Using the GSVD and the lifting technique to find $\{P, k+1\}$ reflexive and anti-reflexive solutions of $AXB = C$

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Abstract

The Generalized Singular Value Decomposition (GSVD) and the lifting technique combined with the Kronecker product are exploited to find reflexive and anti-reflexive (with respect to a generalized $\{k+1\}$-reflection matrix P) solutions of the matrix equation $AXB = C$. An algorithm is presented for both methods. Its computational cost is studied and several numerical examples are analyzed.

Key words: Matrix equation, reflexive solution, potent matrix
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1 Introduction

A matrix $P \in \mathbb{C}^{n\times n}$ is called a generalized reflection matrix if $P^2 = I$ and $P^* = P$, being P^* the conjugate transpose of the matrix P, I the identity matrix of suitable size, and $\mathbb{C}^{n\times n}$ the set of all complex matrices of size $n \times n$. A matrix $X \in \mathbb{C}^{n\times n}$ is called reflexive (anti-reflexive) with respect to a generalized reflection matrix $P \in \mathbb{C}^{n\times n}$ if $PXP = X$ $(PXP = -X)$.

Centrosymmetric and centroskew matrices A (that satisfy $A = J_nAJ_n$ or $A = -J_nAJ_n$, respectively, where J_n denote the $n\times n$ backward identity matrix having the elements 1 along the southwest-northeast diagonal and with the remaining elements being zeros) have been widely discussed. This kind of matrices has important applications in engineering problems, information theory, linear system theory, linear estimation theory, numerical analysis theory, etc.

[1,13]. Peng and Hu studied the existence of reflexive and anti-reflexive solutions to the matrix equation $AX = B$ over the complex field with respect to a generalized reflection matrix P giving its solutions, respectively [11]. Recently, Cvetković-iliić investigated the existence of reflexive solutions of the same matrix equation given necessary and sufficient conditions as a first approach to find more operative conditions [4]. The matrix equation $AXB = C$ in X has been studied in different ways. Some authors have searched the general solution of this problem while others have considered some kind of constraints on the solution, as for example the symmetry, the positive definiteness, etc. This kind of matrices is widely used in engineering and scientific computation, in control theory, etc. Specifically, these matrices are used to solve physical problems related to the altitude estimation of a level network, electric networks and also structural analysis of trusses [7,10,12,15,16].

In this work we analyze the matrix equation $AXB = C$, looking for solutions X that satisfy the constraint $PXP = X$ for a given matrix P under certain conditions. Specifically, we will assume that $P \in \mathbb{C}^{n\times n}$ is a Hermitian and $\{k+1\}$-potent matrix (that is, $P^{k+1} = P = P^*$). In this case, P is said to be a generalized $\{k+1\}$-reflection. Moreover, a matrix $X \in \mathbb{C}^{n\times n}$ is called $\{P, k+1\}$ reflexive with respect to the Hermitian and $\{k+1\}$-potent matrix P if $PXP = X$ and $\{P, k+1\}$ anti-reflexive with respect to the Hermitian and $\{k+1\}$-potent matrix P if $PXP = -X$. Our main goal is to reduce the study to only two cases: $P^2 = P$ and $P^3 = P$. Actually, the reduction of the general case $P^{k+1} = P$ to those cases is crucial in our work because it simplifies substantially the problem. Moreover, clearly our results generalize those given in [12]. On the other hand, there is a relation between $\{k+1\}$-potent matrices and group inverses, that is $P \in \mathbb{C}^{n\times n}$ is $\{k+1\}$-potent if and only if $P^{\#} = P^{k-1}$ for $k \geq 2$. This kind of matrices has been widely used in many topics such as Markov chains, iterative methods, control theory, etc. [5,6,9,14].

Some notation will be used throughout this paper. For a given matrix M, we will denote by $\mathrm{vec}(M)$ the lifting form of M, that is, the result of writing M as a column vector formed by 'stacking' the columns of M into one long column vector [8]. For a given column vector x of length $r \cdot s$, the notation $\mathrm{devec}(x, r, s)$ returns the matrix of size $r \times s$ which first column is defined by the r first elements of x, the second one by the r following elements, and so on until the s-th column. When M is a square matrix, we will denote by $\sigma(M)$ the spectrum of M.

It is known that the generalized singular value decomposition (GSVD) of a pair of matrices $\{M, N\}$, with $M \in \mathbb{C}^{m\times n}$ and $N \in \mathbb{C}^{m\times p}$ matrices having the same number of rows, is given by

$$
\begin{gathered}
M = W\Sigma_M U_M^* \qquad \text{and} \qquad N = W\Sigma_N V_N^* \qquad \text{with} \\
\Sigma_M = \begin{bmatrix} I_M & & \\ & D_M & \\ & & O_M \\ & O & \end{bmatrix}, \qquad \Sigma_N = \begin{bmatrix} O_N & & \\ & D_N & \\ & & I_N \\ & O & \end{bmatrix},
\end{gathered}
\tag{1}
$$

$W \in \mathbb{C}^{m\times m}$ a nonsingular matrix, $U_M \in \mathbb{C}^{n\times n}$ and $V_N \in \mathbb{C}^{p\times p}$ unitary matrices, $D_M, D_N \in \mathbb{C}^{s\times s}$ matrices containing the strictly positive singular values of M and N, respectively, $I_M \in \mathbb{C}^{k\times k}$, $O_M \in \mathbb{C}^{(n-k-s)\times(n-k-s)}$, $I_N \in \mathbb{C}^{(t-k-s)\times(t-k-s)}$, and $O_N \in \mathbb{C}^{k\times(p+k-t)}$ being $t = \operatorname{rank}([\,M\ N\,])$, $k = t - \operatorname{rank}(N)$, and $s = \operatorname{rank}(M) + \operatorname{rank}(N) - t$ [14].

Next, we quote some known results for further references.

Theorem 1 (Theorem 2.1 [3]) *Let $P \in \mathbb{C}^{n\times n}$. Then, the following statements are equivalent:*

1. *P is $\{k+1\}$-potent.*
2. *P is diagonalizable and $\sigma(P) \subseteq \{0\} \cup \Omega_k$, where Ω_k represents the set of all the roots of the unity of order k.*

The following lemma summarizes some useful properties of the Kronecker product, denoted by $\otimes$, and the ones related to the lifting notation (see [8], p. 412).

Lemma 1 *Let $A, C \in \mathbb{C}^{m\times n}$, $B, D \in \mathbb{C}^{n\times l}$, $E \in \mathbb{C}^{p\times q}$, and $X, P \in \mathbb{C}^{n\times n}$. Then*

(a) $\operatorname{vec}(AXB) = (B^* \otimes A)\operatorname{vec}(X)$.
(b) $\operatorname{vec}(A + C) = \operatorname{vec}(A) + \operatorname{vec}(C)$.
(c) $\sigma(P \otimes P) = \{\lambda\mu : \lambda, \mu \in \sigma(P)\}$.
(d) $(A \otimes C)(B \otimes D) = AB \otimes CD$.
(e) $(A \otimes E)^* = A^* \otimes E^*$.

