On the elements $aa^\dagger$ and $a^\dagger a$ in a ring

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Abstract

We study various functions, principal ideals and annihilators depending on the projections $aa^\dagger$ and $a^\dagger a$ for a Moore-Penrose invertible ring element, extending recent work of O.M. Baksalary and G. Trenkler.
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1 Introduction

Throughout this paper, the symbol $\mathcal{R}$ will denote a unital ring (1 will be its unit) with an involution. Let us recall that an *involution* in a ring $\mathcal{R}$ is a map $a \mapsto a^*$ in $\mathcal{R}$ such that $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$, for any $a, b \in \mathcal{R}$.

We say that $a \in \mathcal{R}$ is *regular* if there exists $b \in \mathcal{R}$ such that $aba = a$. It can be proved that for any $a \in \mathcal{R}$, there is at most one $a^\dagger \in \mathcal{R}$ (called the *Moore-Penrose inverse* of a) such that

$$aa^\dagger a = a, \qquad a^\dagger aa^\dagger = a^\dagger, \qquad (aa^\dagger)^* = aa^\dagger, \qquad (a^\dagger a)^* = a^\dagger a.$$

In [8] it was proved that any complex matrix has a unique Moore-Penrose inverse, however, let us notice that the proof given therein is valid to guarantee the uniqueness – if the Moore-Penrose inverse exists – in a ring with involution. If there exists such $a^\dagger$ we will say that a is *Moore-Penrose invertible*. The subset of $\mathcal{R}$ composed of all Moore-Penrose invertible elements will be denote by $\mathcal{R}^\dagger$. We write $\mathcal{R}^{-1}$ for the set of all invertible elements in $\mathcal{R}$. The word *projection* will be reserved for an element q of $\mathcal{R}$ which is self-adjoint and idempotent, that is $q^* = q = q^2$. A ring $\mathcal{R}$ is called $*$-*reducing* if every element a of $\mathcal{R}$ obeys the implication $a^*a = 0 \Rightarrow a = 0$.

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Let $x \in \mathcal{R}$ and let $p \in \mathcal{R}$ be an idempotent ($p = p^2$). Then we can write

$$x = pxp + px(1-p) + (1-p)xp + (1-p)x(1-p)$$

and use the notations

$$x_{11} = pxp, \qquad x_{12} = px(1-p), \qquad x_{21} = (1-p)xp, \qquad x_{22} = (1-p)x(1-p).$$

Every projection $p \in \mathcal{R}$ induces a matrix representation which preserves the involution in $\mathcal{R}$, namely $x \in \mathcal{R}$ can be represented by means of the following matrix:

$$x = \begin{bmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}. \tag{1.1}$$

The purpose of this paper is to study several ideals involving the projections $aa^\dagger$ and $a^\dagger a$, when $a \in \mathcal{R}^\dagger$. We shall consider two kinds of ideals. The *principal ideals* (also called image ideals) generated by $b \in \mathcal{R}$ are defined by $b\mathcal{R} = \{bx : x \in \mathcal{R}\}$ and $\mathcal{R}b = \{xb : x \in \mathcal{R}\}$. The *annihilators* (also called kernel ideals) of $b \in R$ are defined by $b^\circ = \{x \in \mathcal{R} : bx = 0\}$ and ${}^\circ b = \{x \in \mathcal{R} : xb = 0\}$. If $\mathcal{R}$ is a ring with the unit and $p \in \mathcal{R}$, then it is quickly seen that $p\mathcal{R}p = \{pxp : x \in \mathcal{R}\}$ is a sub-ring whose unity is p. From now on, for an arbitrary projection p, we shall denote $\overline{p} = 1 - p$.

The following elementary lemma will be many times used in the sequel.

Lemma 1.1. *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}$. Then*

(i) $a \in \mathcal{R}^\dagger \iff a^* \in \mathcal{R}^\dagger$. *Furthermore,* $(a^*)^\dagger = (a^\dagger)^*$.

(ii) *If $a \in \mathcal{R}^\dagger$, then $a^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger)^\dagger = a$.*

(iii) *If $a \in \mathcal{R}^\dagger$, then $a^*a, aa^* \in \mathcal{R}^\dagger$ and*

$$(a^*a)^\dagger = a^\dagger(a^*)^\dagger, \quad (aa^*)^\dagger = (a^*)^\dagger a^\dagger, \quad a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger, \quad a^* = a^\dagger aa^* = a^*aa^\dagger.$$

(iv) *If $\mathcal{R}$ is $*$-reducing, then $a^*a \in \mathcal{R}^\dagger \Rightarrow a \in \mathcal{R}^\dagger$ and $aa^* \in \mathcal{R}^\dagger \Rightarrow a \in \mathcal{R}^\dagger$.*

Proof. It is evident that (i)-(iii) hold. We will prove only the first implication of (iv) since to prove the other one, it is sufficient to make the same argument for a^* instead of a. Assume that $a^*a \in \mathcal{R}^\dagger$ and let $x = (a^*a)^\dagger a^*$. Observe that the Moore-Penrose inverse of a selfadjoint Moore-Penrose invertible element is again self-adjoint, and thus, $(a^*a)^\dagger$ is self-adjoint. Now $(ax)^* = \left[a(a^*a)^\dagger a^*\right]^* = a(a^*a)^\dagger a^* = ax$; $xa = (a^*a)^\dagger a^*a$ is selfadjoint; $xax = (a^*a)^\dagger a^*a(a^*a)^\dagger a^* = (a^*a)^\dagger a^* = x$. Finally, $a^*axa = a^*a(a^*a)^\dagger a^*a = a^*a$, and since $\mathcal{R}$ is $*$-reducing, we get $axa = a$. $\square$

A consequence of Lemma 1.1 is that

$$\text{if } x \in \mathcal{R}^\dagger \text{ is self-adjoint, then } xx^\dagger = x^\dagger x. \tag{1.2}$$

For the class of Moore-Penrose invertible elements $x \in \mathcal{R}$ such that $xx^\dagger = x^\dagger x$, the reader is referred to [3].

2 Group inverses

Let $\mathcal{R}$ be a ring (possibly without an involution). If $a \in \mathcal{R}$, then there is at most one $x \in \mathcal{R}$ such that

$$axa = a, \qquad xax = x, \qquad ax = xa.$$

When such x exists, we will write $x = a^{\#}$ and we will say that x is the *group inverse* of a and a is *group invertible*. The symbol $\mathcal{R}^{\#}$ will denote the set of all group invertible elements of $\mathcal{R}$.

In this paragraph, let F be a square complex matrix. In [1, p. 10215] it was given a list of several equivalent conditions (involving the orthogonal projectors $FF^{\dagger}$ and $F^{\dagger}F$) for F to has the group inverse. The proof given therein relies in rank matrix theory and a matrix decomposition given by Hartwig and Spindelböck [4]. However, as we shall see, many of these equivalences can be stated in a ring setting, and proved by algebraic reasonings.

We shall use the following result [9, Prop. 8.22], whose proof is included for the convenience of the reader.

Theorem 2.1. *Let $\mathcal{R}$ be a unital ring and $a \in \mathcal{R}$. Then a is group invertible if and only if there exist $x, y \in \mathcal{R}$ such that $a^2x = a$ and $ya^2 = a$.*

Proof. If $a \in \mathcal{R}^{\#}$ we have $a^2a^{\#} = a = a^{\#}a^2$.

Reciprocally, assume that there exist $x, y \in \mathcal{R}$ such that $a^2x = a$ and $ya^2 = a$. We will prove $yax = a^{\#}$. First, let us see that $ax = ya^2x = ya$. Now, $a(yax) = a(ya)x = a^2x^2 = ax$ and $(yax)a = y(ax)a = y^2a^2 = ya$ implies that $a(yax) = (yax)a$. Finally $a(yax)a = ya^2 = a$ and $(yax)a(yax) = yayax = yax$. □

Obviously, Theorem 2.1 implies that in a commutative ring, group invertibility is the same as regularity.

Observe that under the hypothesis of Theorem 2.1, one has

$$a^2x = a \ \text{ and } \ ya^2 = a \qquad \Rightarrow \qquad a^{\#} = yax. \tag{2.1}$$

Let us notice that by Theorem 2.1 one can deduce that for $a \in \mathcal{R}$,

$$a \in \mathcal{R}^{\#} \qquad \Leftrightarrow \qquad a\mathcal{R} = a^2\mathcal{R} \ \text{ and } \ \mathcal{R}a = \mathcal{R}a^2.$$

This latter equivalence can be viewed as a ring version of "for a matrix $F \in \mathbb{C}_{n,n}$, there exists $F^{\#}$ if and only if $\operatorname{rank}(F^2) = \operatorname{rank}(F)$" (see [5, Section 4.4]).

It was mentioned in [1, p. 10215] that for a given square complex matrix F, there exists $F^{\#}$ if and only if $\mathcal{R}(F) \cap \mathcal{N}(F) = \{0\}$, where $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denotes, respectively, the column space and the null space of a matrix. Let us notice that $\mathcal{R}(F) \cap \mathcal{N}(F) = \{0\}$ is equivalent to $\mathcal{N}(F^2) = \mathcal{N}(F)$, and in the matrix setting, this last condition is equivalent to $\operatorname{rank}(F^2) =$

rank(F). However, the things are more complicated in the ring case: if $\mathcal{R}$ is a ring and $a \in \mathcal{R}$, then the following implication is trivial to get

$$a \in \mathcal{R}^{\#} \qquad \Rightarrow \qquad a\mathcal{R} \cap a^{\circ} = \{0\} \ \text{ and } \ \mathcal{R}a \cap {}^{\circ}a = \{0\}.$$

But the opposite implication is false: Take the ring composed of integers numbers, i.e. $\mathbb{Z}$. It is easy to get $\mathbb{Z}^{\#} = \{0, 1, -1\}$ and if $a \in \mathbb{Z} \setminus \{0\}$, then $a^{\circ} = {}^{\circ}a = \{0\}$. However if we assume that any element of $\mathcal{R}$ is Drazin invertible, then it is easy to see that the opposite implication turns it true. Let us remark that any square matrix has Drazin inverse.

