STUDY OF THE KINEMATICS OF RIGID BODY USING THE SLIDING VECTORS' THEORY

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Abstract

The sliding vector theory is a powerful tool for the study of the three parts of Classical Mechanics in vectorial formulation: Kinematics, Statics and Dynamics. Due to the great importance of the Vector Mechanics for their technical applications in engineering, this part of the Physics is studied in the first years of Engineering Degrees, as a fundamental topic included in the subjects of Physics.

The rigid body model is the solid under study in Vectorial Mechanics. Firstly, in Kinematics, its movement is studied regardless of its causes. Afterward, in Dynamics, its motion is related with the forces that cause it. Finally, in Statics, its equilibrium under forces acting is studied.

Rigid body definition as a set of particles in which the distance between them remains constant induces its equiproyectivity velocity field property. The study of elementary movements of rigid body: Translation, Rotation and System of Rotations allow the formal expression of the velocity field.

Taken into account that the equiproyectivity property is characteristic of the moment field of a sliding vectors system, the formal expression of the velocity field of a rigid body coincides with the aforementioned moment field. It can be derived that the instantaneous velocity field of a rigid body can be formulated as the moment field of a sliding vector system. In this case, the rotation vectors act as sliding vectors, and its velocities as the moments at the points of the rigid solid.

In this paper, the theory that allows the study of the velocity field by analogy with the moment field of a sliding vectors system is developed. By means of the rotation and the velocity vectors at a given point, the classification of any the movement of rigid body will be done. Moreover, the plane and spherical movements will defined, which provide any kind of movement of the solid if they are overlapped properly.

Keywords: Rigid body, Sliding vectors, Kinematics, Movements classification, Learning and Teaching Methodologies.

1 INTRODUCTION

Kinematics is the part of mechanics that studies the motion of bodies regardless of the causes that produce it. It aims to establish the equations that describe the position of the bodies in a reference system as a function of time.

The aim of this paper is the study of the motion of bodies that are handle as rigid body (RB) at a given instant (instant analysis). An indeformable system (IS) is defined as any set of points geometrical or materials such that the distance between any two points of the system remains constant. If the system consists solely of material points is called rigid body (RB). It will be useful to speak of both systems, as we shall see, there are geometric points related to a SR which do not belong to it, but are considered of the IS associated with the SR.

For the motion of a system, it is necessary to know the position of all points, i.e. know its configuration. The set of equations that determine the configuration of a system over time are called the system motion equations.

In the IS systems we see that there are links or bonds (relations between the coordinates of its points) that enable to study the system configuration as well as the instantaneous distribution of their kinematics magnitudes easily, based on a minimum number of parameters.

The expressions which are developed allow the instantaneous analysis of the movement and if the magnitudes involved in these are known as a function of time, also allow the study of the temporal evolution of the movement.

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2 MOTION OF RIGID SOLID

2.1 Equiproyective Property

A rigid solid is characterized because the distance between any two points A and B remains constant. This means that $\left|\overrightarrow{AB}\right| = Cte$ or that its square $\left|\overrightarrow{AB}\right|^2 = Cte$. This implies that $\frac{d}{dt}\left|\overrightarrow{AB}\right|^2 = 0$.

Let's see how from the invariance of the distance between two points can be derived a relationship for instantaneous field of velocities.

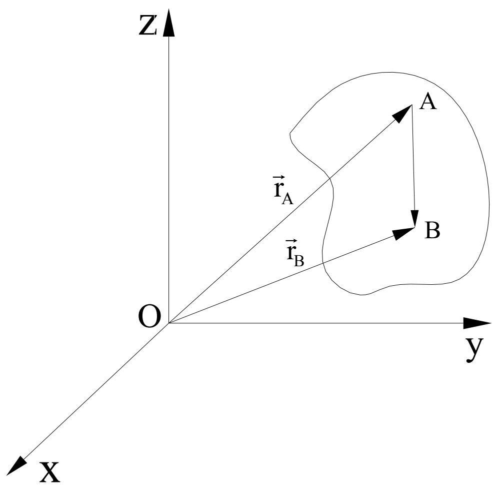


Figure 1.

