Bivariate Nakagami-m distribution with arbitrary fading parameters

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Abstract: The bivariate Nakagami-m distribution with arbitrary fading parameters is derived, obtaining the probability density function (PDF), the cumulative density function (CDF) and the central moments. Additionally, limitations of that distribution are discussed.

Introduction: The bivariate Nakagami-m distribution is derived in [1] for equal fading parameters m in the marginal Nakagami-m distributions. Thus, the performance analysis for diversity combiners with correlated Nakagami-m fading based on [1] are limited to equal fading parameters of signals at the inputs of the combiner. In real mobile radio environments, the fading parameter depends on the characteristic of the propagation path of each contribution [2]. In many combiner structures, as RAKE receivers, that supposition usually is not accomplished.

In this Letter, the bivariate Nakagami-m distribution with arbitrary fading parameters are derived using the characteristic function of the bivariate gamma distribution for equal fading parameters.

Derivation of Bidimensional PDF.- Let s_1 and s_2 be gamma distributed obtained as

$$\begin{aligned} s_1 &= \sum_{k=1}^{m_1} r_{1,k}^2 \\ s_2 &= s_{2a} + s_{2b} = \sum_{k=1}^{m_1} r_{2,k}^2 + \sum_{l=m_1+1}^{m_2} r_{2,l}^2 \end{aligned} \qquad (1)$$

where $r_{1,i}^2 \left(i = 1,...,m_1\right)$ and $r_{2,j}^2 \left(j = 1,...,m_2\right)$ are exponentially distributed with $E\left[r_{1,i}^2\right] = \Omega_1$ and $E\left[r_{2,j}^2\right] = \Omega_2$, $s_{2a} = \sum_{k=1}^{m_1} r_{2,k}^2$, $s_{2b} = \sum_{l=m_1+1}^{m_2} r_{2,l}^2$, $m_2 \geq m_1$ and the correlation coefficient between $r_{1,i}^2$ and $r_{2,j}^2$ for $i = j$ is given by

$$\rho = \frac{E\left[r_{1,i}^2 \cdot r_{2,j}^2\right] - E\left[r_{1,i}^2\right] \cdot E\left[r_{2,j}^2\right]}{\sqrt{\sigma_{r_{1,i}^2}^2 \cdot \sigma_{r_{2,j}^2}^2}} \qquad (2)$$

where $\sigma^2_{r^2_{1,i}}$ and $\sigma^2_{r^2_{2,j}}$ are, respectively, the variances of $r^2_{1,i}$ and $r^2_{2,j}$. For $i \neq j$, ρ equals to 0.

The joint characteristic function of s_1 and s_{2a} is defined as

$$\phi_{s_1,s_{2a}}\left(t_1,t_2\right) = \int_0^\infty \int_0^\infty p_{s_1,s_{2a}}\left(s_1,s_{2a}\right) \mathrm{e}^{-\left(s_1 t_1 + s_{2a} t_2\right)}\, ds_1 ds_{2a} \tag{3}$$

where $p_{s_1,s_{2a}}\left(s_1,s_{2a}\right)$ is the bidimensional probability density function (PDF) of s_1 and s_{2a} defined for $s_1 \geq 0$ and $s_{2a} \geq 0$.

The joint characteristic function of s_1 and s_{2a} was obtained in [1 Eqn. 123] as

$$\phi_{s_1,s_{2a}}\left(t_1,t_2\right) = \frac{1}{\left(\Omega_1\Omega_2\left(1-\rho\right)\right)^{m_1}} \times \frac{1}{\left(\left(t_1 + \frac{1}{\Omega_1\left(1-\rho\right)}\right)\left(t_2 + \frac{1}{\Omega_2\left(1-\rho\right)}\right) - \frac{\rho}{\Omega_1\Omega_2\left(1-\rho\right)^2}\right)^{m_1}} \tag{4}$$

where ρ is given by (2). Nevertheless, it can been verified [1] that ρ is also the correlation coefficient between s_1 and s_{2a}.

