^* K_{\mathcal{F}} - K_{Z_{\mathcal{F}}} \sim \sum_{p \in \mathcal{B}_{\mathcal{F}}} (\nu_p(\mathcal{F}) + \epsilon_p(\mathcal{F}) - 1) E_p^*,$$

which gives the following equivalence

$$(3) \quad K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}} \sim (r+2)L^* - \sum_{p \in \mathcal{B}_{\mathcal{F}}} (\nu_p(\mathcal{F}) + \epsilon_p(\mathcal{F}))E_p^*.$$

Next, we provide the concepts and results that will allow us to state the algorithms that prove theorems 1 to 3 in this paper. For a while, we shall assume that the foliation \mathcal{F} need not to have a rational first integral. Stand $\text{dic}(\mathcal{F})$ for the number of dicritical exceptional divisors appearing in the minimal resolution of \mathcal{F} . The existence of a set of invariant curves for \mathcal{F} as we are going to define is an hypothesis in Theorem 3.

Definition 3. Let \mathcal{F} be a foliation on \mathbb{P}^2 and suppose that $s := \text{dic}(\mathcal{F}) \geq 3$. A $[\text{dic}(\mathcal{F}) - 2]$ -set of independent algebraic solutions of \mathcal{F} is a set $S = \{C_1, C_2, \dots, C_{s-2}\}$ of $s - 2$ integral projective curves on \mathbb{P}^2 , invariant by \mathcal{F} , and such that the family of classes in $A(Z_{\mathcal{F}})$

$$V(S) := \left\{ [\tilde{C}_1], [\tilde{C}_2], \dots, [\tilde{C}_{s-2}], [K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}] \right\} \cup \left\{ [\tilde{E}_p] \mid p \in \mathcal{B}_{\mathcal{F}} \text{ and } E_p \text{ is non-dicritical} \right\}$$

is \mathbb{Q} -linearly independent.

Let us consider the projective space over the field \mathbb{Q} associated with the \mathbb{Q} -vector space $A(Z_{\mathcal{F}})$:

$$\mathbb{P}A(Z_{\mathcal{F}}) := (A(Z_{\mathcal{F}}) \setminus \{0\})/\mathbb{Q},$$

and denote $\mathbb{Q}x$ the element in $\mathbb{P}A(Z_{\mathcal{F}})$ defined by the class $x \in A(Z_{\mathcal{F}})$. The *primitive lattice representative* of an element $\mathbb{Q}x$ in $\mathbb{P}A(Z_{\mathcal{F}})$ is the class in $A(Z_{\mathcal{F}})$ of a divisor $a_0L^* - \sum_{i=1}^n a_i E_{p_i}^*$ included in $\mathbb{Q}x$ satisfying $a_0 > 0$, $a_i \in \mathbb{Z}$ for all i and $\gcd(a_0, a_1, \dots, a_n) = 1$.

Sets S as in Definition 3 determine the following subsets of $\mathbb{P}A(Z_{\mathcal{F}})$, which will be useful in our algorithms:

$$\mathcal{R}_{\mathcal{F}}(S) := \{\mathbb{Q}x \in \mathbb{P}A(Z_{\mathcal{F}}) \mid x^2 = 0 \text{ and } z \cdot x = 0 \text{ for all } z \in V(S)\}.$$

When $\text{dic}(\mathcal{F}) \leq 2$, we shall say that $S = \emptyset$ is a $[\text{dic}(\mathcal{F}) - 2]$ -set of independent algebraic solutions of \mathcal{F} and $\mathcal{R}_{\mathcal{F}}(\emptyset)$ is defined as above, $V(\emptyset)$ being the set of classes

$$\left\{ [K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}] \right\} \cup \left\{ [\tilde{E}_p] \mid p \in \mathcal{B}_{\mathcal{F}} \text{ and } E_p \text{ is non-dicritical} \right\}.$$

Assume again that \mathcal{F} has a (primitive) rational first integral f , let D be a general element of $\mathcal{P}_{\mathcal{F}}$ and suppose that $\pi_{\mathcal{F}}$ is the composition of a sequence as in (1). Notice that $[\tilde{D}] = [D_{\tilde{\mathcal{F}}}]$, where $D_{\tilde{\mathcal{F}}}$ is as in the beginning of this section. Set

$$T_{\mathcal{F}} := s_0L^* - \sum_{i=1}^n s_i E_{p_i}^*$$

such that $[T_{\mathcal{F}}]$ is the primitive lattice representative of $\mathbb{Q}[\tilde{D}]$; Clearly, this implies that $[\tilde{D}] = \gamma[T_{\mathcal{F}}]$ for some positive integer γ . Then, we can state the following

Lemma 1. *Let \mathcal{F} be a foliation on \mathbb{P}^2 having a rational first integral such that it admits a $[\text{dic}(\mathcal{F}) - 2]$ -set of independent algebraic solutions S . Then $\mathbb{Q}[T_{\mathcal{F}}] \in \mathcal{R}_{\mathcal{F}}(S)$ and the cardinality of $\mathcal{R}_{\mathcal{F}}(S)$ is either 1 or 2.*

Proof. The point $\mathbb{Q}[T_{\mathcal{F}}] \in \mathbb{P}A_{Z_{\mathcal{F}}}$ belongs to $\mathcal{R}_{\mathcal{F}}(S)$ as a consequence of the equivalence (2) and clause (b) of Proposition 1. Let us prove the second assertion.