$$\overrightarrow{AB} = \vec{r}_B - \vec{r}_A \quad \rightarrow \quad \frac{d\overrightarrow{AB}}{dt} = \frac{d\vec{r}_B}{dt} - \frac{d\vec{r}_A}{dt} = \vec{v}_B - \vec{v}_A$$

as $\left|\overrightarrow{AB}\right|^2 = \overrightarrow{AB} \cdot \overrightarrow{AB} = Cte$

$$\frac{d\left|\overrightarrow{AB}\right|^2}{dt} = \frac{d}{dt}\left(\overrightarrow{AB} \cdot \overrightarrow{AB}\right) = 2\overrightarrow{AB} \cdot \frac{d\overrightarrow{AB}}{dt} = 0$$

or: $\overrightarrow{AB} \cdot \left(\vec{v}_B - \vec{v}_A\right) = 0$;

dividing by $\left|\overrightarrow{AB}\right|$

$$\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} \cdot \left(\vec{v}_B - \vec{v}_A\right) = 0 \Rightarrow \vec{u}_{\overrightarrow{AB}} \cdot \left(\vec{v}_B - \vec{v}_A\right) = 0$$

That is: $\vec{v}_A \cdot \vec{u}_{\overrightarrow{AB}} = \vec{v}_B \cdot \vec{u}_{\overrightarrow{AB}}$ (1)

The expression (1) is the equiproyective property of the velocity field of a rigid body, as it happens with a moment field of a sliding vector system. It is therefore possible to establish the following relation between the speeds of two points which belong to the system: $\vec{v}_A = \vec{v}_B + \overrightarrow{AB} \wedge \vec{\omega}$ instantaneous velocity field expression where the velocities are the moment vectors of a sliding vectors system (SVS) whose resultant is $\vec{\omega}$ which must have angular velocity dimensions. Vector fields that verify these properties are called antisymmetric due to the cross product is an antisymmetric transformation.

2.2 Elementary Movements

Let's see if all movements can be represented by the moments field of a SVS. We will do an instant analysis of the movement of a body, i.e. a differential time dt, for elemental movements, translation, rotation around a fixed axis and a rotations system.

2.2.1 Translation

It is said that a RB has a translational movement when the speed of all points of the system is the same.

For any two points A and B the system must fulfill $\vec{v}_A = \vec{v}_B$.

It can be deduce that:

$$\vec{v}_B - \vec{v}_A = \frac{d\vec{r}_B}{dt} - \frac{d\vec{r}_A}{dt} = \frac{d}{dt}\left(\vec{r}_B - \vec{r}_A\right) = \frac{d\overrightarrow{AB}}{dt} = \vec{0}$$

The vector $\overrightarrow{AB}$ is constant for any instant when the motion is studied.

The equation $\vec{v}_A = \vec{v}_B$ corresponds to a moment field of a SVS with resultant equal to zero $\vec{\omega} = \vec{0}$

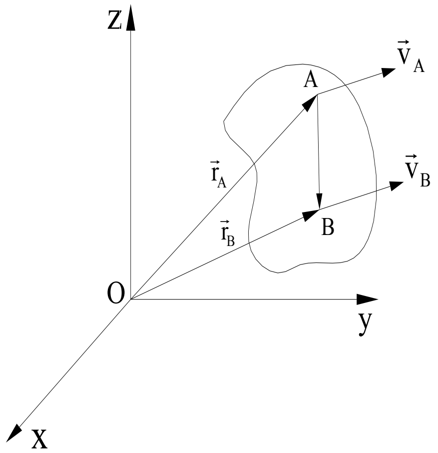


Figure 2.

2.2.2 Rotation

It is said that a RB is subjected to a rotation around a fixed line called rotational axis (RA) when all points have a circular movement, describing trajectories that are circumferences contained in planes perpendicular to the axis of rotation concentric with this axis, with angular velocity $\vec{\omega}$.

