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On stability and reachability of perturbed positive systems

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Abstract

This paper deals mainly with the structural properties of positively reachability and stability. We focus our attention on positive discrete-time systems and analyze the behavior of these systems subject to some perturbation. The effects of permutation and similar transformations are discussed in order to determine the structure of the perturbation such that the closed-loop system is positively reachable and stable. Finally, the results are applied to the Leslie Population Model. It is shown the structure of the perturbation such that the properties of the original system remain and an explicit expression of its set of positively reachable population is given.

Keywords: Positive linear system; M-matrix; Nonnegative matrix; Stability; Reachability; Perturbation

1 Introduction

Systems of difference equations with nonnegative coefficients are used as models in many fields in which the variables are subject to nonnegative restrictions. Examples of such applications can be found in [1, 2, 3, 4, 5, 6, 7]. One of the main aims in the study of real processes is analyze if the system satisfies the stability property. However, many times important properties such as stability and reachability are undetectable. Thus, it is important to know if the process disturbances can be attenuated by a feedback or if a trajectory will reach or not a desired state using nonnegative controls. In [8] some results related to these topics are given.

The problem is that usually the system can be subject to disturbances and it is important to know what conditions must satisfy these disturbances so that the structure and properties that characterize our system remain. In this paper we consider a positive linear discrete-time system, stable and positively reachable. We propose the problem of determining what kind of perturbations can be used so that the closed-loop system maintains stability and positively reachability. Some results on the structure of the disturbances are given. Motivated by the application of the obtained results in some real processes, we focus on the case where the state matrix has a companion structure. In particular, we study the Leslie population model and we give conditions for the system be positively reachable as well as we characterize the collection of perturbations under which the model remains stable.

The rest of the paper is organized as follows. Section 2 presents some results on stability of perturbed positive systems. In Section 3, previous application of a perturbation to the initial system, we analyze how should be the collection of disturbance in order to the closed-loop system remains stability and also be positively reachable. Section 4 gives a real application that illustrates the results provided in the paper. Finally, some final conclusions are given.

Before proceeding, we introduce some notation, definitions and basic results. We recall, see [9], that a matrix M is called nonnegative if all its entries are nonnegative and it is denoted by $M \geq 0$. A matrix M is an M-matrix if $M = sI - A$, where $A \geq 0$ and $s \geq \rho(A)$, where $\rho(\cdot)$ denotes the spectral radius of a matrix, that is the maximum modulus of its eigenvalues.

The stability of a matrix M is equivalent to the condition $\rho(M) < 1$. From the literature this property is also referred to as Schur stable matrix or convergent matrix. In [9] a characterization of this property for nonnegative matrices is given. Thus, a nonnegative matrix M is stable if and only if $(I - M)^{-1} \geq 0$. Finally, the norm $\|M\|_1$ is the maximum absolute column sum of M.

2 Stability of perturbed positive systems

Consider an invariant discrete-time system

$$x(k+1) = Ax(k), \qquad k \geq 0,$$

where the vector $x(k) \in \mathbb{R}^n$ and A is a nonnegative matrix, that is $A \geq O$.

Consider that the system is asymptotically stable, that is $\rho(A) < 1$, and let $\Delta \geq O$ be a perturbation matrix. We can prove that the perturbed system is asymptotically stable, $\rho(A+\Delta) < 1$ if and only if $\rho(\Delta(I-A)^{-1}) < 1$. This is established as follows.

Proposition 1 *Let $A \geq O$ with $\rho(A) < 1$ and $\Delta \geq O$ be. The following assertions are equivalent*

(a) $\Delta(I - (A + \Delta))^{-1} \geq O$.

(b) $\rho(\Delta(I - A)^{-1}) < 1$.

(c) $\rho(A + \Delta) < 1$.

Proof. Since $\rho(A) < 1$, then $H = \Delta(I - A)^{-1} \geq O$.