Set $\mathcal{B}_{\mathcal{F}} = \{p_1, p_2, \dots, p_n\}$ and take projective coordinates $(X_0 : X_1 : \dots : X_n)$ of $\mathbb{P}A(Z_{\mathcal{F}})$ with respect to the basis \mathbf{B} of $A_{Z_{\mathcal{F}}}$. Notice that

$$\mathcal{R}_{\mathcal{F}}(S) = \mathcal{Q} \cap \mathbb{P}(V(S))^{\perp},$$

where \mathcal{Q} is the quadric defined by the equation $X_0^2 - \sum_{i=1}^n X_i^2 = 0$ and $\mathbb{P}\langle V(S) \rangle^\perp$ the projective subspace of $\mathbb{P}A(Z_{\mathcal{F}})$ associated with the orthogonal subspace (with respect to the bilinear form \cdot defined at the beginning of this section) of the linear subspace $\langle V(S) \rangle$ of $A(Z_{\mathcal{F}})$ spanned by $V(S)$.

Let $\mathbb{Q}q$ be a point in $\mathcal{Q} \cap \mathbb{P}\langle V(S) \rangle^\perp$. The polar hyperplane $H_q := \{\mathbb{Q}x \in \mathbb{P}A(Z_{\mathcal{F}}) \mid q \cdot x = 0\}$ of the point $\mathbb{Q}q$ with respect to the quadric \mathcal{Q} is defined by the equation

$$Q_0 X_0 - \sum_{i=1}^n Q_i X_i = 0,$$

where $(Q_0 : Q_1 : \dots : Q_n)$ are projective coordinates of $\mathbb{Q}q$. On the one hand, the hyperplane H_q is tangent to the quadric since $\mathbb{Q}q \in \mathcal{Q}$ and, on the other hand, it contains $\mathbb{P}\langle V(S) \rangle$ because $(L^*)^2 = 1$, $L^* \cdot E_{p_i}^* = 0$, $(E_{p_i}^*)^2 = -1$ for all i and $E_{p_i}^* \cdot E_{p_j}^* = 0$ if $i \neq j$, and because $Q_0[L^*] - \sum_{i=1}^n Q_i[E_{p_i}^*]$ is orthogonal to $V(S)$.

Let $t \leq n$ be a nonnegative integer. Denote by $Gr_t(\mathbb{P}A(Z_{\mathcal{F}}))$ the set of t -dimensional projective linear subspaces of $\mathbb{P}A(Z_{\mathcal{F}})$, that is, the t -Grassmannian variety of $\mathbb{P}A(Z_{\mathcal{F}})$. The polarity map (with respect to the quadric \mathcal{Q}) $\mathbb{Q}p \mapsto H_p$ extends to polarity maps on the Grassmannians

$$G_t : Gr_t(\mathbb{P}A(Z_{\mathcal{F}})) \rightarrow Gr_{n-1-t}(\mathbb{P}A(Z_{\mathcal{F}})).$$

Assume that $\text{dic}(\mathcal{F}) = s > 1$. Then the dimension of $\mathbb{P}\langle V(S) \rangle$ is $n - 2$ and hyperplanes H_q as above (tangent to \mathcal{Q} and containing $\mathbb{P}\langle V(S) \rangle$) are the tangent hyperplanes at the points of intersection between the quadric and the polar variety $W := G_{n-2}(\mathbb{P}\langle V(S) \rangle)$. W is a projective line because it belongs to $Gr_{n-1-(n-2)}(\mathbb{P}A(Z_{\mathcal{F}}))$ and, therefore, the number of intersection points of $\mathbb{P}\langle V(S) \rangle^\perp$ with the quadric \mathcal{Q} must be less than or equal to 2. Since H_q determines $\mathbb{Q}q$ there are, at most, 2 possibilities for such points $\mathbb{Q}q$.

Otherwise $s = 1$ and then $\mathbb{P}\langle V(\emptyset) \rangle$ has dimension $n - 1$, so its polar variety

$$W := G_{n-1}(\mathbb{P}\langle V(\emptyset) \rangle)$$

is a point $\mathbb{Q}q$ of the quadric (which corresponds to $\mathbb{P}\langle V(\emptyset) \rangle^\perp$) and therefore $H_q = \mathbb{P}\langle V(\emptyset) \rangle$. \square

Remark 2. The above proof shows how to compute $\mathcal{R}_{\mathcal{F}}(S)$ (in terms of the basis \mathbf{B}). In the case when $\text{dic}(\mathcal{F}) \leq 2$, one can perform it only using data obtained from the resolution of singularities of \mathcal{F} . Indeed, with these data one is able to compute the class $[K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}]$ (see formula (3)) and, for each $p \in \mathcal{B}_{\mathcal{F}}$, one has $[\tilde{E}_p] = [E_p^*] - \sum_q [E_q^*]$, where q runs over the set of points of $\mathcal{B}_{\mathcal{F}}$ which are proximate to p . When $\text{dic}(\mathcal{F}) > 2$ one also needs the coordinates in the basis \mathbf{B} of the set of classes of strict transforms on $Z_{\mathcal{F}}$ of the invariant by \mathcal{F} curves in S , $\{[\tilde{C}_i]\}_{i=1}^{\text{dic}(\mathcal{F})-2}$.

