Positive finite difference schemes for a partial integro-differential option pricing model

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Abstract

This paper provides a numerical analysis for European options under partial integro-differential Bates model. An explicit finite difference scheme has been used for the differential part, while the integral part has been approximated using the four-points open type formula. The stability and consistency have been studied. Moreover, conditions guaranteeing positivity of the solutions are provided. Illustrative numerical examples are included.

Keywords: Partial integro-differential equation, Bates model, numerical analysis, stability and positivity.

1. Introduction

It is well known that the geometric Brownian motion proposed by Black-Scholes [1] fails to reflect some empirical phenomena such as the volatility smiles and skews in the return distribution and the large random fluctuations as crashes and rallies. There are two ways of developing option pricing models capturing such behavior; firstly adding jumps into the price process for the underlying asset, as proposed by Merton [2] and Kou [3]; secondly, allowing the volatility to evolve stochastically for instant Hull and White [4] and Heston [5].

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Essentially, stochastic volatility appears to be needed to explain the variation in strike at longer time, although it performs poorly across different maturities, especially at shorter time. Adding jumps to the price and/or the volatility provides a great flexibility allowing to explain the strike variation at shorter time, [6, Chap. 14]. In this sense, Cont and Tankov (2003) indicate that a model combining both stochastic volatility and jump diffusion feather provides more reasonable results.

Bates Model [8] combines the Merton and Heston models by adding log-normally distributed jumps to the square root volatility process introduced by Heston. Other further extensions have been studied in [9, 10, 11].

In this paper we deal with the Bates model that describes the behavior of the underlying asset S and its variance ν by the coupled stochastic differential equations:

$$dS(t) = (r - q - \lambda\xi)S(t)dt + \sqrt{\nu(t)}S(t)dW_1 + (\eta - 1)S(t)dZ(t),$$
$$d\nu(t) = \kappa(\theta - \nu(t))dt + \sigma\sqrt{\nu(t)}dW_2,$$
$$dW_1 dW_2 = \rho dt,$$

where W_1 and W_2 are standard Brownian motions, Z is the poisson process. The parameter r is the risk free interest rate, q is the continuous dividend yield, λ is the jump intensity, κ is the mean reversion rate, θ is the long-run variance, σ is the volatility of the variance ν, ρ is the Wiener correlation parameter, η is the jump amplitude of the jump diffusion process and ξ is the expected relative jump size ($\xi = E[\eta - 1]$). By using Itô calculus and standard arbitrage arguments one gets the partial integro-differential equation (PIDE) with the unknown option price $U(S, \nu, \tau)$ [12, 13]

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\nu S^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma\nu S\frac{\partial^2 U}{\partial S\partial\nu} + \frac{1}{2}\sigma^2\nu\frac{\partial^2 U}{\partial \nu^2} + (r - q - \lambda\xi)S\frac{\partial U}{\partial S} + \kappa(\theta - \nu)\frac{\partial U}{\partial \nu} - (r + \lambda)U$$
$$+\lambda\int_0^\infty U(S\eta, \nu, \tau)f(\eta)d\eta, \tag{1}$$

where $\tau = T - t$ is the time to maturity. and the density function $f(\eta)$ is given by

$$f(\eta) = \frac{1}{\sqrt{2\pi}\hat{\sigma}\eta} \exp[-\frac{(\ln\eta - \mu)^2}{2\hat{\sigma}^2}], \tag{2}$$

where μ is the mean of the jump and $\hat{\sigma}$ is the standard deviation. For the European call option we consider the initial condition

$$U(S, \nu, 0) = \max\{S - E, 0\}, \tag{3}$$

where E is the strike price. We assume the boundary conditions applied to the Heston model, see [14], but modified for $\nu = 0$ due to the additional integral term appearing in Bates model. For the boundaries $S = 0$ and $S \to \infty$ one gets

$$\begin{aligned} U(0, \nu, \tau) &= 0, \\ \lim_{S\to\infty} \frac{\partial U}{\partial S}(S, \nu, \tau) &= 1. \end{aligned} \tag{4}$$

Note that this last condition means a linear behavior of the option price for large values of S with slope 1. Based on that fact, we replace it by the following condition, see [15, Chap. 3, pag. 54]

$$U(S, \nu, \tau) \approx e^{-q\tau} S, \text{ as } S \to \infty, \tag{5}$$

with slope $e^{-q\tau}$ for large values of S due to the dividend payment. For $\nu \to \infty$ and $\nu = 0$, the corresponding boundary conditions are imposed as follows

$$\lim_{\nu\to\infty} U(S, \nu, \tau) = S, \tag{6}$$

$$\frac{\partial U}{\partial \tau}(S, 0, \tau) = (r - q - \lambda\xi) S \frac{\partial U}{\partial S}(S, 0, \tau) - (r + \lambda) U(S, 0, \tau) + \kappa\theta \frac{\partial U}{\partial \nu}(S, 0, \tau)$$

$$+ \frac{\lambda}{\sqrt{2\pi}\hat{\sigma}} \int_0^\infty U(\varphi, 0, \tau) \exp[-\frac{(\ln\varphi - \ln S - \hat{\mu})^2}{2\hat{\sigma}^2}] \frac{d\varphi}{\varphi}, \tag{7}$$

where $\varphi = S\eta$.

Some authors used an alternative boundary condition see [21, 23]. Chiarella *et.*

al. [13] used the method of lines to solve the American call option problem for Bates model by discretizing with respect to time and variance variables obtaining a system of first order ordinary differential equations with two unknowns the price and its derivative with respect to asset variable. Then the system is solved using Riccati transformation, see [16]. Final discretization achieves a seven points stencil scheme treated using a linear complementarity problems (LCP). More recently [17] treat also the American call option problem under the Bates model using a full discretization for the spatial variable driving to a seven point finite difference stencil and the quadrature term is discretized using the quadrature rule based on piecewise linear interpolation. The authors use Rannacher scheme [18] for the time-stepping and the resulting LCP problem is solved using an iterative method.

The model (1)-(7) has two challenges from the numerical analysis point of view. Firstly, the presence of a mixed spatial derivative term involves the existence of negative coefficient terms into the numerical scheme deteriorating the quality of the numerical solution such as spurious oscillation and slow convergence, see the introduction of [19]. Secondly, the discretization of the improper integral part should be adequate with the bounded numerical domain and the incorporation of the initial and boundary conditions.

Dealing with prices, guaranty of the positivity of the solutionis is essential. In this paper we construct an explicit difference scheme that guarantees positive solutions. We transform the PIDE (1) into a new PIDE without mixed spatial derivative before the discretization, following the idea of [20], and avoiding the above quoted drawbacks. Furthermore, this strategy has additional computational advantage of the reduction of the stencil scheme points, from nine [21, 22] or seven [13, 17] to just five.

The discrete treatment of the integral part is performed taking into account the chosen boundary numerical domain together with the boundary conditions and using a composite four points integration formula of open type because of the higher order approximation of this rule [24, pp. 92-93].