The RA has the particularity that its points belong to IS associated with the RB because its points may or may not be material points of the system, which have zero velocity $\vec{v}_E = \vec{0}$.

The velocity of any point A of the RB is contained in the plane of its trajectory and is tangent to it. It is given by: $\vec{v}_A = \vec{\omega} \wedge \overrightarrow{EA}$, being the same for any point E which is taken on the axis of rotation. It is shown that the velocity at point A is the moment of the sliding vector ($\vec{\omega}$,RA) at said point A.

For any other point B is: $\vec{v}_B = \vec{\omega} \wedge \overrightarrow{EB}$

If we seek the relationship between the velocities of two points of the indeformable system:

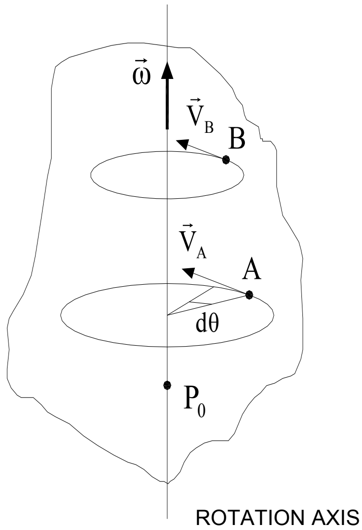


Figure 3

$$\vec{v}_A - \vec{v}_B = \vec{\omega} \wedge \overrightarrow{EA} - (\vec{\omega} \wedge \overrightarrow{EB}) = \vec{\omega} \wedge (\overrightarrow{EA} - \overrightarrow{EB}) = \vec{\omega} \wedge \overrightarrow{BA} \rightarrow \vec{v}_A = \vec{v}_B + \vec{\omega} \wedge \overrightarrow{BA} = \vec{v}_B + \overrightarrow{AB} \wedge \vec{\omega}$$

$$\vec{v}_A = \vec{v}_B + \overrightarrow{AB} \wedge \vec{\omega} \text{ or } \vec{v}_B = \vec{v}_A + \overrightarrow{BA} \wedge \vec{\omega} \qquad (2)$$

Expression (2) represents the moment field of a sliding vector $\left(\vec{\omega},RA\right)$ or $(\vec{\omega},E)$ called rotation vector, being its line of action the rotation axis and its moments the velocities of the points of the rigid body.

2.2.3 Rotations composition

Consider now the instantaneous movement of a SR undergone several rotations.

This movement is more complex and is formed by a system of rotations ($\vec{\omega}_i,P_i$). In that case only one rotation is around a fixed axis, being the other rotations around rotation axes necessarily mobile, which does not remove generality for instantaneous analysis of the movement.

Suppose a RB subjected to two rotations ($\vec{\omega}_1,P_1$) and ($\vec{\omega}_2,P_2$), being the axis e_1 mobile and the axis e_2 fixed.

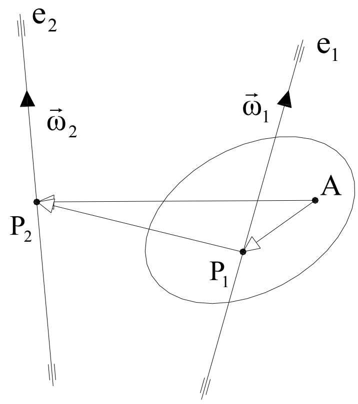


Figure 4.

Point A of the RB will have a velocity that will be the composition of their velocities due to rotations $\vec{\omega}_1$ and $\vec{\omega}_2$ as if both were independent (superposition principle), namely: $\vec{v}_A = \vec{v}_A(\vec{\omega}_1) + \vec{v}_A(\vec{\omega}_2)$.