The following result will permit prove several equivalent conditions for the existence of the group inverse in a ring with involution.

Theorem 2.2. *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger}$. Denote $p = aa^{\dagger}$ and $q = a^{\dagger}a$. Then $pq, qp \in \mathcal{R}^{\dagger}$, $(qp)^{\dagger} = aa^{\#}$, and $(pq)^{\dagger} = (aa^{\#})^*$.*

Proof. Observe that $aq = pa = a$. So $(aa^{\#})(qp) = a^{\#}ap = aa^{\dagger}$ is Hermitian, $(qp)(aa^{\#}) = qaa^{\#} = a^{\dagger}a$ is Hermitian, $(aa^{\#})(qp)(aa^{\#}) = aa^{\dagger}aa^{\#} = aa^{\#}$, and $(qp)(aa^{\#})(qp) = a^{\dagger}aqp = qp$. This proves $qp \in \mathcal{R}^{\dagger}$ and $(qp)^{\dagger} = aa^{\#}$. To finish the proof, let us note that $qp \in \mathcal{R}^{\dagger} \Leftrightarrow (qp)^* \in \mathcal{R}^{\dagger}$ and under this situation one has $[(qp)^{\dagger}]^* = [pq]^{\dagger}$. □

The following result generalizes the considerations concerning group invertible matrices given in [1, p. 10215]. The unique assumption is that the ring is unital and has an involution.

Theorem 2.3. *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}^{\dagger}$. Denote $p = aa^{\dagger}$ and $q = a^{\dagger}a$. Then the following are equivalent:*

(i) $a \in \mathcal{R}^{\#}$,

(ii) $p + q - 1$ *is invertible,*

(iii) $a\mathcal{R} = pq\mathcal{R}$ *and* $\mathcal{R}a = \mathcal{R}pq$,

(iv) $a^*\mathcal{R} = qp\mathcal{R}$ *and* $\mathcal{R}a^* = \mathcal{R}qp$,

(v) $p - q - 1$ *and* $p - q + 1$ *are both invertible.*

Proof. (i) $\Rightarrow$ (ii): Since $p + q - 1$ and $aa^{\#} + (aa^{\#})^* - 1$ are self-adjoint, it is sufficient to check $(p + q - 1)(aa^{\#} + (aa^{\#})^* - 1) = 1$. Observe that

$$aa^{\dagger}(aa^{\#})^* = (aa^{\dagger})^*(aa^{\#})^* = (aa^{\#}aa^{\dagger})^* = aa^{\dagger} \tag{2.2}$$

and

$$a^{\dagger}a(aa^{\#})^* = (a^{\dagger}a)^*(a^{\#}a)^* = (a^{\#}aa^{\dagger}a)^* = (a^{\#}a)^*. \tag{2.3}$$

Hence

$$\begin{aligned}
&(aa^\dagger + a^\dagger a - 1)(aa^\# + (aa^\#)^* - 1)\\
&= aa^\dagger aa^\# + a^\dagger a^2 a^\# - aa^\# + aa^\dagger (aa^\#)^* + a^\dagger a(aa^\#)^* - (aa^\#)^* - aa^\dagger - a^\dagger a + 1\\
&= 1.
\end{aligned}$$

(ii) $\Rightarrow$ (i): Denote $u = p + q - 1$. We have $ua = a^\dagger a^2$ and $au = a^2 a^\dagger$, which implies

$$a = u^{-1}ua = u^{-1}a^\dagger a^2 \quad \text{and} \quad a = auu^{-1} = a^2 a^\dagger u^{-1}. \tag{2.4}$$

Now, by Theorem 2.1 it follows that a is group invertible.

(i) $\Rightarrow$ (iii): The inclusions $pq\mathcal{R} \subseteq a\mathcal{R}$ and $\mathcal{R}pq \subseteq \mathcal{R}a$ are trivial. By Theorem 2.2 we get $a = aa^\# a = (qp)^\dagger a = (qp)^*(qp(qp)^*)^\dagger a = pq(qpq)^\dagger a$, which proves $a\mathcal{R} \subseteq pq\mathcal{R}$. In addition, we have $a = aaa^\# = a(qp)^\dagger = a((qp)^* qp)^\dagger (qp)^* = a(pqp)^\dagger pq$, which proves $\mathcal{R}a \subseteq \mathcal{R}pq$.

(iii) $\Rightarrow$ (i): We shall use Theorem 2.1 to prove the existence of $a^\#$. Since $a \in a\mathcal{R} = pq\mathcal{R}$ and $a \in \mathcal{R}a = \mathcal{R}pq$, there exist $u, v \in \mathcal{R}$ such that $a = pqu$ and $a = vpq$, hence

$$\begin{aligned}
a = pqu = aa^\dagger qu &= aa^*(aa^*)^\dagger qu = a(vpq)^*(aa^*)^\dagger qu\\
&= aqpv^*(aa^*)^\dagger qu = a^2 a^\dagger v^*(aa^*)^\dagger qu
\end{aligned}$$

and

$$\begin{aligned}
a = vpq = vpa^\dagger a &= vp(a^*a)^\dagger a^* a = vp(a^*a)^\dagger (pqu)^* a\\
&= vp(a^*a)^\dagger u^* qpa = vp(a^*a)^\dagger u^* a^\dagger a^2.
\end{aligned}$$

(iii) $\Leftrightarrow$ (iv): It is evident.

(i) $\Rightarrow$ (v): Denote $\pi = aa^\#$. Since $p - q - 1$ and $\pi + \pi^* - 2\pi\pi^* - 1$ are self-adjoint, to prove $(p - q - 1)^{-1} = \pi + \pi^* - 2\pi\pi^* - 1$, it is enough to check $(p - q - 1)(\pi + \pi^* - 2\pi\pi^* - 1) = 1$. To this end, we shall use $p\pi^* = p$, $q\pi^* = \pi^*$ (see (2.2) and (2.3)) and $p\pi = \pi$, $q\pi = q$.

$$\begin{aligned}
&(p - q - 1)(\pi + \pi^* - 2\pi\pi^* - 1)\\
&\quad = p\pi + p\pi^* - 2p\pi\pi^* - p - q\pi - q\pi^* + 2q\pi\pi^* + q - \pi - \pi^* + 2\pi\pi^* + 1\\
&\quad = \pi + p - 2\pi\pi^* - p - q - \pi^* + 2\pi^* + q - \pi - \pi^* + 2\pi\pi^* + 1 = 1.
\end{aligned}$$

Observe that we have proved that for any $b \in \mathcal{R}^\dagger$, the following holds:

$$b \in \mathcal{R}^\# \Rightarrow bb^\dagger - b^\dagger b - 1 \in \mathcal{R}^{-1}. \tag{2.5}$$

Furthermore, since (i) $\Leftrightarrow$ (ii) has been proved, we can use that for any $c \in \mathcal{R}^\dagger$

$$c \in \mathcal{R}^\# \Leftrightarrow cc^\dagger + c^\dagger c - 1 \in \mathcal{R}^{-1}. \tag{2.6}$$

Since $a \in \mathcal{R}^\#$, from (i) $\Rightarrow$ (ii), we get $aa^\dagger + a^\dagger a - 1 \in \mathcal{R}^{-1}$. We can apply (2.6) for $c = a^\dagger$ to get $a^\dagger \in \mathcal{R}^\#$. Now by (2.5) for $b = a^\dagger$ we obtain $q - p - 1 \in \mathcal{R}^{-1}$.

(v) ⇒ (i): Denote $u = p - q - 1$ and $v = p - q + 1$. Observe that $ua = (p - q - 1)a = -a^\dagger a^2$ and $av = a(p - q + 1) = a^2 a^\dagger$. Thus $a = u^{-1}ua = -u^{-1}a^\dagger a^2$ and $a = avv^{-1} = a^2 a^\dagger v^{-1}$. Theorem 2.1 permits assure that $a \in \mathcal{R}^\#$. $\square$

Corollary 2.1. *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$. The following identities hold:*

(i) $(aa^\dagger + a^\dagger a - 1)^{-1} = aa^\# + (aa^\#)^* - 1$,

(ii) $a^\# = (aa^\dagger + a^\dagger a - 1)^{-1} a^\dagger (aa^\dagger + a^\dagger a - 1)^{-1}$,

(iii) $a^\dagger \in \mathcal{R}^\#$ *and* $(a^\dagger)^\# = (aa^\dagger + a^\dagger a - 1)^{-1} a (aa^\dagger + a^\dagger a - 1)^{-1}$,

(iv) $(aa^\dagger - a^\dagger a - 1)^{-1} = aa^\# + (aa^\#)^* - 2aa^\#(aa^\#)^* - 1$,

(v) $a^\# = -(p - q - 1)^{-1} a^\dagger (p - q + 1)^{-1}$,

(vi) $(a^\dagger)^\# = (p - q + 1)^{-1} a (q - p + 1)^{-1}$.