The organization of the paper is as follows. In Section 2 we transform the original problem into a new one without cross derivative term. We also construct the difference scheme including its matrix form that will be used in Section 3 to study positivity and stability. Section 4 is addressed to the study of consistency of the scheme. Numerical examples illustrating the results for Bates European option model are included in Section 5. In Section 6, we consider the Bates model for American option using our finite difference scheme including the comparison with results of other authors.

Here we recall some useful definitions starting with the definition of the norm for vectors and matrices. For a given vector $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} = (v_1, v_2, \dots, v_n)^T$, the infinite norm of $\mathbf{v}$ is denoted by $\|\mathbf{v}\|_\infty$ and is defined as $\|\mathbf{v}\|_\infty = \max\{v_j,\ 1 \leq j \leq n\}$. The vector $\mathbf{v}$ is said to be nonnegative if $v_j \geq 0$ for all $1 \leq j \leq n$, then we denote $\mathbf{v} \geq 0$. For a matrix $B = (b_{ij})_{n\times m}$ in $\mathbb{R}^{m\times n}$, we denote by $\|B\|_\infty = \max_{1\leq i\leq m}\{\sum_{j=1}^{n} |b_{ij}\|\}$. Consequently if A is a block matrix with $n \times m$ block entries A_{ij}, then the infinite norm of A, see [25, Chap. 2],

$$\|A\|_\infty = \max_{1\leq i\leq m}\{\|[A_{i1}\ A_{i2} \dots A_{in}]\|_\infty\}. \tag{8}$$

Matrix A is said to be nonnegative if $a_{ij} \geq 0$ for all $1 \leq i \leq m$, $1 \leq j \leq n$, and we denote $A \geq 0$. For $x \in \mathbb{R}$, the error function of x is denoted by $\mathrm{erf}(x)$ and is given by [26, pag. 93]

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

2. Problem Transformation and Scheme Construction

2.1. The transformation of the problem

We begin this section by eliminating the mixed spatial derivative term of (1), inspired by the reduction of second order linear partial differential equation in two independent variables to canonical form, see [27, Chap. 3] and [20] for details. Let us consider the following transformation

$$x = \tilde{\rho}\sigma \ln S;\ \ y = \rho\sigma \ln S - \nu;\ \ w(x, y, \tau) = e^{(r+\lambda)\tau} U(S, \nu, \tau), \tag{9}$$

where $\tilde{\rho} = \sqrt{1-\rho^2}$, $0 < |\rho| < 1$, obtaining the following transformed equation

$$\frac{\partial w}{\partial \tau} = \frac{\tilde{\rho}^2 \nu \sigma^2}{2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \hat{\delta}\frac{\partial w}{\partial x} + \tilde{\delta}\frac{\partial w}{\partial y} + I(w), \tag{10}$$

with

$$I(w) = \lambda \int_0^\infty w(x + \sigma\tilde{\rho}\ln\eta,\ y + \rho\sigma\ln\eta, \tau) f(\eta) d\eta, \tag{11}$$

where

$$\hat{\delta} = \sigma\tilde{\rho}(\hat{\xi} - \frac{\nu}{2}),\ \tilde{\delta} = \sigma\rho(\hat{\xi} - \frac{\nu}{2}) - \kappa(\theta - \nu) \text{ and } \hat{\xi} = r - q - \lambda\xi. \tag{12}$$

For the sake of convenience in the matching of the further discretization of the differential and integral parts of (10), we consider now the substitution

$$\phi = x + \sigma\tilde{\rho}\ln\eta. \tag{13}$$

Hence from 2 and (11) one gets

$$I(w) = \frac{\lambda}{\sqrt{2\pi}\hat{\sigma}\tilde{\rho}\sigma} \int_{-\infty}^\infty w(\phi, y + m(\phi - x), \tau) \exp\left[\frac{-1}{\hat{\sigma}^2}\left(\frac{\phi - x}{\sigma\tilde{\rho}} - \mu\right)^2\right] d\phi, \tag{14}$$

where $m = \frac{\rho}{\tilde{\rho}}$. Note that from (9), we have

$$y = mx - \nu. \tag{15}$$

The initial and boundary conditions (3)-(7) are transformed into the corresponding conditions using (9) and (13).

$$w(x, y, 0) = \max\{e^{\frac{x}{\sigma\tilde{\rho}}} - E, 0\}, \tag{16}$$

$$\lim_{x\to-\infty} w(x, y, \tau) = 0, \tag{17}$$

$$w(x, y, \tau) \approx \exp\left[\frac{x}{\sigma\tilde{\rho}} + (r - q + \lambda)\tau\right],\ \ x \to \infty, \tag{18}$$

$$w(x,y,\tau) \approx \exp\left[\frac{x}{\sigma\tilde{\rho}} + (r+\lambda)\tau\right],\ mx - y \to \infty, \tag{19}$$

$$\frac{\partial w}{\partial \tau} \approx \sigma\tilde{\rho}\hat{\xi}\frac{\partial w}{\partial x} + (\sigma\rho\hat{\xi} - \kappa\theta)\frac{\partial w}{\partial y}$$

$$+\frac{\lambda}{\sqrt{2\pi}\hat{\sigma}\tilde{\rho}\sigma}\int_{-\infty}^{\infty} w(\phi, m\phi - \nu, \tau)\exp\left[\frac{-1}{\hat{\sigma}^2}\left(\frac{\phi - x}{\sigma\tilde{\rho}} - \mu\right)^2\right]d\phi,\ \nu \to 0. \tag{20}$$

From [28, 29] a suitable bound for the underlying asset variable S is available and generally accepted. In an analogous way, considering an admissible range of the variance ν, we can identify a convenient-bounded numerical domain $\mathcal{R} = [S_1, S_2] \times [\nu_1, \nu_2]$ in the $S - \nu$ plane. Under the transformation (9) as it is shown in [20] the rectangle $\mathcal{R}$ is transformed into the rhomboid $ABCD$ as shown in Fig. 1, where the sides are given by

$$\begin{aligned}
\overline{AB} &= \{(x,y) \in \mathbb{R}^2 |\ a \le x \le b,\ y = mx - \nu_2\},\\
\overline{BC} &= \{(x,y) \in \mathbb{R}^2 |\ x = b,\ y = mb - \nu,\ \nu_1 \le \nu \le \nu_2\},\\
\overline{CD} &= \{(x,y) \in \mathbb{R}^2 |\ a \le x \le b,\ y = mx - \nu_1\},\\
\overline{DA} &= \{(x,y) \in \mathbb{R}^2 |\ x = a,\ y = ma - \nu,\ \nu_1 \le \nu \le \nu_2\},
\end{aligned} \tag{21}$$