Where: $\vec{v}_A(\vec{\omega}_1) = \overrightarrow{AP_1} \wedge \vec{\omega}_1$ and $\vec{v}_A(\vec{\omega}_2) = \overrightarrow{AP_2} \wedge \vec{\omega}_2$ adding and operating that is:

$$\vec{v}_A = \overrightarrow{AP_1} \wedge \vec{\omega}_1 + \overrightarrow{AP_2} \wedge \vec{\omega}_2 = \overrightarrow{AP_1} \wedge \vec{\omega}_1 + (\overrightarrow{AP_1} + \overrightarrow{P_1P_2}) \wedge \vec{\omega}_2 = \overrightarrow{P_1P_2} \wedge \vec{\omega}_2 + \overrightarrow{AP_1} \wedge \left(\vec{\omega}_1 + \vec{\omega}_2\right) \qquad (3)$$

Where the first term is P_1 velocity due to $\vec{\omega}_2$.

If we do the same for a point B, we get the expression: $\vec{v}_B = \overrightarrow{P_1P_2} \wedge \vec{\omega}_2 + \overrightarrow{BP_1} \wedge \left(\vec{\omega}_1 + \vec{\omega}_2\right)$ (4)

Subtracting (4) and (3) $\vec{v}_B - \vec{v}_A = \left(\overrightarrow{BP_1} - \overrightarrow{AP_1}\right) \wedge \left(\vec{\omega}_1 + \vec{\omega}_2\right)$

Operating and simplifying gives rise to the same expression of the moments field of a sliding vectors system ($\vec{\omega}_i, P_i$) being the resultant $\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$ and the velocities their moments

$$\vec{v}_B = \vec{v}_A + \overrightarrow{BA} \wedge \vec{\omega}$$

Expression deduced for any two points of the RB, it is considered that the point B moves with a translation movement equal to the point A, and a rotation around an axis passing through the point A, being the rotation vector, the sliding vector ($\vec{\omega}$, A).

This is therefore the expression of the moment filed associated to the torsor ($\vec{\omega}, \vec{v}_A$), being a vector field satisfies the equiproyective property. Being the free vector $\vec{\omega}$ equal to: $\vec{\omega} = \sum_{i=1}^{n} \vec{\omega}_i$, maintaining the same value at all points of the SR for a given time.

This conclusion can be generalized for instant analysis of the velocity field of a solid subjected to a number n of rotations.

2.3 Velocities field. Kinematic torsor

Since the velocity field of a RB has the same expression than the moment filed of a sliding vector system and having the equiproyective property, this implies property of antisymmetry and allows the study of the velocity field of a RB as the moment field associated with a torsor. The set $\vec{\omega}$ and $\vec{v}_A$ is called kinematic torsor of the indeformable system at point A, $T_{CA}\left(\vec{\omega}, \overrightarrow{v_A}\right)$.

Hence the velocity field of a RB corresponds to the moments field of a SVS.

Next we compare the SVS and the vector system $\vec{\omega}$ and its moments field with their characteristic nomenclatures.