The paper is organized as follows. In Section 2 the problem is stated and some first properties on the solutions X and on the given matrix P are established. Moreover, for the general case $k \geq 2$ it is proved that it is enough to study only the cases $k = 1$ and $k = 2$. In Section 3 and Section 4 the $\{P, 2\}$ reflexive solutions and the $\{P, 3\}$ reflexive solutions are found by using the SVD and the GSVD, respectively, and the lifting technique in both cases. The anti-reflexive solutions are given directly without proofs after the results corresponding to the reflexive ones. Finally, in Section 5, an algorithm to systematize the theoretical procedure developed is presented as well as its computational cost in both cases. Some examples are given to illustrate the results and that the methods work numerically.

2 Statement of the problem and a first approach

For the given matrices $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$, and $P \in \mathbb{C}^{n\times n}$ satisfying $P^{k+1} = P$, for some $k \in \mathbb{N}$, and $P^* = P$, the main purpose of this paper is to solve the equation

$$AXB = C \tag{2}$$

under the condition

$$PXP = X \tag{3}$$

that is, to find $\{P, k+1\}$ reflexive solutions with respect to the Hermitian and $\{k+1\}$-potent matrix P of the matrix equation $AXB = C$.

The anti-reflexive case corresponds to the problem $AXB = C$ with

$$PXP = -X \tag{4}$$

and admits a similar treatment.

From Theorem 1 the following result can be stated.

Lemma 2 *Let $P \in \mathbb{C}^{n\times n}$ a Hermitian matrix. Then P is a $\{k+1\}$-potent matrix if and only if P is idempotent when k is odd or tripotent when k is even. Consequently, $\sigma(P)$ is included in $\{1, 0\}$ or $\{1, -1, 0\}$, respectively.*

Proof. The condition $P^* = P$ assures that the eigenvalues of P are real numbers and P is a unitarily diagonalizable matrix. Moreover, from Theorem 1, the property $P^{k+1} = P$ implies that $\sigma(P) \subseteq \{0\} \cup \Omega_k$. Then, the spectrum $\sigma(P)$ is included in the set $\{1, -1, 0\}$ and there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that the matrix P can be written in the form

$$P = U \begin{bmatrix} I & & \\ & -I & \\ & & O \end{bmatrix} U^*,$$

where some of the diagonal blocks may be absent. Thus, it is easy to see that $P^2 = P$ (when the block $-I$ is absent and k is odd) or $P^3 = P$ (when the block $-I$ is not absent and k is even). The converse is evident. □

Remark 1 *Note that when P is a nonsingular matrix, the condition $P^3 = P$ is equivalent to the simpler condition $P^2 = I$ (studied in [12]) and the condition $P^2 = P$ is equivalent to the trivial case $P = I$.*

One way to solve the equation (2) under the condition (3) is by means of the combination of the Kronecker product with the lifting technique. In order to state a lifting form of the equations $AXB = C$ and $PXP = X$ we will use the notation $\text{vec}(M)$. In fact, both equations can be rewritten in the equivalent form

$$\text{vec}(AXB) = \text{vec}(C), \qquad \text{vec}(PXP) = \text{vec}(X). \tag{5}$$

Then, using the Lemma 1 we get

$$(B^* \otimes A)\text{vec}(X) = \text{vec}(C), \qquad (P^* \otimes P)\text{vec}(X) = \text{vec}(X), \tag{6}$$

which are two linear equations in the unknown $\text{vec}(X)$.

From (6), some additional properties of P can be established.

Proposition 1 *Let $P \in \mathbb{C}^{n\times n}$ a Hermitian matrix (so, there exist a diagonal matrix $D \in \mathbb{C}^{n\times n}$ and a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $P = UDU^*$) and consider the equation (3) in X. The following properties of P and X hold:*

(a) The eigenvectors of the matrix $P \otimes P$ are the columns of the matrix $U \otimes U$.
(b) If P is a $\{k+1\}$-potent matrix, then $\sigma(P \otimes P) \subseteq \{1, 0\}$ when k is odd and $\sigma(P \otimes P) \subseteq \{1, -1, 0\}$ when k is even.
(c) If $1 \notin \sigma(P \otimes P)$, then the equation (3) has not nontrivial solution.
(d) If $1 \in \sigma(P \otimes P)$, then 1 or -1 belong to $\sigma(P)$ and the solution of the equation (3) verifies that $\text{vec}(X)$ is an eigenvector of the matrix $P \otimes P$ associated with the eigenvalue 1.

Proof. From $P = UDU^*$ and Lemma 1 we get $P \otimes P = (U \otimes U)(D \otimes D)(U \otimes U)^*$. So, the statement (a) holds.

The hypothesis that P is a Hermitian and $\{k+1\}$-potent matrix gives $\sigma(P) \subseteq \{1, 0\}$ when k is odd and $\sigma(P) \subseteq \{1, -1, 0\}$ when k is even (see Lemma 2). Using again the Lemma 1, the statement (b) follows directly.

The items (c) and (d) follow as a consequence of applying the lifting form to the equation (3), which gives $(I - P \otimes P)\text{vec}(X) = O$. $\square$

Remark 2 *If in the previous proposition we consider the equation (4) instead of the equation (3), then items (a) and (b) remain invariant while items (c) and (d) change as follows:*

(c') If $-1 \notin \sigma(P \otimes P)$, then the equation (4) has only the trivial solution.
(d') If $-1 \in \sigma(P \otimes P)$, then 1 and -1 belong to $\sigma(P)$ and the solution of the

equation (4) verifies that $\mathrm{vec}(X)$ *is an eigenvector of the matrix* $P \otimes P$ *associated with the eigenvalue* -1.

From Lemma 2, the whole analysis of the stated problem under the condition $P^{k+1} = P$ for any $k \in \mathbb{N}$ can be reduced to the study of the problem of finding X such that $AXB = C$ considering only two cases: $P^2 = P$ and $P^3 = P$. Next, we are going to solve these two cases in order to find $\{P, 2\}$ reflexive solutions and $\{P, 3\}$ reflexive solutions, respectively. The anti-reflexive solutions will be also considered.

3 $\{P, 2\}$ reflexive solutions

In this section we look for $\{P, 2\}$ reflexive solutions of the problem (2), that is, to find matrices $X \in \mathbb{C}^{n\times n}$ such that $AXB = C$ and $X = PXP$ being $P \in \mathbb{C}^{n\times n}$ an idempotent and Hermitian matrix.

From Theorem 1 it is clear that if P is an idempotent matrix then $\sigma(P) \subseteq \{0, 1\}$. Moreover, as P is Hermitian, P is unitarily similar to a diagonal matrix, that is, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$P = UDU^* \qquad \text{with} \qquad D = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \tag{7}$$

where $r = \mathrm{rank}(P)$.