where

$$a = \sigma\tilde{\rho}\ln S_1, \quad b = \sigma\tilde{\rho}\ln S_2. \tag{22}$$

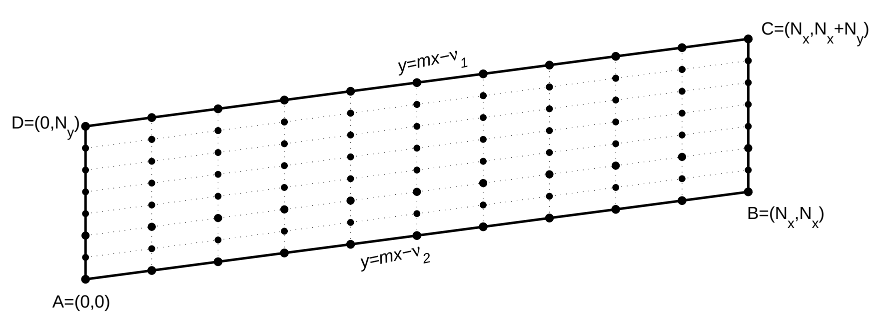

Fig. 1. Rhomboid numerical domain $ABCD$

2.2. The numerical scheme

In light of the transformation (9) and the boundary given by (21), we use a discretization of the numerical domain where the space stepsizes $h = \Delta x$ and $h_y = \Delta y = |m|h$ are related by the slope $m = \frac{\varrho}{\rho}$. Here we subdivide space-time axes into uniform spaced points using

$$\begin{gathered} x_i = a + ih,\ 0 \le i \le N_x, \qquad y_j = y_0 + j|m|h,\ i \le j \le N_y + i, \\ \nu_{i,j} = mx_i - y_j, \qquad \tau^n = nk,\ 0 \le n \le N_\tau, \end{gathered} \tag{23}$$

where $h = \frac{b-a}{N_x}$, $y_0 = ma - \nu_2$, $N_y = \frac{\nu_2 - \nu_1}{|m|h}$ and $k = \frac{T}{N_\tau}$. Note that any mesh point in the computational spatial domain has the form

$$(x_i, y_j) = (a + ih, mx_i - \nu_2 + (j - i)|m|h).$$

The discretization for the boundary points is given by

$$\begin{aligned} P(\overline{AB}) &= \{(x_i, y_i)|\ 0 \le i \le N_x\}, \\ P(\overline{BC}) &= \{(x_{N_x}, y_j)|\ N_x \le j \le N_x + N_y\}, \\ P(\overline{CD}) &= \{(x_i, y_{i+N_y})|\ 0 \le i \le N_x\}, \\ P(\overline{DA}) &= \{(x_0, y_j)|\ 0 \le j \le N_y\}. \end{aligned} \tag{24}$$

Denote the approximate value of w at a representative mesh point $P(x_i, y_j, \tau^n)$ by $W^n_{i,j}$, we implement the center difference approximation for spatial partial derivatives such that

$$\frac{\partial w}{\partial x} \approx \frac{W^n_{i+1,j} - W^n_{i-1,j}}{2h};\ \frac{\partial w}{\partial y} \approx \frac{W^n_{i,j+1} - W^n_{i,j-1}}{2|m|h}, \tag{25}$$

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{W^n_{i+1,j} - 2W^n_{i,j} + W^n_{i-1,j}}{h^2};\ \frac{\partial^2 w}{\partial y^2} \approx \frac{W^n_{i,j+1} - 2W^n_{i,j} + W^n_{i,j-1}}{m^2h^2}, \tag{26}$$

and $\frac{\partial w}{\partial \tau}$ is discretized using the explicit forward approximation

$$\frac{\partial w}{\partial \tau} \approx \frac{W^{n+1}_{i,j} - W^n_{i,j}}{k}. \tag{27}$$

For the approximation of the integral part $I(w)$ in (11), the improper integral is truncated into $[a, b]$ and we implement the composite four points integration formula of open type [24, pp. 92-93] using the same step size for the variable x as in the differential part. Hence the corresponding finite difference equation for (10) is given by

$$W_{i,j}^{n+1} = \beta_{i,j} W_{i,j}^{n} + \hat{\alpha}_{i,j} W_{i+1,j}^{n} + \breve{\alpha}_{i,j} W_{i-1,j}^{n} + \alpha_{i,j} W_{i,j-1}^{n} + \gamma_{i,j} W_{i,j+1}^{n} + \hat{\lambda} J_{i,j}^{n}, \quad (28)$$

$$1 \le i \le N_x - 1,\ i+1 \le j \le N_y + i - 1,\ 0 \le n \le N_\tau - 1,$$

where

$$\begin{aligned}
\beta_{i,j} &= 1 - \tfrac{k\sigma^2}{h^2 m^2} \nu_{i,j} = (1 - \tfrac{k}{h^2} \tilde{a}_{ij}),\\
\hat{\alpha}_{i,j} &= \tfrac{k\sigma\tilde{\rho}}{2h} \left[\tfrac{(2\tilde{\rho}\sigma - h)}{2h} \nu_{i,j} + \hat{\xi} \right] = \tfrac{k}{h} (\tfrac{\rho^2}{2h} \tilde{a}_{ij} + \tilde{b}_{ij})\\
\breve{\alpha}_{i,j} &= \tfrac{k\sigma\tilde{\rho}}{2h} \left[\tfrac{(2\tilde{\rho}\sigma + h)}{2h} \nu_{i,j} - \hat{\xi} \right] = \tfrac{k}{h} (\tfrac{\rho^2}{2h} \tilde{a}_{ij} - \tilde{b}_{ij}),\\
\alpha_{i,j} &= \tfrac{k}{2|m|h} \left[\left(\tfrac{\sigma^2 \tilde{\rho}^2}{|m|h} + \tfrac{\sigma\rho}{2} - \kappa \right) \nu_{i,j} - \sigma\rho\hat{\xi} + \kappa\theta \right] = \tfrac{k}{h} (\tfrac{\tilde{\rho}^2}{2h} \tilde{a}_{ij} - \tfrac{m}{|m|} \tilde{b}_{ij} + \tilde{c}_{ij}),\\
\gamma_{i,j} &= \tfrac{k}{2|m|h} \left[\left(\tfrac{\sigma^2 \tilde{\rho}^2}{|m|h} - \tfrac{\sigma\rho}{2} + \kappa \right) \nu_{i,j} + \sigma\rho\hat{\xi} - \kappa\theta \right] = \tfrac{k}{h} (\tfrac{\tilde{\rho}^2}{2h} \tilde{a}_{ij} + \tfrac{m}{|m|} \tilde{b}_{ij} - \tilde{c}_{ij}),
\end{aligned} \quad (29)$$

$$\hat{\lambda} = \frac{5kh\lambda}{24\sqrt{2\pi}\hat{\sigma}\tilde{\rho}\sigma}, \quad (30)$$

and the integral part is given by

$$J_{i,j}^{n} = \sum_{\ell=0}^{N_x/5-1} \big(11 g_{i,5\ell+1} W_{5\ell+1,5\ell+1+j-i}^{n} + g_{i,5\ell+2} W_{5\ell+2,5\ell+2+j-i}^{n}$$

$$+ g_{i,5\ell+3} W_{5\ell+3,5\ell+3+j-i}^{n} + 11 g_{i,5\ell+4} W_{5\ell+4,5\ell+4+j-i}^{n} \big), \quad (31)$$

assuming that N_x has been previously chosen as a multiple of 5. The weight function $g_{i,\ell}$ is given by

$$g_{i,\ell} \equiv g(x_i, \phi_\ell) = \exp\left[\frac{-1}{2\hat{\sigma}^2} \left(\frac{\phi_\ell - x_i}{\sigma\tilde{\rho}} - \mu \right)^2 \right],\ 0 \le \ell \le N_x. \quad (32)$$