Proof. (i) follows from the proof of (i) ⇒ (ii) of Theorem 2.3. (ii) follows from (2.4) and (2.1). The first part of (iii) follows from (i) ⇔ (ii) of Theorem 2.3, and the last part from (ii). (iv) follows from (i) ⇒ (v) of Theorem 2.3. The proof of (v) ⇒ (i) of Theorem 2.3 distills $a = -(p - q - 1)^{-1} a^\dagger a^2$ and $a = a^2 a^\dagger (p - q + 1)^{-1}$, hence (2.1) permits prove (v). Finally, (vi) follows from (v). $\square$

There is no simple relation (except when a satisfies some concrete relation, see e.g. [7]) between $a^\#$ and $a^\dagger$. One can guess that $(a^\#)^\dagger = (a^\dagger)^\#$. But even in the matrix setting, this expression is false. Take

$$A = \begin{bmatrix} c & s \\ 0 & 0 \end{bmatrix},$$

where $0 < c, s < 1$ and $c^2 + s^2 = 1$. The following equalities can be easily verified:

$$A^\dagger = \begin{bmatrix} c & 0 \\ s & 0 \end{bmatrix}, \quad A^\# = \begin{bmatrix} 1/c & s/c^2 \\ 0 & 0 \end{bmatrix}, \quad (A^\dagger)^\# = \begin{bmatrix} 1/c & 0 \\ s/c^2 & 0 \end{bmatrix}, \quad (A^\#)^\dagger = \begin{bmatrix} c^3 & 0 \\ sc^2 & 0 \end{bmatrix}.$$

3 Expressions involving $aa^\dagger$ and $a^\dagger a$

In this section, we will study several expressions involving $aa^\dagger$ and $a^\dagger a$ when $a \in \mathcal{R}^\dagger$ and $\mathcal{R}$ is a ring with involution. The results from this section are the generalization of some of the results established in [1].

Some facts about projections will be stated here and proved for the sake of completeness.

Lemma 3.1. *Let $\mathcal{R}$ be a ring with involution, $p, q \in \mathcal{R}$ be projections and $x \in \mathcal{R}$ be self-adjoint.*

(i) *If* $pxp \in \mathcal{R}^\dagger$*, then* $(pxp)^\dagger = p(pxp)^\dagger = (pxp)^\dagger p$*,*

(ii) *If* $\mathcal{R}$ *is a* $*$*-reducing ring and* $pqp \in \mathcal{R}^\dagger$*, then* $pq \in \mathcal{R}^\dagger$ *and* $(pqp)(pqp)^\dagger = (pq)(pq)^\dagger$.

Proof. (i): Since pxp is self-adjoint, by (1.2) we have $\overline{p}\,(pxp)^\dagger = \overline{p}\,(pxp)(pxp)^\dagger(pxp)^\dagger = 0$, hence $p(pxp)^\dagger = (pxp)^\dagger$. The equality $(pxp)^\dagger = (pxp)^\dagger p$ can be proved in a similar way.

(ii): Observe that $pqp = (pq)(pq)^*$ holds. By Lemma 1.1 (iv) we get $pq \in \mathcal{R}^\dagger$. Now, by Lemma 1.1 (iii), we have $(pqp)(pqp)^\dagger = pq(pq)^*\,[pq(pq)^*]^\dagger = pq(pq)^\dagger$. □

The following result (interesting in its own) will serve to prove some results.

Theorem 3.1. *Let* $\mathcal{R}$ *be a* $*$*-reducing ring and* $p, q \in \mathcal{R}$ *be two projections such that* $p\overline{q}\,p, \overline{p}\,q\overline{p}$ *are Moore-Penrose invertible. Then* $p+q$ *is Moore-Penrose invertible and*

$$(p+q)(p+q)^\dagger = p + \overline{p}\,q\overline{p}\,(\overline{p}\,q\overline{p}\,)^\dagger.$$

Proof. Let us suppose that the projections p and q are represented by

$$p = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad q = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}. \tag{3.1}$$

By hypothesis one has that $p - a, d \in \mathcal{R}^\dagger$. Since $1 - a = (p - a) + (1 - p)$ and $p - a, \overline{p} \in \mathcal{R}^\dagger$ (observe that since $\overline{p}$ is a projection, obviously $\overline{p} \in \mathcal{R}^\dagger$ and $\overline{p}^{\,\dagger} = \overline{p}\,$) we get $1 - a \in \mathcal{R}^\dagger$. Let

$$x = \frac{1}{2}\left(p + (p-a)(p-a)^\dagger\right) - bd^\dagger - d^\dagger b^* + 2d^\dagger - dd^\dagger. \tag{3.2}$$

We shall prove that $x = (p+q)^\dagger$ by verifying the four conditions of the Moore-Penrose invertibility. We shall decompose x as in (1.1). Obviously we have

$$px = x_{11} + x_{12} \quad \text{and} \quad qx = ax_{11} + bx_{21} + ax_{12} + bx_{22} + b^*x_{11} + dx_{21} + b^*x_{12} + dx_{22},$$

where

$$x_{11} = \frac{1}{2}\left(p + (p-a)(p-a)^\dagger\right), \quad x_{12} = -bd^\dagger, \quad x_{21} = -d^\dagger b^*, \quad x_{22} = 2d^\dagger - dd^\dagger.$$

By Lemma 3.1 (i)

$$\begin{aligned} (p-a)(p-a)^\dagger b &= (p - pqp)(p - pqp)^\dagger pq(1-p) \\ &= -(p(1-q)p)(p(1-q)p)^\dagger p(1-q)(1-p) \\ &= -(p(1-q))(p(1-q))^\dagger p(1-q)(1-p) = b. \end{aligned} \tag{3.3}$$

Similarly, we can prove that

$$bdd^\dagger = b. \tag{3.4}$$

Now, since q is idempotent, we have that $b = ab + bd$, so $bd^\dagger = abd^\dagger + b$, i.e. $b = (p-a)bd^\dagger$. Multiplying the last equality with $(p-a)^\dagger$ from the left side and using (3.3), we get

$$(p-a)^\dagger b = bd^\dagger. \tag{3.5}$$

Observe that (3.3) in conjunction with (3.5) implies that $bd^\dagger - b = abd^\dagger$. Hence by (3.4), we get

$$x_{12} + ax_{12} + bx_{22} = -bd^\dagger - abd^\dagger + b(2d^\dagger - dd^\dagger) = 0.$$

Let us remark that since $p - a$ is self-adjoint, then

$$p - a = (p-a)(p-a)(p-a)^\dagger = (p-a)(p-a)^\dagger - a(p-a)(p-a)^\dagger,$$

and thus by (3.5), $a = a^2 + bb^*$, and the previous computation, we get that $bd^\dagger b^* = (1-a)^\dagger bb^* = (1-a)^\dagger(1-a)a = (1-a)^\dagger(1-a) - (1-a)$. Hence,

$$a - bd^\dagger b^* = 1 - (1-a)(1-a)^\dagger. \tag{3.6}$$

Using the last equality, we get

$$\begin{aligned} (p+a)\left(p + (p-a)(p-a)^\dagger\right) &= p + (p-a)(p-a)^\dagger + a + a(p-a)(p-a)^\dagger \\ &= 2\left[(p-a)(p-a)^\dagger + a\right] \\ &= 2\left[p + bd^\dagger b^*\right]. \end{aligned}$$

Thus,

$$x_{11} + ax_{11} + bx_{21} = \frac{1}{2}(p+a)\left(p + (p-a)(p-a)^\dagger\right) - bd^\dagger b^* = p.$$

Using (3.3) and the self-adjointness of a we get $b^*(p-a)(p-a)^\dagger = b^*$. Furthermore, since $b = pq(1-p)$, we trivially get $b^*p = b$. Now (3.4), yields

$$b^*x_{11} + dx_{21} = \frac{1}{2}b^*\left(p + (p-a)(p-a)^\dagger\right) - dd^\dagger b^* = 0.$$

Since q is self-adjoint, the representation of q given in (3.1) yields that d is self-adjoint, hence $dd^\dagger = d^\dagger d$. In view of $d = d^2 + b^*b$, we have

$$b^*x_{12} + dx_{22} = (d^2 - d)d^\dagger + 2dd^\dagger - d = dd^\dagger.$$

The above computations show that

$$(p+q)x = p + dd^\dagger. \tag{3.7}$$

Thus, $(p+q)x$ is self-adjoint. Since x, $p+q$, and $(p+q)x$ are self-adjoint, fact (1.2) permits get that $x(p+q) = (p+q)x$. By (3.3) and (3.7) we easily have $(p+q)x(p+q) = p+q$ and $x(p+q)x = x$.

Now, since $d = (1-p)q(1-p)$, it is evident that (i) holds. □

Theorem 3.2. *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}^\dagger$, $a \neq 0$. Denote $p = aa^\dagger$ and $q = a^\dagger a$.*

1. (i) $pq = 0 \Leftrightarrow a^2 = 0 \Leftrightarrow qp = 0$,

(ii) $pq \in \mathcal{R}^{-1} \Leftrightarrow a \in \mathcal{R}^{-1} \Leftrightarrow qp \in \mathcal{R}^{-1}$,

(iii) $pq = 1 \Leftrightarrow a \in \mathcal{R}^{-1} \Leftrightarrow qp = 1$.

2. (i) $p + q = 0$ *can never happen if* $\mathrm{char}(\mathcal{R}) \neq 2$,

(ii) $p + q = 1$ *if and only if* $a^2 = 0$ *and* $a\mathcal{R} + a^*\mathcal{R} = \mathcal{R}$,

(iii) *If $\mathcal{R}$ is a $*$-reducing ring and $p\overline{q}\,p, \overline{p}\,q\overline{p} \in \mathcal{R}^\dagger$, then $p + q \in \mathcal{R}^{-1}$ if and only if $a\mathcal{R} + a^*\mathcal{R} = \mathcal{R}$.*

3. (i) $p - q = 0$ *if and only if* $a\mathcal{R} = a^*\mathcal{R}$,

(ii) $p - q = 1$ *can never happen if* $\mathrm{char}(\mathcal{R}) \neq 2$,

(iii) $p - q \in \mathcal{R}^{-1}$ *if and only if* $a\mathcal{R} \oplus a^*\mathcal{R} = \mathcal{R}$.

