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CONVERGENCE OF FUZZY SETS WITH RESPECT TO THE SUPREMUM METRIC

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ABSTRACT. We characterize the convergence of fuzzy sets in the supremum metric given by the supremum of the Hausdorff distances of the α -cuts of the fuzzy sets. We do it by dividing this metric into its lower and upper quasipseudometric parts. This characterization is given in the more general context with no assumption on the fuzzy sets. Furthermore, motivated from the theory of Convex Analysis, we also provide some results about the behaviour of the convergence in the supremum metric with respect to maximizers.

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1. INTRODUCTION

Throughout this paper we will use the letters \mathbb{N} , \mathbb{R} and I to denote the set of positive integer numbers, the set of real numbers and the unit interval, respectively. We also denote by d_e the usual Euclidean metric on \mathbb{R} or on any of its subsets. Furthermore, if (X, d) is a metric space we denote by $B_d(x, \varepsilon)$ (resp. $\overline{B}_d(x, \varepsilon)$) the open ball (resp. the closed ball) with center $x \in X$ and radius $\varepsilon > 0$. We also write $B_d(A, \varepsilon) = \cup_{a \in A} B_d(a, \varepsilon)$ whenever A is a nonempty subset of X .

The supremum metric has its origins in the theory of convergence of fuzzy sets [10, 12, 13, 14, 15, 16, 17, 18, 22, 23, 26]. Recall that a fuzzy set f on a topological space (X, τ) , as originally defined by Zadeh, is a function f from X to the closed unit interval I . The *endograph* or hypograph of f is the set of all the points below its graph, i. e. $\text{end}f = \{(x, \alpha) \in X \times I : \alpha \leq f(x)\}$ whereas the *epigraph* are all the points over the graph, i. e. $\text{epi}f = \{(x, \alpha) \in X \times I : f(x) \leq \alpha\}$. The *superlevel set* of f at height α (or α -cut) is $[f]^\alpha = \{x \in X : \alpha \leq f(x)\}$ where $0 < \alpha \leq 1$. The *support* of f is $[f]^0 = \overline{\cup_{0 < \alpha \leq 1} [f]^\alpha}$ and its supported endograph or *sendograph* is the endograph where the first coordinate is restricted to the support of f , i. e. $\text{send}f = \text{end}f \cap ([f]^0 \times I)$.

Furthermore, we recall that

- the *upper limit* of f in $a \in X$ is given by

$$\overline{f}(a) = \limsup_{x \rightarrow a} f(x) = \inf_{V \in \mathcal{N}(a)} \sup_{x \in V} f(x);$$

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- the *lower limit* of f in $a \in X$ is given by

$$\underline{f}(a) = \liminf_{x \rightarrow a} f(x) = \sup_{V \in \mathcal{N}(a)} \inf_{x \in V} f(x),$$

where $\mathcal{N}(a)$ denotes the neighborhood filter of a .

Then a fuzzy set f is *upper semicontinuous* (resp. *lower semicontinuous*) at a if $f(a) = \limsup_{x \rightarrow a} f(x)$ (resp. $f(a) = \liminf_{x \rightarrow a} f(x)$), equivalently $\limsup_{x \rightarrow a} f(x) \leq f(a)$ (resp. $f(a) \leq \liminf_{x \rightarrow a} f(x)$). We denote by $USC(X)$ the family of all upper semicontinuous fuzzy sets on X . Furthermore, f is continuous at a if and only if $\limsup_{x \rightarrow a} f(x) = \liminf_{x \rightarrow a} f(x)$.

We also notice that $\overline{\text{end}f} = \text{end}\overline{f}$ (resp. $\overline{\text{epi}f} = \text{epi}\overline{f}$), where closure is taken in $\tau \times \tau(d_e)$, so f is upper semicontinuous (resp. lower semicontinuous) if and only if $\text{end}f$ (resp. $\text{epi}f$) is closed.

In the literature, we can find several convergences for fuzzy sets whose properties and relationships have been studied by several authors [14, 17, 25]. One of the most used is the convergence in the supremum metric. Recall that, given a metric space (X, d) , the supremum distance between two fuzzy sets f, g on X is defined as

$$d_\infty(f, g) = \sup_{0 < \alpha \leq 1} H_d([f]^\alpha, [g]^\alpha)$$

where H_d is the Hausdorff extended pseudometric between subsets of X given by

$$H_d(A, B) = \max \{e_d(A, B), e_d(B, A)\}$$

where

$$e_d(A, B) = \begin{cases} \sup_{a \in A} d(a, B) & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}$$

is the excess of A over B .

We also observe that

$$(1) \quad H_d(A, B) = \max \{ \inf \{ \varepsilon > 0 : A \subseteq B_d(B, \varepsilon) \}, \inf \{ \varepsilon > 0 : B \subseteq B_d(A, \varepsilon) \} \}.$$

where we understand that the infimum is $+\infty$ if no such ε exists.

The main purpose of this paper is to obtain a general characterization of d_∞ -convergence in $\mathcal{F}(X)$ (the family of all fuzzy sets over X , i. e. all the $[0, 1]$ -valued functions defined on X). The antecedents of our approach may be found in [14], where the authors gave a characterization of the d_∞ -convergence (or \mathcal{D} -convergence) of upper semicontinuous fuzzy sets f with nonempty compact support and $[f]^1 \neq \emptyset$ (see Corollaries 3 and 6). Under these assumptions, several authors have studied the relationship between this convergence and the hypo-convergence (or Γ -convergence) [14], Hausdorff convergence of supported endographs (or \mathcal{H} -convergence) [20] and Hausdorff convergence of superlevel sets at a fixed height (also named \mathcal{L} -convergence or levelwise convergence) [17]. Nevertheless, we cannot find results which do not impose conditions on the fuzzy sets. To achieve our study, and as it is usual in hyperspace theory, we will split this metric in two halves: the lower supremum (extended) quasi-pseudometric and the upper supremum (extended) quasi-pseudometric. Then, we will study the relationship between the convergences associated to these two (extended) quasi-pseudometrics and some uniform convergences of certain functionals which will allow us to locate these convergences in the lattice of convergence of functions. Finally, we will characterize the aforementioned convergences by means of set-theoretic convergences.

Furthermore, since Γ -convergence is tied to Convex Analysis [2, 3, 4, 5, 6, 8, 21, 24], we study some questions of convergence in the supremum metric from this point of view. Specifically, we analyze how this convergence behaves with respect to maximizers. For example, we prove that if $\{f_\lambda\}_{\lambda \in \Lambda}$ is a net of fuzzy sets converging to an upper semicontinuous fuzzy set f and $\{x_\lambda\}_{\lambda \in \Lambda}$ is a net in X converging to x , where x_λ is a maximizer of f_λ for all $\lambda \in \Lambda$, then x is a maximizer of f .

2. CONVERGENCES ON $[0, 1]$

We will introduce some convergences on the unit interval $I = [0, 1]$ of \mathbb{R} which will be useful in our study. These convergences can also be considered in the general context of the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 1. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in I . Then

- the **lower limit** of $\{x_\lambda\}_{\lambda \in \Lambda}$ is

$$\liminf_{\lambda \in \Lambda} x_\lambda = \sup_{\lambda' \in \Lambda} \inf_{\lambda \geq \lambda'} x_\lambda;$$

- the **upper limit** of $\{x_\lambda\}_{\lambda \in \Lambda}$ is

$$\limsup_{\lambda \in \Lambda} x_\lambda = \inf_{\lambda' \in \Lambda} \sup_{\lambda \geq \lambda'} x_\lambda.$$

Notice that $\lim_{\lambda \in \Lambda} x_\lambda = x$ in the usual topology of I if and only if $\limsup_{\lambda \in \Lambda} x_\lambda = \liminf_{\lambda \in \Lambda} x_\lambda = x$.

From this, we consider the following topological convergences on I .

Definition 2. A net $\{x_\lambda\}_{\lambda \in \Lambda}$ in I is said to be

- **lower convergent** to $x \in I$, and we write $x \in \lim_{\lambda \in \Lambda}^{\mathcal{L}} x_\lambda$, if $x \leq \liminf_{\lambda \in \Lambda} x_\lambda$;
- **upper convergent** to $x \in I$, and we write $x \in \lim_{\lambda \in \Lambda}^{\mathcal{U}} x_\lambda$, if $\limsup_{\lambda \in \Lambda} x_\lambda \leq x$.

Consequently,

$$(2) \quad \liminf_{\lambda \in \Lambda} x_\lambda = \max_{\lambda \in \Lambda} \lim_{\lambda \in \Lambda}^{\mathcal{L}} x_\lambda \text{ and } \limsup_{\lambda \in \Lambda} x_\lambda = \min_{\lambda \in \Lambda} \lim_{\lambda \in \Lambda}^{\mathcal{U}} x_\lambda.$$

Remark 1. It is easy to check that lower convergence (resp. upper convergence) is convergence in the topology of the lower quasi-pseudometric ℓ (resp. upper quasi-pseudometric u) given by $\ell(x, y) = \max\{x - y, 0\}$ (resp. $u(x, y) = \ell(y, x)$) for all $x, y \in I$. Furthermore, the lower convergence (resp. upper convergence) is compatible with the **lower topology** (resp. **upper topology**) on I whose open sets are of the form $(a, 1]$ (resp. $[0, a)$) where $a \in I$.

In the theory of continuous lattices, the lower topology on I is known as the Scott topology [11] and its open sets are characterized by the following: O is Scott-open if and only if O is an upper set and if $\sup D \in O$ then $D \cap O \neq \emptyset$ for every directed set D .

Remark 2. We observe that a fuzzy set f on a metric space (X, d) is upper semicontinuous (resp. lower semicontinuous) at $x \in X$ if and only if f is continuous at x when I is endowed with the upper (resp. lower) quasi-pseudometric. Consequently, f is upper semicontinuous (resp. lower semicontinuous) at x if for every $\varepsilon > 0$ we can find $\delta_\varepsilon > 0$ such that $f(y) - f(x) < \varepsilon$ (resp. $f(x) - f(y) < \varepsilon$) for all $y \in B_d(x, \delta_\varepsilon)$.

Then we say that a fuzzy set f is **uniformly upper semicontinuous** if for every $\varepsilon > 0$ we can find $\delta_\varepsilon > 0$ such that

$$f(y) - f(x) < \varepsilon$$

whenever $d(x, y) < \delta_\varepsilon$.

We denote by $UUSC(X)$ the family of all uniformly upper semicontinuous fuzzy sets on X .

We introduce now two other convergences.

Definition 3. Given $x \in I$, we say that a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in I is:

- \mathcal{S}_\leq -convergent to x if there exists $\lambda_0 \in \Lambda$ such that $x \leq x_\lambda$ for all $\lambda \geq \lambda_0$;
- \mathcal{S}_\geq -convergent to x if there exists $\lambda_0 \in \Lambda$ such that $x \geq x_\lambda$ for all $\lambda \geq \lambda_0$.