The initial condition (16) is discretized into

$$W_{i,j}^0 = \max\{\exp(\frac{x_i}{\sigma\tilde{\rho}}) - E, 0\},\ 0 \le i \le N_x,\ i \le j \le N_y + i, \tag{33}$$

and the two Dirichlet conditions (17) along $\overline{AD}$ and (18) along $\overline{AB}$ take the forms

$$W_{0,j}^n = 0,\ 0 \le j \le N_y - 1,\ 1 \le n \le N_\tau, \tag{34}$$

$$W_{i,i}^n = \exp\left[\frac{x_i}{\sigma\tilde{\rho}} + (r+\lambda)\tau^n\right],\ 1 \le i \le N_x,\ 1 \le n \le N_\tau \tag{35}$$

respectively. For the boundary condition along $\overline{BC}$, x is constant $x = b$ and from (18) one gets

$$W_{N_x,j}^n = \exp\left[\frac{b}{\sigma\tilde{\rho}} + (r+\lambda-q)\tau^n\right],\ N_x+1 \le j \le N_x+N_y,\ 1 \le n \le N_\tau, \tag{36}$$

Note that the boundary condition (20) along the oblique segment $\overline{CD}$ involves $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$. By the way the spatial directional derivative of w for fixed τ along the direction $\overline{CD}$ with unitary vector $\hat{\mathbf{u}} = (\tilde{\rho}, \rho, 0)$ is given by

$$D_{\hat{\mathbf{u}}} w = \nabla w \cdot \hat{\mathbf{u}} = \tilde{\rho}\frac{\partial w}{\partial x} + \rho\frac{\partial w}{\partial y}.$$

The centered finite difference approximation for the directional derivative along CD at the mesh point $(x_i, y_{N_y+i}, \tau^n) \in P(\overline{CD})$ is given by

$$D_{\hat{\mathbf{u}}} w \approx \frac{\tilde{\rho}}{2h}(W_{i+1,N_y+i+1}^n - W_{i-1,N_y+i-1}^n), \tag{37}$$

and the backward difference approximation has been used for the term $\kappa\theta\frac{\partial w}{\partial y}$,

$$\kappa\theta\frac{\partial w}{\partial y} \approx \frac{\kappa\theta}{|m|h}(W_{i,N_y+i}^n - W_{i,N_y+i-1}^n), \tag{38}$$

while the integral part of (20) is approximated using four points open type formula. For the sake of positivity of the coefficients of the scheme, we take the following special approximation of the term $\frac{\partial w}{\partial \tau}$

$$\frac{\partial w}{\partial \tau} \approx \frac{1}{k}\left(W_{i,N_y+i}^{n+1} - \frac{1}{3}\left(W_{i-1,N_y+i-1}^n + W_{i,N_y+i}^n + W_{i+1,N_y+i+1}^n\right)\right). \tag{39}$$

From (37)-(39) the boundary condition (20) is approximated by

$$W^{n+1}_{i,N_y+i} = \hat{a}_1 W^n_{i-1,N_y+i-1} + \hat{a}_2 W^n_{i,N_y+i-1} + \hat{a}_3 W^n_{i,N_y+i} + \hat{a}_4 W^n_{i+1,N_y+i+1} + \hat{\lambda} J^n_{i,N_y+i}, \tag{40}$$

for $1 \le i \le N_x - 1$ and $0 \le n \le N_\tau - 1$, where

$$\hat{a}_1 = \frac{1}{3} - \frac{k\sigma\tilde{\rho}\hat{\xi}}{2h},\ \hat{a}_2 = \frac{k\kappa\theta}{|m|h},\ \hat{a}_3 = \frac{1}{3} - \hat{a}_2,\ \hat{a}_4 = \frac{1}{3} + \frac{k\sigma\tilde{\rho}\hat{\xi}}{2h}, \tag{41}$$

and J^n_{i,N_y+i} is obtained from (31) taking $j = N_y + i$.

In order to study the stability of the numerical scheme (28)-(41), let us write it in a matrix form. It is convenient to write the numerical solutions $\{W^n_{i,j}\}$ in a suitable vector form, following the strategy of [30], let us define the vector $\mathbf{W}^n \in \mathbb{R}^{(N_x+1)(N_y+1)}$ such that

$$\mathbf{W}^n = \left[\begin{array}{cccc} \mathcal{W}^n_0 & \mathcal{W}^n_1 & \dots & \mathcal{W}^n_{N_x} \end{array} \right]^T, \tag{42}$$

where $\mathcal{W}^n_i$ are vectors in $\mathbb{R}^{(N_y+1)}$

$$\mathcal{W}^n_i = \left[\begin{array}{cccc} W^n_{i,i} & W^n_{i,i+1} & \dots & W^n_{i,i+N_y} \end{array} \right].$$

Hence numerical scheme (28)-(41) can be written in a matrix form as

$$\mathbf{W}^{n+1} = (\mathcal{D} + \mathcal{P})\mathbf{W}^n,\ 0 \le n \le N_\tau - 1, \tag{43}$$

where $\mathcal{D}$ and $\mathcal{P}$ are square matrices of size $(N_x+1)(N_y+1) \times (N_x+1)(N_y+1)$ representing the discretization of the differential and integral parts of the scheme (28)-(41) respectively. The block matrix $\mathcal{D}$ can be written in the explicit form

$$\mathcal{D} = \left[\begin{array}{ccccccc} I & \Theta & \Theta & \Theta & \dots & \dots & \Theta \\ \check{C}(1) & B(1) & \hat{C}(1) & \Theta & \dots & \dots & \Theta \\ \Theta & \check{C}(2) & B(2) & \hat{C}(2) & \Theta & \dots & \Theta \\ \vdots & \Theta & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \dots & \dots & \check{C}(N_x-1) & B(N_x-1) & \hat{C}(N_x-1) \\ \Theta & \Theta & \dots & \dots & \dots & \Theta & e^{(r-q+\lambda)k} I \end{array} \right], \tag{44}$$

where I and Θ are the identity and zero matrices in $\mathbb{R}^{(N_y+1)\times(N_y+1)}$. The block entries $\breve{C}(\ell),\ B(\ell)$ and $\hat{C}(\ell)$ are matrices $\in \mathbb{R}^{(N_y+1)\times(N_y+1)}$ such that

$$\breve{c}_{ij}(\ell) = \begin{cases} \breve{\alpha}_{\ell,\ell+i-1}, & i = 2,\dots,N_y,\ j = i+1, \\ \hat{a}_1, & i = j = N_y+1, \\ 0, & \text{otherwise.} \end{cases} \tag{45}$$

$$b_{ij}(\ell) = \begin{cases} e^{(\lambda+r)k}, & i = j = 1, \\ \alpha_{\ell,\ell+i-1}, & j = i-1,\ i = 2,\dots,N_y, \\ \beta_{\ell,\ell+i-1}, & j = i,\ i = 2,\dots,N_y, \\ \gamma_{\ell,\ell+i-1}, & j = i+1,\ i = 2,\dots,N_y, \\ \hat{a}_2, & i = N_y+1,\ j = N_y, \\ \hat{a}_3, & i = j = N_y+1, \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

$$\hat{c}_{ij}(\ell) = \begin{cases} \hat{\alpha}_{\ell,\ell+i-1}, & i = 2,\dots,N_y,\ j = i-1, \\ \hat{a}_4, & i = j = N_y+1, \\ 0, & \text{otherwise.} \end{cases} \tag{47}$$