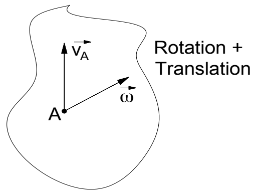
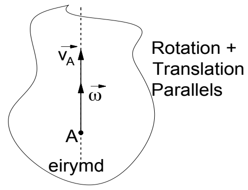
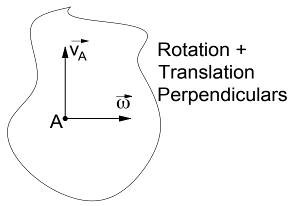
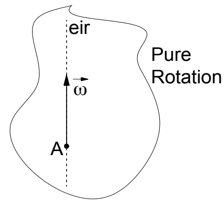
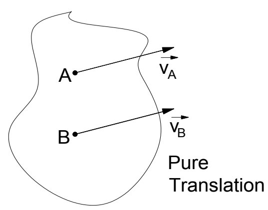
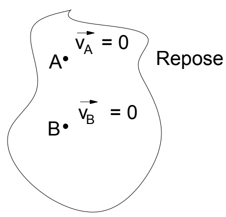
Moment field of a SVS	Velocities field of a Rigid Body
$(\vec{a}_i, P_i)$ SLIDING Vector System $\vec{a}_i$	$(\vec{\omega}_i, P_i)$ ROTATION Vector System $\vec{\omega}_i$
$\vec{M}_A = \sum \overrightarrow{AP}_i \wedge \vec{a}_i$ Moment at point A	$\vec{v}_A = \sum \overrightarrow{AP}_i \wedge \vec{\omega}_i$ Velocity at point A
$\vec{R} = \sum \vec{a}_i$ Resultant	$\vec{\omega} = \sum \vec{\omega}_i$ Rotations resultant or Rotation
$\vec{M}_B = \vec{M}_A + \overrightarrow{BA} \wedge \vec{R}$ Moment field	$\vec{v}_B = \vec{v}_A + \overrightarrow{BA} \wedge \vec{\omega}$ Velocity field
$\vec{M}_A \cdot \vec{u}_{\overrightarrow{AB}} = \vec{M}_B \cdot \vec{u}_{\overrightarrow{AB}}$ Equiproyective property	$\vec{v}_A \cdot \vec{u}_{\overrightarrow{AB}} = \vec{v}_B \cdot \vec{u}_{\overrightarrow{AB}}$ Equiproyective property
INVARIANTS	
$\vec{R}$ Resultant	$\vec{\omega}$ Rotations resultant. If $\vec{\omega} = 0$ Translation Si $\vec{\omega} \neq 0, \begin{cases} \text{Rotation} \\ \text{General Movement} \end{cases}$
$\vec{M} \cdot \vec{R}$ Fundamental Invariant	$\vec{v} \cdot \vec{\omega}$ Fundamental Invariant
$m = \left\lvert \vec{M}_{min} \right\rvert = \frac{\vec{M} \cdot \vec{R}}{\left\lvert \vec{R} \right\rvert}$ Minimum moment	$\left\lvert \vec{v}_{min} \right\rvert = \frac{\vec{v} \cdot \vec{\omega}}{\left\lvert \vec{\omega} \right\rvert}$ Minimum sliding velocity If $\vec{v} \cdot \vec{\omega} = 0 \Rightarrow \left\lvert \vec{v}_{min} \right\rvert = 0$ Rotation If $\vec{v} \cdot \vec{\omega} \neq 0 \Rightarrow \left\lvert \vec{v}_{min} \right\rvert \neq 0$ General Movement
CENTRAL AXIS	
Geometrical place of points with minimum moment Line parallel to the resultant $\vec{R}$ (It does not exist if the resultant is null)	Geometrical place of points with minimum velocity Line parallel to the resultant $\vec{\omega}$. IRA(*) or IRAMS(**) (It does not exist if it movement is a translation)
SPATIAL DISTRIBUTION	
Cylindrical symmetry respect the central axis	Cylindrical symmetry respect the IRA or IRAMS

(*): IRA: Instantaneous Rotation Axis.

(**): IRAMS: Instantaneous Rotation Axis and Minimum Sliding.

3 INSTANTANEOUS ANALYSIS OF MOVEMENT. PARTICULAR CASES

In the following table possible cases of movement are classified according to the point of the RB considered and the kinematic torsor expression at it is obtained.

FUNDAMENTAL INVARIANT	KINEMATIC TORSOR AT POINT A	MOVIMENT
$\vec{v} \cdot \vec{\omega} \neq 0$ Being $\vec{v} \neq 0$ and $\vec{\omega} \neq 0$	A $\notin$to IRAMS $T_A \equiv [\vec{v}_A, \vec{\omega}]$	
$\vec{v} \cdot \vec{\omega} \neq 0$ Being $\vec{v} \neq 0$ and $\vec{\omega} \neq 0$	A $\in$ to IRAMS $T_A \equiv [\vec{v}_A = \vec{v}_{min}, \vec{\omega}]$	
$\vec{v} \cdot \vec{\omega} = 0$ Being $\vec{v} \neq 0$ and $\vec{\omega} \neq 0$	A $\notin$to IRA $T_A \equiv [\vec{v}_A, \vec{\omega}]$	
$\vec{v} \cdot \vec{\omega} = 0$ Being $\vec{v} = 0$ and $\vec{\omega} \neq 0$	A $\in$ to IRA $T_A \equiv [\vec{v}_A = \vec{v}_{min} = 0, \vec{\omega}]$	
$\vec{v} \cdot \vec{\omega} = 0$ Being $\vec{v} \neq 0$ and $\vec{\omega} = 0$	$T_A \equiv [\vec{v}_A, \vec{\omega} = 0]$ $\vec{v}_B = \vec{v}_A + \overrightarrow{BA} \wedge \vec{\omega} = \vec{v}_A$	
$\vec{v} \cdot \vec{\omega} = 0$ Being $\vec{v} = 0$ and $\vec{\omega} = 0$	$T_A \equiv [\vec{v}_A = 0, \vec{\omega} = 0]$ $\vec{v}_B = \vec{v}_A + \overrightarrow{BA} \wedge \vec{\omega} = 0$	