Remark 3. We observe that \mathcal{S}_\leq -convergence coincides with the convergence in the Alexandroff topology associated with the partial order \leq of I , whose open sets constitute the topology

$$\tau_\leq = \{O \subseteq I : \text{if } x, y \in I, x \in O \text{ and } x \leq y \Rightarrow y \in O\} = \{[a, 1] : a \in I\} \cup \{\emptyset\}.$$

We also notice that if $A \subseteq I$ then the closure of A in τ_\leq is $\overline{A} = \downarrow A = \{x \in I : x \leq a \text{ for some } a \in A\}$.

In a similar way, \mathcal{S}_\geq -convergence coincides with the convergence in the Alexandroff topology associated with the partial order \geq of I , whose open sets are

$$\tau_\geq = \{O \subseteq I : \text{if } x, y \in I, x \in O \text{ and } x \geq y \Rightarrow y \in O\} = \{[0, a] : a \in I\} \cup \{\emptyset\}.$$

Furthermore (compare with equation (2))

$$(3) \quad \liminf_{\lambda \in \Lambda} x_\lambda = \sup_{\lambda \in \Lambda} \lim_{\lambda \in \Lambda}^{\tau_\leq} x_\lambda \text{ and } \limsup_{\lambda \in \Lambda} x_\lambda = \inf_{\lambda \in \Lambda} \lim_{\lambda \in \Lambda}^{\tau_\geq} x_\lambda,$$

and, in general, $\{x_\lambda\}_{\lambda \in \Lambda}$ is not τ_\leq -convergent (resp. τ_\geq -convergent) to $\liminf_{\lambda \in \Lambda} x_\lambda$ (resp. $\limsup_{\lambda \in \Lambda} x_\lambda$). Therefore, the Alexandroff topology τ_\leq (resp. τ_\geq) is strictly finer than the lower topology (resp. upper topology).

We introduce now other two convergences which will be useful later on.

Definition 4. Given $x \in I$, we say that a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in I is:

- $\mathcal{S}_<$ -convergent to x if there exists $\lambda_0 \in \Lambda$ such that $x < x_\lambda$ for all $\lambda \geq \lambda_0$;
- $\mathcal{S}_>$ -convergent to x if there exists $\lambda_0 \in \Lambda$ such that $x > x_\lambda$ for all $\lambda \geq \lambda_0$.

Remark 4. Notice that the two above convergences are not topological, i. e. there does not exist any topology whose associated convergence is equivalent to one of the above convergences. This is due to the fact that constant sequences are not convergent in these convergences. Nevertheless, they satisfy the rest of the axioms for being a topological convergence (see [19, page 64]).

Furthermore, the finest topology whose convergence is coarser than $\mathcal{S}_<$ -convergence (resp. $\mathcal{S}_>$ -convergence) is the Alexandroff topology τ_\leq (resp. τ_\geq).

Remark 5. It is obvious that $\mathcal{S}_<$ -convergence implies τ_\leq -convergence which implies lower convergence. In the same way, $\mathcal{S}_>$ -convergence implies τ_\geq -convergence which implies upper convergence.

3. CONVERGENCES OF FUZZY SETS

By $\mathcal{F}(X)$ we denote the family of all fuzzy sets over X , i. e. all the $[0, 1]$ -valued functions defined over X .

3.1. Pointwise convergence.

Definition 5 ([1, Page 71]). Let (X, τ) be a topological space and $\{f_\lambda\}_{\lambda \in \Lambda}$ a net in $\mathcal{F}(X)$, where I is endowed with a convergence \mathcal{S} . We say that $\{f_\lambda\}_{\lambda \in \Lambda}$ is **\mathcal{S} -pointwise convergent** to f at $x \in X$ if the sequence $\{f_\lambda(x)\}_{\lambda \in \Lambda}$ is \mathcal{S} -convergent to $f(x)$.

If $\{f_\lambda\}_{\lambda \in \Lambda}$ is \mathcal{S} -pointwise convergent to f at every $x \in X$, we just say that $\{f_\lambda\}_{\lambda \in \Lambda}$ is \mathcal{S} -pointwise convergent to f .

When we particularize the above definition for the lower and upper convergences we have the following:

Definition 6. Let (X, τ) be a topological space. A net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is said to be:

- **lower pointwise convergent** to f at $x \in X$ if $\liminf_{\lambda \in \Lambda} f_\lambda(x) \geq f(x)$;
- **upper pointwise convergent** to f at $x \in X$ if $\limsup_{\lambda \in \Lambda} f_\lambda(x) \leq f(x)$;
- **pointwise convergent** to f at $x \in X$ if it is both lower and upper pointwise convergent to f at x , i. e. $\lim_{\lambda \in \Lambda} f_\lambda(x) = f(x)$.

When the above conditions are satisfied for all $x \in X$ we just say that $\{f_\lambda\}_{\lambda \in \Lambda}$ is lower pointwise convergent (resp. upper pointwise convergent, pointwise convergent) to f .

Remark 6. Notice that $\{f_\lambda\}_{\lambda \in \Lambda}$ is lower pointwise convergent to f if and only if $\text{end } f \subseteq \text{end li } f_\lambda$ where $(\text{li } f_\lambda)(x) = \liminf_{\lambda \in \Lambda} f_\lambda(x)$ for all $x \in X$.

In a similar way, $\{f_\lambda\}_{\lambda \in \Lambda}$ is upper pointwise convergent to f if and only if $\text{end ls } f_\lambda \subseteq \text{end } f$ where $(\text{ls } f_\lambda)(x) = \limsup_{\lambda \in \Lambda} f_\lambda(x)$.

Consequently, $\{f_\lambda\}_{\lambda \in \Lambda}$ is pointwise convergent to f if and only if $\text{end ls } f_\lambda = \text{end li } f_\lambda = \text{end } f$. This statement is also valid considering epigraphs instead of endographs.

If we consider in Definition 5 the τ_{\leq} -convergence and τ_{\geq} -convergence we have the following:

Definition 7. Let (X, τ) be a topological space. A net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is said to be:

- **τ_{\leq} -pointwise convergent** to f if given $x \in X$ there exists $\lambda_x \in \Lambda$ such that $f(x) \leq f_\lambda(x)$ for all $\lambda \geq \lambda_x$;
- **τ_{\geq} -pointwise convergent** to f if given $x \in X$ there exists $\lambda_x \in \Lambda$ such that $f(x) \geq f_\lambda(x)$ for all $\lambda \geq \lambda_x$.

If the above inequalities are strict we say that $\{f_\lambda\}_{\lambda \in \Lambda}$ is $\mathcal{S}_{<}$ -pointwise convergent (resp. $\mathcal{S}_{>}$ -pointwise convergent) to f .

3.2. Γ -convergence. The origins of the Γ -convergence, and its counterpart epi-convergence, are due to Wijsman [27] when he introduced the so-called *infimal convergence* for convex functions. The relationship between this convergence and Kuratowski-Painlevé convergence of epigraphs was explicitly observed by Klee in the review of Wijsman's paper, although it is implicit in the results of the paper.

Later on, Γ -convergence appeared again in nonconvex problems of minimization by De Giorgi and Franzoni [7].

Recall that if (X, τ) is a topological space and $\{A_\lambda\}_{\lambda \in \Lambda}$ is a net of subsets of X then

- the *lower limit* of $\{A_\lambda\}_{\lambda \in \Lambda}$ is the set

$$\text{Li}A_\lambda = \{x \in X : A_\lambda \cap V \neq \emptyset \text{ residually, for all } V \in \mathcal{N}(x)\};$$

- the *upper limit* of $\{A_\lambda\}_{\lambda \in \Lambda}$ is the set

$$\text{Ls}A_\lambda = \{x \in X : A_\lambda \cap V \neq \emptyset \text{ cofinally, for all } V \in \mathcal{N}(x)\}.$$

In this way, a net $\{A_\lambda\}_{\lambda \in \Lambda}$ of subsets of X is *lower Kuratowski-Painlevé convergent* (resp. *upper Kuratowski-Painlevé convergent*; *Kuratowski-Painlevé convergent*) to A [2] if $A \subseteq \text{Li}A_\lambda$ (resp. $\text{Ls}A_\lambda \subseteq A$; $A = \text{Li}A_\lambda = \text{Ls}A_\lambda$). We also observe (see [2, Proposition 5.2.5]) that lower Kuratowski-Painlevé convergence is convergence in the lower Vietoris topology so it is always topological.

We recall that Γ -convergence is Kuratowski-Painlevé convergence of endographs of functions. Consequently, a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is said to be Γ^- -convergent to the fuzzy set f if $\text{end } f \subseteq \text{Li end } f_\lambda$. It can be proved (see [2, Lemmas 5.3.3 and 5.3.4],[9, Proposition 1.9] for the epi-convergence version) that $\text{Li end } f_\lambda$ is the endograph of the fuzzy set given by

$$(4) \quad (\text{Li}f_\lambda)(x) = \inf_{V \in \mathcal{N}(x)} \liminf_{\lambda \in \Lambda} M_{f_\lambda}(V),$$

where $M_{f_\lambda}(V) = \sup_{v \in V} f_\lambda(v)$.

Furthermore, the net $\{f_\lambda\}_{\lambda \in \Lambda}$ is Γ^+ -convergent to f if $\text{Ls end } f_\lambda \subseteq \text{end } f$. The upper limit of the endographs of the elements of the net $\{f_\lambda\}_{\lambda \in \Lambda}$ is the endograph of the fuzzy set

$$(5) \quad (\text{Ls}f_\lambda)(x) = \inf_{V \in \mathcal{N}(x)} \limsup_{\lambda \in \Lambda} M_{f_\lambda}(V).$$

Remark 7. We notice that if (X, d) is a metric space and $\{f_\lambda\}_{\lambda \in \Lambda}$ is a net in $\mathcal{F}(X)$ then

$$\begin{aligned} (\text{Li}f_\lambda)(x) &= \inf_{\varepsilon > 0} \liminf_{\lambda \in \Lambda} M_{f_\lambda}(B_d(x, \varepsilon)), \\ (\text{Ls}f_\lambda)(x) &= \inf_{\varepsilon > 0} \limsup_{\lambda \in \Lambda} M_{f_\lambda}(B_d(x, \varepsilon)). \end{aligned}$$

It is not difficult to prove that if $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of fuzzy sets in $\mathcal{F}(X)$ then

$$\begin{aligned} (\text{Li}f_n)(x) &= \sup_{\{x_n\}_{n \in \mathbb{N}} \rightarrow x} \liminf_{n \rightarrow +\infty} f_n(x_n), \\ (\text{Ls}f_n)(x) &= \sup_{\{x_n\}_{n \in \mathbb{N}} \rightarrow x} \limsup_{n \rightarrow +\infty} f_n(x_n). \end{aligned}$$

We observe that, in contraposition with Γ^- -convergence, Γ^+ -convergence is not topological in general. In fact, the sequence of fuzzy sets $\{f_n\}_{n \in \mathbb{N}}$ over I endowed with the Euclidean topology whose elements are equal to the characteristic function f of the set $]1/2, 1]$ is not Γ^+ -convergent to f since $\inf_{\varepsilon > 0} \limsup_{n \in \mathbb{N}} M_{f_n}(B_d(1/2, \varepsilon)) = 1 > f(1/2) = 0$. The sequence $\{f_n\}_{n \in \mathbb{N}}$ is Γ^+ -convergent to every fuzzy set greater than or equal to the characteristic function of the closed interval $[1/2, 1]$.