With respect to the matrix $\mathcal{P}$, we denote its block entries by $\mathcal{P}_{\ell s}$ such that

$$\mathcal{P}_{\ell s} = \begin{cases} \Theta, & \ell = 1 \text{ and } N_x+1,\ \text{for } s = 1,\dots,N_x+1, \\ P^{(s)}(\ell-1), & \ell = 2,\dots,N_x,\ s = 1,\dots,N_x+1, \end{cases} \tag{48}$$

where $P^{(s)}(\ell-1)$ are matrices in $\mathbb{R}^{(N_y+1)\times(N_y+1)}$ their elements are denoted by $P^s_{ij}(\ell-1)$. Note that from the periodic weight structure $(\{0,11,1,1,11,0,\dots\})$ of four points open type formula (31), one gets

$$P^{(s)}(\ell-1) = \Theta,\ s = 1,6,\dots,N_x+1, \tag{49}$$

for $s = 2,7,\dots,N_x-3$ and $s = 5,10,\dots,N_x$, we have

$$P^{(s)}_{ij}(\ell-1) = \begin{cases} 11\hat{\lambda} g_{\ell-1,s-1}, & i = 2,3,\dots,N_y,N_y+1,\ i = j \\ 0, & \text{otherwise.} \end{cases} \tag{50}$$

Finally for $s = 3,8,\dots,N_x-2$ and $s = 4,9,\dots,N_x-1$,

$$P^{(s)}_{ij}(\ell-1) = \begin{cases} \hat{\lambda} g_{\ell-1,s-1}, & i = 2,3,\dots,N_y,N_y+1,\ i = j \\ 0, & \text{otherwise.} \end{cases} \tag{51}$$

Thus the matrix representation of the scheme (31)-(41) has been detailed in (43-51).

3. Numerical properties of the scheme

3.1. Positivity of the solution

We start this section by providing suitable conditions on the step sizes that guarantee the positivity of the numerical solution $\{W_{i,j}^n\}$ of scheme (28)-(41). First let us present the following lemma

lemma 1. *Let* $f(z) = \frac{z}{|\alpha z+\beta|}$, $z \in I = [z_1, z_2]$ *and* $\alpha\beta \neq 0$ *then the minimum of* $f(z)$ *in* $0 < z_1 \leq z \leq z_2$ *is achieved in one of the extremum of* I, *i.e.,*

$$\min_{z\in I} f(z) = \min\left\{\frac{z_i}{|\alpha z_i + \beta|},\ i = 1, 2\right\}. \tag{52}$$

Proof. If $\alpha z + \beta \neq 0$ for all $z_1 < z < z_2$, then $f(z)$ is a monotonic function, consequently (52) holds. Otherwise there exists a value $z_0 = \frac{-\beta}{\alpha}$ such that $f(z)$ is increasing in $[z_1, z_0[$ and decreasing in $]z_0, z_2]$ and then (52) also holds true. □

Note that as $\nu_{i,j}$ defined in (23) satisfy $0 < \nu_1 \leq \nu_{i,j} \leq \nu_2$, the coefficient $\beta_{i,j}$ of (29) is nonnegative under the following condition

$$\frac{k}{h^2} < \frac{m^2}{\sigma^2\nu_2}. \tag{53}$$

Note also from (29) that coefficients $\hat{\alpha}_{i,j}$ and $\breve{\alpha}_{i,j}$ are simultaneously nonnegative provided that

$$|\tilde{b}_{ij}| \leq \frac{\rho^2}{2h}\tilde{a}_{ij}. \tag{54}$$

If $\tilde{b}_{ij} = 0$, then (54) holds for any value of the step size h. Otherwise (54) can be written in the following form

$$h \leq \frac{2\sigma\tilde{\rho}\nu_{i,j}}{|2\hat{\xi} - \nu_{i,j}|}, \tag{55}$$

and from lemma 1 for $z = \nu_{i,j}$, $\alpha = -1$ and $\beta = 2\hat{\xi}$, $z_i = \nu_i$, $i = 1, 2$, one gets that (55) is verified under condition

$$h \leq h_1 = \min\left\{ \frac{2\sigma\tilde{\rho}\nu_i}{|2\hat{\xi} - \nu_i|},\ i = 1, 2 \right\}. \tag{56}$$

Similarly, one guarantees the simultaneous positivity of the coefficients $\alpha_{i,j}$ and $\gamma_{i,j}$ under the condition

$$h \leq \frac{\tilde{\rho}^2\sigma^2\nu_{i,j}}{2m^2\left|\frac{|m|}{m}\tilde{b}_{ij} - \tilde{c}_{ij}\right|}. \tag{57}$$

From (29), we have

$$\frac{|m|}{m}\tilde{b}_{ij} - \tilde{c}_{ij} = \left(\frac{\kappa}{2|m|} - \frac{\sigma\rho}{4|m|}\right)\nu_{i,j} + \frac{\sigma\rho}{2|m|}\hat{\xi} - \frac{\kappa\theta}{2|m|} = \alpha\nu_{i,j} + \beta, \tag{58}$$

and from lemma 1, (57) holds true under the condition

$$h \leq h_2 = \min\left\{ \frac{\sigma^2\tilde{\rho}^2\nu_i}{2m^2|\alpha\nu_i + \beta|},\ i = 1, 2 \right\}, \tag{59}$$

where α and β are defined in (58). Then by incorporating the conditions (56) and (59) one gets

$$h \leq \min\{h_1, h_2\}. \tag{60}$$

To guarantee the positivity of the numerical solution on boundary of the domain, it is sufficient to put condition on the coefficients $\hat{a}_i$ of (40) defined in (41) in terms of h and k. This condition is

$$k \leq \min\left\{ \frac{2h}{3\sigma\tilde{\rho}|\hat{\xi}|},\ \frac{|m|h}{3\kappa\theta} \right\}. \tag{61}$$

The entries of matrix $\mathcal{P}$ are nonnegative since the coefficients of the integral part of the scheme given by (28) are nonnegative. On the other hand under conditions (53),(56),(59) and (61), the entries of matrix $\mathcal{D}$ are also nonnegative and then the following theorem is established.

Theorem 1. *With previous notation, if stepsizes h and k satisfy*

C1. $$h \leq \min\left\{ \frac{2\sigma\tilde{\rho}\nu_i}{|2\hat{\xi} - \nu_i|},\ \frac{\sigma^2\tilde{\rho}^2\nu_i}{2m^2|\alpha\nu_i + \beta|},\ i = 1, 2 \right\}$$

C2. $$k \leq \min\left\{ \frac{m^2h^2}{\sigma^2\nu_2},\ \frac{2h}{3\sigma\tilde{\rho}|\hat{\xi}|},\ \frac{|m|h}{3\kappa\theta} \right\},$$

then the numerical solution $\{W^n_{i,j}\}$ of the scheme (28)-(41) is nonnegative.