Remark 3. If we do not assume that a foliation \mathcal{F} has a rational first integral, then it also holds that the cardinality of the set $\mathcal{R}_{\mathcal{F}}(S)$ is less than or equal to 2; moreover, in this case, $\mathcal{R}_{\mathcal{F}}(S)$ may be empty. In addition, if $\text{dic}(\mathcal{F}) = 1$, $\mathcal{R}_{\mathcal{F}}(S)$ is either empty or its unique element is $\mathbb{P}\langle V(\emptyset) \rangle^\perp$. These facts are straightforward from the proof of Lemma 1.

The following algorithm is also a *proof of clause b) of Theorems 2 and 3*. It can be applied to foliations \mathcal{F} admitting a $[\text{dic}(\mathcal{F}) - 2]$ -set of independent algebraic solutions and it decides about existence of a rational first integral of \mathcal{F} of a prefixed genus $g \neq 1$ (computing it in the affirmative case). The algorithm works because Proposition 1 and Lemma 1 hold and, when \mathcal{F} admits a rational first integral, the divisor $D_{\tilde{\mathcal{F}}}$ in Proposition 1 must satisfy the Adjunction formula.

Algorithm 1.

Input: A projective differential 1-form Ω defining \mathcal{F} , a non-negative integer $g \neq 1$, the configuration $\mathcal{B}_{\mathcal{F}}$ and a $[\text{dic}(\mathcal{F}) - 2]$ -set $S = \{C_1, C_2, \dots, C_{s-2}\}$ of independent algebraic solutions of \mathcal{F} .

Output: Either a primitive rational first integral of \mathcal{F} of genus g or “0” (which means that such a first integral does not exist).

1. If $\text{dic}(\mathcal{F}) \leq 2$, define $S := \emptyset$.
2. Compute the set $\mathcal{R}_{\mathcal{F}}(S)$. If $\mathcal{R}_{\mathcal{F}}(S) = \emptyset$ then return “0”. Else, let $\mathcal{L} := \mathcal{R}_{\mathcal{F}}(S)$.
3. While $\mathcal{L} \neq \emptyset$:
 - 3.1. Choose $\ell = \mathbb{Q}q \in \mathcal{L}$ and let $\mathcal{L} := \mathcal{L} \setminus \{\ell\}$.
 - 3.2. Compute coordinates in the basis \mathbf{B} , $(d, -m_1, -m_2, \dots, -m_n)$, of the primitive lattice representative of ℓ . If $m_i < 0$ for some i , then go to Step 3.
 - 3.3. Compute

$$\alpha := \frac{2(g-1)}{-3d + \sum_{i=1}^n m_i}.$$

If α is not a positive integer then go to Step 3.

- 3.4. Compute the space of global sections

$$H^0(\mathbb{P}^2, \pi_{\mathcal{F}*} \mathcal{O}_{Z_{\mathcal{F}}}(\alpha T)) \subseteq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(\alpha d)),$$

where $T := dL^* - \sum_{i=1}^n m_i E_i^*$.

- 3.5. If the dimension of the above space is not 2, then go to Step 3. Else, choose two homogeneous polynomials F and G of degree αd generating that space.
- 3.6. If $\Omega \wedge (GdF - FdG) = 0$ then the rational map $\mathbb{P}^2 \cdots \rightarrow \mathbb{P}^1$ whose components are F and G is a primitive rational first integral of \mathcal{F} ; return it. Else, go to step 3.
4. Return “0”. \square

Remark 4. The points of the configuration $\mathcal{B}_{\mathcal{F}}$ are used in the last steps (from 3.4 to 3.6) of the previous algorithm because, there, it is required to compute and use global sections of sheaves on \mathbb{P}^2 obtained by pushing forward invertible sheaves on $Z_{\mathcal{F}}$. To perform the remaining steps the algorithm only requires the following data: the degree of \mathcal{F} , the genus g of a general invariant curve, the proximity relations among the points of the configuration $\mathcal{B}_{\mathcal{F}}$, the above defined numbers $\nu_p(\mathcal{F})$ and $\epsilon_p(\mathcal{F})$ for each point $p \in \mathcal{B}_{\mathcal{F}}$ and (only when $\text{dic}(\mathcal{F}) \geq 3$) the degrees of the curves in S and their multiplicities at the points of $\mathcal{B}_{\mathcal{F}}$.