Remark 8. Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a net of fuzzy sets defined on a topological space. We observe that:

$$\text{end li } f_\lambda \subseteq \text{Li end } f_\lambda = \text{end Li}f_\lambda \quad \text{and} \quad \text{end ls } f_\lambda \subseteq \text{Ls end } f_\lambda = \text{end Ls}f_\lambda.$$

Suppose that $(x, \alpha) \in \text{end li } f_\lambda$, i. e. $\alpha \leq \liminf_{\lambda \in \Lambda} f_\lambda(x)$. Let $\varepsilon > 0$. Then $\alpha - \varepsilon/2 < \inf_{\lambda \geq \lambda'} f_\lambda(x)$ for all λ' greater than a certain λ_ε . Consequently, $(x, \alpha -$

$\varepsilon/2) \in \text{end } f_\lambda \cap (V \times (\alpha - \varepsilon, \alpha + \varepsilon))$ for all $\lambda \geq \lambda_\varepsilon$ and all neighborhood V of x . Therefore, $(x, \alpha) \in \text{Li end } f_\lambda$.

To prove the other inclusion we can proceed in a similar way but taking into account that if $\alpha \leq \limsup_\lambda f_\lambda(x)$ then for each $\lambda \in \Lambda$ we can find $\lambda' \geq \lambda$ such that $\alpha - \varepsilon/2 < f_{\lambda'}(x)$.

For first countable spaces, Γ -convergence admits a very useful characterization.

Theorem 1 (cf. [2, Theorem 5.3.5], [21, Propositions 8.6.1 and 8.6.2], [24, Proposition 7.2]). Given a first countable topological space (X, τ) , a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}(X)$ is

- Γ^- -convergent to f if and only if for every $x \in X$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ convergent to x such that $f(x) \leq \liminf_{n \rightarrow +\infty} f_n(x_n)$;
- Γ^+ -convergent to f if and only if $\limsup_{n \rightarrow +\infty} f_n(x_n) \leq f(x)$ for every sequence $\{x_n\}_{n \in \mathbb{N}}$ convergent to x .

Although in general there is no relationship between Γ -convergence and pointwise convergence [2, p. 156], [9, Theorem 2.18], by the above remark we can obtain the following:

Proposition 1. Let (X, τ) be a topological space and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{F}(X)$. Then:

- (1) if $\{f_\lambda\}_{\lambda \in \Lambda}$ is lower pointwise convergent to f then it is Γ^- -convergent to f ;
- (2) if $\{f_\lambda\}_{\lambda \in \Lambda}$ is Γ^+ -convergent to f then it is upper pointwise convergent to f ;
- (3) the set of limit points of the net $\{f_\lambda\}_{\lambda \in \Lambda}$ in the lower pointwise convergence and in the Γ^- -convergence coincide if and only if $(\text{li } f_\lambda)(x) = (\text{Li } f_\lambda)(x)$ for all $x \in X$;
- (4) the set of limit points of the net $\{f_\lambda\}_{\lambda \in \Lambda}$ in the upper pointwise convergence and in the Γ^+ -convergence coincide if and only if $(\text{ls } f_\lambda)(x) = (\text{Ls } f_\lambda)(x)$ for all $x \in X$;

Proof. The proof follows from Remarks 6 and 8. □

The next example shows limiting counterexamples for the preceding result.

Example 1. Let us consider the sequence $\{f_n\}_{n \in \mathbb{N}}$ of spike functions on I given by

$$f_n(x) = \begin{cases} 2nx & \text{if } 0 \leq x < \frac{1}{2n} \\ 2 - 2nx & \text{if } \frac{1}{2n} \leq x < \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}.$$

If we endow I with the Euclidean metric, then it is clear that $\{f_n\}_{n \in \mathbb{N}}$ is pointwise convergent to the zero fuzzy set on I , so $(\text{li } f_n)(x) = \liminf_{n \rightarrow +\infty} f_n(x) = 0$ for all $x \in I$. Nevertheless,

$$(\text{Li } f_n)(x) = (\text{Ls } f_n)(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}.$$

Consequently, $\text{Li end } f_n = \text{Ls end } f_n = (\{0\} \times I) \cup ([0, 1] \times \{0\})$. From this, we can deduce that the sequence $\{f_n\}_{n \in \mathbb{N}}$ is Γ^- -convergent to the characteristic function of the singleton $\{0\}$ but it is not lower pointwise convergent to this fuzzy set. Furthermore, the sequence is upper pointwise convergent to the zero fuzzy set but it is not Γ^+ -convergent to that fuzzy set.

3.3. Uniform convergence. Recall that a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is uniform convergent to a fuzzy set f if for each $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \Lambda$ such that for all $x \in X$

$$|f(x) - f_\lambda(x)| < \varepsilon \text{ for all } \lambda \geq \lambda_\varepsilon.$$

Definition 6 splits the pointwise convergence in two convergences. The same can be made with the uniform convergence.

Definition 8. Let (X, τ) be a topological space and let f be a fuzzy set on X . A net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is:

- **lower uniformly convergent** to f if for each $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \Lambda$ such that for all $x \in X$

$$f(x) - f_\lambda(x) < \varepsilon \text{ for all } \lambda \geq \lambda_\varepsilon;$$

- **upper uniformly convergent** to f if for each $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \Lambda$ such that for all $x \in X$

$$f_\lambda(x) - f(x) < \varepsilon \text{ for all } \lambda \geq \lambda_\varepsilon.$$

From this it is clear that $\{f_\lambda\}_{\lambda \in \Lambda}$ is uniformly convergent to f if it is both lower and upper uniformly convergent to f .

We next introduce other uniform convergences related to some convergences introduced in Section 2.

Definition 9. Let (X, τ) be a topological space and let f be a fuzzy set on X . A net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is:

- τ_{\leq} -**uniformly convergent** to f if there exists $\lambda_0 \in \Lambda$ such that for each $x \in X$

$$f(x) \leq f_\lambda(x) \text{ for all } \lambda \geq \lambda_0;$$

- τ_{\geq} -**uniformly convergent** to f there exists $\lambda_0 \in \Lambda$ such that for each $x \in X$

$$f_\lambda(x) \leq f(x) \text{ for all } \lambda \geq \lambda_0.$$

If the above inequalities are strict we say that $\{f_\lambda\}_{\lambda \in \Lambda}$ is $\mathcal{S}_{<}$ -uniformly convergent (resp. $\mathcal{S}_{>}$ -uniformly convergent) to f .

Observe that simultaneous τ_{\leq} -uniform convergence and τ_{\geq} -uniform convergence of a net $\{f_\lambda\}_{\lambda \in \Lambda}$ to a function f obviously implies that $f = f_\lambda$ residually. Furthermore, it is clear that τ_{\leq} -uniform convergence (resp. τ_{\geq} -uniform convergence) implies lower uniform convergence (resp. upper uniform convergence). It is easy to construct examples to check that the converse is not true in general.

4. THE SUPREMUM METRIC

Let (X, d) be a metric space. The **supremum metric** d_∞ [16, 17, 22] on $\mathcal{F}(X)$ is given by

$$d_\infty(f, g) = \sup_{0 < \alpha \leq 1} H_d([f]^\alpha, [g]^\alpha) = \sup_{0 < \alpha \leq 1} \max\left\{ \sup_{y \in [g]^\alpha} d([f]^\alpha, y), \sup_{x \in [f]^\alpha} d(x, [g]^\alpha) \right\}.$$

Actually, d_∞ is an extended pseudometric (for example, $d_\infty(f, g) = 0$ where f, g are the characteristic functions of $\mathbf{Q} \cap I$ and $\mathbf{I} \cap I$, respectively) although we will refer to it as a metric. Furthermore, if d_∞ is restricted to $USC(X)$, the family of all upper semicontinuous fuzzy sets on X , then we obtain an extended metric since superlevel sets are closed.

Remark 9. Notice that

$$\sup_{0 < \alpha \leq 1} H_d([f]^\alpha, [g]^\alpha) = \sup_{0 \leq \alpha \leq 1} H_d([f]^\alpha, [g]^\alpha).$$

Indeed, it is obvious that $d_\infty(f, g) = \sup_{0 < \alpha \leq 1} H_d([f]^\alpha, [g]^\alpha) \leq \sup_{0 \leq \alpha \leq 1} H_d([f]^\alpha, [g]^\alpha)$. On the other hand, suppose that $d_\infty(f, g) = \varepsilon$ (if $d_\infty(f, g) = +\infty$ the conclusion is obvious). Given $x \in [f]^0$ and $n \in \mathbb{N}$ we can find $\alpha_n \in (0, 1]$ and $y \in [f]^{\alpha_n}$ such that $d(x, y) < 1/(2n)$. By assumption, $H_d([f]^{\alpha_n}, [g]^{\alpha_n}) \leq \varepsilon$ so there exists $z \in [g]^{\alpha_n} \subseteq [g]^0$ with $d(y, z) \leq \varepsilon + 1/(2n)$. Consequently, $d(x, z) < \varepsilon + 1/n$ so $d(x, [g]^0) < \varepsilon + 1/n$. Hence $d(x, [g]^0) \leq \varepsilon$ for all $x \in [f]^0$.

In a similar way, it can be proved that $d(x, [f]^0) \leq \varepsilon$ for every $x \in [g]^0$. Therefore, $H_d([f]^0, [g]^0) \leq d_\infty(f, g)$.

The supremum metric can be split in two (extended) quasi-pseudometrics given by

$$d_\infty^-(f, g) = \sup_{0 < \alpha \leq 1} e_d([f]^\alpha, [g]^\alpha),$$

$$d_\infty^+(f, g) = \sup_{0 < \alpha \leq 1} e_d([g]^\alpha, [f]^\alpha),$$

where $e_d([f]^\alpha, [g]^\alpha)$ is the excess of $[f]^\alpha$ over $[g]^\alpha$. We call d_∞^- (resp. d_∞^+) the *lower supremum quasi-pseudometric* (resp. the *upper supremum quasi-pseudometric*). We emphasize that we use the term ‘‘quasi’’ because d_∞^- and d_∞^+ are not in general symmetric. It is easy to check that $d_\infty(f, g) = \max\{d_\infty^-(f, g), d_\infty^+(f, g)\}$.

Moreover, it is straightforward to prove (see equation (1)) that a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is

- d_∞^- -convergent to $f \in \mathcal{F}(X)$ if and only if given $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \Lambda$ such that $[f]^\alpha \subseteq B_d([f_\lambda]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$ and all $\alpha \in (0, 1]$;
- d_∞^+ -convergent to $f \in \mathcal{F}(X)$ if and only if given $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \Lambda$ such that $[f_\lambda]^\alpha \subseteq B_d([f]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$ and all $\alpha \in (0, 1]$.