Proof. (1.i): If $aa^\dagger a^\dagger a = 0$, then $0 = a^\dagger(aa^\dagger a^\dagger a)a^\dagger = (a^\dagger aa^\dagger)(a^\dagger aa^\dagger) = (a^\dagger)^2$. Denote $b = a^\dagger$. We get $a^2 = b^\dagger b^\dagger = (b^*b)^\dagger b^* b^*(bb^*)^\dagger = (b^*b)^\dagger (b^2)^*(bb^*)^\dagger = 0$. If $a^2 = 0$, then $pq = aa^\dagger a^\dagger a = a(a^*a)^\dagger a^* a^*(aa^*)^\dagger a = 0$. The remaining equivalence of (1.i) is trivial.

(1.ii): If $pq \in \mathcal{R}^{-1}$, then there exists $b \in \mathcal{R}$ such that $aa^\dagger a^\dagger ab = 1$ and $baa^\dagger a^\dagger a = 1$. Now, $a^\dagger = a^\dagger(aa^\dagger a^\dagger ab) = a^\dagger a^\dagger ab$, thus $1 = aa^\dagger a^\dagger ab = aa^\dagger$. Similarly, $a^\dagger = (baa^\dagger a^\dagger a)a^\dagger = baa^\dagger a^\dagger$, hence $1 = baa^\dagger a^\dagger a = a^\dagger a$. If $a \in \mathcal{R}^{-1}$, then it is trivial $a^\dagger = a^{-1}$, thus $pq = 1$. The remaining equivalence of (1.ii) can be proved by taking adjoint.

(1.iii) follows from (1.ii).

(2.i): If $aa^\dagger + a^\dagger a = 0$, then $0 = a^\dagger(aa^\dagger + a^\dagger a) = a^\dagger + a^\dagger a^\dagger a$. Thus, $0 = (a^\dagger + a^\dagger a^\dagger a)a^\dagger = 2a^\dagger a^\dagger$. Since $\mathrm{char}(\mathcal{R}) \neq 2$, then $a^\dagger a^\dagger = 0$. Substituting it into $0 = a^\dagger + a^\dagger a^\dagger a$ leads to $a^\dagger = 0$, which cannot happen in view of the hypotheses.

(2.ii): Assume $p+q = 1$. Premultiplying by p leads to $pq = 0$, and by (1.i) we get $a^2 = 0$. Since $1 = p + q = aa^\dagger + a^*(aa^*)^\dagger a \in a\mathcal{R} + a^*\mathcal{R}$, then $\mathcal{R} = a\mathcal{R} + a^*\mathcal{R}$.

Assume $a^2 = 0$ and $\mathcal{R} = a\mathcal{R} + a^*\mathcal{R}$. To prove $p + q = 1$, by [3, Th. 5], it is sufficient to prove $a\mathcal{R} \perp a^*\mathcal{R}$. In fact, if $y, z \in \mathcal{R}$, then $(ay)^*(a^*z) = y^*(a^2)^*z = 0$.

(2.iii): If $p + q$ is invertible, then there exists $y \in \mathcal{R}$ such that $(p + q)y = 1$, hence $1 = aa^\dagger y + a^*(aa^*)^\dagger ay \in a\mathcal{R} + a^*\mathcal{R}$, which shows $\mathcal{R} = a\mathcal{R} + a^*\mathcal{R}$.

If $a\mathcal{R} + a^*\mathcal{R} = \mathcal{R}$, then there exists $u, v \in \mathcal{R}$ such that $1 = au + a^*v$. Hence

$$1 = au + a^*v = aa^\dagger au + a^\dagger aa^*v = pau + qa^*v.$$

From this, we get $p = pau + pqa^*v$, hence $1 = p - pqa^*v + qa^*v = p + \overline{p}\,qa^*v$. By Theorem 3.1 and Lemma 3.1 we have

$$(p + q)(p + q)^\dagger = \left[p + \overline{p}\,q\overline{p}\,(\overline{p}\,q\overline{p}\,)^\dagger\right][p + \overline{p}\,qa^*v] = p + \overline{p}\,qa^*v = 1. \tag{3.8}$$

Since $p+q$ is self-adjoint, then $(p+q)^\dagger$ is also self-adjoint, and thus from (3.8) we get $(p+q)^\dagger(p+q)=1$. Therefore, $p+q\in\mathcal{R}^{-1}$.

(3.i): Assume $aa^\dagger=a^\dagger a$. The equalities $a=aa^\dagger a=a^\dagger aa=a^*(aa^*)^\dagger a^2$ imply $a\mathcal{R}\subseteq a^*\mathcal{R}$. Now, $a^*=a^\dagger aa^*=aa^\dagger a^*$ yields $a^*\mathcal{R}\subseteq a\mathcal{R}$.

Assume $a\mathcal{R}=a^*\mathcal{R}$. Since $p\in a\mathcal{R}=a^*\mathcal{R}$, there exists $u\in\mathcal{R}$ such that $p=a^*u$. So, $qp=a^\dagger aa^*u=a^*u=p$. Since $q\in a^*\mathcal{R}=a\mathcal{R}$, there exists $v\in\mathcal{R}$ such that $q=av$. So, $pq=aa^\dagger av=av=q$. Now, $p=qp=(pq)^*=q^*=q$.

(3.ii): Assume that $p-q=1$. By [3, Th. 3] and [3, Cor. 4(ii)] we get that there exists an idempotent $h\in\mathcal{R}$ such that $ha=a$, $ha^*=0$ and $2=h+h^*$. Squaring the last equality yields $4=h+h^*+hh^*+h^*h$, and thus, $2=hh^*+h^*h=h(2-h)+(2-h)h=2h$. We deduce that $h=1$, which contradicts $ha^*=0$ and $a\neq 0$.

(3.iii) See [3, Th. 3]. $\square$

Let us recall that the elements $a\in\mathcal{R}^\dagger$ such that $aa^\dagger-a^\dagger a=0$ was also studied in [2, Th. 2.1] (the setting of this paper is a C^*-algebra, but the proof of [2, Th. 2.1] works in a ring with involution). Also, further characterizations of the invertibility of $aa^\dagger-a^\dagger a$ were given in [3, Th. 3].

Remark 3.1. The hypothesis "$\mathcal{R}$ is $*$-reducing" in item (2.iii) of the former theorem cannot be removed as the following example shows. Let $\mathcal{R}=\mathbb{Z}/4\mathbb{Z}$ and $a=[1]$. Trivially, $a\mathcal{R}+a^*\mathcal{R}=\mathcal{R}$ and $aa^\dagger+a^\dagger a=[2]$ is not invertible in $\mathcal{R}$ (because $[2][2]=[0]$). Observe that this latter equality implies also that $\mathcal{R}$ is not $*$-reducing.

The following result extends [1, Th. 4].

Theorem 3.3. *Let $\mathcal{R}$ be a ring with involution and $a\in\mathcal{R}^\dagger$, $a\neq 0$. Denote $p=aa^\dagger$ and $q=a^\dagger a$.*

1. (i) *If $\mathcal{R}$ is a $*$-reducing ring, then $pqp=0\Leftrightarrow a^2=0\Leftrightarrow qpq=0$,*

(ii) *$pqp\in\mathcal{R}^{-1}\Leftrightarrow a\in\mathcal{R}^{-1}\Leftrightarrow qpq\in\mathcal{R}^{-1}$,*

(iii) *$pqp=1\Leftrightarrow a\in\mathcal{R}^{-1}\Leftrightarrow qpq=1$,*

(iv) *If $\mathcal{R}$ is a $*$-reducing ring, then pqp is idempotent $\Leftrightarrow pq=qp\Leftrightarrow qpq$ is idempotent.*

2. (i) *$1-pq=0\Leftrightarrow a\in\mathcal{R}^{-1}\Leftrightarrow 1-qp=0$,*

(ii) *$1-pq=1\Leftrightarrow a^2=0\Leftrightarrow 1-qp=1$,*

(iii) *$1-pq\in\mathcal{R}^{-1}\Leftrightarrow p\overline{q}\,p\in(p\mathcal{R}p)^{-1}$.*

(iv) *If $p\overline{q}\,p,\overline{p}\,q\overline{p}\in\mathcal{R}^\dagger$, then $1-pq\in\mathcal{R}^{-1}\Leftrightarrow a\mathcal{R}\cap a^*\mathcal{R}=\{0\}$,*

(v) *If $\mathcal{R}$ is a $*$-reducing ring, then $1-pq$ is idempotent $\Leftrightarrow pq=qp$.*

3. (i) *$pq-qp=1$ can never happen,*

(ii) *$pq-qp\in\mathcal{R}^{-1}\Leftrightarrow a\in\mathcal{R}^{\#}$ and $a\mathcal{R}\oplus a^*\mathcal{R}=\mathcal{R}$,*

(iii) *$pq - qp$ is idempotent* $\Leftrightarrow pq = qp$.

4. (i) *If* $\mathrm{char}(\mathcal{R}) \neq 2$, *then* $pq + qp = 0$ *if and only if* $a^2 = 0$,

(ii) $pq + qp = 1$ *can never happen,*

(iii) *If* $\mathcal{R}$ *is a* $*$*-reducing ring and* $p\overline{q}p, \overline{p}q\overline{p} \in \mathcal{R}^\dagger$, *then* $pq + qp \in \mathcal{R}^{-1}$ *if and only if* $a \in \mathcal{R}^\#$ *and* $a\mathcal{R} + a^*\mathcal{R} = \mathcal{R}$,

(iv) *If* $\mathrm{char}(\mathcal{R}) \neq 2$, *then* $pq + qp$ *is idempotent if and only if* $a^2 = 0$.