To end this section *we shall prove clause a) of Theorems 2 and 3*. In both cases, this clause is an easy consequence of Algorithm 1. Indeed, if G is a bound on the genus of the rational first integral of a foliation \mathcal{F} (assuming that it is different from 1), then the degree of the first integral can be bounded by the maximum of the numbers $2(G-1)d / (\sum_{i=1}^n m_i - 3d)$ corresponding to coordinates of primitive lattice representatives $(d, -m_1, -m_2, \dots, -m_n)$ of the elements in $\mathcal{R}_{\mathcal{F}}(S)$ determined by a $[\text{dic}(\mathcal{F}) - 2]$ -set S of algebraic solutions (notice that, by Lemma 1, there are, at most, two possibilities for these elements). To compute this bound, one needs the following data: the bound G , the degree of \mathcal{F} , the degrees of the curves in S and their multiplicities at the points of $\mathcal{B}_{\mathcal{F}}$ (only when $\text{dic}(\mathcal{F}) \geq 3$), the proximity relations among the points of the configuration $\mathcal{B}_{\mathcal{F}}$ and the above defined numbers $\nu_p(\mathcal{F})$ and $\epsilon_p(\mathcal{F})$ for each point $p \in \mathcal{B}_{\mathcal{F}}$. Since the last

two data only depend on the local analytic type of the dicritical singularities of \mathcal{F} , we conclude clause a) of the mentioned theorems. \square

4. FOLIATIONS WITH ONLY ONE DICRITICAL DIVISOR

In this section we shall prove the main result of this paper (Theorem 1), which for foliations \mathcal{F} on \mathbb{P}^2 such that $\text{dic}(\mathcal{F}) = 1$ solves the Poincaré problem and gives an algorithm to decide algebraic integrability. First we shall show the algorithm (which proves clause b)) and, then, we shall deduce clause a) as a consequence of the results used to justify it.

For a start, fix a foliation \mathcal{F} of degree r , having a rational first integral and such that $\text{dic}(\mathcal{F}) = 1$. To avoid trivialities, we also assume that the cardinality of $\mathcal{B}_{\mathcal{F}}$ is greater than 1 (note that otherwise the foliation is defined by a pencil of lines). The following results will allow us state the mentioned algorithm which, as we shall see, consists, in fact, of two algorithms that must be applied consecutively.

Lemma 2. *Let \mathcal{F} be as above. All the curves in the irreducible pencil $\mathcal{P}_{\mathcal{F}}$ defined in Section 2 are irreducible and, at most two of them, are non-reduced.*

Proof. From [24, Corollary 2] and the subsequent remark, it can be deduced that the cardinality of the set of dicritical exceptional divisors $\text{dic}(\mathcal{F})$ attached to a foliation \mathcal{F} satisfies the following inequality

$$1 + \sum (e_R - 1) \leq \text{dic}(\mathcal{F}),$$

where the sum is taken over the set of curves R in the pencil $\mathcal{P}_{\mathcal{F}}$ and e_R stands for the number of different integral components of R . As a consequence any curve in $\mathcal{P}_{\mathcal{F}}$ is irreducible because $\text{dic}(\mathcal{F}) = 1$. The second part of the statement follows from a result of Poincaré in [34, page 187 of I]. \square

Proposition 2. *Let \mathcal{F} and $\mathcal{P}_{\mathcal{F}}$ be as in Lemma 2. Let \mathcal{A} be the set of integral components of the non-reduced curves in $\mathcal{P}_{\mathcal{F}}$. Then $\deg(A) < \deg(\mathcal{F}) + 2$ for all $A \in \mathcal{A}$. Moreover, if \mathcal{A} has two elements (say A_1 and A_2) then $\deg(A_1) + \deg(A_2) = \deg(\mathcal{F}) + 2$.*

Proof. Let δ be the degree of a primitive rational first integral of \mathcal{F} and let $r = \deg(\mathcal{F})$. Set θ the number of non-reduced curves in the pencil $\mathcal{P}_{\mathcal{F}}$ and χ the sum of the degrees of the integral components of these curves. Notice that $\theta \leq 2$ by Lemma 2.

Taking, in (2), intersection products with the total transform of a general line of \mathbb{P}^2 , one has that

$$(4) \quad 2\delta - r - 2 = \sum (e_R - 1) \deg(R),$$

where the sum is taken over the set of integral components R of the curves in $\mathcal{P}_{\mathcal{F}}$ and e_R denotes the multiplicity of R as a component of such curves. Therefore

$$2\delta - r - 2 = \theta\delta - \chi.$$

This concludes the proof of the first assertion because $\theta = 1$ implies $\delta < r + 2$ and $\theta = 2$ shows $\chi = r + 2$, and in both cases $r + 2$ is a strict upper bound of the degrees of the mentioned integral components. The last assertion holds because $\theta = 2$ and it is the equality $\chi = r + 2$. \square

Next we shall define, for an arbitrary foliation \mathcal{F} (which needs not to have a rational first integral), a set of divisors on $Z_{\mathcal{F}}$ that will be useful to state our last result before

giving the algorithms that will prove Theorem 1. Let x be a positive integer and denote by $\Gamma(x)$ the (finite) set of divisors $C = xL^* - \sum_{p \in \mathcal{B}_{\mathcal{F}}} y_p E_p^*$ satisfying the following conditions:

- (a) $0 \leq y_p \leq x$ for all $p \in \mathcal{B}_{\mathcal{F}}$.
- (b) $C \cdot \tilde{E}_p \geq 0$ for all $p \in \mathcal{B}_{\mathcal{F}}$.
- (c) Either $C^2 = K_{Z_{\mathcal{F}}} \cdot C = -1$, or $C^2 \leq 0$, $K_{Z_{\mathcal{F}}} \cdot C \geq 0$ and $C^2 + K_{Z_{\mathcal{F}}} \cdot C \geq -2$.
- (d) The complete linear system $|C|$ has (projective) dimension 0.