We will use this characterization of these two convergences in the sequel.

Remark 10. Let (X, d) be a metric space and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{F}(X)$. Given $f, F \in \mathcal{F}(X)$ such that $f(x) \leq \inf_{\lambda \in \Lambda} f_\lambda(x)$ and $F(x) \geq \sup_{\lambda \in \Lambda} f_\lambda(x)$ for all $x \in X$, then it is obvious that $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^- -convergent (resp. d_∞^+ -convergent) to f (resp. F). In particular $\{f_\lambda\}_{\lambda \in \Lambda}$ is always d_∞^- -convergent to the identically 0 fuzzy set and d_∞^+ -convergent to the constant fuzzy set 1.

Furthermore, we know by [2, Lemma 1.5.1] that $e_d(A, B) = \sup_{x \in X} d(B, x) - d(A, x)$ so we have that

$$d_\infty^-(f, g) = \sup_{0 < \alpha \leq 1} \sup_{x \in X} d([g]^\alpha, x) - d([f]^\alpha, x),$$

$$d_\infty^+(f, g) = \sup_{0 < \alpha \leq 1} \sup_{x \in X} d([f]^\alpha, x) - d([g]^\alpha, x).$$

Therefore,

$$d_\infty(f, g) = \sup_{0 < \alpha \leq 1} d_{uc}(d([f]^\alpha, \cdot), d([g]^\alpha, \cdot))$$

where d_{uc} denotes the metric compatible with the topology of uniform convergence given by

$$d_{uc}(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

The proof of the next proposition follows from the above observations.

Proposition 2. *Let (X, d) be a metric space. Then a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is d_∞ -convergent to f if and only if given $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \Lambda$ such that*

$$|d([f_\lambda]^\alpha, x) - d([f]^\alpha, x)| < \varepsilon$$

for all $\lambda \geq \lambda_\varepsilon$, $x \in X$ and $\alpha \in (0, 1]$.

In the following, we try to give a formula which allows us to obtain, as in equations (4) and (5) for Γ -convergence, the limit of a net of functions convergent with respect to the lower or upper supremum quasi-pseudometrics. We also intend to establish a characterization of the convergence in these two quasi-pseudometrics in the spirit of Theorem 1.

4.1. Convergence in the lower supremum quasi-pseudometric. Let (X, d) be a metric space. Recall that if $f \in \mathcal{F}(X)$, then $M_f(A) = \sup_{a \in A} f(a)$, where $\emptyset \neq A \subseteq X$. The role of this functional in the study of the d_∞^- -convergence is crucial as the next proposition shows.

Proposition 3 (cf. [14]). *Let (X, d) be a metric space and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{F}(X)$. Consider the following statements:*

- (1) $\{f_\lambda\}_{\lambda \in \Lambda}$ is τ_\leq -uniformly convergent to f ;
- (2) $\{M_{f_\lambda}(B_d(\cdot, \varepsilon))\}_{\lambda \in \Lambda}$ is $\mathcal{S}_<$ -uniformly convergent to f for all $\varepsilon > 0$;
- (3) $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^- -convergent to f ;
- (4) $\{M_{f_\lambda}(B_d(\cdot, \varepsilon))\}_{\lambda \in \Lambda}$ is τ_\leq -uniformly convergent to f for all $\varepsilon > 0$;
- (5) $\{f_\lambda\}_{\lambda \in \Lambda}$ is Γ^- -convergent to f .

We have the following implications:

$$\begin{array}{c} (1) \Downarrow \\ (3) \Rightarrow (4) \Rightarrow (5) \\ (2) \Updownarrow \end{array}$$

Proof. (1) \Rightarrow (3) Suppose that $\{f_\lambda\}_{\lambda \in \Lambda}$ is τ_\leq -uniformly convergent to f . Then there exists $\lambda_0 \in \Lambda$ such that $f(x) \leq f_\lambda(x)$ for all $\lambda \geq \lambda_0$ and all $x \in X$. Therefore, given $\alpha \in (0, 1]$ and $x \in [f]^\alpha$ then $\alpha \leq f(x) \leq f_\lambda(x)$ so $[f]^\alpha \subseteq B_d([f_\lambda]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_0$. Consequently, $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^- -convergent to f .

(2) \Rightarrow (3) Let $\varepsilon > 0$. By assumption, we can find $\lambda_\varepsilon \in \Lambda$ such that $f(x) < M_{f_\lambda}(B_d(x, \varepsilon))$ for all $\lambda \geq \lambda_\varepsilon$ and all $x \in X$. Consequently, given $\lambda \geq \lambda_\varepsilon$ and $\alpha \in (0, 1]$, if $x \in [f]^\alpha$ we can find $x_\lambda \in B_d(x, \varepsilon)$ such that $\alpha \leq f(x) \leq f_\lambda(x_\lambda)$. Therefore, $x \in B_d([f_\lambda]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$, which concludes this implication.

(3) \Rightarrow (4) Since $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^- -convergent to f , given $\varepsilon > 0$ we can find $\lambda_\varepsilon \in \Lambda$ such that $[f]^{f(x)} \subseteq B_d([f_\lambda]^{f(x)}, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$ and all $x \in X$ (notice that by Remark 9 there is no need to ask that $f(x) > 0$). Given $x \in X$, if $f(x) = 0$ then $f(x) = 0 \leq f_\lambda(x) \leq M_{f_\lambda}(B_d(x, \varepsilon))$ for all $\lambda \in \Lambda$. On the other hand, if $f(x) > 0$ we can find $x_\lambda \in B_d(x, \varepsilon)$ such that $f(x) \leq f_\lambda(x_\lambda)$ for all $\lambda \geq \lambda_\varepsilon$. Hence $f(x) \leq M_{f_\lambda}(B_d(x, \varepsilon))$ for all $\lambda \geq \lambda_\varepsilon$ and all $x \in X$.

(4) \Rightarrow (5) Given $\varepsilon > 0$ then $f(x) \leq M_{f_\lambda}(B_d(x, \varepsilon))$ for all $\lambda \geq \lambda_\varepsilon$ and all $x \in X$. Hence we deduce that $f(x) \leq \inf_{\lambda \geq \lambda_\varepsilon} M_{f_\lambda}(B_d(x, \varepsilon))$ for all $x \in X$. Consequently, $f(x) \leq \inf_{\varepsilon > 0} \liminf_{\lambda \in \Lambda} M_{f_\lambda}(B_d(x, \varepsilon))$ so $\{f_\lambda\}_{\lambda \in \Lambda}$ is Γ^- -convergent to f . \square

Remark 11. *One of the main uses of the Γ -convergence appears in the context of variational analysis where questions about the maximum or minimum of a function relative to a certain set are fundamental. In this way, a number of basic questions with respect to the stability of the set of minimizers or maximizers appear in a natural way. For example, if a net $\{f_\lambda\}_{\lambda \in \Lambda}$ converges to f in some sense does the net $\{M_{f_\lambda}(X)\}_{\lambda \in \Lambda}$ converge to $M_f(X)$? If $x_\lambda \in \text{Argmax}_X f_\lambda = \{x \in X : f_\lambda(x) = M_{f_\lambda}(X)\}$ for all $\lambda \in \Lambda$ and $\{x_\lambda\}_{\lambda \in \Lambda}$ converges to x , is x in $\text{Argmax}_X f$? The dual version of these questions for minimizers and others are developed deeply in [24]. Although these results deal with lower semicontinuous functions, it is not difficult to translate them to the context of fuzzy sets.*

For example, it is not hard to prove that (cf. [24, Proposition 7.29 (b)]) a net of fuzzy sets $\{f_\lambda\}_{\lambda \in \Lambda}$ is Γ^- -convergent to f if and only if $M_f(B_d(x, \varepsilon)) \leq \liminf_{\lambda \in \Lambda} M_{f_\lambda}(B_d(x, \varepsilon))$ for every $x \in X$ and $\varepsilon > 0$. Hence $M_f(X) \leq \liminf_{\lambda \in \Lambda} M_{f_\lambda}(X)$ (cf. [2, Proposition 1.3.5]).

The previous proposition gives a result in this direction when considering the d_∞^- -convergence. In this way, from (4) we deduce that $f(x) \leq \liminf_{\lambda \in \Lambda} M_{f_\lambda}(B_d(x, \varepsilon))$ for all $x \in X$ and all $\varepsilon > 0$ and $M_f(X) \leq M_{f_\lambda}(X)$ residually. Taking into account this and the above comments, we can easily deduce that d_∞^- convergence implies Γ^- -convergence.

At this point, it is natural to wonder whether it is possible to obtain a characterization of the d_∞^- -convergence similar to the characterization of the Γ^- -convergence that we have just mentioned. We prove it in Corollary 4.

From this, it is also easy to obtain (4) \Rightarrow (5) of the above proposition.

Now we provide several examples showing that the implications in the above proposition cannot be reversed in general. In all of them, we will consider that the real line and its subsets are endowed with the Euclidean metric d_e .

Example 2 ((3) \Rightarrow (1)). *Consider the sequence of fuzzy sets on I given by*

$$f_n(x) = \begin{cases} x - \frac{1}{3n} & \text{if } \frac{1}{3n} < x < 1 - \frac{1}{3n}, \\ 1 & \text{otherwise} \end{cases},$$

for all $n \in \mathbb{N}$. Let f be the identity function on I . Given $\varepsilon > 0$, take $n_\varepsilon \in \mathbb{N}$ with $1/n_\varepsilon < \varepsilon$. Let $\alpha \in (0, 1]$ and $x \in [f]^\alpha$, i. e. $\alpha \leq f(x) = x$. Pick $n \geq n_\varepsilon$. If $x + 1/(3n) \in [0, 1]$ then $\alpha \leq x \leq f_n(x + 1/(3n))$. Since $x + 1/(3n) \in B_{d_e}(x, \varepsilon)$ then $x \in B_{d_e}([f_n]^\alpha, \varepsilon)$. If $x + 1/(3n) \notin I$ then $f_n(x) = 1 \geq \alpha$ so $x \in B_{d_e}([f_n]^\alpha, \varepsilon)$. Consequently, $\{f_n\}_{n \in \mathbb{N}}$ is d_∞^- -convergent to f .

Nevertheless, the sequence $\{f_n\}_{n \in \mathbb{N}}$ is obviously not τ_{\leq} -uniformly convergent to f .