Therefore, the relation $X = PXP$ can be written as $X = UDU^*XUDU^*$. So, we can construct the matrix $\tilde{X} = U^*XU$, which verifies $\tilde{X} = D\tilde{X}D$. By splitting the matrix $\tilde{X}$ into appropriate size blocks X_{ij}, $1 \le i, j \le 2$, according to the matrix blocks in D, the last equality becomes

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

and after making operations we get

$$\tilde{X} = \begin{bmatrix} X_{11} & O \\ O & O \end{bmatrix}. \tag{8}$$

Remark 3 *Notice that some blocks of* $\tilde{X}$ *in (8) may be absent because some diagonal blocks of* D *may not be in the diagonal blocks of* P. *This occurs when the spectrum of* P *is a proper subset of* $\{0, 1\}$. *In particular, if* X_{11} *is absent* *then the matrix X is null because $P = O$. The case $\sigma(P) = \{1\}$ is similar to the case $\sigma(P) = \{0, 1\}$, which is analyzed below.*

By taking into account that $X = U\tilde{X}U^*$, the matrix equation $AXB = C$ becomes

$$\tilde{A}\tilde{X}\tilde{B} = C \qquad \text{where} \qquad \tilde{A} = AU \qquad \text{and} \qquad \tilde{B} = U^*B.$$

By splitting into blocks the matrices $\tilde{A}$ and $\tilde{B}$ we have:

$$\tilde{A} = \begin{bmatrix} A_1^* & A_2^* \end{bmatrix} \qquad \text{and} \qquad \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{9}$$

and by substituting into the matrix equation $\tilde{A}\tilde{X}\tilde{B} = C$ we get

$$\begin{bmatrix} A_1^* & A_2^* \end{bmatrix} \begin{bmatrix} X_{11} & O \\ O & O \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = C,$$

that is

$$A_1^* X_{11} B_1 = C. \tag{10}$$

Remark 4 *Notice that in the anti-reflexive case, a similar reasoning leads to the trivial solution, which implies that the equation $AXB = C$ will have no $\{P, 2\}$ anti-reflexive solutions if $C \neq O$.*

3.1 Using the Singular Value Decomposition (SVD)

By applying the singular value decomposition to the matrices $A_1^* \in \mathbb{C}^{m\times r}$ and $B_1^* \in \mathbb{C}^{l\times r}$ we get the form of the solution X. In fact,

$$A_1^* = W_A \Sigma_{1A} U_A^*, \qquad B_1^* = W_B \Sigma_{1B} U_B^*, \tag{11}$$

where $W_A \in \mathbb{C}^{m\times m}$, $W_B \in \mathbb{C}^{l\times l}$, $U_A \in \mathbb{C}^{r\times r}$, and $U_B \in \mathbb{C}^{r\times r}$ are unitary matrices and

$$\Sigma_{1A} = \begin{bmatrix} D_{1A} & O \\ O & O \end{bmatrix}, \qquad \Sigma_{1B} = \begin{bmatrix} D_{1B} & O \\ O & O \end{bmatrix}, \tag{12}$$

with $D_{1A} \in \mathbb{C}^{a\times a}$ and $D_{1B} \in \mathbb{C}^{b\times b}$ nonsingular diagonal matrices, being $a = \mathrm{rank}(A_1)$ and $b = \mathrm{rank}(B_1)$. Then, the equation (10) becomes

$$\Sigma_{1A}(U_A^* X_{11} U_B)\Sigma_{1B}^* = W_A^{-1} C W_B^{-*}. \tag{13}$$

By splitting into the following blocks

$$U_A^* X_{11} U_B = \begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} \\ \bar{X}_{21} & \bar{X}_{22} \end{bmatrix}, \qquad W_A^{-1} C W_B^{-*} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \tag{14}$$

the equation (13) holds if and only if the blocks C_{12}, C_{21}, and C_{22} are null and $\bar{X}_{11} = D_{1A}^{-1} C_{11} D_{1B}^{-*}$.

We obtain the following result as a summary of this reasoning.

Theorem 2 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$. For a Hermitian and idempotent matrix $P \in \mathbb{C}^{n\times n}$, the matrix equations*

$$AXB = C \qquad \textit{and} \qquad PXP = X$$

have solution $X \in \mathbb{C}^{n\times n}$ if and only if the matrix $W_A^{-1} C W_B^{-*}$ has the following form

$$W_A^{-1} C W_B^{-*} = \begin{bmatrix} C_{11} & O \\ O & O \end{bmatrix}$$

where W_A and W_B are the unitary matrices appearing in the singular value decomposition of the matrices A_1^* and B_1^*, respectively, and C_{11} an arbitrary matrix of size $a \times b$, being $a = \operatorname{rank}(A_1)$ and $b = \operatorname{rank}(B_1)$.

In this case the general solution can be expressed as

$$X = U \begin{bmatrix} U_A \begin{bmatrix} D_{1A}^{-1} C_{11} D_{1B}^{-*} & \bar{X}_{12} \\ \bar{X}_{21} & \bar{X}_{22} \end{bmatrix} U_B^* & O \\ O & O \end{bmatrix} U^*. \tag{15}$$

3.2 Applying the lifting technique

By applying the lifting technique to the equation (10) we get

$$\operatorname{vec}(A_1^* X_{11} B_1) = \operatorname{vec}(C)$$

and the properties given in Lemma 1 allow to write

$$(B_1^* \otimes A_1^*)\operatorname{vec}(X_{11}) = \operatorname{vec}(C). \tag{16}$$

Then, by using generalized inverses (see [2]), we have the following result.

Theorem 3 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$. For a given Hermitian and idempotent matrix $P \in \mathbb{C}^{n\times n}$, the matrix equations*

$$AXB = C, \qquad PXP = X$$

have solution if and only if any of the following statements holds:

(a) $\mathrm{vec}(C) \in \mathcal{R}(B_1^* \otimes A_1^*)$*, where $\mathcal{R}(\cdot)$ denotes the range of $(\cdot)$, and A_1 and B_1 are given in (9).*
(b) there exists a $\{1\}$-inverse M^- of the matrix $M = B_1^* \otimes A_1^*$ such that $\mathrm{vec}(C) \in \mathcal{N}(I - MM^-)$*, where $\mathcal{N}(\cdot)$ denotes the null space of $(\cdot)$.*

Then, the general solution is given by

$$X = U \begin{bmatrix} \mathrm{devec}(\mathrm{vec}(X_{11}), r, r) & O \\ O & O \end{bmatrix} U^* \tag{17}$$

where $r = \mathrm{rank}(P)$ *and* $\mathrm{vec}(X_{11})$ *is obtained by solving (16). In the case (b),*

$$\mathrm{vec}(X_{11}) = M^-\mathrm{vec}(C) + Y - M^-MY$$

with Y an arbitrary matrix.

4 $\{P, 3\}$ reflexive solutions

In this section we look for $\{P, 3\}$ reflexive solutions $X \in \mathbb{C}^{n\times n}$ with respect to the Hermitian and tripotent matrix $P \in \mathbb{C}^{n\times n}$ of the matrix equation $AXB = C$.

From Theorem 1 it is clear that if P is a tripotent matrix then $\sigma(P) \subseteq \{0, 1, -1\}$. Moreover, as P is Hermitian, P is diagonalizable by a unitary matrix, that is, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$P = UDU^* \qquad \text{with} \qquad D = \begin{bmatrix} I_\alpha & O & O \\ O & -I_\beta & O \\ O & O & O \end{bmatrix} \tag{18}$$

being $\alpha + \beta = \mathrm{rank}(P)$.