Lemma 3. *Let \mathcal{F} be as in Lemma 2. Any integral component of a non-reduced curve in $\mathcal{P}_{\mathcal{F}}$ is the push-forward $\pi_{\mathcal{F}*}|C|$ for some divisor C on $Z_{\mathcal{F}}$ which belongs to $\bigcup_{x < r+2} \Gamma(x)$.*

Proof. Let H be an integral component of a non-reduced curve of the pencil $\mathcal{P}_{\mathcal{F}}$. Let x be the degree of H and, for each $p \in \mathcal{B}_{\mathcal{F}}$, denote by y_p the multiplicity at p of the strict transform of H on the surface to which p belongs. Then, it holds the following linear equivalence between divisors on $Z_{\mathcal{F}}$:

$$\tilde{H} \sim C := xL^* - \sum_{p \in \mathcal{B}_{\mathcal{F}}} y_p E_p^*,$$

and it happens that $H = \pi_{\mathcal{F}*}|C|$. Let us see that C belongs to $\Gamma(x)$. Indeed, condition (a) of the definition of $\Gamma(x)$ is clear, (b) is true because \tilde{H} is irreducible and non-exceptional, (c) follows from statement (c) in Proposition 1 and the Adjunction formula and (d) holds because the integral components of the curves in $\pi_{\mathcal{F}*}|C|$ are also integral components of the curves in the pencil $\mathcal{P}_{\mathcal{F}}$. This concludes the proof because $x < r + 2$ by Proposition 2. \square

Now we shall give two algorithms (Algorithm 2 and Algorithm 3) that, successively applied, *prove part b) of Theorem 1*.

Firstly we shall describe the ideas supporting Algorithm 2. Consider an arbitrary foliation \mathcal{F} on \mathbb{P}^2 such that $\text{dic}(\mathcal{F}) = 1$ and consider the set $\mathcal{R}_{\mathcal{F}}(\emptyset)$. By Remark 3, this set must have cardinality ≤ 1 . When $\mathcal{R}_{\mathcal{F}}(\emptyset) = \emptyset$ we can ensure that \mathcal{F} has no rational first integral. Otherwise, it is straightforward to obtain the unique candidate T for the divisor $T_{\mathcal{F}}$ defined before Lemma 1 (see Remark 3). If \mathcal{F} has a rational first integral, then, by Lemma 2, there exist, at most, two integral components of non-reduced curves in $\mathcal{P}_{\mathcal{F}}$. Moreover, by Lemma 3, the classes in $A(Z_{\mathcal{F}})$ of their strict transforms on $Z_{\mathcal{F}}$ must be in the finite set $\bigcup_{x < \text{deg}(\mathcal{F})+2} \Gamma(x)$ and furthermore by Proposition 1, they must be orthogonal to T . Taking into account these considerations, it can be computed a set \mathcal{A} such that, if \mathcal{F} had rational first integral, its elements would be exactly the integral components of the non-reduced curves in $\mathcal{P}_{\mathcal{F}}$. The precise algorithm is the following one:

Algorithm 2.

Input: A projective differential 1-form Ω defining a foliation \mathcal{F} on \mathbb{P}^2 of degree r such that $\text{dic}(\mathcal{F}) = 1$ and the configuration $\mathcal{B}_{\mathcal{F}}$.

Output: Either a pair (T, \mathcal{A}) , where T is a candidate for the divisor $T_{\mathcal{F}}$ and \mathcal{A} is a candidate set for the set of integral components of the non-reduced curves of $\mathcal{P}_{\mathcal{F}}$ (in case \mathcal{F} being algebraically integrable), or “0” (which implies that \mathcal{F} is not algebraically integrable).

1. Compute the set $\mathcal{R}_{\mathcal{F}}(\emptyset)$. If $\mathcal{R}_{\mathcal{F}}(\emptyset) = \emptyset$ then return “0”.
2. Take the unique element $\ell \in \mathcal{R}_{\mathcal{F}}(\emptyset)$.

3. Set $(d, -m_1, -m_2, \dots, -m_n)$ coordinates in the basis \mathbf{B} of the primitive lattice representative of ℓ . If $m_i < 0$ for some i , then return “0”. Else, let $T := dL^* - \sum_{i=1}^n m_i E_i^*$.
4. Seek $C \in \bigcup_{x < \deg(\mathcal{F})+2} \Gamma(x)$ such that $T \cdot C = 0$ and $\pi_{\mathcal{F}*}|C|$ is an invariant curve. Perform this in the following manner: order the divisors in $[T]^\perp \cap \left(\bigcup_{x < \deg(\mathcal{F})+2} \Gamma(x) \right)$ in a list C_1, C_2, \dots satisfying the following implication:
$$i < j \Rightarrow C_i \in \Gamma(x_i) \text{ and } C_j \in \Gamma(x_j) \text{ with } x_i \leq x_j;$$
then, set $C := C_{i_0}$, i_0 being the minimum index i such that $\pi_{\mathcal{F}*}|C_i|$ is an invariant by \mathcal{F} curve.
5. If such a divisor does not exist, then return (T, \emptyset) . Else,
 - 5.1. Set x such that $C \in \Gamma(x)$.
 - 5.2. If $x > \frac{r+2}{2}$, then return $(T, \{C\})$. Else,
 - 5.2.1. Find $W \in [T]^\perp \cap \Gamma(r+2-x)$ such that the curve $\pi_{\mathcal{F}*}|W|$ is invariant and it has not C as integral component.
 - 5.2.2. If such a curve W does not exist, then return $(T, \{C\})$. Else return $(T, \{C, W\})$.