(a) $\Rightarrow$ *(b)* As $H \geq O$, $r = \rho(H)$ is an eigenvalue of H with a nonnegative eigenvector v. From $Hv = rv$ and noting that $\Delta(I - (A + \Delta))^{-1} = H(I - H)^{-1}$ we get to $\Delta(I - (A + \Delta))^{-1}v = \frac{r}{1-r}v \geq 0$. Thus, $1 - \rho(H) > 0$.

(b)$\Rightarrow$ *(c)* If $\rho(\Delta(I - A)^{-1}) < 1$, then $I - (A + \Delta) = (I - \Delta(I - A)^{-1})(I - A)$ is an invertible M-matrix whose inverse matrix is nonnegative, $I - A$ and $I - \Delta(I - A)^{-1}$ are M-matrices. Then, $\rho(A + \Delta) < 1$.

(c) $\Rightarrow$ *(a)* It is straightforward since $\rho(A + \Delta) < 1$ and $\Delta \geq O$. □

When the matrix A has a companion structure, under similarity, we can take the entries of A of the upper diagonal equal to 1,

$$A = \begin{pmatrix} a_1 & 1 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ a_{n-1} & 0 & \cdots & 1 \\ a_n & 0 & \cdots & 0 \end{pmatrix}. \tag{1}$$

This matrix satisfies $|I - A| = 1 - \sum_{j=1}^{n} a_j$.

If the entries $\{a_i \geq 0,\ i = 1, \ldots n\}$ are perturbed, $a_i + \delta_i$ with $\delta_i \geq 0$, $i = 1, \ldots, n$, the new perturbed matrix $A + \Delta$, with

$$\Delta = \begin{pmatrix} \delta_1 & 0 & \cdots & 0 \\ \delta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ \delta_{n-1} & 0 & \cdots & 0 \\ \delta_n & 0 & \cdots & 0 \end{pmatrix}, \tag{2}$$

satisfies the following result.

Proposition 2 *Consider matrix A as (1) and matrix $\Delta \geq O$ as (2), then $A + \Delta$ is asymptotically stable if and only if $\|\Delta\|_1 < 1 - \|A\|_1$.*

Proof. From structure of matrices A and Δ and by a simple calculation we check

$$\rho(\Delta(I - A)^{-1}) = \frac{\sum_{j=1}^{n} \delta_j}{|I - A|}.$$

Hence, $A + \Delta$ is asymptotically stable if and only if $\displaystyle\sum_{j=1}^{n} \delta_j < 1 - \sum_{j=1}^{n} a_j$, that is $\|\Delta\|_1 < 1 - \|A\|_1$. □

Note that, if $\delta_i = 1,\ i = 1, \ldots, n$, then $A + \Delta$ is not asymptotically stable. By definition of A and Δ , we can check that the matrix $\Delta(I - (A + \Delta))^{-1}$ is not a nonnegative matrix since all its entries are equal to $\dfrac{-1}{(n-1) + a_1 + \cdots + a_n} < 0$. From Proposition 1, $\rho(A + \Delta) \geq 1$, then $A + \Delta$ is not asymptotically stable. On the other hand, the characterization of Proposition 2 suggests that the parameters of the perturbation must satisfy $0 \leq \delta_i < 1,\ i = 1, \ldots, n$.

3 Stability and positive reachability of perturbed positive systems

Now we fix our attention in a positive control discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \qquad k \geq 0,$$

where the state vector $x(k) \in \mathbb{R}^n$, the control vector $u(k) \in \mathbb{R}^m$ and A and B are nonnegative matrices, that is $A,\ B \geq O$. This system is denoted by $(A, B) \geq O$ and it is a positive system since for all nonnegative initial state $x(0) \geq 0$ and for all nonnegative control or input sequences $\{u(j)\} \geq 0,\ j \geq 0$, the trajectory of the system is nonnegative.

Using a nonnegative feedback $u(k) = \Delta x(k)$, $\Delta \geq O$, the closed-loop system is given by the state matrix $A + B\Delta$. If the initial system is asymptotically stable, $\rho(A) < 1$, we want obtain conditions on Δ in order to ensure that the new closed-loop system is also asymptotically stable. Initially we have the straightforward result from the Proposition 1.