Since $X = PXP$, we get $X = UDU^*XUDU^*$. Premultiplying by U^* and postmultiplying by U we construct the matrix $\tilde{X} = U^*XU$ that satisfies $\tilde{X} =$

$D\tilde{X}D$. By splitting the matrix $\tilde{X}$ into appropriate size blocks X_{ij}, $1 \leq i, j \leq 3$, according to the matrix blocks of D, the last equality becomes

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} = \begin{bmatrix} I_\alpha & O & O \\ O & -I_\beta & O \\ O & O & O \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \begin{bmatrix} I_\alpha & O & O \\ O & -I_\beta & O \\ O & O & O \end{bmatrix}$$

and by making operations we get

$$\tilde{X} = \begin{bmatrix} X_{11} & O & O \\ O & X_{22} & O \\ O & O & O \end{bmatrix}. \tag{19}$$

Remark 5 *Notice that some blocks of $\tilde{X}$ in (19) may be absent because some diagonal blocks of D may be absent. This happens when $\sigma(P)$ is a proper subset of $\{0, 1, -1\}$.*

From (6), (18), and (19) a first result on tripotent matrices can be stated.

Theorem 4 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$. If $P \in \mathbb{C}^{n\times n}$ is a Hermitian and tripotent matrix, then the solution of the matrix equation system*

$$AXB = C, \qquad PXP = X \tag{20}$$

is given by solving the linear system

$$(B^* \otimes A)(U \otimes U)\mathrm{vec}(\tilde{X}) = \mathrm{vec}(C)$$

where U and $\tilde{X}$ have been introduced in (18) and (19), respectively.

Proof. Writing the matrix P in the form (18), the equivalence between the system (20) and the equation $\tilde{A}\tilde{X}\tilde{B} = C$ with $\tilde{X}$ in the form (19), where $\tilde{A} = AU$ and $\tilde{B} = U^*B$ is obtained. By applying the lifting technique to the last equation we get $\mathrm{vec}(\tilde{A}\tilde{X}\tilde{B}) = \mathrm{vec}(C)$. By using Lemma 1, $\tilde{A} = AU$ and $\tilde{B} = U^*B$, we arrive to the linear system $(B^* \otimes A)(U \otimes U)\mathrm{vec}(\tilde{X}) = \mathrm{vec}(C)$. □

On the other hand, by taking into account that $X = U\tilde{X}U^*$, the matrix equation $AXB = C$ becomes

$$\tilde{A}\tilde{X}\tilde{B} = C \qquad \text{where} \qquad \tilde{A} = AU \qquad \text{and} \qquad \tilde{B} = U^*B.$$

By partitioning the matrices $\tilde{A}$ and $\tilde{B}$ we have:

$$\tilde{A} = \begin{bmatrix} A_1^* \ A_2^* \ A_3^* \end{bmatrix} \qquad \text{and} \qquad \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \tag{21}$$

and by substituting into the matrix equation $\tilde{A}\tilde{X}\tilde{B} = C$ we obtain

$$\begin{bmatrix} A_1^* \ A_2^* \ A_3^* \end{bmatrix} \begin{bmatrix} X_{11} & O & O \\ O & X_{22} & O \\ O & O & O \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = C.$$

The corresponding block products lead to the reduced expression

$$A_1^* X_{11} B_1 + A_2^* X_{22} B_2 = C. \tag{22}$$

Remark 6 *By a similar treatment for the anti-reflexive case we obtain a similar result as in Theorem 4 where the condition $PXP = X$ and the definition of $\tilde{X}$ must be changed by $PXP = -X$ and*

$$\tilde{X} = \begin{bmatrix} O & X_{12} & O \\ X_{21} & O & O \\ O & O & O \end{bmatrix},$$

respectively. Moreover, in this case, the condition (22) becomes

$$A_1^* X_{12} B_2 + A_2^* X_{21} B_1 = C. \tag{23}$$

From now on, in order to solve the stated problem by using the above simplification we will consider different techniques: the GSVD (when X_{11} and X_{22} are not absent) or the SVD (when one of these blocks is absent as in the subsection 3.1), and the lifting technique. These two techniques allow to give the solution in terms of blocks, while the Kronecker properties applied as in Theorem 4 give the solution in terms of the original matrices.

4.1 Using the GSVD

By applying the generalized singular value decomposition to the pairs of matrices $\{A_1^*, A_2^*\}$ and $\{B_1^*, B_2^*\}$, the form of the solution X is obtained. In fact,

$$A_1^* = W_A \Sigma_{1A} U_A^*, \qquad A_2^* = W_A \Sigma_{2A} V_A^* \tag{24}$$

and

$$B_1^* = W_B \Sigma_{1B} U_B^*, \qquad B_2^* = W_B \Sigma_{2B} V_B^*, \tag{25}$$

where the involved matrices satisfy the conditions given in (1). By substituting both expressions into the equality (22) we arrive to:

$$\Sigma_{1A}(U_A^* X_{11} U_B)\Sigma_{1B}^* + \Sigma_{2A}(V_A^* X_{22} V_B)\Sigma_{2B}^* = W_A^{-1} C W_B^{-*} \tag{26}$$

where the nonsingularity of the matrices W_A and W_B^* has been used. By splitting into blocks of suitable sizes the matrices in brackets we get:

$$U_A^* X_{11} U_B = \begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} & \bar{X}_{13} \\ \bar{X}_{21} & \bar{X}_{22} & \bar{X}_{23} \\ \bar{X}_{31} & \bar{X}_{32} & \bar{X}_{33} \end{bmatrix}, \qquad V_A^* X_{22} V_B = \begin{bmatrix} \bar{X}_{44} & \bar{X}_{45} & \bar{X}_{46} \\ \bar{X}_{54} & \bar{X}_{55} & \bar{X}_{56} \\ \bar{X}_{64} & \bar{X}_{65} & \bar{X}_{66} \end{bmatrix}$$

and moreover, we can write:

$$W_A^{-1} C W_B^{-*} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}. \tag{27}$$

By substituting these last three expressions into (26) and computing the products we obtain:

$$\begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} D_{1B} & O & O \\ D_{1A}\bar{X}_{21} & D_{1A}\bar{X}_{22} D_{1B} + D_{2A}\bar{X}_{55} D_{2B} & D_{2A}\bar{X}_{56} & O \\ O & \bar{X}_{65} D_{2B} & \bar{X}_{66} & O \\ O & O & O & O \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$$

where we have used that Σ_{1A}, Σ_{2A}, Σ_{1B}, and Σ_{2B} are split as in (1).

Consequently, this last equality holds if and only if the blocks

$$C_{14}, C_{24}, C_{34}, C_{13}, C_{31}, C_{41}, C_{42}, C_{43}, C_{44}$$

are null matrices of appropriate sizes.

As a summary, we have proved the following result.