Finally, we are going to give Algorithm 3, which combined with Algorithm 2 gives rise to our above mentioned algorithm to decide algebraic integrability. Its inputs will be a differential 1-form Ω defining \mathcal{F} and the candidate pair provided by Algorithm 2; the output will be either a primitive rational first integral for \mathcal{F} or “0” (which means that \mathcal{F} is not algebraically integrable). Algorithm 3 is supported on the following result:

Proposition 3. *Let \mathcal{F} be a foliation on \mathbb{P}^2 such that $\text{dic}(\mathcal{F}) = 1$ and it has a rational first integral f . Let \tilde{D}_f be a general fiber of $\tilde{f} = f \circ \pi_{\mathcal{F}}$ and γ the positive integer such that $[\tilde{D}_f] = \gamma[T_{\mathcal{F}}]$, $[T_{\mathcal{F}}]$ being the primitive lattice representative of $\mathbb{Q}[\tilde{D}_f]$. Let \mathcal{A} be the set of integral components of the non-reduced curves in $\mathcal{P}_{\mathcal{F}}$. Then, the following statements hold:*

- (a) *If $\mathcal{A} = \emptyset$, then $\gamma = \frac{r+2}{2s_0}$, s_0 being the first coordinate of the class $[T_{\mathcal{F}}]$ in the basis \mathbf{B} .*
- (b) *If $\mathcal{A} = \{A_1\}$, then $\gamma = \frac{r+2-\deg(A_1)}{s_0}$.*
- (c) *Otherwise $\mathcal{A} = \{A_1, A_2\}$ and then $\gamma = \frac{\text{lcm}(\deg(A_1), \deg(A_2))}{s_0}$.*

Proof. (a) and (b) are direct consequences of equality (4). To prove (c) observe first that, by Lemma 2, there exist positive integers n_1, n_2 such that the pencil $\mathcal{P}_{\mathcal{F}}$ is spanned by homogeneous polynomials giving equations of $n_1 A_1$ and $n_2 A_2$. Moreover n_1 and n_2 are relatively primes because the pencil is irreducible. Since $n_1 \deg(A_1) = n_2 \deg(A_2)$ we have that $n_1 = \frac{\deg(A_2)}{\text{gcd}(\deg(A_1), \deg(A_2))}$ and, therefore, the degree of a general integral invariant curve is

$$\frac{\deg(A_1) \deg(A_2)}{\text{gcd}(\deg(A_1), \deg(A_2))} = \text{lcm}(\deg(A_1), \deg(A_2)).$$

□

Algorithm 3.

Input: A projective differential 1-form Ω defining a foliation \mathcal{F} on \mathbb{P}^2 such that $\text{dic}(\mathcal{F}) = 1$ and a candidate pair (T, \mathcal{A}) given by the output of Algorithm 2.

Output: Either a rational first integral of \mathcal{F} or “0” (which means that \mathcal{F} has no such a first integral).

1. Compute γ according with the values provided in Proposition 3 (taking the set \mathcal{A} and the divisor T instead of $T_{\mathcal{F}}$). If either γ is not an integer or γT is not a divisor then return “0”.
2. Compute a basis of the space of global sections $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\gamma T))$.
3. If $h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\gamma T)) \neq 2$, then return “0”.
4. Else, take a basis $\{F, G\}$ and check the equality

$$\Omega \wedge (GdF - FdG) = 0.$$

If it is satisfied, then the rational map $\mathbb{P}^2 \cdots \rightarrow \mathbb{P}^1$ whose components are F and G is a primitive rational first integral of \mathcal{F} ; return it. Else, return “0”.

Notice that to run Algorithms 2 and 3, one only performs very simple integer arithmetics and resolution of systems of linear equations.

We finish this section by giving the *proof of clause a) of Theorem 1*. That is, we are going to prove the inequality

$$d \leq \frac{(r+2)^2}{4},$$

where r is the degree of a foliation \mathcal{F} on \mathbb{P}^2 having a rational first integral of degree d and such that $\text{dic}(\mathcal{F}) = 1$. Let \mathcal{A} be as in Proposition 3. If either this set is empty or its cardinality is 1, the above inequality is trivially satisfied by clauses (a) and (b) of Proposition 3. Therefore, let us take $\mathcal{A} = \{A_1, A_2\}$. Applying Proposition 2 and clause (c) of Proposition 3 one has that

$$d = \text{lcm}(\deg(A_1), r+2 - \deg(A_1)) \leq \deg(A_1)(r+2 - \deg(A_1)) \leq \frac{(r+2)^2}{4},$$

completing the proof of Theorem 1. \square

Remark 5. Let \mathcal{F} be a foliation on \mathbb{P}^2 such that $\text{dic}(\mathcal{F}) = 1$ and the coefficients of a differential 1-form Ω providing \mathcal{F} are integer numbers. Then, since we have a bound on the degree of the first integral (if it exists), an alternative algorithm to compute that integral is that described in [10] which relies on the factorization of the exactic curves studied in [31] (see also [13]). Nevertheless, to check whether $\text{dic}(\mathcal{F}) = 1$, a resolution of the singularities of \mathcal{F} is needed.