5. (i) $p + q - pq = 0$ *can never happen,*

(ii) *If* $\mathcal{R}$ *is a* $*$*-reducing ring and* $p\overline{q}p, \overline{p}q\overline{p} \in \mathcal{R}^\dagger$, *then* $p + q - pq = 1$ *if and only if* $pq = qp$ *and* $a\mathcal{R} + a^*\mathcal{R} = \mathcal{R}$,

(iii) *If* $\mathcal{R}$ *is a* $*$*-reducing ring and* $p\overline{q}p, \overline{p}q\overline{p} \in \mathcal{R}^\dagger$, *then* $p + q - pq \in \mathcal{R}^{-1}$ *if and only if* $a\mathcal{R} + a^*\mathcal{R} = \mathcal{R}$,

(iv) *If* $\mathcal{R}$ *is* $*$*-reducing, then* $p + q - pq$ *is idempotent if and only if* $pq = qp$.

Proof. (1.i): If $a^2 = 0$, then Theorem 3.2 (1.i) yields $pq = qp = 0$.

Since $pqp = (pq)(qp) = (pq)(pq)^*$, then $0 = pqp$ implies $0 = pq$, and Theorem 3.2 (i) leads to $a^2 = 0$. Similarly, since $qpq = (qp)(qp)^*$, then $qpq = 0$ implies $a^2 = 0$.

(1.ii) and (1.iii): If $pqp \in \mathcal{R}^{-1}$, there exists $b \in \mathcal{R}$ such that $pqpb = bpqp = 1$. Now, $a^\dagger = a^\dagger pqpb = a^\dagger qpb$, which implies $1 = pqpb = aa^\dagger qpb = aa^\dagger = p$. Hence, $1 = pqpb = qb$, which by premultiplying by a leads to $a = aqb = ab$, hence $1 = qb = a^\dagger ab = a^\dagger a$. Since $1 = aa^\dagger = a^\dagger a$, then $a \in \mathcal{R}^{-1}$. Similarly, we can prove $qpq \in \mathcal{R}^{-1} \Rightarrow a \in \mathcal{R}^{-1}$. The implications $a \in \mathcal{R}^{-1} \Rightarrow pqp = 1$ and $a \in \mathcal{R}^{-1} \Rightarrow qpq = 1$ are evident.

(1.iv): Assume that pqp is idempotent. Since $pq\overline{p}qp = pq(1-p)qp = pqp - (pqp)^2 = 0$ and $pq\overline{p}qp = (pq\overline{p})(pq\overline{p})^*$, then $pq\overline{p} = 0$, hence $pq = pqp$. By taking $*$ we get $qp = pqp$, and therefore, $pq = qp$. The proof of $(qpq)^2 = qpq \Rightarrow pq = qp$ is similar. The remaining implications are evident.

(2.i) and (2.ii): They follow from Theorem 3.2, items (iii) and (i).

(2.iii): Observe that

$$1 - pq = p + \overline{p} - pqp - pq\overline{p} = p\overline{q}p - pq\overline{p} + \overline{p}. \tag{3.9}$$

If $1 - pq \in \mathcal{R}^{-1}$, then there exists $x \in \mathcal{R}$ such that $(1-pq)x = 1 = x(1-pq)$. Using (3.9) we get $1 = (p\overline{q}p - pq\overline{p} + \overline{p})x$. If the last equality is pre-multiplied by $\overline{p}$ and post-multiplied by p, then one obtains $0 = \overline{p}xp$. Pre-multiplying and post-multiplying by p the equality $1 = (p\overline{q}p - pq\overline{p} + \overline{p})x$ and using $0 = \overline{p}xp$ lead to $p = (p\overline{q}p)(pxp)$. Using $1 = x(1-pq)$ and a similar technique we get $(pxp)(p\overline{q}p) = p$. Hence, $p\overline{q}p \in (p\mathcal{R}p)^{-1}$.

If $p\overline{q}p \in (p\mathcal{R}p)^{-1}$, then there exists $x \in \mathcal{R}$ such that $(p\overline{q}p)(pxp) = (pxp)(p\overline{q}p) = p$. The equalities $(1-pq)(pxp + pxpq\overline{p} + \overline{p}) = 1$ and $(pxp + pxpq\overline{p} + \overline{p})(1-pq) = 1$ are now easy to prove.

(2.iv): By Theorem 3.1 we have $(\overline{p}+\overline{q})(\overline{p}+\overline{q})^\dagger = \overline{p} + p\overline{q}p(p\overline{q}p)^\dagger$. By item (iii) of this theorem and Theorem 3.1 we have $1 - pq \in \mathcal{R}^{-1} \Leftrightarrow \overline{p}+\overline{q} \in \mathcal{R}^{-1}$.

Assume $1 - pq \in \mathcal{R}^{-1}$. If $y \in a\mathcal{R} \cap a^*\mathcal{R}$, there exist $u, v \in \mathcal{R}$ such that $y = au = a^*v$. Now $py = y$ and $qy = y$. Since $p\overline{q}p$ is self-adjoint, then $p\overline{q}p$ commutes with its Moore-Penrose inverse, and $y = py = (p\overline{q}p)^\dagger p\overline{q}py = 0$.

Let $a\mathcal{R} \cap a^*\mathcal{R} = \{0\}$. If $z = p - p\overline{q}p(p\overline{q}p)^\dagger$, then obviously, $z \in a\mathcal{R}$ and $pz = z$. By Lemma 3.1 (ii) we get $zp\overline{q} = 0$, and by taking $*$ and considering that z is self-adjoint, we obtain $\overline{q}pz = 0$, i.e. $qz = z$, which leads $z = a^\dagger az = a^*(aa^*)^\dagger az$. Thus, $z \in a\mathcal{R} \cap a^*\mathcal{R} = \{0\}$.

(2.v): A straightforward computation shows that $1 - pq$ is idempotent if and only if $pqpq = pq$. If $pqpq = pq$, then it is easy to see that $(pqp)(pqp) = pqp$, and by item (1.iv) of this theorem we get $pq = qp$. Reciprocally, if $pq = qp$, evidently we have $pqpq = pq$.

(3.i): Pre-multiplying and post-multiplying $pq - qp = 1$ by p lead to $p = 0$. Thus $0 = pa = a$, which contradicts the hypotheses.

(3.ii): Let us observe that

$$(p + q - 1)(q - p) = pq - qp. \tag{3.10}$$

Assume that $a \in \mathcal{R}^\#$ and $a\mathcal{R} \oplus a^*\mathcal{R} = \mathcal{R}$. By Theorem 2.3 and Theorem 3.2 (3.iii) we have $p + q - 1, p - q \in \mathcal{R}^{-1}$. Expression (3.10) permits assure that $pq - qp \in \mathcal{R}^{-1}$.

Assume that $pq - qp \in \mathcal{R}^{-1}$. From (3.10) there exists $x \in \mathcal{R}$ such that

$$(p + q - 1)(q - p)x = 1 \qquad \text{and} \qquad x(p + q - 1)(q - p) = 1. \tag{3.11}$$

To prove $p + q - 1 \in \mathcal{R}^{-1}$, in view of the first equality of (3.11) it is sufficient to prove $(q-p)x(p+q-1) = 1$. In fact: Since $(p-q)(pq-qp) = pq-pqp-qpq+qp = (pq-qp)(q-p)$, we get $x(p - q) = (q - p)x$. Thus, $(q - p)x(p + q - 1) = x(p - q)(p + q - 1) = x(pq - qp) = 1$, which implies $p + q - 1 \in \mathcal{R}^{-1}$. Observe that this last computation and the second equality of (3.11) prove $q - p \in \mathcal{R}^{-1}$. Since $p + q - 1, p - q \in \mathcal{R}^{-1}$, by Theorem 2.3 and Theorem 3.2 (3.iii) we get $a \in \mathcal{R}^\#$ and $a\mathcal{R} \oplus a^*\mathcal{R} = \mathcal{R}$.

(3.iii): A straightforward computation shows that $pq - qp$ is idempotent if and only if

$$pqpq - pqp - qpq + qpqp = pq - qp. \tag{3.12}$$

If $pq = qp$, then obviously $pq - qp$ is idempotent. If $pq - qp$ is idempotent, then by pre-multiplying and post-multiplying (3.12) by p one gets $pqpqp = pq$ and $pqpqp = 2pqp - qp$, respectively. Therefore, $2pqp = pq + qp$. Again, by pre-multiplying and post-multiplying the last equality by p, we get $pq = qp$.

(4.i): Assume $pq + qp = 0$. By pre- and post-multiplying $pq + qp = 0$ by p, one gets $pq + pqp = 0 = pqp + qp$, hence $pq = qp$. Inserting this last equality into $pq + qp = 0$ and using $2 \in \mathcal{R}^{-1}$ lead to $pq = 0$. Theorem 3.2 (i) allows to deduce $a^2 = 0$. The reciprocal is evident by using again Theorem 3.2 (i).

(4.ii): Assume $pq + qp = 1$. By pre- and post-multiplying $pq + qp = 1$ by p, we have $2pqp = p$ and by pre- and post-multiplying $pq + qp = 1$ by q, we get $2qpq = q$. Now, $pq = p(2qpq) = (2pqp)q = p^2 = p$. Using again $2pqp = p$ leads to $2p = p$, which yields $p = 0$. Thus $a = 0$, which is unfeasible.