Theorem 5 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$. For a given Hermitian and tripotent matrix $P \in \mathbb{C}^{n\times n}$, the matrix equations*

$$AXB = C \qquad \textit{and} \qquad PXP = X$$

have solution $X \in \mathbb{C}^{n\times n}$ *if and only if*

$$C_{14}, C_{24}, C_{34}, C_{13}, C_{31}, C_{41}, C_{42}, C_{43}, C_{44}$$

are null blocks of appropriate sizes, where

$$W_A^{-1} C W_B^{-*} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$$

being W_A *and* W_B *the nonsingular matrices appearing when applying the GSVD to the pairs of matrices* $\{A_1^*, A_2^*\}$ *and* $\{B_1^*, B_2^*\}$*, respectively (that is,* $A_1^* = W_A \Sigma_{1A} U_A^*$*,* $A_2^* = W_A \Sigma_{2A} V_A^*$ *and* $B_1^* = W_B \Sigma_{1B} U_B^*$*,* $B_2^* = W_B \Sigma_{2B} V_B^*$*).*

In this case, the general solution can be expressed as

$$X = U \begin{bmatrix} U_A^{-*} \mathcal{X}_1 U_B^{-1} & O & O \\ O & V_A^{-*} \mathcal{X}_2 V_B^{-1} & O \\ O & O & O \end{bmatrix} U^* \tag{28}$$

where

$$\mathcal{X}_1 = \begin{bmatrix} C_{11} & C_{12} D_{1B}^{-1} & \bar{X}_{13} \\ D_{1A}^{-1} C_{21} & \bar{X}_{22} & \bar{X}_{23} \\ \bar{X}_{31} & \bar{X}_{32} & \bar{X}_{33} \end{bmatrix}$$

and

$$\mathcal{X}_2 = \begin{bmatrix} \bar{X}_{44} & \bar{X}_{45} & \bar{X}_{46} \\ \bar{X}_{54} & D_{2A}^{-1}(C_{22} - D_{1A} \bar{X}_{22} D_{1B}) D_{2B}^{-1} & D_{2A}^{-1} C_{23} \\ \bar{X}_{64} & C_{32} D_{2B}^{-1} & C_{33} \end{bmatrix},$$

being $\bar{X}_{ij}$ *arbitrary matrices of suitable sizes.*

Remark 7 *For the anti-reflexive case, we obtain a similar result as in Theorem 5 where the expression (28) must be changed by*

$$X = U \begin{bmatrix} O & U_A^{-*} \mathcal{X}_2 V_B^{-1} & O \\ V_A^{-*} \mathcal{X}_1 U_B^{-1} & O & O \\ O & O & O \end{bmatrix} U^*.$$

4.2 Applying the lifting technique

Another way to solve the problem of finding $\{P,3\}$ reflexive solutions is the lifting technique as described before. In fact, by applying this technique to the equation (22) we get

$$\mathrm{vec}(A_1^* X_{11} B_1 + A_2^* X_{22} B_2) = \mathrm{vec}(C)$$

and the properties given in Lemma 1 allow to write

$$\mathrm{vec}(A_1^* X_{11} B_1) + \mathrm{vec}(A_2^* X_{22} B_2) = \mathrm{vec}(C),$$

which implies

$$(B_1^* \otimes A_1^*)\mathrm{vec}(X_{11}) + (B_2^* \otimes A_2^*)\mathrm{vec}(X_{22}) = \mathrm{vec}(C)$$

and so

$$\begin{bmatrix} B_1^* \otimes A_1^* & B_2^* \otimes A_2^* \end{bmatrix} \begin{bmatrix} \mathrm{vec}(X_{11}) \\ \mathrm{vec}(X_{22}) \end{bmatrix} = \mathrm{vec}(C). \tag{29}$$

Then, by using generalized inverses (see [2]), we have the following result.

Theorem 6 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$. For a given Hermitian and tripotent matrix $P \in \mathbb{C}^{n\times n}$, the matrix equations*

$$AXB = C, \qquad PXP = X \tag{30}$$

have solution if and only if any of the following statements holds:

(a) $\mathrm{vec}(C) \in \mathcal{R}\left(\begin{bmatrix} B_1^* \otimes A_1^* & B_2^* \otimes A_2^* \end{bmatrix}\right)$*, where A_1, A_2, B_1, and B_2 are given in (21).*

(b) there exists a $\{1\}$-inverse M^- of the matrix $M = \begin{bmatrix} B_1^* \otimes A_1^* & B_2^* \otimes A_2^* \end{bmatrix}$ such that $\mathrm{vec}(C) \in \mathcal{N}(I - MM^-)$.

Then, the general solution is given by

$$X = U \begin{bmatrix} \mathrm{devec}(\mathrm{vec}(X_{11}), \alpha, \alpha) & O & O \\ O & \mathrm{devec}(\mathrm{vec}(X_{22}), \beta, \beta) & O \\ O & O & O \end{bmatrix} U^* \tag{31}$$

where α, β are the sizes indicated in (18), and $\mathrm{vec}(X_{11})$ *and* $\mathrm{vec}(X_{22})$ *are* *obtained by solving (29). In the case (b),*

$$\begin{bmatrix} \mathrm{vec}(X_{11}) \\ \mathrm{vec}(X_{22}) \end{bmatrix} = M^{-}\mathrm{vec}(C) + Y - M^{-}MY$$

with Y an arbitrary matrix.

Remark 8 *When $PXP = -X$, we obtain a similar result as in Theorem 6 where A_1 and A_2 must be interchanged. Moreover, the expression (31) is changed by*

$$\tilde{X} = \begin{bmatrix} O & \mathrm{devec}(\mathrm{vec}(X_{12}), \alpha, \alpha) & O \\ \mathrm{devec}(\mathrm{vec}(X_{21}), \beta, \beta) & O & O \\ O & O & O \end{bmatrix}.$$

5 Algorithm and Examples

The algorithm below constructs reflexive solutions for the stated problem in Section 2.

Algorithm

Inputs: $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$, $P \in \mathbb{C}^{n\times n}$, $k \in \mathbb{N}$, and Method.

Outputs: $X \in \mathbb{C}^{n\times n}$ such that $AXB = C$ and $PXP = X$.

Step 1 If $P = O$ then go to Step 9.
Step 2 Compute P^*. If $P^* \neq P$ then go to Step 10.
Step 3 If k is odd then
  Step 3.1 Compute P^2.
  Step 3.2 If $P^2 = P$ then go to Step 5 else go to Step 10.
Step 4 If k is even then
  Step 4.1 Compute P^3.
  Step 4.2 If $P^3 = P$ then go to Step 5 else go to Step 10.
Step 5 Diagonalize P as $P = UDU^*$. Then $\sigma(P \otimes P) = \mathrm{diag}(D \otimes D)$.
Step 6 If k is odd then
  Step 6.1 Compute $\tilde{A}$ and $\tilde{B}$ as in (9).
  Step 6.2 If 'Method = SVD'
    Step 6.2.1 Decompose (SVD) A_1^* and B_1^* as in (11) and (12).
    Step 6.2.2 Compute $W_A^{-1} C W_B^{-*}$ and split it as in (14).
    Step 6.2.3 If $C_{12} \neq O$, $C_{21} \neq O$ or $C_{22} \neq O$ then go to Step 11.

Step 6.2.4 If C_{12}, C_{21} and C_{22} are null matrices then the $\{P, 2\}$ reflexive solutions are given by (15). Go to End.

Step 6.3 If 'Method = Lifting'

Step 6.3.1 If $\mathrm{vec}(C) \notin \mathcal{R}(B_1^* \otimes A_1^*)$ then go to Step 11, else the general solution is given by (17). Go to End.

Step 7 If k is even then

Step 7.1 Compute $\tilde{A}$ and $\tilde{B}$ as in (21).

Step 7.2 If 'Method = GSVD'

Step 8.2.1 Decompose (GSVD) $\{A_1^*, A_2^*\}$ and $\{B_1^*, B_2^*\}$ as in (24) and (25).

Step 7.2.2 Compute $W_A^{-1} C W_B^{-*}$ and split it as in (27).

Step 7.2.3 $J := \{(1,4), (2,4), (3,4), (1,3), (3,1), (4,1), (4,2), (4,3), (4,4)\}$.