Example 3 ((3) \Rightarrow (2)). *Just consider the constant sequence $\{f_n\}_{n \in \mathbb{N}}$ of fuzzy sets on \mathbb{R} such that f_n is the function identically 1. Then $f_n(x) = M_{f_n}(B_d(x, \varepsilon)) = 1$ for all $x \in \mathbb{R}$ and all $\varepsilon > 0$. It is obvious that $\{f_n\}_{n \in \mathbb{N}}$ is d_∞^- -convergent to the constant fuzzy set 1 but $\{M_{f_n}(B_{d_e}(\cdot, \varepsilon))\}_{n \in \mathbb{N}}$ is not $\mathcal{S}_{<}$ -pointwise convergent to that fuzzy set for any $\varepsilon > 0$.*

Example 4 ((4) \Rightarrow (3) and (5) \Rightarrow (3)). *Let us consider the space ℓ_1 of all real sequences such that $\|x\|_1 = \sum_{n=1}^{\infty} |x(n)| < \infty$. Let $\{f_n\}_{n \in \mathbb{N}}$ be the sequence of fuzzy sets on ℓ_1 given by*

$$f_n(x) = \begin{cases} 1 - \frac{1}{k} & \text{if } x = \frac{1}{n}e_k, \\ 0 & \text{otherwise} \end{cases},$$

where e_k is the sequence whose terms are always 0 except the k^{th} term which is 1, for all $k \in \mathbb{N}$. Let f be the characteristic function of the constant zero sequence $\mathbf{0}$. It is easy to see that f_n is upper semicontinuous for all $n \in \mathbb{N}$ and f so is. It is clear that $\{M_{f_n}(B_{\|\cdot\|_1}(\cdot, \varepsilon))\}_{n \in \mathbb{N}}$ is τ_{\leq} -uniformly convergent to f for all $\varepsilon > 0$. In fact, given $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ such that $1/n_0 < \varepsilon$. Then, given $x \in \ell_1$ we have that

$$\begin{cases} f(x) = 0 \leq f_n(x) \leq M_{f_n}(B_{\|\cdot\|_1}(x, \varepsilon)) & \text{if } x \neq \mathbf{0} \\ f(x) \leq M_{f_n}(B_{\|\cdot\|_1}(x, \varepsilon)) = M_{f_n}(\{\frac{1}{n}e_k : k \in \mathbb{N}\}) = 1 & \text{if } x = \mathbf{0} \end{cases}$$

for all $n \geq n_0$. Nevertheless, $\{f_n\}_{n \in \mathbb{N}}$ is not d_{∞}^- -convergent to f since $[f]^1 = \{\mathbf{0}\}$ but $[f_n]^1 = \emptyset$ for all $n \in \mathbb{N}$.

Example 5 ((5) \Rightarrow (4)). Let us define on \mathbb{R} the sequence of fuzzy sets $\{f_n\}_{n \in \mathbb{N}}$ where

$$f_n(x) = \begin{cases} \frac{n-1}{n^3}(n^2 - x^2) & \text{if } -n \leq x \leq n \\ 0 & \text{otherwise} \end{cases}.$$

It is not difficult to see that

$$\inf_{\varepsilon > 0} \liminf_{n \rightarrow +\infty} M_{f_n}(B_{d_e}(a, \varepsilon)) = \liminf_{n \rightarrow +\infty} M_{f_n}(B_{d_e}(a, \delta)) = 1$$

for all $a \in \mathbb{R}$ and all $\delta > 0$. Therefore, $\{f_n\}_{n \in \mathbb{N}}$ is Γ^- -convergent to the constant function $f(x) = 1$ for all $x \in \mathbb{R}$. Nevertheless, given $\varepsilon > 0$ the sequence $\{M_{f_n}(B_{d_e}(\cdot, \varepsilon))\}_{n \in \mathbb{N}}$ is not τ_{\leq} -pointwise convergent to f since $M_{f_n}(B_{d_e}(x, \varepsilon)) < 1$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$.

The next two examples show that, in general, there is no relationship between the lower uniform convergence and the d_{∞}^- -convergence.

Example 6. Let us consider the sequence $\{f_n\}_{n \in \mathbb{N}}$ of fuzzy sets on \mathbb{R} given by $f_n(x) = 1 - 1/n$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. Then it is obvious that this sequence is (lower-)uniformly convergent to the constant fuzzy set $f(x) = 1$ for all $x \in \mathbb{R}$. However, $\{f_n\}_{n \in \mathbb{N}}$ is not d_{∞}^- -convergent to f since $[f]^1 = \mathbb{R} \not\subseteq B_d([f_n]^1, \varepsilon) = \emptyset$ for all $\varepsilon > 0$ and all $n \in \mathbb{N}$.

We now exhibit that d_{∞}^- -convergence does not imply lower uniform convergence (compare with Proposition 5).

Example 7. In the real line, consider, for all $n \in \mathbb{N}$, the following fuzzy sets:

$$f_n(x) = \begin{cases} -n\frac{x}{2} + \frac{1}{2} & \text{if } -\frac{1}{n} \leq x < 0 \\ n\frac{x}{2} + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}.$$

Let $f(x) = 1$ for all $x \in \mathbb{R}$. Then, given $\varepsilon > 0$ choose $n_{\varepsilon} \in \mathbb{N}$ such that $1/n_{\varepsilon} < \varepsilon/2$. Given $x \in \mathbb{R}$, if $x \notin [-\frac{1}{n_{\varepsilon}}, \frac{1}{n_{\varepsilon}}]$ then $f_n(x) = 1$ for all $n \geq n_{\varepsilon}$. If $x \in [-\frac{1}{n}, \frac{1}{n}]$ for some $n \geq n_{\varepsilon}$ then $x - \varepsilon/2 \notin [-\frac{1}{n}, \frac{1}{n}]$ or $x + \varepsilon/2 \notin [-\frac{1}{n}, \frac{1}{n}]$. In any case, given $n \geq n_{\varepsilon}$, $f_n(x') = 1$ for some $x' \in B_{d_e}(x, \varepsilon)$. This proves that $\{f_n\}_{n \in \mathbb{N}}$ is d_{∞}^- -convergent to f .

However, this sequence is not lower uniformly convergent to f , since $f_n(0) = 1/2$ for all $n \in \mathbb{N}$ but $f(0) = 1$.

Corollary 1. *Let (X, d) be a metric space and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{F}(X)$. If $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^- -convergent to f then, given $x \in X$,*

$$f(x) \leq \inf_{\varepsilon > 0} \liminf_{\lambda \in \Lambda} M_{f_\lambda}(B_d(x, \varepsilon)).$$

Proof. This is a direct consequence of Proposition 3 and equation 4. \square

In [14, Lemma 3.3] it is proved a nice characterization for the convergence in the supremum metric of sequences of upper semicontinuous fuzzy sets with compact support, which can be considered as a bilateral converse of implication (3) \Rightarrow (4) in Proposition 3. We can reformulate that result for the lower supremum quasipseudometric in our terms in the following way. Given a metric space (X, d) , let us denote by $SUSC(X)$ the family of all upper semicontinuous fuzzy sets on X with compact support. Then the result asserts that a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $SUSC(X)$ is d_∞^- -convergent to $f \in SUSC(X)$ if and only if $\{M_{f_n}(\overline{B}_d(x, \varepsilon))\}_{n \in \mathbb{N}}$ is τ_\leq -uniformly convergent to f for all $\varepsilon > 0$. Therefore, this lemma shows that the converse of implication (3) \Rightarrow (4) of Proposition 3 is true under these additional hypotheses. Nevertheless, this result is not valid in general as Example 4 exhibits. Notice that in this example, the sequence $\{f_n\}_{n \in \mathbb{N}}$ is d_∞^- -convergent to every fuzzy set f verifying that $\text{end}f \subseteq [(\ell_1 - \{\mathbf{0}\}) \times \{0\}] \cup [\{\mathbf{0}\} \times [0, 1]] = A$. However, the set A does not correspond to the endograph of a fuzzy set. This shows that there is no hope to obtain a characterization of the d_∞^- -convergence in the spirit of the characterization of the Γ^- -convergence given by equation (4) in Section 3.2.

However, we can provide a purely set-theoretic characterization of the d_∞^- -convergence. To achieve this, we need to introduce an appropriate concept of uniform convergence for sets.

Let X be a nonempty set and let $\mathcal{P}(X)$ be the family of all subsets of X . Recall that a net $\{A_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{P}(X)$ is **lower convergent** to a set $A \in \mathcal{P}(X)$ if

$$A \subseteq \liminf_{\lambda \in \Lambda} A_\lambda := \bigcup_{\lambda_0 \in \Lambda} \left(\bigcap_{\lambda \geq \lambda_0} A_\lambda \right),$$

that is, we can find $\lambda_0 \in \Lambda$ such that

$$A \subseteq \bigcap_{\lambda \geq \lambda_0} A_\lambda.$$

Notice that if (X, τ) is a topological space then $\liminf_{\lambda \in \Lambda} A_\lambda \subseteq \text{Li}A_\lambda$ and the two limits coincide when τ is the discrete topology.

The previous set-convergence allows us to define a convergence for a net of multifunctions. The following definition seems to be a natural uniform version for this convergence of multifunctions.

Definition 10. *Let X be a nonempty set and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a net of multifunctions $f_\lambda : X \rightrightarrows I$. We say that the net $\{f_\lambda\}_{\lambda \in \Lambda}$ is **lower uniformly convergent** to the multifunction $f : X \rightrightarrows I$ if we can find $\lambda_0 \in \Lambda$ such that*

$$f(x) \subseteq \bigcap_{\lambda \geq \lambda_0} f_\lambda(x) \text{ for all } x \in X.$$

The following result characterizes d_∞^- -convergence in the most general context.

Theorem 2. *Let (X, d) be a metric space and let us consider that I is endowed with the Alexandroff topology τ_\leq . Then a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is d_∞^- -convergent*

to a fuzzy set f if and only if the net of multifunctions $\{f_\lambda^\varepsilon\}_{\lambda \in \Lambda}$ is lower uniformly convergent to f for all $\varepsilon > 0$, where $f_\lambda^\varepsilon : X \rightrightarrows I$ is given by $f_\lambda^\varepsilon(x) = \overline{f_\lambda(B_d(x, \varepsilon))}^{\tau \leq}$.

Proof. Suppose that $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^- -convergent to f . Let $\varepsilon > 0$. Given $x \in X$, if $f(x) = 0$ it is obvious that $f(x) \in \overline{f_\lambda(B_d(x, \varepsilon))}^{\tau \leq} = \downarrow f_\lambda(B_d(x, \varepsilon))$ for all $\lambda \in \Lambda$ (see Remark 3). Otherwise, by assumption, there exists $\lambda_\varepsilon \in \Lambda$ such that $[f]^\alpha \subseteq B_d([f_{\lambda_\varepsilon}]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$ and all $\alpha \in (0, 1]$. In particular, taking $\alpha = f(x) > 0$, there exists $x_\lambda \in B_d(x, \varepsilon)$ such that $f(x) \leq f_\lambda(x_\lambda)$ for all $\lambda \geq \lambda_\varepsilon$. Hence $f(x) \in \bigcap_{\lambda \geq \lambda_\varepsilon} \overline{f_\lambda(B_d(x, \varepsilon))}^{\tau \leq}$. Since λ_ε does not depend on x , we have proved the necessity.