(4.iii): By Theorem 2.3 and Theorem 3.2 (2.iii) we have $a \in \mathcal{R}^{\#} \Leftrightarrow p + q - 1 \in \mathcal{R}^{-1}$ and $a\mathcal{R} + a^*\mathcal{R} = \mathcal{R} \Leftrightarrow p + q \in \mathcal{R}^{-1}$. Furthermore, let us observe that $(p + q - 1)(p + q) = pq + qp$. Hence we have proved $\left[a \in \mathcal{R}^{\#} \text{ and } a\mathcal{R} + a^*\mathcal{R} = \mathcal{R}\right] \Rightarrow pq + qp \in \mathcal{R}^{-1}$.

Assume that $pq+qp \in \mathcal{R}^{-1}$ and let us define $x = (p+q)(pq+qp)^{-1}$. Since $(pq+qp)(p+q) = (p+q)(pq+qp)$ we have $(p+q)(pq+qp)^{-1} = (pq+qp)^{-1}(p+q)$. From $(p+q-1)(p+q) = pq+qp$ we get $(p + q - 1)x = 1$. Now

$$x(p + q - 1) = (p + q)(pq + qp)^{-1}(p + q - 1) = (pq + qp)^{-1}(p + q)(p + q - 1) = 1;$$

which yields $p + q - 1 \in \mathcal{R}^{-1}$. If we define $y = (pq + qp)^{-1}(p + q - 1)$, then similarly we can prove $(p + q)y = y(p + q) = 1$.

(4.iv): We shall prove $pq + qp$ is idempotent if and only if $pq + qp = 0$, which in view of item (4.i), will prove this item. Obviously, the implication $pq + qp = 0 \Rightarrow (pq + qp)^2 = pq + qp$ is evident. Let us prove the opposite one: Since $\overline{p}\,(pq + qp)\overline{p} = 0$, $\overline{p}\,(pq + qp)^2\overline{p} = \overline{p}\,qpq\overline{p}$ and the idempotency of $pq + qp$ we get $0 = \overline{p}\,qpq\overline{p} = (\overline{p}\,qp)(\overline{p}\,qp)^*$. Hence $0 = \overline{p}\,qp$, or equivalently, $qp = pqp$. By inverting the roles of p and q we have $pq = qpq$. Now, $(pq+qp)^2 = pqpq + pqp + qpq + qpqp = 2pq + 2qp$, which in view of the idempotency of $pq + qp$, leads to $pq + qp = 0$.

(5.i): If $p + q = pq$, by pre-multiplying by p, we get $p = 0$, which implies $a = 0$.

(5.ii): If $p + q - pq = 1$, by post-multiplying by p, then we get $qp = pqp$, which by taking $*$ leads to $pq = pqp$, therefore $pq = qp$. Also $1 = p(1 - q) + q \in a\mathcal{R} + a^*\mathcal{R}$, which entails $\mathcal{R} = a\mathcal{R} + a^*\mathcal{R}$.

Assume that $pq = qp$ and $\mathcal{R} = a\mathcal{R} + a^*\mathcal{R}$. The last hypothesis, in view of Theorem 3.2 (2.iii) is equivalent to $p + q \in \mathcal{R}^{-1}$. It is easy to see that from $pq = qp$ we can get $(p + q)(p + q - pq - 1) = 0$, which in conjunction with $p + q \in \mathcal{R}^{-1}$ yields $p + q - pq - 1 = 0$.

(5.iii): If $p+q-pq \in \mathcal{R}^{-1}$, there exists $x \in \mathcal{R}$ such that $1 = (p+q-pq)x = p(x-qx)+qx \in a\mathcal{R} + a^*\mathcal{R}$, hence $\mathcal{R} = a\mathcal{R} + a^*\mathcal{R}$.

If $\mathcal{R} = a\mathcal{R} + a^*\mathcal{R}$, then by Theorem 3.2 2. (iii), $p + q \in \mathcal{R}^{-1}$. Now, by (1.2), Theorem 3.1, and by denoting $u = (\overline{p}\,q\overline{p}\,)^{\dagger}$, we get $\overline{p}\,q\overline{p}\,u = u\overline{p}\,q\overline{p} = \overline{p}$ (these two last relations express that $\overline{p}\,q\overline{p}$ is invertible in $\overline{p}\,\mathcal{R}\overline{p}$ and u is the inverse of $\overline{p}\,q\overline{p}$ in such subring), or equivalently, $\overline{p}\,qu = uq\overline{p} = \overline{p}$. Let us remark two simple things: $u \in \overline{p}\,\mathcal{R}\overline{p}$ and $p + q - pq = p + \overline{p}\,q$. Now, it is easy to prove $(p + \overline{p}\,q)(p - uqp + u) = (p - uqp + u)(p + \overline{p}\,q) = 1$.

(5.iv): A straightforward computation shows that $p + q - pq$ is idempotent if and only if $pqpq + qp = qpq + pqp$. Hence, if $pq = qp$, then obviously $p + q - pq$ is idempotent. If $pqpq + qp = qpq + pqp$, by pre- and post-multiplying by $\overline{p}$ one gets $0 = \overline{p}\,qpq\overline{p} = \overline{p}\,qp(\overline{p}\,qp)^*$, hence $0 = \overline{p}\,qp$, or equivalently, $qp = pqp$. By taking $*$ we get $pq = pqp$. Thus, $pq = qp$. $\square$

The following results extends [1, Th. 3].

Theorem 3.4. *Let $\mathcal{R}$ be a unital ring with involution and $a \in \mathcal{R}^\dagger$, $a \neq 0$. Denote $p = aa^\dagger$ and $q = a^\dagger a$.*

1. (i) *pq is a projection if and only if $pq = qp$.*

(ii) *If $\mathrm{char}(\mathcal{R}) \neq 2$, then $p + q$ is a projection if and only if $a^2 = 0$.*

(iii) *If $\mathrm{char}(\mathcal{R}) \neq 2$, then $p - q$ is a projection if and only if $ap = a$.*

2. (i) *$\overline{p}\,q$ is a projection if and only if $pq = qp$.*

(ii) *If $\mathrm{char}(\mathcal{R}) \neq 2$, then $\overline{p} + q$ is a projection if and only if $ap = a$.*

(iii) *$\overline{p} - q$ is a projection if and only $a^2 = 0$.*

3. (i) *$p\overline{q}$ is a projection if and only if $pq = qp$.*

(ii) *If $\mathrm{char}(\mathcal{R}) \neq 2$, then $p + \overline{q}$ is a projection if and only if $a = qa$.*

(iii) *If $\mathrm{char}(\mathcal{R}) \neq 2$, then $p - \overline{q}$ is a projection if and only if $pq = qp$ and $a\mathcal{R} + a^*\mathcal{R} = \mathcal{R}$.*

4. (i) *$\overline{p}\;\overline{q}$ is a projection if and only if $pq = qp$.*

(ii) *If $\mathrm{char}(\mathcal{R}) \neq 2$, then $\overline{p} + \overline{q}$ is a projection if and only if $pq = qp$ and $a\mathcal{R} + a^*\mathcal{R} = \mathcal{R}$.*

(iii) *If $\mathrm{char}(\mathcal{R}) \neq 2$, then $\overline{p} - \overline{q}$ is a projection if and only if $a = qa$.*

Proof. To prove items (i), it is enough to observe that any of the following conditions: $(pq)^* = pq$, $(\overline{p}\,q)^* = \overline{p}\,q$, $(p\overline{q}\,)^* = p\overline{q}\,$, $(\overline{p}\,\overline{q}\,)^* = \overline{p}\,\overline{q}$ is equivalent to $pq = qp$.

(1.ii): It follows from Theorem 3.3 (4.i).

(1.iii): Obviously $p - q$ is a projection if and only if $2q = pq + qp$.

If $2q = pq + qp$, then by pre-multiplying by p one gets $pq = pqp$, which by taking $*$ leads to $pq = qp$. Using again $2q = pq + qp$ gets $q = qp$, which by premultiplying by a leads to $a = ap$.

Assume $ap = a$. If we multiply the last equality by $a^\dagger$ from the left side, we get $qp = q$. Now we use Lemma 1.1 to get $pa^\dagger = pa^*(aa^*)^\dagger = (ap)^*(aa^*)^\dagger = a^*(aa^*)^\dagger = a^\dagger$, which by post-multiplying by a yields $pq = q$. Obviously, we have $2q = pq + qp$.

(2.ii): It follows from $(\overline{p} + q)^2 - (\overline{p} + q) = \overline{p}\,q + q\overline{p} = 2q - pq - qp$ and the proof of (1.iii).

(2.iii): Observe that $(\overline{p} - q)^2 - (\overline{p} - q) = pq + qp$. Hence, Theorem 3.3 (4.i) leads to $\overline{p} - q$ is a projection if and only if $a^2 = 0$.

(3.ii): First observe that $p + \overline{q}$ is a projection if and only if $2p = pq + qp$. If $2p = pq + qp$, by pre-multiplying by q we get $qp = qpq$, which by taking $*$ yields $pq = qpq$. Hence $pq = qp$ which implies that $p = qp$. Thus, $a = qa$.

If $a = qa$, then $p = aa^\dagger = qaa^\dagger = qp$. By taking adjoint of the last equality, we get $pq = qp = p$, hence $2p = pq + qp$.