Proposition 3 *Let $A,\ B \geq O$ with $\rho(A) < 1$ and $\Delta = \delta S \geq 0$ where $S = (I_m\ O)$, then $A + B\Delta$ is asymptotically stable if and only if $\delta < \dfrac{1}{R}$ with $R = \rho(BS(I - A)^{-1})$.*

In several applications will be important to reach a given state using an adequate control sequence. Thus, (A, B) is reachable if for every final state $x_f \in \mathbb{R}^n$ there exists a finite input sequence transferring the initial state to x_f. This property is known as *reachability property* and it is characterized from the range of reachability matrix $\mathrm{R}(A,\ B) = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$. Thus, (A, B) is reachable if and only if the matrix $\mathrm{R}(A,\ B)$ has full rank. The set of all reachable states is the subspace generate by the independent linear columns of $\mathrm{R}(A,\ B)$. When the system is reachable this subspace is the space $\mathbb{R}^n$. But when the nonnegative restrictions are imposed new features arise and we have the concept of *positive reachability property*. The interest in this property is motivated by the large number of fields (like bioengineering, economic modelling, biology and behavioral science) in which it is always necessary that the inputs u are also nonnegative. Thus, the system $(A, B) \geq 0$ is positively reachable if for every final state $x_f \in \mathbb{R}^n_+$ there exists a finite nonnegative input sequence transferring the initial state to x_f.

This property was studied in [10], [11]. Some results given in these works establish that this property holds if and only if the reachability matrix contains a monomial submatrix of order n. Recall that, a monomial vector is a (nonzero) multiple of some unit basis vector, and a monomial matrix M is a matrix whose columns are distinct monomial vectors, and can be decomposed as $M = DP$ where D is a diagonal matrix and P is a permutation matrix. In this case, the set of all positively reachable nonnegative states is the cone generate by the independent monomial columns of $\mathrm{R}(A,\ B)$. When the system is positively reachable this cone is $\mathbb{R}^n_+$.

It is widely know two systems are similar if we can obtain one of the other one by a change of base, $x(k) = T\hat{x}(k)$. Thus, system (A, B) is similar to system $(\hat{A}, \hat{B})$ if there exists a nonsingular matrix T such that $\hat{A} = T^{-1}AT$ and $\hat{B} = T^{-1}B$.

The general reachability property is preserved under similarity transformations however, two similar positive systems are not necessarily both positively reachable. Then, the concept of *positive similar* is introduced in the following way. Two positive systems (A, B) and $(\hat{A}, \hat{B})$ are positively similar if there exists a square nonnegative monomial matrix M satisfying $\hat{A} = M^{-1}AM$ and $\hat{B} = M^{-1}B$. In [12] is established that the positive reachability property is transferred under positive similarity.

Moreover, in [10] the authors gave a positive reachability canonical form. This canonical form has a upper triangular block structure where the diagonal blocks are formed by cyclic, nilpotent and companion submatrices. Using this canonical structure we consider the pair (A, B)

$$A = \begin{pmatrix} A_1 & \Phi & \cdots & \Phi \\ O & A_2 & \cdots & \Phi \\ \vdots & \vdots & \dots & \vdots \\ O & O & \cdots & \Phi \\ O & O & \cdots & A_h \end{pmatrix}, \ B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_{h-1} \\ B_h \end{pmatrix} \tag{3}$$

with $A_j \in \mathbb{R}^{n_j \times n_j}_+$ companion matrix as (1) whose entries of the first column are $\{a^j_i,\ i = 1, \dots, n_j\}$, $B_j \in \mathbb{R}^{n_j \times h}_+$ has all entries zero except the entry of position

$(n_j, h-j+1)$ denoted by b_j, for all $j = 1, \ldots, h$ and $\sum_{j=1}^{h} n_j = n$. Moreover $\Phi \geq O$ only can have nonzero entries in the first column. This system is positively reachable since satisfies the structure of the canonical form and it is easy to prove that the reachability matrix contains a monomial matrix of order n.

From now on, without loss of generality we assume that the initial time is zero, because otherwise we just need to perform a change of variables first to transfer the initial state to zero.