Step 7.2.4 If there exists $(i, j) \in J$ such that $C_{ij} \neq O$ then go to Step 11.

Step 7.2.5 If $C_{ij} = O$ for each $(i, j) \in J$ then the $\{P, 3\}$ reflexive solutions are given by (28). Go to End.

Step 7.3 If 'Method = Lifting'

Step 7.3.1 If $\mathrm{vec}(C) \notin \mathcal{R}(M)$ then go to Step 11, else the general solution is given by (31). Go to End.

Step 8 Display *'The solution is $X = O$ when $C = O$ and there is no solution when $C \neq O$'*. Go to End.

Step 9 Display *'The matrix P does not satisfy the required hypothesis'*. Go to End.

Step 10 Display *'There is no solution'*.

End

A similar algorithm for the anti-reflexive case can be developed.

An analysis and comparison of the computational cost of the algorithm is presented for both methods in what follows. The first part of the algorithm (until Step 5 included) is shared by both methods and it requires a computational cost $O(n^3)$. Now, we analyze the Step 6. The part corresponding to the SVD method requires $O(mr^2 + lr^2)$ for the SVD decompositions and $O(m^2 l + ml^2)$ for the Step 6.2.2. In the lifting method the computational cost is at most of $O(mlr^2)$. Then, in case of $n \gg m$ and $n \gg l$, both methods require a computational cost at most of $O(n^3)$. Next, we study the Step 7. The part corresponding to the GSVD method requires $O(m^3 + m(\alpha+\beta)\max(m, \alpha+\beta))$ for the GSVD decompositions and $O(m^2 l + ml^2)$ for the Step 7.2.2. The lifting technique costs about $O(ml(\alpha+\beta)^2)$. So, in case of $n \gg m$ and $n \gg l$, the cost of both methods is dominated by the first part, that is $O(n^3)$.

Next, we illustrate the obtained results with some examples. As we have shown, it is enough to give examples finding $\{P, 2\}$ and $\{P, 3\}$ reflexive solutions.

Example 1 *Consider the matrices*

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.288920346061937 & 0 & 0 \\ -0.308655868523851 & 0 & 0 \\ -0.906232083109640 & 0 & 0 \end{bmatrix}$$

and the Hermitian and idempotent matrix

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From Theorem 2 and Theorem 3, the $\{P, 2\}$ *reflexive solutions of the matrix equation* $AXB = C$ *are, for each* $\alpha, \beta \in \mathbb{C}$,

$$X = \begin{bmatrix} 0.7420 & 0.7420 & 0.7420\alpha - 0.6462\beta \\ 0.7420 & 0.7420 & 0.2870\alpha - 0.6462\beta \\ -0.5976 & -0.5976 & 0.9139\alpha + 0.4060\beta \end{bmatrix}.$$

Note that we exhibit the results rounded to four decimals and so, the solution presented is the same for both techniques. However, working with the MATLAB precision we obtain that $\|AXB - C\|_F = 1.7953 \cdot 10^{-15}$ *for the SVD technique and* $\|AXB - C\|_F = 4.0792 \cdot 10^{-16}$ *for the lifting technique.*

Example 2 *Consider the random matrices*

$$A = \begin{bmatrix} 0.5828 & 0.2091 & 0.4154 & 0.2140 & 0.6833 & 0.4514 & 0.6085 & 0.0841 & 0.1210 & 0.2319 \\ 0.4235 & 0.3798 & 0.3050 & 0.6435 & 0.2126 & 0.0439 & 0.0158 & 0.4544 & 0.4508 & 0.2393 \\ 0.5155 & 0.7833 & 0.8744 & 0.3200 & 0.8392 & 0.0272 & 0.0164 & 0.4418 & 0.7159 & 0.0498 \\ 0.3340 & 0.6808 & 0.0150 & 0.9601 & 0.6288 & 0.3127 & 0.1901 & 0.3533 & 0.8928 & 0.0784 \\ 0.4329 & 0.4611 & 0.7680 & 0.7266 & 0.1338 & 0.0129 & 0.5869 & 0.1536 & 0.2731 & 0.6408 \\ 0.2259 & 0.5678 & 0.9708 & 0.4120 & 0.2071 & 0.3840 & 0.0576 & 0.6756 & 0.2548 & 0.1909 \\ 0.5798 & 0.7942 & 0.9901 & 0.7446 & 0.6072 & 0.6831 & 0.3676 & 0.6992 & 0.8656 & 0.8439 \\ 0.7604 & 0.0592 & 0.7889 & 0.2679 & 0.6299 & 0.0928 & 0.6315 & 0.7275 & 0.2324 & 0.1739 \\ 0.5298 & 0.6029 & 0.4387 & 0.4399 & 0.3705 & 0.0353 & 0.7176 & 0.4784 & 0.8049 & 0.1708 \\ 0.6405 & 0.0503 & 0.4983 & 0.9334 & 0.5751 & 0.6124 & 0.6927 & 0.5548 & 0.9084 & 0.9943 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4398 & 0.9342 & 0.1370 & 0.4225 & 0.2974 & 0.3759 & 0.1939 & 0.6273 & 0.7165 & 0.1146 \\ 0.3400 & 0.2644 & 0.8188 & 0.8560 & 0.0492 & 0.0099 & 0.9048 & 0.6991 & 0.5113 & 0.6649 \\ 0.3142 & 0.1603 & 0.4302 & 0.4902 & 0.6932 & 0.4199 & 0.5692 & 0.3972 & 0.7764 & 0.3654 \\ 0.3651 & 0.8729 & 0.8903 & 0.8159 & 0.6501 & 0.7537 & 0.6318 & 0.4136 & 0.4893 & 0.1400 \\ 0.3932 & 0.2379 & 0.7349 & 0.4608 & 0.9830 & 0.7939 & 0.2344 & 0.6552 & 0.1859 & 0.5668 \\ 0.5915 & 0.6458 & 0.6873 & 0.4574 & 0.5527 & 0.9200 & 0.5488 & 0.8376 & 0.7006 & 0.8230 \\ 0.1197 & 0.9669 & 0.3461 & 0.4507 & 0.4001 & 0.8447 & 0.9316 & 0.3716 & 0.9827 & 0.6739 \\ 0.0381 & 0.6649 & 0.1660 & 0.4122 & 0.1988 & 0.3678 & 0.3352 & 0.4253 & 0.8066 & 0.9994 \\ 0.4586 & 0.8704 & 0.1556 & 0.9016 & 0.6252 & 0.6208 & 0.6555 & 0.5947 & 0.7036 & 0.9616 \\ 0.8699 & 0.0099 & 0.1911 & 0.0056 & 0.7334 & 0.7313 & 0.3919 & 0.5657 & 0.4850 & 0.0589 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.2333 & 0.1042 & 0.8555 & 0.4361 & 0.6314 & 0.6509 & 0.3687 & 0.3442 & -0.1395 & 0.3309 \\ 0.3684 & 0.1645 & 1.3506 & 0.6885 & 0.9968 & 1.0276 & 0.5820 & 0.5434 & -0.2202 & 0.5224 \\ 0.8056 & 0.3598 & 2.9535 & 1.5056 & 2.1797 & 2.2471 & 1.2727 & 1.1883 & -0.4815 & 1.1425 \\ 0.4719 & 0.2107 & 1.7300 & 0.8819 & 1.2768 & 1.3162 & 0.7455 & 0.6961 & -0.2820 & 0.6692 \\ 0.5642 & 0.2520 & 2.0687 & 1.0545 & 1.5267 & 1.5739 & 0.8914 & 0.8323 & -0.3372 & 0.8002 \\ 1.0121 & 0.4521 & 3.7109 & 1.8917 & 2.7387 & 2.8234 & 1.5990 & 1.4931 & -0.6050 & 1.4355 \\ 1.0740 & 0.4797 & 3.9376 & 2.0072 & 2.9060 & 2.9959 & 1.6967 & 1.5843 & -0.6419 & 1.5232 \\ 0.5346 & 0.2388 & 1.9600 & 0.9991 & 1.4465 & 1.4913 & 0.8446 & 0.7886 & -0.3195 & 0.7582 \\ 0.5545 & 0.2477 & 2.0331 & 1.0364 & 1.5005 & 1.5469 & 0.8761 & 0.8181 & -0.3315 & 0.7865 \\ 0.4845 & 0.2164 & 1.7764 & 0.9055 & 1.3110 & 1.3515 & 0.7654 & 0.7147 & -0.2896 & 0.6872 \end{bmatrix},$$