3.2. Stability of the scheme

For the sake of clarity in the presentation and as one finds many definitions of stability in the literature, we introduce the following definition

Definition 1. *Let $\left\{W_{i,j}^n\right\}$ be a numerical solution of the PIDE computed from the scheme (28)-(41) with stepsizes $h = \Delta x$, $h_y = mh$ in a rhomboid computational domain bounded by (24) and $k = \Delta\tau$ in $[0, T]$. We say that $\left\{W_{i,j}^n\right\}$ is strongly uniformly $\|\cdot\|_\infty$ stable, if the corresponding vector solution $\mathbf{W}^n$ given by (42) and (43) satisfies*

$$\|W^n\|_\infty \leq \Lambda \left\|W^0\right\|_\infty , \qquad 0 \leq n \leq N_\tau , \tag{62}$$

where $\Lambda > 0$ is independent of n, h and k.

We begin here by providing bounds for the infinite norm of $\mathcal{D}$ and $\mathcal{P}$. From (29) and (41), under the positivity conditions of theorem 1, we have

$$\alpha_{i,j} + \hat{\alpha}_{i,j} + \breve{\alpha}_{i,j} + \beta_{i,j} + \gamma_{i,j} = 1, \quad \sum_{s=1}^{4} \hat{a}_s = 1. \tag{63}$$

From (63) and the structure of matrices $\breve{\mathcal{C}}$, $\mathcal{B}$ and $\hat{\mathcal{C}}$, given by (45)-(47) it follows that

$$\|[\breve{C}(\ell)\ B(\ell)\ \hat{C}(\ell)]\|_\infty = \max\{e^{(\lambda+r)k}, 1\} = e^{(\lambda+r)k}. \tag{64}$$

From the definition of $\mathcal{D}$ (44), property of infinite norm of the block matrices (8) and (64), one gets

$$\|\mathcal{D}\|_\infty = \max\left\{1, \max_{1\leq\ell\leq N_x-1} \left\{\|[\breve{C}(\ell)\ B(\ell)\ \hat{C}(\ell)]\|_\infty\right\},\ e^{(r-q+\lambda)k}\right\} = e^{(\lambda+r)k}. \tag{65}$$

In order to bound the norm of the matrix $\mathcal{P}$ (48)-(51), let i_m be the row that coincides with the infinite norm of $\mathcal{P}$, therefore

$$\|\mathcal{P}\|_\infty = \frac{5h\lambda k}{24\sqrt{2\pi}\hat{\sigma}\tilde{\rho}\sigma} \sum_{\ell=0}^{N_x/5-1} \left(11 g_{i_m,5\ell+1} + g_{i_m,5\ell+2} + g_{i_m,5\ell+3} + 11 g_{i_m,5\ell+4}\right). \tag{66}$$

Since the right hand side of (66) represents the approximation of

$$k\lambda I_1 = \frac{k\lambda}{\sqrt{2\pi}\hat{\sigma}\tilde{\rho}\sigma} \int_a^b g(x_{i_m}, \phi) d\phi,$$

see (32), its value is given by

$$k\lambda I_1 = \frac{k\lambda}{2}\left(\operatorname{erf}\left(\frac{x_{i_m} - a + \mu\sigma\tilde{\rho}}{\sqrt{2}\sigma\tilde{\rho}\hat{\sigma}}\right) - \operatorname{erf}\left(\frac{x_{i_m} - b + \mu\sigma\tilde{\rho}}{\sqrt{2}\sigma\tilde{\rho}\hat{\sigma}}\right)\right). \tag{67}$$

Then for small enough h, we have

$$\|\mathcal{P}\|_\infty < k\lambda(I_1 + 1) = k\lambda_1, \tag{68}$$

and from (42) it follows that

$$\|\mathbf{W}^n\|_\infty \leq (\|\mathcal{D}\|_\infty + \|\mathcal{P}\|_\infty)\|\mathbf{W}^{n-1}\|_\infty, \tag{69}$$

and from (66) and (67), one gets

$$\begin{aligned}\frac{\|\mathbf{W}^n\|_\infty}{\|\mathbf{W}^0\|_\infty} &\leq \left(e^{(r+\lambda)k} + k\lambda_1\right)^n = e^{(r+\lambda)T}\left(1 + k\lambda_1 e^{-(r+\lambda)k}\right)^n \\ &\leq e^{(r+\lambda)T}\left(1 + k\lambda_1\right)^n \leq \exp\left((r + \lambda + \lambda_1)T\right).\end{aligned} \tag{70}$$

Summarizing, according to definition (1), a conditional strong uniform stable scheme is established.

4. Consistency

Let us denote the local truncation error $T_{i,j}^n(w)$ as

$$T_{i,j}^n(w) = F(W_{i,j}^n) - (L(w_{i,j}^n) - I(w_{i,j}^n)), \tag{71}$$

where w is the exact theoretical solution for the PIDE (10), ($w_{i,j}^n = w(x_i, y_j, \tau^n)$), $F(W_{i,j}^n) = 0$ represent the approximating finite difference equation (28), $L(w)$ is the differential operator of (10) and $I(w)$ is the integral part given by (14). Based on the definition of consistency of [31] and [32], a numerical scheme is consistent with a PIDE if an exact theoretical solution of the PIDE approximates well the difference scheme as the stepsizes discretization tend to zero, i.e., the proposed scheme (28)-(41) is consistent with the PIDE (10) if $T_{i,j}^n \to 0$ as $h \to 0$, $h_y \to 0$ and $k \to 0$.

Let w be a continuous function of x, y and τ with continuous derivatives of order four with respect to x and y and of order two with respect to τ. By using Taylor expansion about (x_i, y_j, τ^n), we have

$$\frac{w_{i,j}^{n+1} - w_{i,j}^n}{k} = \frac{\partial w}{\partial \tau}(x_i, y_j, \tau^n) + kE_{i,j}^n(1), \tag{72}$$

where

$$E_{i,j}^n(1) = \frac{1}{2}\frac{\partial^2 w}{\partial \tau^2}(x_i, y_j, \chi), \;\; nk < \chi < (n+1)k,$$

$$\left|E_{i,j}^n(1)\right| \leq \frac{1}{2}\max\left\{\left|\frac{\partial^2 w}{\partial \tau^2}(x_i, y_j, \tau)\right|, \;\; \tau^n \leq \tau \leq \tau^{n+1}\right\} = \frac{1}{2}D^n(1). \tag{73}$$