Returning to the initial approach we want study the invariance of both properties, stability and positive reachability when the system is subjected to perturbations.

Proposition 4 *Let $A,\ B \geq O$ be given as in (3). Consider the perturbation matrix $\Delta = (\Delta_1\ \ldots \Delta_h)$ being $\Delta_j = \delta_j S_j \geq O$ and $S_j \in \mathbb{R}_+^{h \times n_j}$ has all entries zero except the entry of position $(h-j+1, n_j)$ which is equal to 1, $j = 1, \ldots, h$. Then*

(a) The perturbed system $(A + B\Delta,\ B)$ is also positively reachable from zero.

(b) If the system (A, B) is asymptotically stable the perturbed system $(A + B\Delta,\ B)$ is asymptotically stable if and only if $\delta_j < \frac{1-\|A_j\|_1}{b_j}$, $j = 1, \ldots, h$.

Proof.

(a) To prove the positive reachability of the new system $(A + B\Delta)$, we construct its reachability matrix and it is easy to check that it has a monomial matrix of size $n \times n$.

(b) By structure of $A,\ B$ and Δ we have

$$\rho(B\Delta(I-A)^{-1}) = \max_{1 \leq j \leq n} \rho(B_j \Delta_j (I - A_j)^{-1}) = \max_{1 \leq j \leq n} \frac{b_j \delta_j}{|I - A_j|}.$$

By Propositions 1 and 3 we have that the new system $(A + B\Delta)$ is asymptotically stable if and only if $\delta_j < \frac{1-\|A_j\|_1}{b_j}$, for all $j = 1, \ldots, h$. □

4 Application to the Leslie's Population Model

Leslie matrix is a discrete, age-structured model of population growth. It is used to model the changes in a population of organisms over a period of time. For that, it is widely used in population ecology and demography to determine the growth of a population, as well as the age distribution within the population over time. There are a lot of studies on this matrix. To obtain more information on some applications from population matrix models in ecological and evolutionary studies see [13] and the references therein.

The Leslie model combines births and deaths in a single model and it is based on these hypotheses: (i) The age x is a variable starting from 0 and subdivided into n discrete age classes. The age class i corresponds to the ensemble of individuals whose ages satisfies $i - 1 \leq x < i$, $i = 1, \ldots, n$. (ii) Time is a discrete variable denoted by k and the time-step is equal to the duration of each age class. That is, from k to $k + 1$ all individuals go from class i to $i + 1$.

If we denote by $x(k)$ the number of individuals in each age class at time k and $u(k)$ the measure of immigration or stocking rate, then the Leslie model is given by

$$x(k+1) = \bar{A}x(k) + \bar{B}u(k)$$

where $\bar{A}$ represents the $n \times n$ Leslie matrix and $\bar{B}$ represents the number of individuals of age i entering the system per unit of control,

$$\bar{A} = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ s_1 & & & 0 \\ & \ddots & & \vdots \\ & & s_{n-1} & 0 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the entries of the first row of the matrix $\bar{A}$ are given by the fertility, f_i and the sub-diagonal is given by the survival, s_i and there are zeros elsewhere [14]. The fertility and survival rates are generally referred to as vital rates. And the entry b represents the fertility from an extern input. [15]

The eigenstructure of the matrix $\bar{A}$ gives many information on the model. Thus, the dominant eigenvalue λ determines the population growth in the long run. The other eigenvalues determine transient dynamics of the population. When $\lambda = 1$ the population is stationary, $\lambda > 1$ there is an over-population and when $\lambda < 1$ the population diminish. On the other hand, the right eigenvectors include the stable age distribution and the left eigenvectors include the reproductive value [16].

In addition, the pair $(\bar{A}, \bar{B})$ is similar to the pair (A, B) via the diagonal matrix $S = \mathrm{diag}(1,\ s_1,\ s_1 s_2,\ \ldots,\ s_1 \cdots s_{n-1})$ being $A = S^{-1}\bar{A}S$ and $B = S^{-1}\bar{B}$. This process is represented by the following discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

with

$$A = \begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ 1 & \cdots & 0 & 0 \\ \vdots & \ldots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4}$$

where $a_1 = f_1$, $a_j = f_j \prod_{i=1}^{j-1} s_i$, $j = 2, \ldots, n$.