Since s_{2b} is gamma distributed, the characteristic function of s_{2b} defined as $\phi_{s_{2b}}\left(t_2\right) = \int_0^\infty p_{s_{2b}}\left(s_{2b}\right) e^{-s_{2b}} ds_{2b}$ is given by

$$\phi_{s_{2b}}\left(t_2\right) = \frac{1}{\Omega_2^{m_2-m_1}\left(t_2 + 1/\Omega_2\right)^{m_2-m_1}} \tag{5}$$

Since s_{2a} and s_{2b} are independent, the joint characteristic function of s_1 and s_2 can be written as

$$\phi_{s_1,s_2}(t_1,t_2)=\int_0^\infty\int_0^\infty p_{s_1,s_2}(s_1,s_2)\mathrm{e}^{-(s_1t_1+s_{2a}t_2+s_{2b}t_2)}\,ds_1ds_2=\phi_{s_1,s_{2a}}(t_1,t_2)\cdot\phi_{s_{2b}}(t_2) \tag{6}$$

Substituting (5) into (6), it yields

$$\phi_{s_1,s_2}(t_1,t_2)=\frac{1}{\Omega_1^{m_1}\Omega_2^{m_2}(1-\rho)^{m_1}\left(\left(t_1+\frac{1}{\Omega_1(1-\rho)}\right)\left(t_2+\frac{1}{\Omega_2(1-\rho)}\right)-\frac{\rho}{\Omega_1\Omega_2(1-\rho)^2}\right)^{m_1}}\times$$
$$\times\frac{1}{\left(t_2+\frac{1}{\Omega_2}\right)^{m_2-m_1}} \tag{7}$$

In order to derive the PDF of s_1 and s_2 we can expand (7) in series form as

$$\phi_{s_1,s_2}(t_1,t_2)=\frac{1}{\Omega_1^{m_1}\Omega_2^{m_2}(1-\rho)^{m_1}\left(\left(t_1+\frac{1}{\Omega_1(1-\rho)}\right)\left(t_2+\frac{1}{\Omega_2(1-\rho)}\right)\right)^{m_1}\left(t_2+\frac{1}{\Omega_2}\right)^{m_2-m_1}}\times$$
$$\sum_{k=0}^\infty\frac{(m_1)_k}{k!}\frac{\rho^k}{\Omega_1^k\Omega_2^k(1-\rho)^{2k}}\frac{1}{\left(\left(t_1+\frac{1}{\Omega_1(1-\rho)}\right)\left(t_2+\frac{1}{\Omega_2(1-\rho)}\right)\right)^k} \tag{8}$$

where $(m_1)_k$ is the Pochhammer symbol and $\Gamma(\cdot)$ is the gamma function

The bidimensional PDF of s_1 and s_2 is obtained from

$$p_{s_1,s_2}(s_1,s_2)=\int_{-\infty}^\infty\int_{-\infty}^\infty\phi_{s_1,s_2}(t_1,t_2)e^{s_1t_1+s_2t_2}ds_1ds_2 \tag{9}$$

Inserting (8) into (9), integral (9) can be solved as

$$p_{s_1,s_2}\left(s_1,s_2\right)=(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{\left(m_1\right)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k} s_1^{m_1+k-1}\frac{\exp\left(-\frac{s_1}{\Omega_1(1-\rho)}\right)}{\Gamma\left(m_1+k\right)}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k}\times \tag{10}$$
$$\frac{s_2^{m_2+k-1}\exp\left(-\frac{s_2}{\Omega_2(1-\rho)}\right)}{\Gamma\left(m_2+k\right)}{}_1F_1\left(m_2-m_1,m_2+k;\frac{\rho}{\Omega_2(1-\rho)}s_2\right),\quad s_1,s_2\geq 0$$

where ${}_1F_1(\cdot,\cdot;\cdot)$ is the confluent hypergeometric function [3 eqn. 13.1.2]. Note that (10) is the bidimensional PDF of two gamma variates with different fading parameters and marginal PDFs given by

$$p_{s_j}\left(s_j\right)=\left(\frac{1}{\Omega_j}\right)^{m_j}\frac{s_j^{m_j-1}}{\Gamma\left(m_j\right)}\exp\left(-\frac{s_j}{\Omega_j}\right),\quad \mathrm{s}_j\geq 0\quad ,j=1,2 \tag{11}$$

Using the transformation $r_1=\sqrt{s_1/m_1},\ \ r_2=\sqrt{s_2/m_2}$, the Nakagami-$m$ bidimensional PDF with arbitrary fading parameters is derived as

$$p_{r_1,r_2}\left(r_1,r_2\right)=4(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{\left(m_1\right)_k}{k!}\rho^k\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} r_1^{2(m_1+k)-1}\frac{\exp\left(-\frac{m_1r_1^2}{\Omega_1(1-\rho)}\right)}{\Gamma\left(m_1+k\right)}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k}\times \tag{12}$$
$$\frac{r_2^{2(m_2+k)-1}\exp\left(-\frac{m_2r_2^2}{\Omega_2(1-\rho)}\right)}{\Gamma\left(m_2+k\right)}{}_1F_1\left(m_2-m_1,m_2+k;\frac{m_2\rho}{\Omega_2(1-\rho)}r_2^2\right),\quad r_1,r_2\geq 0$$