Conversely, let $\varepsilon > 0$ and $\alpha \in (0, 1]$. Suppose that $x_0 \in [f]^\alpha$. By hypothesis, there exists $\lambda_\varepsilon \in \Lambda$ such that $f(x) \subseteq \bigcap_{\lambda \geq \lambda_\varepsilon} \overline{f_\lambda(B_d(x, \varepsilon))}^{\tau \leq}$ for all $x \in X$. In particular, $f(x_0) \in \overline{f_\lambda(B_d(x_0, \varepsilon))}^{\tau \leq}$ for all $\lambda \geq \lambda_\varepsilon$ so $x_0 \in B_d([f_\lambda]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$ which finishes the proof. \square

Corollary 2 (cf. Equation 4). *Let (X, d) be a metric space. If $\{f_\lambda\}_{\lambda \in \Lambda}$ is a net in $\mathcal{F}(X)$ d_∞^- -convergent to f then*

$$f(x) \in \bigcap_{\varepsilon > 0} \liminf_{\lambda \in \Lambda} \overline{f_\lambda(B_d(x, \varepsilon))}^{\tau \leq} \text{ for all } x \in X.$$

Remark 12. *Given a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$, it is easy to check that*

$$\sup_{\varepsilon > 0} \bigcap_{\lambda \in \Lambda} \liminf_{\lambda \in \Lambda} \overline{f_\lambda(B_d(x, \varepsilon))}^{\tau \leq} = \inf_{\varepsilon > 0} \liminf_{\lambda \in \Lambda} M_{f_\lambda}(B_d(x, \varepsilon))$$

from which we again deduce immediately that d_∞^- -convergence implies Γ^- -convergence.

Corollary 3 (cf. [14, Lemma 3.3]). *Let (X, d) be a metric space. Then a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $SUSC(X)$ is d_∞^- -convergent to $f \in SUSC(X)$ if and only if $\{M_{f_\lambda}(B_d(\cdot, \varepsilon))\}_{\lambda \in \Lambda}$ is τ_\leq -uniformly convergent to f for all $\varepsilon > 0$.*

Proof. By Proposition 3, we only have to prove the sufficiency.

Let $\varepsilon > 0$. By assumption, we can find λ_ε such that $f(x) \leq M_{f_\lambda}(B_d(x, \varepsilon/2))$ for all $\lambda \geq \lambda_\varepsilon$ and all $x \in X$. We show that $[f]^\alpha \subseteq B_d([f_\lambda]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$ and all $\alpha \in (0, 1]$. Let $x \in [f]^\alpha$. Given $\lambda \geq \lambda_\varepsilon$, if $\alpha \leq f(x) < M_{f_\lambda}(B_d(x, \varepsilon/2))$ then it is obvious that $x \in B_d([f_\lambda]^\alpha, \varepsilon)$. Otherwise, suppose that $f(x) = M_{f_\lambda}(B_d(x, \varepsilon/2)) > 0$ (if $f(x) = 0$ the conclusion is obvious). Then we can find a sequence $\{a_n\}_{n \in \mathbb{N}}$ in $B_d(x, \varepsilon/2)$ such that $f_\lambda(a_n) > 0$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow +\infty} f_\lambda(a_n) = f(x)$. Since $[f_\lambda]^0$ is compact, we can suppose without loss of generality that the sequence $\{a_n\}_{n \in \mathbb{N}}$ is convergent to $a \in \overline{B_d(x, \varepsilon/2)}$. Since f_λ is upper semicontinuous we have that $\limsup_{n \rightarrow +\infty} f_\lambda(a_n) = f(x) \leq f_\lambda(a)$. We conclude again that $x \in B_d([f_\lambda]^\alpha, \varepsilon)$. Consequently, $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^- -convergent to f . \square

Corollary 4. *Let (X, d) be a metric space. Then a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is d_∞^- -convergent to f if and only if for each $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \Lambda$ such that*

$$f(x) \leq M_{f_\lambda}(B_d(x, \varepsilon)) \text{ for all } \lambda \geq \lambda_\varepsilon \text{ and all } x \in X$$

and

$$\text{if given } \lambda \geq \lambda_\varepsilon, f(x) = M_{f_\lambda}(B_d(x, \varepsilon)) \text{ then } \text{Argmax}_{B_d(x, \varepsilon)} f_\lambda \neq \emptyset.$$

Proof. It easily follows from Theorem 2. \square

For convergence of sequences, we can give the following characterization of the d_∞^- -convergence (cf. Theorem 1).

Theorem 3. *Let (X, d) be a metric space and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F}(X)$. Then $\{f_n\}_{n \in \mathbb{N}}$ is d_∞^- -convergent to a fuzzy set f if and only if there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$ of self-maps in X uniformly convergent to id_X such that $\{f_n \circ g_n\}_{n \in \mathbb{N}}$ is τ_\leq -uniformly convergent to f .*

Proof. Suppose first that $\{f_n\}_{n \in \mathbb{N}}$ is d_∞^- -convergent to f . Therefore, given $k \in \mathbb{N}$ we can find $n_k \in \mathbb{N}$ such that $[f]^{f(x)} \subseteq B_d([f_n]^{f(x)}, 1/k)$ for all $n \geq n_k$ and all $x \in X$. We can assume without loss of generality that the sequence $\{n_k\}_{k \in \mathbb{N}}$ is strictly increasing. Given $x \in [f]^0$, since $x \in [f]^{f(x)}$ (see Remark 9), for each $n \geq n_k$ pick $x_{nk} \in B_d(x, 1/k)$ such that $f(x) \leq f_n(x_{nk})$. For each $n \in \mathbb{N}$, define $g_n(x) = x_{nk}$ when $n_k \leq n < n_{k+1}$ and $g_n(x) = x_0$ when $n < n_1$ for some fixed point $x_0 \in X$. On the other hand, if $x \notin [f]^0$ define $g_n(x) = x$ for all $n \in \mathbb{N}$. Then $\{g_n\}_{n \in \mathbb{N}}$ is the desired sequence. In fact, given $k \in \mathbb{N}$ then $d(x, g_n(x)) \leq d(x, x_{nk}) < 1/k$ for all $n \geq n_k$ and all $x \in X$ so $\{g_n\}_{n \in \mathbb{N}}$ is uniformly convergent to id_X . Furthermore, $f(x) \leq f_n(g_n(x))$ for all $n \geq n_k$ and all $x \in X$.

Conversely, let $\varepsilon > 0$, $\alpha \in (0, 1]$ and $x_0 \in X$ such that $\alpha \leq f(x_0)$. By assumption, there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x, g_n(x)) < \varepsilon$ and $f(x) \leq f_n(g_n(x))$ for all $n \geq n_\varepsilon$ and all $x \in X$. This means that $x_0 \in B_d([f_n]^\alpha, \varepsilon)$ for all $n \geq n_\varepsilon$. Since n_ε does not depend on α the proof is finished. \square

Remark 13. *By Theorem 1 we deduce that a sequence $\{f_n\}_{n \in \mathbb{N}}$ of fuzzy sets in a first countable topological space is Γ^- -convergent to a fuzzy set f if and only if there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$ of self-maps on X pointwise convergent to id_X such that $\{f_n \circ g_n\}_{n \in \mathbb{N}}$ is lower pointwise convergent to f . From this and the above result, we obtain again that d_∞^- -convergence of sequences implies Γ^- -convergence.*

4.2. Convergence in the upper supremum quasi-pseudometric. Now, we center our attention into the upper part.

Proposition 4. *Let (X, d) be a metric space and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{F}(X)$. Consider the following statements:*

- (1) $\{f_\lambda\}_{\lambda \in \Lambda}$ is $\mathcal{S}_>$ -uniformly convergent to $M_f(B_d(\cdot, \varepsilon))$ for all $\varepsilon > 0$;
- (2) $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^+ -convergent to f ;
- (3) $\{f_\lambda\}_{\lambda \in \Lambda}$ is τ_\geq -uniformly convergent to $M_f(B_d(\cdot, \varepsilon))$ for all $\varepsilon > 0$.

Then the following implications hold:

$$(1) \Rightarrow (2) \Rightarrow (3).$$

Proof. (1) \Rightarrow (2) Given $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \Lambda$ such that $f_\lambda(x) < M_f(B_d(x, \varepsilon))$ for all $\lambda \geq \lambda_\varepsilon$ and all $x \in X$. Let $\alpha \in (0, 1]$. Given $\lambda \geq \lambda_\varepsilon$, if $x \in [f_\lambda]^\alpha$ then $\alpha \leq f_\lambda(x) < M_f(B_d(x, \varepsilon))$ so we can find $x_\lambda \in B_d(x, \varepsilon)$ such that $f_\lambda(x) < f(x_\lambda)$. Consequently, $x \in B_d([f]^\alpha, \varepsilon)$. Therefore, $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^+ -convergent to f .

(2) \Rightarrow (3) If $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^+ -convergent to f then, given $\varepsilon > 0$ we can find $\lambda_\varepsilon \in \Lambda$ such that $[f_\lambda]^\alpha \subseteq B_d([f]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$ and all $\alpha \in (0, 1]$. Fix $x \in X$ and $\lambda \geq \lambda_\varepsilon$. If $f_\lambda(x) = 0$ it is clear that $f_\lambda(x) \leq M_f(B_d(x, \varepsilon))$. If $f_\lambda(x) > 0$, then $[f_\lambda]^{f_\lambda(x)} \subseteq B_d([f]^{f_\lambda(x)}, \varepsilon)$ so $f_\lambda(x) \leq M_f(B_d(x, \varepsilon))$, which concludes the proof. \square

Remark 14. *Observe that if $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^+ -convergent to f and x is a local maximum of f then $\{f_\lambda(x)\}_{\lambda \in \Lambda}$ is τ_\geq -convergent to $f(x)$. In fact, we can find $\varepsilon > 0$ such that $M_f(B_d(x, \varepsilon)) = f(x)$ and, by (3) of the above result, $\lambda_0 \in \Lambda$ such that*

$$f_\lambda(x) \leq M_f(B_d(x, \varepsilon)) = f(x),$$

for all $\lambda \geq \lambda_0$ so the conclusion follows.

In particular, if f is a constant fuzzy set then $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^+ -convergent to f if and only if it is τ_\geq -uniformly convergent to f .

It is clear that the converse of implication (1) \Rightarrow (2) of the above proposition is not true in general since $\mathcal{S}_>$ -uniform convergence is not topological but d_∞^+ -convergence so is (just consider the constant sequence whose elements are the zero function). Furthermore, (2) \Rightarrow (3) cannot be reversed either as the next example shows.

Example 8. For each $n \in \mathbb{N}$ define the fuzzy sets $f_n(x) = 1 - x$ for all $x \in I$ and

$$f(x) = \begin{cases} 1 - x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

It is easy to see that $\{f_n\}_{n \in \mathbb{N}}$ is τ_\geq -uniformly convergent to $M_f(B_d(\cdot, \varepsilon))$ for all $\varepsilon > 0$ (however, observe that $\{f_n(0)\}_{n \in \mathbb{N}}$ is not convergent to $f(0)$). Nevertheless, $[f_n]^1 = \{0\}$ for all $n \in \mathbb{N}$ and $[f]^1 = \emptyset$. Consequently, $\{f_n\}_{n \in \mathbb{N}}$ is not d_∞^+ -convergent to f .

