Full rank factorization in quasi-LDU form of totally nonpositive rectangular matrices ☆

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Abstract

Let $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ be a totally nonpositive matrix with $\operatorname{rank}(A) = r \leq \min\{n, m\}$ and $a_{11} = 0$. In this paper we obtain a characterization in terms of the full rank factorization in quasi-LDU form, that is, $A = \tilde{L}DU$ where $\tilde{L} \in \mathbb{R}^{n \times r}$ is a block lower echelon matrix, $U \in \mathbb{R}^{r \times m}$ is a unit upper echelon totally positive matrix and $D \in \mathbb{R}^{r \times r}$ is a diagonal matrix, with $\operatorname{rank}(\tilde{L}) = \operatorname{rank}(U) = \operatorname{rank}(D) = r$. We use this quasi-$LDU$ decomposition to construct the quasi-bidiagonal factorization of A. Moreover, some properties about these matrices are studied.

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1. Introduction

A matrix is called *totally nonpositive (negative)* if all its minors are nonpositive (negative) and it is abbreviated as t.n.p. (t.n.) see, for instance, [3, 4, 6, 8, 14, 15, 18]. These matrices can be considered as a generalization of the partially negative matrices, that is, matrices with all its principal minors negative. The partially negative matrices are called N-matrices in economic models [2, 16]. If, instead, all minors of a matrix are nonnegative (positive) the matrix is called *totally nonnegative (totally positive)* and it is abbreviated as TN (TP). These classes of matrices have been studied by several authors [1, 5, 7, 9, 10, 11, 13, 17] obtaining properties, the Jordan structure and characterizations by applying the Gaussian or Neville elimination.

The nonsingular t.n.p. matrices with a negative $(1, 1)$ entry have been characterized in terms of the factors of their LDU factorization in [3]. This factorization provides a criteria to determine if a matrix is t.n.p. and allows us to reduce the number of minors to be checked to decide the total nonpositivity of a nonsingular matrix with a negative $(1, 1)$ entry. When the $(1, 1)$ entry is

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equal to zero but the (n,n) entry is negative we can obtain a UDL factorization of this nonsingular t.n.p. matrix by permutation similarity. Then, we have studied in [4] the extension of the same characterization to rectangular t.n.p. matrices, obtaining a full rank LDU factorization in echelon form of this class of matrices and other characterization by means of its thin QR factorization. This QR characterization is similar to the one obtained in [5] for rectangular TN matrices and it is an extension of the result for square TN matrices given in [10].

When the nonsingular t.n.p. matrix has the $(1,1)$ and (n,n) entries equal to zero a characterization, in terms of the signs of minors with consecutive initial rows or consecutive initial columns, is obtained in [14]. Furthermore, in [6] the authors characterize the nonsingular t.n.p. matrices with the $(1,1)$ entry equal to zero in terms of a quasi-LDU factorization, that is, a $\tilde{L}DU$ factorization, where $\tilde{L}$ is a block lower triangular matrix, U is a unit upper triangular TN matrix and D is a diagonal matrix. This result holds when the (n,n) entry is equal to zero or when it is negative but the permutation similarity is not used.

The main goal of this paper is to conclude the characterization of any t.n.p. matrix A by using a LDU factorization of A or a quasi-LDU factorization in the cases when the $(1,1)$ entry of A is equal to zero. To finish this process, we need to characterize the rectangular t.n.p. matrices with the $(1,1)$ entry equal to zero in terms of a $\tilde{L}DU$ full rank factorization, where $\tilde{L}$ is a block lower echelon matrix, U is a unit upper echelon TN matrix and D is a diagonal matrix (see Theorem 4 when A has full row rank, and Theorem 7 when A has arbitrary rank). This quasi-LDU factorization will be used to construct a quasi-bidiagonal factorization of this class of matrices.

We recall that a matrix is an upper echelon matrix if the first nonzero entry in each row (leading entry) is to the right of all leading entry in the row above it and all zero rows are at the bottom. If, in addition, each leading entry is the only nonzero entry in its column it is called upper reduced echelon matrix. A matrix is a lower (lower reduced) echelon matrix if its transpose is an upper (upper reduced) echelon matrix. When all of the leading entries are equal to 1, this matrix is called *unit upper (lower) echelon matrix* if it is in echelon form, or *unit upper (lower) reduced echelon matrix* if it is in reduced echelon form.