and the Hermitian and idempotent matrix

$$P = \begin{bmatrix} 0.8168 & -0.1573 & 0.0074 & -0.0460 & -0.1325 & 0.0516 & 0.0735 & -0.0785 & 0.2277 & 0.1977 \\ -0.1573 & 0.2110 & 0.1248 & 0.1844 & 0.0324 & 0.0318 & 0.1786 & 0.2018 & -0.0348 & 0.1276 \\ 0.0074 & 0.1248 & 0.5859 & -0.1944 & 0.0845 & 0.1101 & -0.0397 & 0.3559 & 0.2038 & -0.0117 \\ -0.0460 & 0.1844 & -0.1944 & 0.4375 & 0.1472 & 0.0913 & 0.3016 & -0.0446 & -0.0502 & 0.2161 \\ -0.1325 & 0.0324 & 0.0845 & 0.1472 & 0.5766 & 0.3141 & -0.0121 & -0.1969 & 0.2244 & -0.0936 \\ 0.0516 & 0.0318 & 0.1101 & 0.0913 & 0.3141 & 0.2008 & 0.0327 & -0.0693 & 0.1783 & 0.0032 \\ 0.0735 & 0.1786 & -0.0397 & 0.3016 & -0.0121 & 0.0327 & 0.2746 & 0.1027 & -0.0297 & 0.2381 \\ -0.0785 & 0.2018 & 0.3559 & -0.0446 & -0.1969 & -0.0693 & 0.1027 & 0.4218 & -0.0309 & 0.1152 \\ 0.2277 & -0.0348 & 0.2038 & -0.0502 & 0.2244 & 0.1783 & -0.0297 & -0.0309 & 0.2377 & -0.0071 \\ 0.1977 & 0.1276 & -0.0117 & 0.2161 & -0.0936 & 0.0032 & 0.2381 & 0.1152 & -0.0071 & 0.2372 \end{bmatrix}.$$

Now, the matrix

$$X = \begin{bmatrix} 1.1686 & -0.2756 & 0.5163 & -0.9468 & -1.5349 & -0.6638 & -0.1943 & 0.7513 & -0.0701 & 0.2196 \\ -0.7661 & 0.1807 & -0.3384 & 0.6207 & 1.0062 & 0.4352 & 0.1274 & -0.4925 & 0.0460 & -0.1440 \\ -0.9853 & 0.2324 & -0.4353 & 0.7983 & 1.2941 & 0.5597 & 0.1639 & -0.6335 & 0.0591 & -0.1852 \\ -0.2002 & 0.0472 & -0.0884 & 0.1622 & 0.2629 & 0.1137 & 0.0333 & -0.1287 & 0.0120 & -0.0376 \\ -0.3172 & 0.0748 & -0.1401 & 0.2570 & 0.4166 & 0.1802 & 0.0527 & -0.2039 & 0.0190 & -0.0596 \\ -0.1548 & 0.0365 & -0.0684 & 0.1254 & 0.2033 & 0.0879 & 0.0257 & -0.0995 & 0.0093 & -0.0291 \\ -0.2546 & 0.0601 & -0.1125 & 0.2063 & 0.3345 & 0.1447 & 0.0424 & -0.1637 & 0.0153 & -0.0479 \\ -0.9489 & 0.2238 & -0.4192 & 0.7688 & 1.2463 & 0.5390 & 0.1578 & -0.6101 & 0.0569 & -0.1783 \\ 0.0530 & -0.0125 & 0.0234 & -0.0429 & -0.0696 & -0.0301 & -0.0088 & 0.0341 & -0.0032 & 0.0100 \\ -0.0519 & 0.0122 & -0.0229 & 0.0420 & 0.0681 & 0.0295 & 0.0086 & -0.0333 & 0.0031 & -0.0097 \end{bmatrix}$$

is a $\{P, 2\}$ *reflexive solution. Again, we exhibit the results rounded to four decimals and the solution presented is as well the same for both techniques. However, working with the MATLAB precision we obtain that* $\|AXB - C\|_F = 8.0855 \cdot 10^{-15}$ *for the SVD technique and* $\|AXB - C\|_F = 3.5463 \cdot 10^{-15}$ *for the lifting technique.*

Example 3 *Consider the matrices*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix},$$

and the Hermitian and tripotent matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, the $\{P,3\}$ reflexive solutions are

$$X = \begin{bmatrix} 2-1.4142\alpha & 1-\beta & 1-\beta & 0 \\ 0.7071\alpha & 0.5(\beta+\gamma) & 0.5(\beta-\gamma) & 0 \\ 0.7071\alpha & 0.5(\beta-\gamma) & 0.5(\beta+\gamma) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{for each } \alpha,\beta,\gamma\in\mathbb{C}.$$

Here, working with the MATLAB precision, $\|AXB-C\|_F = 1.1254\cdot 10^{-15}$ for the GSVD technique and $\|AXB-C\|_F = 8.8818\cdot 10^{-16}$ for the lifting technique.