5. EXAMPLES

This last section is devoted to provide some examples that show how our algorithms work. For a start, we shall use Algorithm 1 to get a rational first integral of a foliation of degree 4.

Example 1. Consider the singular algebraic foliation \mathcal{F} given by the differential 1-form $\Omega = (2X_1X_2^5) dX_0 + (-7X_1^5X_2 - 3X_0X_2^5 + X_1X_2^5) dX_1 + (7X_1^6 + X_0X_1X_2^4 - X_1^2X_2^4) dX_2$.

From the minimal resolution of singularities we compute the configuration of dicritical points $\mathcal{B}_{\mathcal{F}}$. It has 13 points, $\{p_i\}_{i=1}^{13}$, and its proximity graph is displayed in Figure 1. We recall that the vertices of this graph represent the points in $\mathcal{B}_{\mathcal{F}}$, and two vertices, $p_i, p_j \in \mathcal{B}_{\mathcal{F}}$, are joined by an edge if p_i belongs to the strict transform of the exceptional divisor E_{p_j} . This edge is curved-dotted except when p_i belongs to the first infinitesimal neighborhood of p_j (here the edge is straight-continuous). For simplicity's sake, we delete

those edges which can be deduced from others (for instance, we have deleted the curved-dotted edges joining p_7 and p_6 with p_4 since there is an edge joining p_8 and p_4). From the local differential 1-forms defining the transformed foliations in the resolution process, it can be easily deduced that the unique dicritical divisors are E_{p_3} and $E_{p_{13}}$. Therefore $\text{dic}(\mathcal{F}) = 2$. Then we can use Algorithm 1 to check whether \mathcal{F} has a rational first integral of genus $g = 0$ because $g \neq 1$ and \mathcal{F} admits an empty $[\text{dic}(\mathcal{F}) - 2]$ -set of independent algebraic solutions. From the minimal resolution, we can obtain the divisor class

$$[K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}] = 7[L^*] - [E_{p_1}^*] - [E_{p_2}^*] - 2[E_{p_3}^*] - 5[E_{p_4}^*] - 2 \sum_{i=5}^8 [E_{p_i}^*] - \sum_{i=9}^{12} [E_{p_i}^*] - 2[E_{p_{13}}^*]$$

and the set $\mathcal{R}_{\mathcal{F}}(\emptyset)$:

$$\begin{aligned} \mathcal{R}_{\mathcal{F}}(\emptyset) = \{ & (10 : -2 : -1 : -1 : -8 : -2 : -2 : -2 : -2 : -2 : -2 : -2 : -1 : -1), \\ & (2770 : -762 : -381 : -381 : -2152 : -538 : \\ & -538 : -538 : -538 : -538 : -538 : -538 : -269 : -269) \}, \end{aligned}$$

where we have taken projective coordinates with respect to the basis \mathbf{B} . Following Algorithm 1, we must consider the first element in $\mathcal{R}_{\mathcal{F}}(\emptyset)$ and compute the value α in step 3.3. Here $\alpha = 1$,

$$T = 10L^* - 2E_{p_1}^* - E_{p_2}^* - E_{p_3}^* - 8E_{p_4}^* - 2 \sum_{i=5}^{11} E_{p_i}^* - E_{p_{12}}^* - E_{p_{13}}^*$$

and the dimension of the vector space $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*} \mathcal{O}_{Z_{\mathcal{F}}}(T))$ is two, being $F = X_1^3 X_2^7$ and $G = X_1^{10} - 2X_0 X_1^5 X_2^4 + 2X_1^6 X_2^4 + X_0^2 X_2^8 - 2X_0 X_1 X_2^8 + X_1^2 X_2^8$ a basis of this vector space. Finally, \mathcal{F} has a rational first integral given by F and G because $\Omega \wedge (GdF - FdG) = 0$. Notice that this example is [16, Example 2], where we proved the same result with a different procedure.

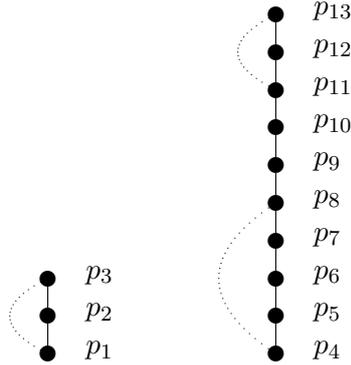


FIGURE 1. Proximity graph of $\mathcal{B}_{\mathcal{F}}$ in Example 1

Example 2. Set \mathcal{F} the foliation attached to the differential 1-form

$$\Omega = (3X_0^2 X_2^3) dX_0 - (5X_1^4 X_2) dX_1 + (5X_1^5 - 3X_0^3 X_2^2) dX_2.$$