We follow the notation given in [1]. For $k,n \in \mathbb{N}$, $1 \leq k \leq n$, $\mathcal{Q}_{k,n}$ denotes the set of all increasing sequences of k natural numbers less than or equal to n. If A is an $n \times m$ matrix, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathcal{Q}_{k,n}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathcal{Q}_{k,m}$, $A[\alpha|\beta]$ denotes the $k \times k$ submatrix of A lying in rows α_i and columns β_i, $i = 1, 2, \ldots, k$. The principal submatrix $A[\alpha|\alpha]$ is abbreviated as $A[\alpha]$. Note that, an $n \times m$ matrix A is a t.n.p. matrix if $\det A[\alpha|\beta] \leq 0$, $\forall \alpha \in \mathcal{Q}_{k,n}$ and $\forall \beta \in \mathcal{Q}_{k,m}$, with $k = 1, 2, \ldots, n$. Moreover, we represent by A_j, $j = 1, 2, \ldots, m$, its jth-column and by $A^{(i)}$, $i = 1, 2, \ldots, n$, its ith-row and we denote by $E_i(x)$ and $F_i(x)$ bidiagonal matrices which differ from the identity matrix only in its $(i, i-1)$ and $(i-1, i)$ entry x, respectively.

We denote by $F_n^{\{j_1, j_2, \ldots, j_k\}}$ ($C_n^{\{j_1, j_2, \ldots, j_k\}}$) the matrix obtained from the $n \times n$ identity matrix by deleting the columns (rows) $j_1, j_2, \ldots, j_k$ [4]. These matrices allow us to suppose, without loss of generality, that $A \in \mathbb{R}^{n\times m}$ has nonzero rows or columns. In other case, if A has the $j_1, j_2, \cdots, j_s$ zero rows and the $i_1, i_2, \cdots, i_r$ zero columns, $1 \le s \le n$, $1 \le r \le m$, with $F_n^{\{j_1,j_2,\cdots,j_s\}}$ and $C_m^{\{i_1,i_2,\cdots,i_p\}}$ we obtain

$$A = F_n^{\{j_1,j_2,\cdots,j_s\}} S C_m^{\{i_1,i_2,\cdots,i_p\}}$$

where $S \in \mathbb{R}^{(n-s)\times(m-p)}$ has nonzero rows or columns. If $\operatorname{rank}(S) = r$ and S has a quasi full rank factorization in echelon form $S = \tilde{L}_S D_S U_S$, then A has the following quasi full rank factorization in echelon form

$$A = \left\{ F_n^{\{j_1,j_2,\cdots,j_s\}} \tilde{L}_S \right\} D_S \left\{ U_S C_m^{\{i_1,i_2,\cdots,i_r\}} \right\} = \tilde{L} D U$$

where $\tilde{L} \in \mathbb{R}^{n\times r}$ is a block lower echelon matrix, $D \in \mathbb{R}^{r\times r}$ is a nonsingular diagonal matrix, $U \in \mathbb{R}^{r\times m}$ is an upper echelon matrix and $\operatorname{rank}(\tilde{L}) = \operatorname{rank}(U) = r$.

Therefore, from now on and without loss of generality, we work with matrices which have nonzero rows and nonzero columns.

2. Properties of the t.n.p. matrices

In this section we study some properties of the row and column entries of any rectangular t.n.p. matrix without zero rows and columns.

Proposition 1. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times m}$ be a t.n.p. matrix with nonzero rows or columns and $a_{11} = 0$.*

1. *If a_{1r}, $2 < r \le m$, is the first nonzero entry in the first row of A then, $A_j = \alpha_j A_1$, with $\alpha_j > 0$, for $j = 2, 3, \ldots, r-1$.*
2. *If a_{r1}, $2 < r \le n$, is the first nonzero entry in the first column of A, then $A^{(i)} = \beta_i A^{(1)}$, with $\beta_i > 0$, for $i = 2, 3, \ldots, r-1$.*

Proof. 1. Let a_{1r}, $r > 2$, be the first nonzero entry in the first row. Since A has nonzero columns, let a_{i1}, $2 \le i \le n$ be the first nonzero entry in the first column. If $i > 2$, for $j = 2, 3, \ldots, i-1$, and $s = 2, 3, \ldots, r-1$, we have that

$$\det A[1, j, i | 1, s, r] = \det \begin{bmatrix} 0 & 0 & a_{1r} \\ 0 & a_{js} & a_{jr} \\ a_{i1} & a_{is} & a_{ir} \end{bmatrix} = -a_{1r} a_{js} a_{i1} \ge 0,$$

so, $a_{js} = 0$, for $j = 2, 3, \ldots, i-1$, and $s = 2, 3, \ldots, r-1$.