For the second partial derivatives with respect to the spatial variables x and y, the Taylor's expansions are given by

$$\frac{w_{i+1,j}^n - 2w_{i,j}^n + w_{i-1,j}^n}{h^2} = \frac{\partial^2 w}{\partial x^2}(x_i, y_j, \tau^n) + h^2 E_{i,j}^n(2), \tag{74}$$

$$E_{i,j}^n(2) = \frac{1}{12}\frac{\partial^4 w}{\partial x^4}(\chi_1, y_j, \tau^n), \;\; x_i - h < \chi_1 < x_i + h,$$

$$\left|E_{i,j}^n(2)\right| \leq \frac{1}{12}\max\left\{\left|\frac{\partial^4 w}{\partial x^4}(x, y_j, \tau^n)\right|, \;\; a \leq x \leq b\right\} = \frac{1}{12}D_j^n(2), \tag{75}$$

and

$$\frac{w_{i,j+1}^n - 2w_{i,j}^n + w_{i,j-1}^n}{h_y^2} = \frac{\partial^2 w}{\partial y^2}(x_i, y_j, \tau^n) + h_y^2 E_{i,j}^n(3), \tag{76}$$

$$E_{i,j}^n(3) = \frac{1}{12}\frac{\partial^4 w}{\partial y^4}(x_i, \chi_2, \tau^n), \;\; y_j - h_y < \chi_2 < y_j + h_y,$$

$$\left|E_{i,j}^n(3)\right| \leq \frac{1}{12}\max\left\{\left|\frac{\partial^4 w}{\partial y^4}(x_i, y, \tau^n)\right|, \;\; mx_i - \nu_2 \leq y \leq mx_i - \nu_1\right\} = \frac{1}{12}D_i^n(3). \tag{77}$$

The expansions for the first partial derivatives with respect to x and y are given by

$$\frac{w_{i+1,j}^n - w_{i-1,j}^n}{2h} = \frac{\partial w}{\partial x}(x_i, y_j, \tau^n) + h^2 E_{i,j}^n(4), \tag{78}$$

$$E_{i,j}^n(4) = \frac{1}{6}\frac{\partial^3 w}{\partial x^3}(\chi_3, y_j, \tau^n), \;\; x_i - h < \chi_3 < x_i + h,$$

$$\left|E_{i,j}^n(4)\right| \leq \frac{1}{6} \max\left\{\left|\frac{\partial^3 w}{\partial x^3}(x, y_j, \tau^n)\right|,\ a \leq x \leq b\right\} = \frac{1}{6} D_j^n(4) \tag{79}$$

$$\frac{w_{i,j+1}^n - w_{i,j-1}^n}{2h_y} = \frac{\partial w}{\partial x}(x_i, y_j, \tau^n) + h_y^2 E_{i,j}^n(5), \tag{80}$$

$$E_{i,j}^n(5) = \frac{1}{6}\frac{\partial^3 w}{\partial y^3}(x_i, \chi_4, \tau^n),\ y_j - h_y < \chi_4 < y_j + h_y,$$

$$\left|E_{i,j}^n(5)\right| \leq \frac{1}{6} \max\left\{\left|\frac{\partial^3 w}{\partial y^3}(x_i, y, \tau^n)\right|,\ mx_i - \nu_2 \leq y \leq mx_i - \nu_1\right\} = \frac{1}{6} D_i^n(5). \tag{81}$$

On the other hand for the integral part, there are two error sources; the first coming from the truncation of improper integral into a bounded one (a, b) and the second coming from the numerical approximation of the finite integral using the four point open type formula. Let $\mathcal{T}_{i,j}^n(w)$ denote the total truncation error for the integral part such that

$$\begin{aligned} \mathcal{T}_{i,j}^n(w) = I(w_{i,j}^n) - \hat{\lambda} J_{i,j}^n &= (I(w_{i,j}^n) - I_{ab}(w_{i,j}^n)) + (I_{ab}(w_{i,j}^n) - \hat{\lambda} J_{i,j}^n) \\ &= \mathcal{H}_{i,j}^n(w) + \mathcal{Y}_{i,j}^n(w) \end{aligned} \tag{82}$$

where $I_{ab}(w) = \frac{\lambda}{\sqrt{2\pi}\hat{\sigma}\tilde{\rho}\sigma} \int_a^b g(x, \phi) w(x, y + m(\phi - x), \tau) d\phi$, the truncation error $\mathcal{H}_{i,j}^n(w) = I(w) - I_{ab}(w)$ and the error due to the numerical integration $\mathcal{Y}_{i,j}^n(w) = I_{ab}(w) - \hat{\lambda} J_{i,j}^n$.

According to Briani *et. al.* [33], since the integral part contains the Gaussian function, then the absolute value of $\mathcal{H}_{i,j}^n(w)$ can be controlled using a tolerance parameter error $\varepsilon > 0$ by choosing

$$b = \sqrt{-2\hat{\sigma}^2 \ln(\varepsilon \hat{\sigma} \sqrt{2\pi})},\ a = -b. \tag{83}$$

Furthermore, due to the symmetric property of the probability measure of Gaussian distribution, one can assume that the option price w satisfies the Lipschitz condition with respect to the spacial variables, then one has [33],

$$\left|\mathcal{H}_{i,j}^n(w)\right| < 2\hat{\sigma}^2 \varepsilon. \tag{84}$$

Finally, from [24, 95],

$$\left|\mathcal{Y}_{i,j}^{n}(w)\right| \leq \frac{90h^4}{144} D_{i,j}^{n}(6), \tag{85}$$

where

$$D_{i,j}^{n}(6) = \max\left\{ \left(w(\phi, y_j + m(\phi - x_i), \tau^n)\right)^{(4)},\ a \leq \phi \leq b \right\}, \tag{86}$$

and the fourth derivative of the function $w(\phi, y_j + m(\phi - x_i), \tau^n)$ is taken with respect to ϕ. Hence the total error for the integral part $|\mathcal{T}_{i,j}^{n}|$ satisfies

$$\left|\mathcal{T}_{i,j}^{n}\right| < 2\hat{\sigma}^2\varepsilon + \frac{90h^4}{144} D_{i,j}^{n}(6). \tag{87}$$

From (72), (74), (76), (78), (80), (82) and (71), the local truncation error has the following form

$$T_{i,j}^{n} = kE_{i,j}^{n}(1) - \frac{\tilde{\rho}^2 \nu_{i,j} \sigma^2}{2} \left(h^2 E_{i,j}^{n}(2) + m^2 h^2 E_{i,j}^{n}(3)\right) - \hat{\delta}_{i,j} h^2 E_{i,j}^{n}(4)$$

$$-\tilde{\delta}_{i,j} m^2 h^2 E_{i,j}^{n}(5) - \mathcal{T}_{i,j}^{n}(w), \tag{88}$$

where $\hat{\delta}_{i,j}$ and $\tilde{\delta}_{i,j}$ correspond to expressions appearing in (12) when replacing ν by $\nu_{i,j}$. Finally, from (73), (75), (77), (79), (81), (87) and (88), we have

$$\left|T_{i,j}^{n}\right| \leq \frac{k}{2} D^n(1) + \left|\frac{\tilde{\rho}^2 \nu_{i,j} \sigma^2}{24}\right| \left(D_j^n(2) + m^2 D_i^n(3)\right) h^2 + \left(|\hat{\delta}_{i,j}| D_j^n(4) + m^2 |\tilde{\delta}_{i,j}| D_i^n(5)\right) \frac{h^2}{6}$$

$$+ \frac{90h^4}{144} D_{i,j}^{n}(6) + 2\hat{\sigma}^2\varepsilon, \tag{89}$$

Therefore

$$\left|T_{i,j}^{n}\right| \leq \mathcal{O}(k) + \mathcal{O}(h^2) + \mathcal{O}(\varepsilon). \tag{90}$$

Summarizing, the consistency for the scheme is established.