(3.iii): We have

$$(p-\overline{q}\,)^2-(p-\overline{q}\,)=2-2p-2q+pq+qp. \tag{3.13}$$

If $p-\overline{q}$ is a projection, then by pre- and post-multiplying $2+pq+qp=2p+2q$ by p we obtain $pq=qp$. Also, we have $1=2\cdot 2^{-1}=p(2-q)2^{-1}+q(2-p)2^{-1}\in a\mathcal{R}+a^*\mathcal{R}$. Hence, $\mathcal{R}=a\mathcal{R}+a^*\mathcal{R}$.

Assume that $pq=qp$ and $a\mathcal{R}+a^*\mathcal{R}=\mathcal{R}$. We shall use [6, Cor. 3.8] to prove $p+q\in\mathcal{R}^{-1}$. A simple computation proves $(p+q)(p+q-\frac{3}{2}pq)(p+q)=p+q$, hence $p+q$ is regular. Let $x\in p\mathcal{R}\cap q(1-p)\mathcal{R}$. From $x\in p\mathcal{R}$ we get $px=x$, while from $x\in q(1-p)\mathcal{R}$ and $pq=qp$ we get $px=0$. Therefore $p\mathcal{R}\cap q(1-p)\mathcal{R}=\{0\}$. Now, pick any $y\in p^\circ\cap q^\circ$. Since $y\in\mathcal{R}=a\mathcal{R}+a^*\mathcal{R}$, there exist $b,c\in\mathcal{R}$ such that $y=ab+a^*c$. Combining this last equality with $py=0$ and $pa=a$ leads to

$$0=ab+pa^*c. \tag{3.14}$$

By $y=ab+a^*c$, $qy=0$, and $qa^*=a^*$ we get

$$0=qab+a^*c. \tag{3.15}$$

Thus, (3.14), (3.15), $pq=qp$, and $pa=a$ yield

$$y=ab+a^*c=-pa^*c+a^*c=(1-p)a^*c=(p-1)qab=q(p-1)ab=0,$$

i.e. $p^\circ\cap q^\circ=\{0\}$. From Corollary 3.8 [6] we get $p+q\in\mathcal{R}^{-1}$. Let us remind that we have proved $(p+q)(p+q-\frac{3}{2}pq)(p+q)=p+q$, which entails $(p+q)(p+q-\frac{3}{2}pq)=1$. By doing elementary algebra (let us recall that we can use $pq=qp$) we get $p+q-pq=1$. By (3.13) we obtain that $p-\overline{q}$ is a projection.

(4.ii): A straightforward computation show that $(\overline{p}+\overline{q}\,)^2-(\overline{p}+\overline{q}\,)=2-2p-2q+pq+qp$. Now the proof follows from (3.13) and the proof of (3.iii).

(4.iii): Trivially we have that $\overline{p}-\overline{q}$ is a projection if and only if $2p=pq+qp$. Now, the proof follows from the proof of (3.ii). □

If we assume that $a\in\mathcal{R}^{\#}$, then some conditions of Theorem 3.4 can be written in a simpler form:

Theorem 3.5. *Let $\mathcal{R}$ be a unital ring with involution and $a\in\mathcal{R}^\dagger\cap\mathcal{R}^{\#}$, $a\neq 0$. Denote $p=aa^\dagger$ and $q=a^\dagger a$. Then $ap=a\Leftrightarrow p=q\Leftrightarrow qa=a$.*

Proof. Obviously $p=q$ implies $ap=a$ and $qa=a$.

Assume $ap=a$. As we have shown in the proof of Theorem 3.4 (1.iii), we can deduce $pq=qp=q$. By Theorem 2.2, we have $aa^{\#}=q$, and Corollary 2.1 (i) yields (observe that we use $(2q-1)^2=1$)

$$p+q-1=(aa^{\#}+(aa^{\#})^*-1)^{-1}=(2q-1)^{-1}=2q-1.$$

Thus, $p=q$. The proof of $qa=a\Rightarrow p=q$ is similar. □

Remark 3.2. If the ring is $\mathbb{C}_{n,n}$ and if $F \in \mathbb{C}_{n,n}$ satisfies $F^2F^\dagger = F$ (this equality is the matrix version of $ap = a$), then it is evident that $\mathcal{R}(F) = \mathcal{R}(F^2)$, and this set equality is equivalent to the existence of $F^\#$. On the other hand, if $F \in \mathbb{C}_{n,n}$ satisfies $F^\dagger F^2 = F$, then $\mathcal{N}(F) = \mathcal{N}(F^2)$, and this implies again that $F^\#$ exists. Therefore, Theorem 3.5 proves that $F^2F^\dagger = F \Rightarrow FF^\dagger = F^\dagger F$ and $F^\dagger F^2 = F \Rightarrow FF^\dagger = F^\dagger F$. A matrix F such that $FF^\dagger = F^\dagger F$ is called EP-matrix.

Theorem 3.6. *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}^\dagger$, $a \neq 0$. Denote $p = aa^\dagger$ and $q = a^\dagger a$. Then*

(i) $(p - q)\mathcal{R} = (a\mathcal{R} + a^*\mathcal{R}) \cap (a^\circ + (a^*)^\circ)$.

(ii) $(pq - qp)\mathcal{R} \in [a\mathcal{R} + a^*\mathcal{R}] \cap [a\mathcal{R} + a^\circ] \cap [a^*\mathcal{R} + (a^*)^\circ] \cap [a^\circ + (a^*)^\circ]$.

Proof. It will be useful recall the following formulas easy to prove

$$a^\circ = q^\circ = (1 - q)\mathcal{R}, \qquad (a^*)^\circ = p^\circ = (1 - p)\mathcal{R}, \qquad p\mathcal{R} = a\mathcal{R}, \qquad q\mathcal{R} = a^*\mathcal{R}. \tag{3.16}$$

(i $\subseteq$): Let $x \in (p-q)\mathcal{R}$. There exists $u \in \mathcal{R}$ such that $x = (p-q)u$. Evidently, $x \in a\mathcal{R}+a^*\mathcal{R}$ and $x = (1 - p)(-u) + (1 - q)u \in (1 - p)\mathcal{R} + (1 - q)\mathcal{R}$.

(i $\supseteq$): Let $x \in (a\mathcal{R} + a^*\mathcal{R}) \cap (a^\circ + (a^*)^\circ)$. There exist $u, v \in \mathcal{R}$, $y \in a^\circ$ and $z \in (a^*)^\circ$ such that $x = au + a^*v = y + z$. Since $y \in a^\circ$ we get $qy = 0$. Since $z \in (a^*)^0$ we get $pz = 0$. Since $au - z = y - a^*v$, then $x = aa^\dagger au + a^\dagger aa^*v = p(au - z) - q(y - a^*v) \in (p - q)\mathcal{R}$.

(ii $\subseteq$): Let $x \in \mathcal{R}$. Obviously, $(pq - qp)x \in p\mathcal{R} + q\mathcal{R} = a\mathcal{R} + a^*\mathcal{R}$. Moreover, $(pq - qp)x = p(qx - x) + (1 - q)px \in p\mathcal{R} + (1 - q)\mathcal{R} = a\mathcal{R} + a^\circ$ and $(pq - qp)x = q(x - px) + (1 - p)q(-x) \in q\mathcal{R}+(1-p)\mathcal{R} = a^*\mathcal{R}+(a^*)^\circ$. Finally, by (i), $(pq-qp)x = (1-p)q(-x)+(1-q)px+(p-q)(-x) \in (1 - p)\mathcal{R} + (1 - q)\mathcal{R} + (p - q)\mathcal{R} \subseteq a^\circ + (a^*)^\circ$.

(ii $\supseteq$): Let $x, y, u, v, z, w, s, t \in \mathcal{R}$ such that

$$px + qy = pu + v = qz + w = s + t, \qquad qv = pw = qs = pt = 0. \tag{3.17}$$

We will prove $px + qy \in (pq - qp)\mathcal{R}$. From (3.17) we have $px + pqy = pqz$ and $qpx + qy = qpu$. Hence,

$$px + qy = pqz - pqy + qpu - qpx = pq\theta - qp\psi,$$

where $\theta = qz - qy$ and $\psi = px - pu$. Let us define $\eta = v - s$ and $\mu = t - w$, which by (3.17) we get $q\eta = p\mu = 0$. Furthermore, by (3.17) we have

$$\theta + \eta = qz - qy + v - s = px - w + t - pu = \psi + \mu.$$

All these computations prove $px + qy = pq(\theta + \eta) - qp(\psi + \mu) \in (pq - qp)\mathcal{R}$. □

Remark 3.3. In [1], the authors gave expressions for the range space of several matrices depending on $\mathbf{FF}^\dagger$ and $\mathbf{F}^\dagger\mathbf{F}$. We shall show by examples that some of these relations do not hold in arbitrary rings with an involution. In what follows $\mathcal{R}$ will be a ring with involution, $a \in \mathcal{R}^\dagger$, and $p = aa^\dagger$, $q = a^\dagger a$.

The equality $pq\mathcal{R} = a\mathcal{R} \cap (a^\circ + (a^*)^\circ)$ is not true in general. Let $\mathcal{R}$ be commutative and take $a \in \mathcal{R}^{-1}$. Obviously , $a^\circ = (a^*)^\circ = \{0\}$ and $p = q = 1$.

The equalities $(p+q)\mathcal{R} = a\mathcal{R}+a^*\mathcal{R}$ and $(pq+qp)\mathcal{R} = (a\mathcal{R}+a^*\mathcal{R})+(a\mathcal{R}+a^\circ)+(a^*\mathcal{R}+(a^*)^\circ)$ do not hold in an arbitrary ring. To see this, it is sufficient take $\mathcal{R} = \mathbb{Z}$ and $a = 1$.