Since A and B are nonnegative matrices we have a positive system. In addition, the system is reachable since the reachability matrix $\mathrm{R}(A,\ B)$ has full rank but the problem is that we can not ensure that the used control is nonnegative. That is, we can not assert that we can achieve a certain population from nonnegative controls. Maybe some nonnegative states can be reached by means of nonnegative inputs but not all because the system is not positively reachable, since $\mathrm{R}(A,\ B)$ does not contain a monomial submatrix of order n. To analyze this problem we will use the results of above sections. First, we make some comments about the transformations that allows us to obtain the system of interest.

We define $T_z = (I - N)$ and $T = PT_z$ where P is the antidiagonal permutation matrix, being

$$N = \begin{pmatrix} 0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{5}$$

Since N is a nilpotent matrix, then $\rho(N) < 1$ and T_z is an invertible M-matrix with $T_z^{-1} = \sum_{i=0}^{n-1} N^i$. Thus, T_z^{-1} and T^{-1} are nonnegative matrices.

The set of positively reachable states is given in the following result.

Proposition 5 *The set of population states which can be obtained in the Leslie's Population Model from a nonnegative control sequence is the cone*

$$\mathcal{X} =< T_z^{-1}e_1,\ \ldots,\ T_z^{-1}e_n > \tag{6}$$

where e_i *is the ith-canonical vector and* $T_z = (I - N)$ *with* N *as (5).*

Proof. First, we observe that the system (A, B) given in (4) is similar to the system $(\hat{A},\ \hat{B})$ where

$$\hat{A} = \begin{pmatrix} a_1 & 1 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ a_{n-1} & 0 & \cdots & 1 \\ a_n & 0 & \cdots & 0 \end{pmatrix},\ \hat{B} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b \end{pmatrix} \tag{7}$$

by means of the transformation matrix T with $T = PT_z$ where P is the antidiagonal permutation matrix.

As the reachability matrix of this system $(\hat{A},\ \hat{B})$ contains a monomial submatrix of order n then the system is positively reachable. Hence, we can ensure that for all nonnegative state $\hat{x}$ there exists a nonnegative sequence of control $\mathbf{u} = (u(n-1)\ \ldots\ u(1)\ u(0))^T \geq O$ such that

$$\mathrm{R}\left(\hat{B},\ \hat{A}\right)\mathbf{u} = \hat{x}.$$

Then $\mathrm{R}\,(B,\ A)\,\mathbf{u} = T^{-1}\hat{x}$. So, x is reachable by means of a nonnegative control sequence if and only if Tx is nonnegative.

Summarizing the previous comments, in the system $(A,\ B)$, a state x is reachable by means of a nonnegative control sequence if and only if there exists $\hat{x} \geq 0$ such that $x = T^{-1}\hat{x}$. Thus, the set of positively reachable states is the image of the application T^{-1} restricted on $\mathcal{R}^n_+$. From $T_z = (I - N)$ and $T_z \geq O$ then $\{T^{-1}e_j,\ j = 1, \ldots, n\}$ are independent linear nonnegative vectors and they generate the cone of the reachable states by a nonnegative control sequence

$$\mathcal{X} = \{x \in \mathcal{R}^n \;/\; \exists \hat{x} \in \mathcal{R}^n_+, \;\; x = T^{-1}\hat{x}\} = \\ = < T^{-1}e_1, \; \ldots, \; T^{-1}e_n > = \;\; < T_z^{-1}e_1, \; \ldots, \; T_z^{-1}e_n > .$$

□

Note that using the expression of matrix $T_z^{-1} = \sum_{i=0}^{n-1} N^i$ we can write

$$\mathcal{X} = < e_1, \; (I+N)e_2, \; \ldots, \sum_{j=0}^{i-1} N^j e_i, \; \ldots, \; \sum_{j=0}^{n-1} N^j e_n > . \tag{8}$$