For $i \ge 2$, $t = i+1, i+2, \ldots, n$, and $s = 2, 3, \ldots, r-1$, it is satisfied that

$$\det A[1, i, t | 1, s, r] = \det \begin{bmatrix} 0 & 0 & a_{1r} \\ a_{i1} & a_{is} & a_{ir} \\ a_{t1} & a_{ts} & a_{tr} \end{bmatrix} = a_{1r} \det A[i, t | 1, s] \ge 0,$$

which implies that $\det A[i, t | 1, s] = 0$, for $t = i+1, i+2, \ldots, n$ and $s = 2, 3, \ldots, r-1$. That is, $A_s = \alpha_s A_1$, for $s = 2, 3, \ldots, r-1$.

2. The result is obtained working with the transpose of A. $\square$

Taking into account the definition of t.n.p. matrices it is easy to prove the following properties.

Proposition 2. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times m}$ be a t.n.p. matrix with nonzero rows or columns. The following statements are verified.*

1. *If $a_{12} < 0$, then $a_{1j} < 0$ for $j = 3, 4, \ldots, m$.*
2. *If $a_{21} < 0$, then $a_{i1} < 0$ for $i = 3, 4, \ldots, n$.*
3. *If $a_{nm} < 0$, then $a_{im} < 0$ and $a_{nj} < 0$, for $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m-1$.*

Proposition 3. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times m}$ be a t.n.p. matrix with nonzero rows or columns. The following statements are verified.*

1. *If there exists an index r, $1 < r < m$, such that $a_{nr} = 0$, then $a_{nj} = 0$ for $j = r+1, r+2, \ldots, m$. Moreover, $A_j = \alpha_j A_r$, with $\alpha_j > 0$, for $j = r+1, r+2, \ldots, m$.*
2. *If there exists an index s, $1 < s < n$, such that $a_{sm} = 0$, then $a_{im} = 0$ for $i = s+1, s+2, \ldots, n$. Moreover, $A^{(i)} = \beta_i A^{(s)}$, with $\beta_i > 0$, for $i = s+1, s+2, \ldots, n$.*

Proof. 1. Suppose that $a_{nr} = 0$, with $1 < r < m$. Since A has nonzero columns, let i be the first index such that $a_{ir} < 0$, $1 \leq i \leq n-1$. Then, for $j = r+1, r+2, \ldots, m$, we have that

$$\begin{aligned} \det A[i,n|r,j] &= \det \begin{bmatrix} a_{ir} & a_{ij} \\ 0 & a_{nj} \end{bmatrix} = a_{ir}a_{nj} \geq 0 \\ &\Longrightarrow \; a_{nj} = 0,\; j = r+1, r+2, \ldots, m. \end{aligned}$$

Analogously, if $i < n-1$, for $q = i+1, i+2, \ldots, n$ and $j = r+1, r+2, \ldots, m$, we have that

$$\det A[i,q|r,j] = \det \begin{bmatrix} a_{ir} & a_{ij} \\ 0 & a_{qj} \end{bmatrix} = a_{ir}a_{qj} \geq 0 \;\Longrightarrow\; a_{qj} = 0.$$

Moreover, since A has nonzero rows there exists an entry $a_{nt} < 0$, with $1 \leq t < r$. Therefore, for $j = r+1, r+2, \ldots, m$ and $s = 1, 2, \ldots, n-1$,

$$\det A[s,i,n|t,r,j] = \det \begin{bmatrix} a_{st} & a_{sr} & a_{sj} \\ a_{it} & a_{ir} & a_{ij} \\ a_{nt} & 0 & 0 \end{bmatrix} = a_{nt} \det A[s,i|r,j] \geq 0,$$

from which we deduce that $\det A[s,i|r,j] = 0$. So, $A_j = \alpha_j A_r$, with $\alpha_j > 0$, for $j = r+1, r+2, \ldots, m$.