Pure rotational and translation movements are both cases of plane motion, which is that in which the velocities are contained in a plane (plane director of the movement), and in any plane parallel to this, the distribution of velocities are the same.

The case of pure rotational motion also includes a rigid solid motion with a fixed point (spherical motion), corresponding to the velocity field of a concurrent rotation system, which can be reduced to a single rotation at the fixed point.

From the foregoing, any three-dimensional movement of the rigid body can be studied by superimposing plane motion (translation) and a movement with a fixed point (rotation).

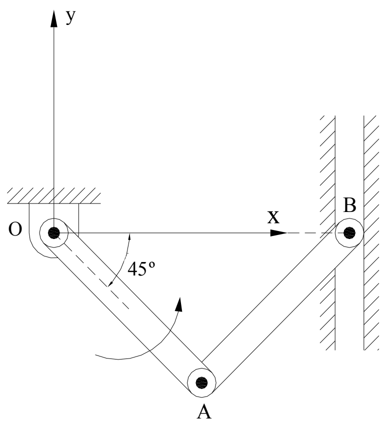
4 APLICATION

The plane mechanism of figure comprises two bars OA and AB with same length L. OA rod is hinged at the fixed point O and its end A is articulated at the end of the bar AB. The end B of the bar AB is constrained to move on a vertical guide. Knowing that bar OA is subjected to a rotation $\vec{\omega} = \omega_0 \vec{k}$, being ω_0 constant, for the movement of the bar AB in the position shown is required:

a) Point A velocity, $\vec{v}_A$.

b) Rotation vector, $\vec{\omega}_{AB}$ and point B velocity, $\vec{v}_B$ respectively.

c) Point plane belonging to the instantaneous axis of rotation in which the movement is reduced to the rotation $\vec{\omega}_{AB}$ (instantaneous rotation center).



SOLUTION

SECTION	RESULT
a) Point A velocity, $\vec{v}_A$.	$\vec{v}_A = \frac{L\omega_0}{\sqrt{2}}\vec{i} + \frac{L\omega_0}{\sqrt{2}}\vec{j}$
b) Rotation vector, $\vec{\omega}_{AB}$ and point B velocity, $\vec{v}_B$ respectively.	$\vec{\omega}_{AB} = \omega_0 \vec{k}$ $\vec{v}_B = \frac{2L\omega_0}{\sqrt{2}}\vec{j}$
c) Point plane belonging to the instantaneous axis of rotation in which the movement is reduced to the rotation $\vec{\omega}_{AB}$ (instantaneous rotation center).	$O(0,0)$

5 CONCLUSIONS

The conclusions of this work we can enumerate the following:

1. Sliding Vectors theory provides a simple management tool for the study of the three parts of classical mechanics of rigid solid, Statics, Kinematics and Dynamics, in their formulation vector.

2. The study of the kinematics from the velocity field is established from equiproyective property and its formal expression as an antisymmetric field coinciding with the moments field of a system of sliding vectors.

3. The consideration of the translation and rotation movements as plane motion is a great simplification for study.

4. Instantaneous expressions deduced are valid for the temporal analysis if known the magnitudes involved in function of time

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