Moreover, a specific population $x = (x_1 \; x_2 \; \ldots \; x_n)^T$ can be obtained in the Leslie's Population Model from a nonnegative control sequence if and only if $x_n \geq 0$ and

$$x_{n-i} \geq \sum_{k=1}^{i} a_k x_{n-i+k}, \; i = 1, \; \ldots, \; n-1, \tag{9}$$

Now, we study the Leslie model submitted to some kind of perturbations and we analyze the reachability and stability properties. At this point should discuss how the structure of the disturbance is such that the properties of the initial system remain: be stable and have the same set of positively reachable states.

If we want keep the same set of the reachable states using a nonnegative control sequence, then we only can consider perturbations of the kind $\Delta = (0 \; 0 \; \ldots \; \delta)$, with $\delta \geq 0$. Thereby we have that the similar perturbed system $(\hat{A} + \hat{B}\hat{\Delta})$, with $\hat{A} = TAT^{-1}$, $\hat{B} = TB$ and $\hat{\Delta} = \Delta T^{-1}$, has a structure as (7). Then, it is sufficient apply Proposition 4 for $h = 1$ to prove that the positive reachability property is preserved. In the same way of Proposition 5 we can establish that the set of positively reachable population is given by (6).

To study the stability of the closed-loop perturbed system it is sufficient to analyze the spectral radius of the matrix $\hat{A} + \hat{B}\hat{\Delta}$. Applying the item (b) of Proposition 4 to this matrix we obtain that $\rho(A + B\Delta) = \rho(\hat{A} + \hat{B}\hat{\Delta}) < 1$ if and only if $\delta < \frac{|I-A|}{b}$.

These results are summarized in the following proposition.

Proposition 6 *Let $A, \; B \geq O$ be given as in (4). Consider the perturbation matrix $\Delta = (0 \; 0 \; \ldots \; \delta)$ such that $\delta \geq 0$. Then*

(a) The perturbed system $(A + B\Delta, \; B)$ has the same cone of positively reachable states from zero than $(A, \; B)$.

(b) If the system (A, B) is asymptotically stable then the perturbed system $(A + B\Delta, \; B)$ is asymptotically stable if and only if $\delta < \frac{|I-A|}{b}$.

The obtained results in this section can be extended to a population with several groups or types of individuals where the group G_i can also receive births from the rest of the groups $G_j, \; j > i$. Without loss of generality, we can see the results for the case of a species with two types or groups of individuals G_1 and G_2 so that one of them, the G_2 group, also provides G_1 group births. Then, we obtain that the process is modeled by a system $(\bar{A}, \bar{B})$

$$\bar{A} = \begin{pmatrix} \bar{A}_1 & \bar{\Phi} \\ O & \bar{A}_2 \end{pmatrix}, \qquad \bar{B} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix},$$

where the matrix blocks are defined as (4) and $\bar{\Phi}$ represents the connection between the two groups. After applying the appropriate transformation $S = \text{diag}(S_1,\ S_2)$ where $S_j,\ j = 1, 2$, is constructed as the transformation matrix used from system (4) to (4), the system $(\bar{A}, \bar{B})$ is transformed in the system (A, B)

$$A = \begin{pmatrix} A_1 & \Phi \\ O & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{10}$$

where for each $j = 1, 2$, $A_j \in \mathbb{R}_+^{n_j \times n_j}$ is a companion matrix as (4) whose entries of the first row are $\{a_i^j,\ i = 1, \ldots, n_j\}$, $a_1^j = f_1^j$,$a_l^j = f_l^j \prod_{i=1}^{l-1} s_i^j$, $l = 2, \ldots, n_j$ being $\{f_i^j,\ i = 1, \ldots, n_j\}$ and $\{s_i^j,\ i = 1, \ldots, n_{j-1}\}$ the fertility and survival coefficients of the type or group G_j, respectively. Moreover, $B_j \in \mathbb{R}_+^{n_j \times 2}$ has all entries zero except the $(n_{j-1} + 1, 3 - j)$-entry denoted by b_j and $n_1 + n_2 = n$. Moreover, in this case we consider the matrix $\Phi \geq O$ has only one nonzero element in position $(1, n_1)$, given by $\varphi \prod_{i=1}^{n_2-1} s_i^j$ with φ the fertility coefficient from the last age class of the group G_2 to group G_1.