The configuration $\mathcal{B}_{\mathcal{F}}$ has 19 points $\{p_i\}_{i=1}^{19}$ and only one dicritical divisor: $E_{p_{19}}$. We show the corresponding proximity graph in Figure 2. From the resolution of singularities it is

deduced that

$$[K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}] = 6[L^*] - 2[E_{p_1}^*] - 3[E_{p_2}^*] - 2[E_{p_3}^*] - 2[E_{p_4}^*] - \sum_{i=5}^{18} [E_{p_i}^*] - 2[E_{p_{19}}^*]$$

and, moreover, it can be checked that

$$(5 : -2 : -2 : -1 : -1 : \cdots : -1)$$

are the projective coordinates with respect to the basis \mathbf{B} of the unique element of the set $\mathcal{R}_{\mathcal{F}}(\emptyset)$. Applying Algorithm 2 we get the pair (T, \mathcal{A}) , T being the divisor $5L^* - 2E_{p_1}^* - 2E_{p_2}^* - \sum_{i=3}^{19} E_{p_i}^*$ and $\mathcal{A} = \{(X_2 = 0)\}$. Now, applying Algorithm 3 we compute $\gamma = \frac{r+2-\deg(A_1)}{s_0} = \frac{4+2-1}{5} = 1$ (step 1). Performing the remaining steps of the algorithm we conclude that \mathcal{F} admits a rational first integral given by $F = X_1^5 - X_0^3 X_2^2$ and $G = X_2^5$.

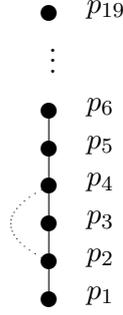


FIGURE 2. Proximity graph of $\mathcal{B}_{\mathcal{F}}$ in Examples 2 and 4

Example 3. Consider now the foliation \mathcal{F} defined by the projective differential 1-form $\Omega = AdX_0 + BdX_1 + CdX_2$, where

$$\begin{aligned} A &= 8X_0^4 X_1^2 + 10X_0 X_1^5 + 2X_0^5 X_2 - 4X_0^2 X_1^3 X_2 - 4X_0^3 X_1 X_2^2 - 4X_1^4 X_2^2 + 2X_0 X_1^2 X_2^3, \\ B &= -8X_0^5 X_1 - 10X_0^2 X_1^4 + 10X_0^3 X_1^2 X_2 + 5X_1^5 X_2 - X_0^4 X_2^2 - 2X_0 X_1^3 X_2^2 + 2X_0^2 X_1 X_2^3 - X_1^2 X_2^4, \\ C &= -2X_0^6 - 6X_0^3 X_1^3 - 5X_1^6 + 5X_0^4 X_1 X_0 + 6X_0 X_1^4 X_2 - 4X_0^2 X_1^2 X_2^2 + X_1^3 X_2^3. \end{aligned}$$

$\mathcal{B}_{\mathcal{F}} = \{p_i\}_{i=1}^{10}$ and the reader can see its proximity graph in Figure 3. Algorithm 2 gives the pair (T, \mathcal{A}) , where

$$T = 10L^* - 4 \sum_{i=1}^6 E_{p_i}^* - \sum_{i=7}^{10} E_{p_i}^*$$

and $\mathcal{A} = \{(F_1 = 0), (F_2 = 0)\}$, being

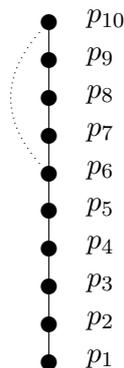
$$F_1 = X_1 X_2 - X_0^2 \text{ and } F_2 = 2X_0^3 X_1^2 + X_1^5 + X_0^4 X_2 - 2X_0 X_1^3 X_2 - 2X_0^2 X_1 X_2^2 + X_1^2 X_2^3.$$

The value γ in Algorithm 3 is $\gamma = \text{lcm}(2, 5)/10 = 1$ and a basis of $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*} \mathcal{O}_{Z_{\mathcal{F}}}(T))$ is $\{F := F_1^5, G := F_2^2\}$, which defines a rational first integral of \mathcal{F} since $\Omega \wedge (GdF - FdG) = 0$.

Example 4. Let \mathcal{F} be the foliation given by the differential 1-form

$$\Omega = (3X_0^2 X_2^3 - X_1^2 X_2^3) dX_0 - (5X_1^4 X_2 - X_0 X_1 X_2^3) dX_1 + (5X_1^5 - 3X_0^3 X_2^2) dX_2.$$

The configuration $\mathcal{B}_{\mathcal{F}}$ has 19 points $\{p_i\}_{i=1}^{19}$, only one dicritical divisor, $E_{p_{19}}$, and its proximity graph is that of Figure 2. Applying Algorithm 2 one obtains the pair (T, \mathcal{A}) , where T has the same expression than the one of Example 2 and the unique element of \mathcal{A} is also the line $(X_2 = 0)$. Then the value γ given in Algorithm 3 is also $\gamma = 1$. It can be

FIGURE 3. The proximity graph of $\mathcal{B}_{\mathcal{F}}$ in Example 3

checked that $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(T))$ is the pencil generated by $X_1X_2^4$ and X_2^5 , which is not irreducible and, therefore, does not provide a primitive rational first integral of \mathcal{F} . As a consequence, \mathcal{F} does not admit a rational first integral.

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