In next result we shall extend some equalities of Theorem 5 of [1] involving kernel ideals. We shall introduce the notion of positivity in rings with involution (this notion is borrowed from the C^*-algebra theory). Let $\mathcal{R}$ be a ring with involution. An element $x \in \mathcal{R}$ is said to be *positive* (denoted by $0 \leq x$) if exists $k \in \mathcal{R}$ such that $x = kk^*$. We write $x \leq y$ if and only if $0 \leq y - x$. In other words,

$$x \leq y \qquad \Longleftrightarrow \qquad \exists k \in \mathcal{R} : y - x = kk^*. \tag{3.18}$$

It is evident that the usual order in $\mathbb{Z}$ coincides with (3.18) and this order is antisymmetric. Also, by Corollary 5.4 of [10], it follows that the relation (3.18) defined in any C^*-algebra is antisymmetric. But in general this is not the case.

Theorem 3.7. *Let $\mathcal{R}$ be a ring with involution, $a \in \mathcal{R}^\dagger$ and $p = aa^\dagger$, $q = a^\dagger a$. Then*

(i) $(p - q)^\circ = [a\mathcal{R} \cap a^*\mathcal{R}] + [a^\circ \cap (a^*)^\circ]$.

(ii) *If $\mathcal{R}$ is *-reducing and the relation (3.18) is antisymmetric, then $(p + q)^\circ = a^\circ \cap (a^*)^\circ$.*

(iii) *If $\mathcal{R}$ is *-reducing and the relation (3.18) is antisymmetric, then $(pq + qp)^\circ = [a\mathcal{R} \cap a^\circ] + [a^*\mathcal{R} \cap (a^*)^\circ] + [a^\circ \cap (a^*)^\circ]$.*

(iv) $(pq - qp)^\circ = [a\mathcal{R} \cap a^*\mathcal{R}] + [a\mathcal{R} \cap a^\circ] + [a^*\mathcal{R} \cap (a^*)^\circ] + [a^\circ \cap (a^*)^\circ]$.

Proof. We will use (3.16).

(i $\subseteq$): Let $x \in (p-q)^\circ$, i.e., $px = qx$. We decompose x as $x = px+(1-p)x$. Now, observe that $px \in p\mathcal{R}$, $px = qx \in q\mathcal{R}$, $(1 - p)x \in (1 - p)\mathcal{R}$, and $(1 - p)x = (1 - q)x \in (1 - q)\mathcal{R}$.

(i $\supseteq$): Let $x \in a\mathcal{R} \cap a^*\mathcal{R}$ and $y \in a^\circ \cap (a^*)^\circ$. We have to prove that $x + y \in (p - q)^\circ$. We have $x = au = a^*v$ for some $u, v \in \mathcal{R}$, and $ay = a^*y = 0$. Now $px = pau = au = x$; $qx = qa^*v = a^*v = x$, $py = aa^\dagger y = 0$, and $qy = 0$. These calculations prove $(p-q)(x+y) = 0$.

(ii): Obviously, $a^\circ \cap (a^*)^\circ \subseteq (p + q)^\circ$. To prove the opposite inclusion, pick $x \in \mathcal{R}$ such that $px+qx = 0$. We get $x^*px+x^*qx = 0$. Furthermore, x^*px and x^*qx are positive elements because $x^*px = (px)^*(px)$ and $x^*qx = (qx)^*(qx)$. Hence $0 \leq x^*px$ and $x^*px \leq x^*px + x^*qx = 0$. Since the relation (3.18) is antisymmetric, then $x^*px = 0$. Hence $(px)^*(px) = 0$. Since $\mathcal{R}$ is *-reducing we get $px = 0$. Analogously, $qx = 0$ holds. Therefore, $x \in p^\circ \cap q^\circ$.

(iii $\subseteq$): Let $x \in (pq+qp)^\circ$. Notice that $(p+q)(p+q-1)x = (pq+qp)x = 0$, hence $(p+q-1)x \in (p+q)^\circ$. By item (ii) we get $(p+q-1)x \in a^\circ \cap (a^*)^\circ$, and thus, $0 = a(p+q-1)x = apx$ and $0 = a^*(p+q-1)x = a^*qx$. The decomposition $x = px + qx + (1-p-q)x$ permits prove the required inclusion because $px \in a\mathcal{R} \cap a^\circ$; $qx \in a^* \cap (a^*)^\circ$ and $(1-p-q)x \in a^\circ \cap (a^*)^\circ$.

(iii $\supseteq$): Let $x = au + a^*v + w$, where $u, v, w \in \mathcal{R}$ satisfy $a(au) = 0$, $a^*(a^*v) = 0$, and $aw = a^*w = 0$. Then $pqau = 0$; $qpau = 0$; $pqa^*v = pa^*v = a(a^*a)^\dagger (a^*)^2 v = 0$; $qpa^*v = qaa^\dagger a^* v = qa(a^*a)^\dagger (a^*)^2 v = 0$; $pqw = 0$ and $qpw = 0$. All these computations prove $(pq+qp)x = 0$.

(iv $\subseteq$): Let $x \in (pq-qp)^\circ$ and denote $u = pqx = qpx$. We shall see that the decomposition $x = u + (px-u) + (qx-u) + (x+u-px-qx)$ permits prove the inclusion. Observe that $u = pqx = qpx \in a\mathcal{R} \cap a^*\mathcal{R}$. In addition, $px - u \in a\mathcal{R} \cap a^\circ$ since

$$px - u = p(x-qx) \in p\mathcal{R}, \qquad a(px-u) = apx - aqpx = apx - apx = 0.$$

Similarly, $qx - u \in a^*\mathcal{R} \cap (a^*)^\circ$ because

$$qx - u = q(x-px) \in q\mathcal{R}, \qquad a^*(qx-u) = a^*qx - a^*pqu = a^*qx - a^*qx = 0.$$

Finally, $x + u - px - qx \in a^\circ \cap (a^*)^\circ$ because $x + u - px - qx = (1-q)x + (u-px) \in a^\circ$ and $x + u - px - qx = (1-p)x + (u-qx) \in (a^*)^\circ$.

(iv $\supseteq$): By (i), it is sufficient to prove $[a\mathcal{R} \cap a^\circ] + [a^*\mathcal{R} \cap (a^*)^\circ] \subset (pq-qp)^\circ$. Let $x \in \mathcal{R}$ satisfy $ax = 0$ and $x = au$ for some $u \in \mathcal{R}$. Now, $pqx = pa^\dagger ax = 0$ and $qpx = qpau = qau = qx = 0$, so $a\mathcal{R} \cap a^\circ \subset (pq-qp)^\circ$.

Let $y \in \mathcal{R}$ satisfy $a^*y = 0$ and let $y = a^*v$ for some $v \in \mathcal{R}$. Now, $pqy = pqa^*v = pa^*v = py = a(a^*a)^\dagger a^* y = 0$ and $y \in (a^*)^\circ = p^\circ \subseteq (qp)^\circ$. Thus, $a^*\mathcal{R} \cap (a^*)^\circ \subset (pq-qp)^\circ$. □

If we do not assume that (3.18) is antisymmetric, we need impose another condition in order that items (ii) and (iii) of Theorem 3.7 hold.

Theorem 3.8. *Let $\mathcal{R}$ be a ring with involution and $*$-reducing, $a \in \mathcal{R}^\dagger$ and $p = aa^\dagger$, $q = a^\dagger a$. If $p\overline{q}\,p, \overline{p}\,q\overline{p} \in \mathcal{R}^\dagger$, then*

(i) $(p+q)^\circ = a^\circ \cap (a^*)^\circ$.

(ii) $(pq+qp)^\circ = [a\mathcal{R} \cap a^\circ] + [a^*\mathcal{R} \cap (a^*)^\circ] + [a^\circ \cap (a^*)^\circ]$.

Proof. By (3.16), to prove (i), it is sufficient to prove $(p+q)^\circ = p^\circ \cap q^\circ$. Since $p^\circ \cap q^\circ \subseteq (p+q)^\circ$ is evident, we will only prove the opposite inclusion. Pick $x \in (p+q)^\circ$. By Theorem 3.1 we get $px + \overline{p}\,q\overline{p}\,(\overline{p}\,q\overline{p}\,)^\dagger x = 0$, which by premultiplying by p leads to $px = 0$. By inverting the roles of p, q we get $qx = 0$. The proof of (ii) is the same as the corresponding item in Theorem 3.7. □

Remark 3.4. The equalities

$$(p+q)^\circ = a^\circ \cap (a^*)^\circ \qquad \text{and} \qquad (pq+qp)^\circ \subset (a\mathcal{R} \cap a^\circ) + (a^*\mathcal{R} \cap (a^*)^\circ) + (a^\circ \cap (a^*)^\circ)$$

do not hold in an arbitrary ring: take $\mathcal{R} = \mathbb{Z}/4\mathbb{Z}$ and $a = [1]$. Evidently, $p = q = [1]$, and $[2] \in (p+q)^\circ = (pq+qp)^\circ$, and however $a^\circ = (a^*)^\circ = \{[0]\}$.

Remark 3.5. In [1] the authors gave an expression for the null space of $(\mathbf{F}\mathbf{F}^\dagger)(\mathbf{F}^\dagger\mathbf{F})$ in terms of the range space and the null space of $\mathbf{F}$ and $\mathbf{F}^*$, when $\mathbf{F}$ is a square complex matrix. We shall see that the corresponding ring version does not hold. More precisely, the relation $(pq)^\circ = a^*\mathcal{R} + [a^\circ \cap (a^*)^\circ]$ is not generally true when $a \in \mathcal{R}^\dagger$, $p = aa^\dagger$, $q = a^\dagger a$, and $\mathcal{R}$ is a ring with an involution. To see this, it is enough to take $\mathcal{R} = \mathbb{Z}$ and $a = 1$.

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