Using the transformation matrix $T = \text{diag}(T_1,\ T_2)$ where $T_j = P(I - N_j)$, $j = 1, 2$ we obtain a system $(\hat{A},\ \hat{B})$ as (3). Then, applying the results on positive reachability and stability when the system is submitted to a perturbation we obtain that the set $\mathcal{X}$ of population states of the Leslie's Population Model (10) which can be reached from zero using a nonnegative control sequence is the cone

$$\begin{aligned} &\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \\ &\mathcal{X}_1 = \{(x_1^T\ 0)^T\ /\ x_1 \in \tilde{\mathcal{X}}_1\} \text{ and } \quad \mathcal{X}_2 = \{(0\ x_2^T)^T\ /\ x_2 \in \tilde{\mathcal{X}}_2\} \end{aligned} \tag{11}$$

being $\tilde{\mathcal{X}}_j$, constructed as in (8), $\tilde{\mathcal{X}}_j =< e_1,\ (I + N_j)e_2,\ \ldots,\ \sum_{i=0}^{n_j-1} N_1^i e_{n_j} >$, where $T_j = P(I - N_j)$, $j = 1, 2$.

And if we consider the perturbation matrix $\Delta = (\Delta_1\ \Delta_2)$ being $\Delta_j = \delta_j S_j \geq O$ such that $S_j \in \mathbb{R}_+^{2 \times n_j}$ has only one nonzero element in position $(3 - j, n_j)$ equal to 1, $j = 1, 2$, then

(a) The perturbed system $(A + B\Delta,\ B)$ has the same cone of positively reachable states from zero than $(A,\ B)$.

(b) If the system (A, B) is asymptotically stable then the perturbed system $(A + B\Delta,\ B)$ is asymptotically stable if and only if $\delta_j < \frac{|I - A_j|}{b_j}$, $j = 1, 2$.

To clarify we give the following example.

Example 1 *Consider a population with two groups or types of individuals where the type or group G_1 can also receive births from the group G_2, where each of them has four age classes with the following fertility f_l^j, survival s_l^j and fertility from an extern input b_j, coefficients*

$$\begin{aligned} &f_1^1 = 0, \quad f_2^1 = 1, \quad f_3^1 = 4, \quad s_1^1 = 0.3, \quad s_2^1 = 0.2, \quad s_3^1 = 0.4, \quad b_1 = 1 \\ &f_1^2 = 0, \quad f_2^2 = 1, \quad f_3^2 = 3, \quad s_1^2 = 0.4, \quad s_2^2 = 0.2, \quad s_3^2 = 0.3, \quad b_2 = 1, \end{aligned}$$

and the fertility coefficient from the last age class of the group G_2 to group G_1 is $\varphi = 3$.

By the above results, a state is positively reachable if $x - A^{n-1}x_0$ is in the cone (11). For instance, if we consider a initial population

$$x_0 = (330\ 290\ 210\ 1030\ 300\ 200)^T$$

in the Leslie's Model (10), a population state is positively reachable if there exists a nonnegative control such that $x = A^{n-1}x_0 + \mathrm{R}(A, B)u$, or equivalently, if $x - A^{n-1}x_0$ is in the cone

$$\mathcal{X} = \{(x_1^T\ 0)^T\ /\ x_1 \in \tilde{\mathcal{X}}_1\} \oplus \{(0\ x_2^T)^T\ /\ x_2 \in \tilde{\mathcal{X}}_2\}$$

$$\tilde{\mathcal{X}}_1 = < e_1,\ e_2,\ 0.3e_1 + e_3 >$$
$$\tilde{\mathcal{X}}_2 = < e_1,\ e_2,\ 0.4e_1 + e_3 >$$