Remark 15. Observe that by statement (3) of Proposition 4, if $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^+ -convergent to f then

$$\limsup_{\lambda \in \Lambda} f_\lambda(x) \leq \overline{f}(x) = \limsup_{x' \rightarrow x} f(x'),$$

for all $x \in X$. Nevertheless, we cannot obtain the converse (see Example 10).

Furthermore, we can also deduce that $M_{f_\lambda}(X) \leq M_f(X)$ residually.

Remark 16. Notice that to obtain the equivalence between (2) and (3) of the above proposition we need to consider at least those fuzzy sets f which verify that given $x \in X$ if the net $\{M_f(B_d(x, \delta))\}_{\delta > 0}$ is residually constant then the net $\{\text{Argmax}_{B_d(x, \delta)} f\}_{\delta > 0}$ is residually nonempty (see Example 8). Otherwise, we can find $x_0 \in X$ and $\varepsilon_0 > 0$ such that $\text{Argmax}_{B_d(x_0, \varepsilon_0)} f = \emptyset$ but for every $\varepsilon > 0$ there exists $\delta > \delta_\varepsilon > 0$ such that $M_f(B_d(x_0, \delta_\varepsilon)) = M_f(B_d(x_0, \varepsilon_0))$. For each $\delta > 0$ define f_δ by $f_\delta(x) = M_f(B_d(x, \delta))$ for all $x \in X$. It is clear that $\{f_\delta\}_{\delta > 0}$ is τ_\geq -uniformly convergent to $M_f(B_d(\cdot, \varepsilon))$ for all $\varepsilon > 0$. Nevertheless, if $\{f_\delta\}_{\delta > 0}$ is d_∞^+ -convergent to f then we can find $\gamma_{\varepsilon_0} > 0$ such that $[f_\delta]^\alpha \subseteq B_d([f]^\alpha, \varepsilon_0)$ for all $\delta \leq \gamma_{\varepsilon_0}$ and for all $\alpha \in (0, 1]$. However, $x_0 \in [f_{\delta_{\gamma_{\varepsilon_0}}}]^{M_f(B_d(x_0, \delta_{\gamma_{\varepsilon_0}}))}$ but since $\text{Argmax}_{B_d(x_0, \varepsilon_0)} f = \emptyset$ and $M_f(B_d(x_0, \varepsilon_0)) = M_f(B_d(x_0, \delta_{\gamma_{\varepsilon_0}}))$ then $x_0 \notin B_d([f]^{M_f(B_d(x_0, \delta_{\gamma_{\varepsilon_0}}))}, \varepsilon_0)$.

Proposition 5. Let (X, d) be a metric space, let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{F}(X)$ and let $f \in UUSC(X)$. Consider the following statements:

- (1) $\{f_\lambda\}_{\lambda \in \Lambda}$ is τ_\geq -uniformly convergent to f ;
- (2) $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^+ -convergent to f ;
- (3) $\{f_\lambda\}_{\lambda \in \Lambda}$ is upper uniformly convergent to f ;
- (4) $\{f_\lambda\}_{\lambda \in \Lambda}$ is Γ^+ -convergent to f .

Then the following implications hold:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

Proof. (1) \Rightarrow (2) If $\{f_\lambda\}_{\lambda \in \Lambda}$ is τ_\geq -uniformly convergent to f , there exists $\lambda_0 \in \Lambda$ such that $f_\lambda(x) \leq f(x)$ for all $\lambda \geq \lambda_0$ and all $x \in X$. Obviously, this implies that $[f_\lambda]^\alpha \subseteq B_d([f]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_0$, $\varepsilon > 0$ and $\alpha \in (0, 1]$.

(2) \Rightarrow (3) By Proposition 4 we know that $\{f_\lambda\}_{\lambda \in \Lambda}$ is τ_{\geq} -uniformly convergent to $M_f(B_d(\cdot, \varepsilon))$ for all $\varepsilon > 0$. Since f is uniformly upper semicontinuous we can find $\delta_\varepsilon > 0$ such that

$$f(y) < f(x) + \varepsilon,$$

whenever $d(x, y) < \delta_\varepsilon$. Therefore, given $x \in X$ we have that

$$M_f(B_d(x, \delta_\varepsilon)) < f(x) + \varepsilon.$$

Furthermore, there exists $\lambda_{\delta_\varepsilon} \in \Lambda$ such that, $f_\lambda(x) \leq M_f(B_d(x, \delta_\varepsilon)) < f(x) + \varepsilon$ for all $\lambda \geq \lambda_{\delta_\varepsilon}$ and all $x \in X$. Consequently, $\{f_\lambda\}_{\lambda \in \Lambda}$ is upper uniformly convergent to f .

(3) \Rightarrow (4) If $\{f_\lambda\}_{\lambda \in \Lambda}$ is upper uniformly convergent to f , given $\delta > 0$ we can find $\lambda_\delta \in \Lambda$ such that $f_\lambda(x) \leq f(x) + \delta$ for all $\lambda \geq \lambda_\delta$ and all $x \in X$. Hence, given $\varepsilon > 0$, then $M_{f_\lambda}(B_d(x, \varepsilon)) \leq M_f(B_d(x, \varepsilon)) + \delta$ for all $\lambda \geq \lambda_\delta$, so $\limsup_{\lambda \in \Lambda} M_{f_\lambda}(B_d(x, \varepsilon)) \leq M_f(B_d(x, \varepsilon)) + \delta$. Since δ is arbitrary we deduce that $\limsup_{\lambda \in \Lambda} M_{f_\lambda}(B_d(x, \varepsilon)) \leq M_f(B_d(x, \varepsilon))$, so

$$\inf_{\varepsilon > 0} \limsup_{\lambda \in \Lambda} M_{f_\lambda}(B_d(x, \varepsilon)) \leq \inf_{\varepsilon > 0} M_f(B_d(x, \varepsilon)) = \limsup_{x' \rightarrow x} f(x') = f(x).$$

Therefore, $\{f_\lambda\}_{\lambda \in \Lambda}$ is Γ^+ -convergent to f . \square

Next, we provide some examples showing that the converses of the implications of the above proposition are not true in general. Again, we suppose that \mathbb{R} and all its subsets are endowed with the Euclidean metric d_e .

Example 9 ((2) \Rightarrow (1)). Consider the sequence $\{f_n\}_{n \in \mathbb{N}}$ of upper semicontinuous fuzzy sets on I given by

$$f_n(x) = \begin{cases} x + \frac{1}{n} & \text{if } \frac{1}{3n} \leq x \leq 1 - \frac{1}{3n}, \\ 0 & \text{otherwise} \end{cases},$$

for all $n \in \mathbb{N}$. Let f be the identity fuzzy set on I . Given $\varepsilon > 0$, take $n_\varepsilon \in \mathbb{N}$ with $1/n_\varepsilon < \varepsilon$. Let $\alpha \in (0, 1]$, $n \geq n_\varepsilon$ and $x \in [f_n]^\alpha$, i. e. $\alpha \leq f_n(x) = x + 1/n = f(x) + 1/n$. Since $x + 1/n \in B_{d_e}(x, \varepsilon)$ then $x \in B_{d_e}([f]^\alpha, \varepsilon)$. Consequently, $\{f_n\}_{n \in \mathbb{N}}$ is d_∞^+ -convergent to f .

On the other hand, it is clear that $\{f_n\}_{n \in \mathbb{N}}$ is not τ_{\geq} -uniformly convergent to f since $f(x) < f_n(x)$ for all $x \in [\frac{1}{3}, \frac{2}{3}]$ and all $n \in \mathbb{N}$.

Example 10 ((3) \Rightarrow (2)). Consider the fuzzy sets $f_n(x) = 1/2 + 1/(n+1)$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. It is clear that the sequence $\{f_n\}_{n \in \mathbb{N}}$ is (upper-)uniformly convergent to the fuzzy set $f(x) = 1/2$ for all $x \in \mathbb{R}$. Nevertheless, given $\varepsilon > 0$ and $n \in \mathbb{N}$ we can find $\alpha \in (0, 1]$ such that $1/2 < \alpha < 1/2 + 1/(n+1)$ so $[f_n]^\alpha = \mathbb{R} \not\subseteq B_{d_e}([f]^\alpha, \varepsilon) = \emptyset$. Consequently, $\{f_n\}_{n \in \mathbb{N}}$ is not d_∞^+ -convergent to f .

Example 11 ((4) \Rightarrow (3)). For each $n \in \mathbb{N}$, define $f_n : I \rightarrow I$ given by

$$f_n(x) = \begin{cases} \frac{1}{2} & \text{if } x \geq \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}.$$

Since $\inf_{\varepsilon > 0} \limsup_{n \in \mathbb{N}} M_{f_n}(B_d(x, \varepsilon)) = 1/2$ for all $x \in I$, we deduce that $\{f_n\}_{n \in \mathbb{N}}$ is Γ^+ -convergent to the identically $1/2$ fuzzy set f .

On the other hand, given $0 < \varepsilon < 1/2$ and $n \in \mathbb{N}$ then

$$f_n(1/(n+1)) - f(1/(n+1)) = 1/2 > \varepsilon,$$

so $\{f_n\}_{n \in \mathbb{N}}$ is not upper uniformly convergent to f .

We notice that, in general, d_∞^+ -convergence does not imply Γ^+ -convergence. In fact, Γ^+ -convergence is not topological in general. For example, the constant sequence of fuzzy sets defined over \mathbb{R} whose elements are the characteristic function of $\mathbb{R} - \{0\}$ is not Γ^+ -convergent to itself. Observe that these fuzzy sets are not upper semicontinuous.

However, as in the lower part, we can provide a purely set-theoretic characterization of this convergence. Let X be a nonempty set and let $\mathcal{P}(X)$ be the family of all subsets of X . Recall that a net $\{A_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{P}(X)$ is *upper convergent* to a set $A \in \mathcal{P}(X)$ if

$$\limsup_{\lambda \in \Lambda} A_\lambda := \bigcap_{\lambda_0 \in \Lambda} \left(\bigcup_{\lambda \geq \lambda_0} A_\lambda \right) \subseteq A.$$

Notice that if (X, τ) is a topological space then $\limsup_{\lambda \in \Lambda} A_\lambda \subseteq \text{Ls}A_\lambda$. In contraposition with the lower part, $\limsup_{\lambda \in \Lambda} A_\lambda$ and $\text{Ls}A_\lambda$ can be different even for discrete topological spaces.

This allows us to define the following uniform convergence for multifunctions:

Definition 11. Let X be a nonempty set and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a net of multifunctions $f_\lambda : X \rightrightarrows I$. We say that the net $\{f_\lambda\}_{\lambda \in \Lambda}$ is **upper uniformly convergent** to the multifunction $f : X \rightrightarrows I$ if we can find $\lambda_0 \in \Lambda$ such that

$$\bigcup_{\lambda \geq \lambda_0} f_\lambda(x) \subseteq f(x) \text{ for all } x \in X.$$

The following result characterizes d_∞^+ -convergence in the most general context, where we use the fact that a fuzzy set f on a metric space (X, d) can be considered as a multifunction $f' : X \rightrightarrows I$ such that $f'(x) = \{f(x)\}$. In fact, we will not distinguish between f and f' .