It can be verified that marginal distributions from (12) follow Nakagami-m distributions with PDFs given by

$$p_{r_j}\left(r_j\right)=2\left(\frac{m_j}{\Omega_j}\right)^{m_j}\frac{r_j^{2m_j-1}}{\Gamma\left(m_j\right)}\exp\left(-\frac{m_jr_j^2}{\Omega_j}\right),\quad r_j\geq 0\quad ,j=1,2 \tag{13}$$

Note that bidimensional PDFs (10) and (12) are restricted to $m_2 \geq m_1$. In particular, for $m_2 = m_1 = m$ (10) agrees with [1 eqn. 136]. The results (10) and

(12) like derivation of bidimensional Nakagami PDF with equal fading parameters in [1] can be extended to m_1 and m_2 non-integers satisfying $m_2 \geq m_1 \geq$ $\frac{1}{2}$.

Bivariate cumulative density function (CDF) and joint moments.- The bivariate Nakagami-m CDF with arbitrary fading parameters can be derived by integrating (12) as

$$F_{r_1,r_2}\left(r_1,r_2\right) = \int_0^{r_1}\int_0^{r_2} p_{r_1,r_2}\left(r_1,r_2\right) dr_1 dr_2 \tag{14}$$

Substituting (12) into (14) and using [3 eqn. 3.381 1], one can obtain

$$\begin{aligned} F_{r_1,r_2}\left(r_1,r_2\right) &= \frac{(1-\rho)^{m_2}}{\Gamma\left(m_1\right)} \sum_{k=0}^{\infty} \rho^k \frac{\gamma\left(m_1+k, \dfrac{m_1 r_1^2}{\Omega_1(1-\rho)}\right)}{k!} \times \\ & \sum_{l=0}^{\infty} \frac{\Gamma\left(m_2-m_1+l\right)}{\Gamma\left(m_2-m_1\right)\Gamma\left(m_2+k+l\right)l!} \rho^l \gamma\left(m_2+k+l, \frac{m_2 r_2^2}{\Omega_2(1-\rho)}\right), \qquad r_1,r_2 \geq 0 \end{aligned} \tag{15}$$

where $\gamma(\cdot,\cdot)$ is the incomplete gamma function [3 eqn. 6.5.2].

Eqn. 15 agrees with [4] for $m_2 = m_1 = m$.

The joint central moments of the bivariate Nakagami-m distribution with arbitrary fading parameters can be derived as

$$E\left[r_1^n \cdot r_2^l\right] = \left(\frac{\Omega_1}{m_1}\right)^{\frac{n}{2}} \left(\frac{\Omega_2}{m_2}\right)^{\frac{l}{2}} \frac{\Gamma\left(m_1+\dfrac{n}{2}\right)\Gamma\left(m_2+\dfrac{l}{2}\right)}{\Gamma(m_1)\Gamma(m_2)} {}_2F_1\left(-\frac{n}{2},-\frac{l}{2};m_2;\rho\right) \tag{16}$$

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the Gauss hypergeometric function [3 eqn. 15.1.1]. Eqn. 16 is equivalent to [1 eqn. 137] for $m_1 = m_2 = m$.

From (16) and the central moments of the univariate Nakagami-m distribution defined in [1 eqn. 17], the correlation coefficient between s_1 and s_2 is calculated as $\rho_{m_1,m_2} = \rho\sqrt{m_1/m_2}$, where ρ is defined in (2) and used in (10) and (12).

The correlation coefficient in power terms between two Nakagami-m variates generated using (1) is limited therefore to $\left|\rho_{m_1,m_2}\right| < \sqrt{m_1/m_2}$.

In urban environment, the variation of fading parameters m is reduced, e.g. in [2] a mean value of m equal to 1.56 and standard deviation of 0.34 are reported. Thus, in several real environments Nakagami-m bidimensional probability density functions using (12) can be achieved for high correlation coefficients.

Conclusions.- In this Letter, the Nakagami-m bivariate distribution with arbitrary fading parameters has been derived analytically. The probability density function, the cumulative density function and the central moments have been obtained from the characteristic function. The main constraint of this joint distribution is the limited range of correlation coefficients values between both

Nakagami-m variates. Nevertheless, in many practical cases the Nakagami-m bivariate distribution with arbitrary fading parameters can be achieved for high correlation coefficients.

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