Note that to obtain de cone $\mathcal{X}$ we use conditions given in (9). And graphically the cones $\tilde{\mathcal{X}}_1$ and $\tilde{\mathcal{X}}_2$ would be the following

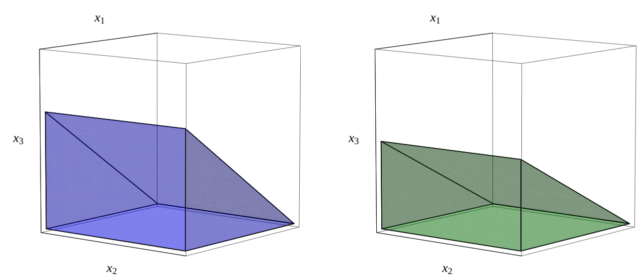

Figure 1 *Positively reachable population from 0 in the Leslie's Model (10).*
Positive reachability cone of G_1: $\{x \in \mathbb{R}^3 :\ x_1 \geq 0.3x_3,\ x_3, x_2 \geq 0\}$ *Positive reachability cone of G_2:* $\{x \in \mathbb{R}^3 :\ x_1 \geq 0.4x_3,\ x_3, x_2 \geq 0\}$

Using the transformation matrix T we obtain the system $(\hat{A}, \hat{B})$ given by (10)

$$\hat{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0.3 & 0 & 1 \\ 0.24 & 0 & 0 \end{pmatrix}, \quad \hat{A}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0.4 & 0 & 1 \\ 0.24 & 0 & 0 \end{pmatrix},$$

$$\hat{B}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{B}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In fact this system is positively reachable and the nonnegative sequence control to reach any state of $\mathcal{X}$ can be calculate using this system. If, for example, we want to reach this population

$$x = (490\ 380\ 380\ 1300\ 600\ 1100)^T,$$

which satisfies that $\hat{x} - \hat{A}^2\hat{x}_0 \geq 0$, it is sufficient increase births by means of an input calculate as

$$\mathbf{u} = \mathrm{R}^{-1}\left(\hat{B},\ \hat{A}\right)(\hat{x} - \hat{A}^2\hat{x}_0),$$

where $\mathbf{u} = (u^T(2)\ u^T(1)\ u^T(0))^T$ *with* $u(j)) = (u_1^T(j)\ u_2^T(j),\ j = 0, 1, 2$. *Then, the desired population is obtained using the following control or input of births at step* j *and each group* G_j

j	G_1	G_2
0	70	50
1	432	195
2	788	234

Now, we observe that the system $(A,\ B)$ *is asymptotically stable since* $\rho(A) = 0.83 < 1$. *If we consider a perturbation*

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \delta_2 \\ 0 & 0 & \delta_1 & 0 & 0 & 0 \end{pmatrix},$$

the system $(A+B\Delta,\ B)$ *has the same cone of positively reachable states than* $(A,\ B)$ *and taking* $\delta_1 < \frac{|I-A_1|}{b_1} = 7$ *and* $\delta_2 < \frac{|I-A_2|}{b_2} = 4$ *we can ensure that the perturbed system* $(A+B\Delta,\ B)$ *is also asymptotically stable. For instance if* $\delta_1 = 6$ *and* $\delta_2 = 3$ *we have* $\rho(A + B\Delta) = 0.96 < 1$.

5 Conclusions

Discrete-time positive systems are quite frequent in science and engineering. We consider the problem of determining the structure of a perturbation such that a perturbed positive discrete-time system has the positively reachability and stability properties. In the general model, to solve this problem the structure of the positive reachability canonical form introduced in [10] and positive similarity transformation are used. The Leslie's Population model is analyzed. It is a discrete-time positive system, it is reachable, since the reachability matrix $\mathrm{R}(A,\ B)$ has full rank, but it is not positively reachable, since $\mathrm{R}(A,\ B)$ does not contain a monomial submatrix of order n. It is shown the explicit expression of the cone of population states which can be obtained in the Leslie's Population Model from a nonnegative control sequence. Finally, a numerical example is given to clarify the obtained results.

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