Example 4 *As before, for the random matrices*

$$A = \begin{bmatrix} 0.6154 & 0.4103 & 0.1987 & 0.4660 & 0.6813 & 0.1934 & 0.8537 & 0.3420 & 0.3704 & 0.5226 \\ 0.7919 & 0.8936 & 0.6038 & 0.4186 & 0.3795 & 0.6822 & 0.5936 & 0.2897 & 0.7027 & 0.8801 \\ 0.9218 & 0.0579 & 0.2722 & 0.8462 & 0.8318 & 0.3028 & 0.4966 & 0.3412 & 0.5466 & 0.1730 \\ 0.7382 & 0.3529 & 0.1988 & 0.5252 & 0.5028 & 0.5417 & 0.8998 & 0.5341 & 0.4449 & 0.9797 \\ 0.1763 & 0.8132 & 0.0153 & 0.2026 & 0.7095 & 0.1509 & 0.8216 & 0.7271 & 0.6946 & 0.2714 \\ 0.4057 & 0.0099 & 0.7468 & 0.6721 & 0.4289 & 0.6979 & 0.6449 & 0.3093 & 0.6213 & 0.2523 \\ 0.9355 & 0.1389 & 0.4451 & 0.8381 & 0.3046 & 0.3784 & 0.8180 & 0.8385 & 0.7948 & 0.8757 \\ 0.9169 & 0.2028 & 0.9318 & 0.0196 & 0.1897 & 0.8600 & 0.6602 & 0.5681 & 0.9568 & 0.7373 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1365 & 0.5828 & 0.2091 & 0.4154 & 0.2140 & 0.6833 & 0.4514 \\ 0.0118 & 0.4235 & 0.3798 & 0.3050 & 0.6435 & 0.2126 & 0.0439 \\ 0.8939 & 0.5155 & 0.7833 & 0.8744 & 0.3200 & 0.8392 & 0.0272 \\ 0.1991 & 0.3340 & 0.6808 & 0.0150 & 0.9601 & 0.6288 & 0.3127 \\ 0.2987 & 0.4329 & 0.4611 & 0.7680 & 0.7266 & 0.1338 & 0.0129 \\ 0.6614 & 0.2259 & 0.5678 & 0.9708 & 0.4120 & 0.2071 & 0.3840 \\ 0.2844 & 0.5798 & 0.7942 & 0.9901 & 0.7446 & 0.6072 & 0.6831 \\ 0.4692 & 0.7604 & 0.0592 & 0.7889 & 0.2679 & 0.6299 & 0.0928 \\ 0.0648 & 0.5298 & 0.6029 & 0.4387 & 0.4399 & 0.3705 & 0.0353 \\ 0.9883 & 0.6405 & 0.0503 & 0.4983 & 0.9334 & 0.5751 & 0.6124 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.4478 & 0.9400 & 2.2716 & 0.8773 & 1.6894 & 2.3220 & 1.5612 \\ 1.8992 & 1.3869 & 2.8472 & 1.3636 & 2.1633 & 2.8669 & 1.6062 \\ 1.1724 & 0.6586 & 1.9750 & 0.2172 & 1.0931 & 2.5130 & 1.8356 \\ 1.5764 & 1.0632 & 2.4714 & 0.8199 & 1.7904 & 2.8025 & 1.8403 \\ 1.4621 & 0.9341 & 2.1214 & 1.5195 & 2.1737 & 1.3448 & 0.6762 \\ 1.4792 & 1.0831 & 2.1849 & 1.0146 & 1.6160 & 2.4034 & 1.4099 \\ 1.7878 & 1.3211 & 2.6720 & 1.2667 & 2.1843 & 3.0792 & 2.2203 \\ 1.0587 & 0.7952 & 1.6485 & 0.5918 & 1.2886 & 1.8156 & 0.9326 \end{bmatrix},$$

and

$$P = \begin{bmatrix} -0.1608 & -0.2824 & -0.7414 & -0.1340 & -0.2457 & -0.2101 & -0.1258 & -0.0051 & -0.2264 & -0.3330 \\ -0.2824 & 0.9313 & -0.1804 & -0.0326 & -0.0598 & -0.0511 & -0.0306 & -0.0012 & -0.0551 & -0.0810 \\ -0.7414 & -0.1804 & 0.5265 & -0.0856 & -0.1569 & -0.1342 & -0.0804 & -0.0033 & -0.1446 & -0.2127 \\ -0.1340 & -0.0326 & -0.0856 & -0.8333 & 0.3057 & 0.2613 & 0.1565 & 0.0063 & 0.2817 & 0.0449 \\ -0.2457 & -0.0598 & -0.1569 & 0.3057 & -0.4394 & 0.4793 & 0.2871 & 0.0116 & 0.5166 & 0.0823 \\ -0.2101 & -0.0511 & -0.1342 & 0.2613 & 0.4793 & -0.5902 & 0.2455 & 0.0100 & 0.4417 & 0.0704 \\ -0.1258 & -0.0306 & -0.0804 & 0.1565 & 0.2871 & 0.2455 & -0.8530 & 0.0060 & 0.2646 & 0.0422 \\ -0.0051 & -0.0012 & -0.0033 & 0.0063 & 0.0116 & 0.0100 & 0.0060 & -0.9998 & 0.0107 & 0.0017 \\ -0.2264 & -0.0551 & -0.1446 & 0.2817 & 0.5166 & 0.4417 & 0.2646 & 0.0107 & -0.5239 & 0.0759 \\ -0.3330 & -0.0810 & -0.2127 & 0.0449 & 0.0823 & 0.0704 & 0.0422 & 0.0017 & 0.0759 & -0.0574 \end{bmatrix},$$

the matrix

$$X = \begin{bmatrix} 0.1764 & -0.5966 & -0.6825 & 1.1554 & -2.4395 & 0.9344 & 0.0475 & 0.7460 & 0.4751 & -0.2111 \\ -0.6394 & 0.3682 & 0.3367 & 0.4406 & -0.3009 & 0.4775 & 0.1614 & 0.1875 & 0.3852 & 0.0946 \\ -0.4470 & 0.2424 & 0.1213 & 0.7731 & -1.4937 & 0.6519 & 0.0633 & 0.4778 & 0.3628 & -0.1027 \\ -0.1253 & 0.1638 & 0.1589 & 1.3121 & -2.2482 & 1.2247 & 0.7551 & 0.6690 & 0.2171 & 0.0289 \\ -1.1893 & 0.0670 & -0.3214 & 1.2953 & -2.6149 & 1.1899 & -0.3600 & 0.9324 & 0.7773 & -0.3960 \\ -1.2833 & -0.0075 & -0.4449 & 1.2588 & -2.3965 & 0.8802 & -0.1246 & 1.7775 & 0.5869 & -0.4633 \\ 0.7953 & 0.3760 & 0.7324 & -0.4978 & 1.4190 & -0.4448 & 0.3523 & -1.1224 & -0.3169 & 0.4545 \\ -0.8159 & -0.1911 & -0.5120 & 1.2313 & -1.2150 & 0.4958 & 0.2129 & 1.0018 & -0.5118 & -0.3786 \\ 0.9629 & 0.5626 & 1.0189 & -2.2368 & 4.3739 & -1.7882 & 0.0238 & -1.9852 & -0.7891 & 0.5987 \\ -0.2976 & 0.1054 & 0.0286 & 0.5841 & -1.0623 & 0.5054 & 0.0630 & 0.3508 & 0.2956 & -0.0591 \end{bmatrix}$$

is a $\{P,3\}$ *reflexive solution. With the MATLAB precision we obtain* $\|AXB-$

$C\|_F = 5.7935 \cdot 10^{-15}$ *for the GSVD technique and* $\|AXB - C\|_F = 4.3470 \cdot 10^{-15}$ *for the lifting technique.*

The algorithms were implemented in MATLAB, version 7.1., and used to test the results on several numerical experiments. In all of them we obtain error bounds about $O(10^{-15})$ in both methods when matrices have larger sizes and for smaller sizes the lifting technique can be improved to $O(10^{-16})$. Next, in Table 1 we summarize some of the obtained numerical results.

Example	Lifting	SVD	GSVD
1	10^{-16}	10^{-15}	–
2	10^{-15}	10^{-15}	–
3	10^{-16}	–	10^{-15}
4	10^{-15}	–	10^{-15}

Table 1
Error bounds for the numerical experiments: $\|AXB - C\|_F$

These experiments show that the algorithm implemented works well for numerical examples.

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