5. Numerical Examples

After removing the mixed derivative of the PIDE (1) for Bates model, a finite difference scheme has been constructed to obtain a numerical approximation for the option price. Furthermore, the positivity conditions are provided, also stability and consistency have been studied. In this section, several examples are provided to study the behavior of the option price obtained by the proposed scheme using Matlab. The used computer has Microprocessor 3.4 GHz Intel Core i7. The following example reveals the importance of the positivity conditions (60) and (61) on the stepsizes h and k.

Example 1. Consider an European call option under Bates model with the following parameters $T = 0.5$, $E = 100$, $r = 0.05$, $q = 0$, $\theta = 0.05$, $\kappa = 2.5$, $\sigma = 0.25$, $\hat{\sigma} = 0.7$, $\mu = 0.5$, $\lambda = 0.2$ $\nu_1 = 0.1$, $\nu_2 = 1$ and $\rho = -0.5$ with a tolerance error $\varepsilon = 10^{-3}$. In Figure 2, the solid curve represents the option price as a function of the underlying asset S when the positivity conditions hold for $(N_x, N_y, N_\tau) = (100, 45, 150)$ corresponding to $h = 0.05$ and $k = 0.0033$, while the dashed curve represents the option price when the positivity conditions are broken for $(N_x, N_y, N_\tau) = (100, 45, 50)$ corresponding to $h = 0.05$ and $k = 0.01$.

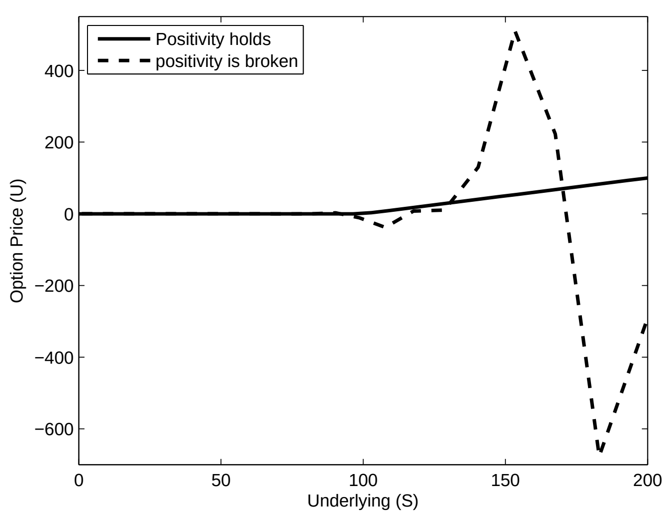

Fig. 2. The effect of positivity conditions on the option price U

The next example investigates the associated error for the scheme (28)-(41) when $\lambda = 0$, i.e., for European option under Heston model. Considering the strike price $E = 100$, the numerical solutions for the set of underlying assets $\mathcal{S} =$

$\{80, 90, 100, 110, 120\}$ are obtained. In order to evaluate the error, a Matlab code for the closed form solution is used [34] obtaining the set of corresponding reference option price values $\mathcal{U} = \{0.207581, 4.889877, 10.488226, 16.503506, 22.856611\}$. The root mean square relative error (**RMSRE**) is calculated based on the equation

$$\textbf{RMSRE} = \sqrt{\frac{1}{5}\sum_{i=1}^{5}\left(\frac{\mathcal{U}(\mathcal{S}_i, \nu_0, T) - U(\mathcal{S}_i, \nu_0, T)}{\mathcal{U}(\mathcal{S}_i, \nu_0, T)}\right)^2}, \tag{91}$$

where $U(\mathcal{S}_i, \nu_0, T)$ is the numerical solution at spot variance $\nu_0 = 0.4$.

Example 2. Here the parameters are chosen as follows $T = 0.5$, $E = 100$, $r = 0.05$, $q = 0$, $\theta = 0.05$, $\kappa = 2$, $\sigma = 0.3$, and $\rho = -0.5$. The computational domain is $[a, b] = [-0.5, 1.5]$, $\nu_1 = 0.1$ and $\nu_2 = 1$. Table 1 exhibits the variation of **RMSRE** for several values of N_τ while $N_x = 70$ and $N_y = 16$, the numerical order of error and CPU time in seconds.

N_τ	**RMSRE**	Ratio	CPU (sec)
200	1.764×10^{-3}	–	1.01
400	9.387×10^{-4}	1.88	1.05
800	4.581×10^{-4}	2.05	1.17
1600	2.371×10^{-4}	1.93	1.19
3200	1.191×10^{-4}	1.99	1.32

Table (1): The associated **RMSRE** for several values of N_τ.

In Table 2, the variation of error due to the change of the spatial step sizes, while $N_\tau = 500$ has been studied.

(N_x, N_y)	**RMSRE**	Ratio	CPU (sec)
$(40, 9)$	4.166×10^{-3}	–	0.11
$(60, 14)$	2.986×10^{-3}	1.395	0.71
$(80, 18)$	9.367×10^{-4}	3.188	2.52
$(100, 23)$	3.861×10^{-4}	2.426	7.476
$(120, 27)$	9.287×10^{-5}	4.157	19.53

Table (2): The associated **RMSRE** for different values of (N_x, N_y).

The aim of the last example is to study the variation of the resultant error for European option under Bates model.

Example 3. The parameters are selected as follows $T = 0.5$, $E = 100$, $r = 0.05$, $q = 0$, $\theta = 0.05$, $\kappa = 2.0$, $\sigma = 0.3$, $\hat{\sigma} = 0.35$, $\mu = -0.5$, $\lambda = 0.2$ and $\rho = -0.5$ with a tolerance error $\varepsilon = 10^{-4}$. The boundary points a and b of the spatial computational domain are obtained from (83), while $\nu_1 = 0.1$ and $\nu_2 = 1$. Table 3 shows the variation of the **RMSRE** for several values of the time step sizes, for fixed $N_x = 70$ and $N_y = 35$, with respect to reference values computed at $(N_x, N_y, N_\tau) = (500, 146, 7000)$.

N_τ	**RMSRE**	Ratio	CPU (sec)
500	2.485×10^{-3}	–	6.66
1000	1.322×10^{-3}	1.88	6.94
2000	6.429×10^{-4}	2.06	7.28
4000	3.296×10^{-4}	1.95	7.69
8000	1.569×10^{-4}	2.10	7.91

Table (3): The associated **RMSRE** for several values of N_τ.

The variation of error due to the change of the spatial step sizes, while $N_\tau = 500$ has been presented in Table 4.

(N_x, N_y)	**RMSRE**	Ratio	CPU (sec)
$(40, 20)$	1.526×10^{-2}	–	0.32
$(60, 30)$	3.459×10^{-3}	4.412	1.83
$(80, 40)$	9.271×10^{-4}	3.371	6.95
$(100, 50)$	3.589×10^{-4}	2.583	19.64
$(120, 60)$	8.473×10^{-5}	4.236	46.72

Table (4): The associated **RMSRE** for different values of (N_x, N_y).

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