Theorem 4. Let (X, d) be a metric space. Then a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is d_∞^+ -convergent to $f \in \mathcal{F}(X)$ if and only if the net $\{f_\lambda\}_{\lambda \in \Lambda}$ considered as a net of multifunctions is upper uniformly convergent to the multifunction $f^\varepsilon : X \rightrightarrows I$ for all $\varepsilon > 0$ where $f^\varepsilon(x) = \overline{f(B_d(x, \varepsilon))}^{\tau \leq}$.

Proof. Suppose that $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^+ -convergent to f . Given $\varepsilon > 0$ we can find $\lambda_\varepsilon \in \Lambda$ such that $[f_\lambda]^\alpha \subseteq B_d([f]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$ and all $\alpha \in (0, 1]$. Fix $x \in X$ and $\lambda \geq \lambda_\varepsilon$. If $f_\lambda(x) = 0$ then it is obvious that $f_\lambda(x) \in f^\varepsilon(x) = \downarrow f(B_d(x, \varepsilon))$. If $f_\lambda(x) \neq 0$ then $[f_\lambda]^{f_\lambda(x)} \subseteq B_d([f]^{f_\lambda(x)}, \varepsilon)$ so we deduce that $f_\lambda(x) \in f^\varepsilon(x)$ which finishes this implication.

Now suppose that $\{f_\lambda\}_{\lambda \in \Lambda}$ is upper uniformly convergent to f^ε for all $\varepsilon > 0$. Then, given $\varepsilon > 0$, there exists $\lambda_\varepsilon \in \Lambda$ such that $f_\lambda(x) \in f^\varepsilon(x)$ for all $\lambda \geq \lambda_\varepsilon$. Let $\alpha \in (0, 1]$, $\lambda \geq \lambda_\varepsilon$ and $x \in [f_\lambda]^\alpha$. Since $f_\lambda(x) \in f^\varepsilon(x)$ we can find $x_\lambda \in B_d(x, \varepsilon)$ such that $f_\lambda(x) \leq f(x_\lambda)$. Therefore $x \in B_d([f]^\alpha, \varepsilon)$. This concludes the proof. \square

Corollary 5. Let (X, d) be a metric space. If the net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is d_∞^+ -convergent to $f \in \mathcal{F}(X)$ then

$$\limsup_{\lambda \in \Lambda} \text{end} f_\lambda \subseteq \text{end} \overline{f}.$$

Proof. If $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^+ -convergent to f , by the above theorem $\{f_\lambda\}_{\lambda \in \Lambda}$ is upper uniformly convergent to f^ε for all $\varepsilon > 0$. Therefore, given $\varepsilon > 0$ we can find

$\lambda_\varepsilon \in \Lambda$ such that $f_\lambda(x) \leq M_f(B_d(x, \varepsilon))$ for all $\lambda \geq \lambda_\varepsilon$ and all $x \in X$. Hence, $\cup_{\lambda \geq \lambda_\varepsilon} \text{end } f_\lambda \subseteq \text{end } M_f(B_d(\cdot, \varepsilon))$ so

$$\limsup_{\lambda \in \Lambda} \text{end } f_\lambda \subseteq \bigcap_{\varepsilon > 0} \bigcup_{\lambda \geq \lambda_\varepsilon} \text{end } f_\lambda \subseteq \bigcap_{\varepsilon > 0} \text{end } M_f(B_d(\cdot, \varepsilon)) = \text{end } \bar{f}.$$

□

Observe that the converse of the previous result is not true in general (see Example 10).

Remark 17. Notice that when we consider upper semicontinuous fuzzy sets, the above result can be improved since $\{f_\lambda\}_{\lambda \in \Lambda}$ is Γ^+ -convergent to f so

$$\text{Ls end } f_\lambda \subseteq \text{end } f.$$

Corollary 6 (cf. [14, Lemma 3.3]). Let (X, d) be a metric space. Then a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $SUSC(X)$ is d_∞^+ -convergent to $f \in SUSC(X)$ if and only if $\{f_\lambda\}_{\lambda \in \Lambda}$ is τ_\geq -uniformly convergent to $M_f(B_d(\cdot, \varepsilon))$ for all $\varepsilon > 0$.

Proof. By Proposition 4, we only have to prove the sufficiency.

Let $\varepsilon > 0$. By assumption, we can find λ_ε such that $f_\lambda(x) \leq M_f(B_d(x, \varepsilon/2))$ for all $\lambda \geq \lambda_\varepsilon$ and all $x \in X$. We show that $[f_\lambda]^\alpha \subseteq B_d([f]^\alpha, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$ and all $\alpha \in (0, 1]$. Fix $\lambda \geq \lambda_\varepsilon$ and $\alpha \in (0, 1]$. Let $x \in [f_\lambda]^\alpha$. If $\alpha \leq f_\lambda(x) < M_f(B_d(x, \varepsilon/2))$ then it is obvious that $x \in B_d([f]^\alpha, \varepsilon)$. Otherwise, suppose that $f_\lambda(x) = M_f(B_d(x, \varepsilon/2)) > 0$ (if $f_\lambda(x) = 0$ the conclusion is obvious). Then we can find a sequence $\{a_n\}_{n \in \mathbb{N}}$ in $B_d(x, \varepsilon/2)$ such that $f(a_n) > 0$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow +\infty} f(a_n) = f_\lambda(x)$. Since $[f]^0$ is compact, we can suppose without loss of generality that the sequence $\{a_n\}_{n \in \mathbb{N}}$ is convergent to an $a \in \bar{B}_d(x, \varepsilon/2)$. Since f is upper semicontinuous we have that $\limsup_{n \rightarrow +\infty} f(a_n) = f(a) \leq f(x)$. We conclude again that $x \in B_d([f]^\alpha, \varepsilon)$. Consequently, $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞^+ -convergent to f . □

Corollary 7. Let (X, d) be a metric space. Then a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is d_∞^+ -convergent to f if and only if for all $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \Lambda$ such that

$$f_\lambda(x) \leq M_f(B_d(x, \varepsilon)) \text{ for all } \lambda \geq \lambda_\varepsilon \text{ and all } x \in X$$

and

$$\text{if given } \lambda \geq \lambda_\varepsilon, f_\lambda(x) = M_f(B_d(x, \varepsilon)) \text{ then } \text{Argmax}_{B_d(x, \varepsilon)} f \neq \emptyset.$$

4.3. Convergence in the supremum metric. From the results obtained in the last two sections, it is easy to state results about the convergence in the supremum metric only combining the corresponding fact about the lower and upper parts. For instance:

Theorem 5. Let (X, d) be a metric space. Then a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is d_∞ -convergent to $f \in \mathcal{F}(X)$ if and only if:

- (1) the net of multifunctions $\{f_\lambda^\varepsilon\}_{\lambda \in \Lambda}$ is lower uniformly convergent to f for all $\varepsilon > 0$, where $f_\lambda^\varepsilon : X \rightrightarrows I$ is given by $f_\lambda^\varepsilon(x) = \overline{f_\lambda(B_d(x, \varepsilon))}^{\tau \leq}$;
- (2) the net $\{f_\lambda\}_{\lambda \in \Lambda}$ considered as a net of multifunctions is upper uniformly convergent to the multifunction $f^\varepsilon : X \rightrightarrows I$ for all $\varepsilon > 0$ where $f^\varepsilon(x) = \overline{f(B_d(x, \varepsilon))}^{\tau \leq}$.

Corollary 8 ([14, Lemma 3.3]). Let (X, d) be a metric space. Then a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $SUSC(X)$ is d_∞ -convergent to $f \in SUSC(X)$ if and only if:

- (1) $\{M_{f_\lambda}(B_d(\cdot, \varepsilon))\}_{\lambda \in \Lambda}$ is τ_{\leq} -uniformly convergent to f for all $\varepsilon > 0$;
(2) $\{f_\lambda\}_{\lambda \in \Lambda}$ is τ_{\geq} -uniformly convergent to $M_f(B_d(\cdot, \varepsilon))$ for all $\varepsilon > 0$.

Finally we will treat explicitly some questions about maximization when considering the d_∞ -convergence.

Proposition 6. *Let (X, d) be a metric space. If a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ is d_∞ -convergent to f then $\{M_{f_\lambda}(X)\}_{\lambda \in \Lambda}$ is residually constant and equal to $M_f(X)$.*

Proof. This is a consequence of Remarks 11 and 15. \square

Proposition 7. *Let (X, d) be a metric space and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{F}(X)$ d_∞ -convergent to an upper semicontinuous fuzzy set f . If $x_\lambda \in \text{Argmax}_X f_\lambda$ for all $\lambda \in \Lambda$ and $\{x_\lambda\}_{\lambda \in \Lambda}$ is convergent to x_0 then $x_0 \in \text{Argmax}_X f$.*

Proof. Suppose that $x_\lambda \in \text{Argmax}_X f_\lambda$ for all $\lambda \in \Lambda$ and that $\{x_\lambda\}_{\lambda \in \Lambda}$ is convergent to x_0 . Since $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞ -convergent to f , by Proposition 4 given $\varepsilon > 0$ we can find $\lambda_\varepsilon \in \Lambda$ such that

$$f_\lambda(x) \leq M_f(B_d(x_\lambda, \varepsilon/2)) \text{ for all } x \in X, \lambda \geq \lambda_\varepsilon,$$

and

$$d(x_0, x_\lambda) < \varepsilon/2 \text{ for all } \lambda \geq \lambda_\varepsilon.$$

In particular, and since $B_d(x_\lambda, \varepsilon/2) \subseteq B_d(x_0, \varepsilon)$ for all $\lambda \geq \lambda_\varepsilon$, we have that

$$f_\lambda(x_\lambda) = M_{f_\lambda}(X) \leq M_f(B_d(x_\lambda, \varepsilon/2)) \leq M_f(B_d(x_0, \varepsilon)) \leq M_f(X)$$

for all $\lambda \geq \lambda_\varepsilon$. By the above proposition we know that $M_f(X) = M_{f_\lambda}(X)$ residually so

$$M_{f_\lambda}(X) = M_f(B_d(x_0, \varepsilon)) = M_f(X) \text{ residually.}$$

Since this is valid for all $\varepsilon > 0$ and f is upper semicontinuous then $\limsup_{x \rightarrow x_0} f(x) = \inf_{\varepsilon > 0} M_f(B_d(x_0, \varepsilon)) = f(x_0) = M_f(X)$. \square

Remark 18. *Notice that the previous proposition is not true if we don't consider upper semicontinuous fuzzy sets. For example, consider the net of fuzzy sets $\{f_\lambda\}_{\lambda \in \Lambda}$ on I which are equally constant to 1 and the fuzzy set in I given by*

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 1/2 & \text{if } x = 0 \end{cases}.$$

It is obvious that $\{f_\lambda\}_{\lambda \in \Lambda}$ is d_∞ -convergent to f . Nevertheless, $0 \in \text{Argmax}_X f_\lambda$ for all $\lambda \in \Lambda$ but $0 \notin \text{Argmax}_X f$.

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