2. The result is obtained working with the transpose of A. $\square$

Consequently, from now on, by the previous propositions we can consider a t.n.p. matrix $A = (a_{ij}) \in \mathbb{R}^{n\times m}$ with $a_{ij} < 0$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, except for $a_{11} = 0$ and $a_{nm} \leq 0$. Furthermore, we can consider $n < m$, because otherwise we work with the transpose matrix.

3. Matrices with full row rank

In this section we characterize the rectangular t.n.p. matrices with the $(1,1)$ entry equal to zero and full row rank in terms of their quasi-LDU factorization. This characterization is an extension of the decomposition obtained for nonsingular t.n.p. matrices [6].

Theorem 1. *Let A be an $n \times m$ t.n.p. matrix with negative entries except for $a_{11} = 0$ and $a_{nm} \leq 0$, and full row rank. Then, A has a unique factorization $\tilde{L}DU$, where $U \in \mathbb{R}^{n\times m}$ is a unit upper echelon TN matrix with positive entries from the leading entry in each row and the two first columns linearly independent, $D = diag(-d_1, -d_2, d_3, \ldots, d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$ and $\tilde{L} \in \mathbb{R}^{n\times n}$ is a block lower triangular matrix*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the entries in the first column of $\tilde{L}_{21}$ are positive and in the second one nonpositive, $\tilde{L}_{22}$ is a unit lower triangular TN matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n}, \; k = 2, 3, \ldots, n.$$

Proof. Let $\bar{A} = A[1, 2, \ldots, n|1, 2, s_3, \ldots, s_n] \in \mathbb{R}^{n\times n}$ be the matrix formed by the n first linearly independent columns of A. Then, there exists a unique unit upper reduced echelon matrix C such that $A = \bar{A}C$. Since $\bar{A}$ is a nonsingular t.n.p. matrix with the $(1,1)$ entry equal to zero, by [6, Theorem 1] it admits the unique quasi-LDU factorization $\bar{A} = \tilde{L}_{\bar{A}} D_{\bar{A}} U_{\bar{A}}$, where $\tilde{L}_{\bar{A}}$ is a block lower triangular matrix, with $\tilde{l}_{i1} > 0$ for $i = 3, 4, \ldots, n$ and

$$\det \tilde{L}_{\bar{A}}[\alpha|1, 2, \ldots, k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n}, \; k = 2, 3, \ldots, n.$$

$D_{\bar{A}} = \mathrm{diag}(-d_1, -d_2, d_3, \ldots, d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$, and $U_{\bar{A}}$ is a unit upper triangular TN matrix, with positive entries above the main diagonal. Then, A admits the unique quasi-LDU factorization

$$A = \bar{A}C = \left(\tilde{L}_{\bar{A}} D_{\bar{A}} U_{\bar{A}}\right) C = \tilde{L}_{\bar{A}} D_{\bar{A}} \left(U_{\bar{A}} C\right) = \tilde{L}DU,$$

where $L = \tilde{L}_{\bar{A}}$, $D = D_{\bar{A}}$ and by [5, Proposition 2] $U = U_{\bar{A}}C$ is a unit upper echelon TN matrix with positive entries from the leading entry in each row. □

The converse of Theorem 1 is not true in general, as the next example shows.

Example 1. *The matrix*

$$\begin{aligned} A = \tilde{L}DU \quad &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} -10 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2 & -4 & -6 \\ -10 & -10 & -10 & -10 \\ -20 & -14 & -7 & 2 \end{bmatrix}, \end{aligned}$$

is not a t.n.p. matrix although the matrices $\tilde{L}$, D and U satisfy the conditions of Theorem 1.

Now, we study necessary conditions for a product $\tilde{L}DU$ to be a t.n.p. matrix. Suppose that $A = (a_{ij}) = \tilde{L}DU \in \mathbb{R}^{n\times m}$, with $\tilde{L}$, D and U verifying the conditions of Theorem 1. In order to apply the results for square matrices given in [6], we construct from $L = P\tilde{L} \in \mathbb{R}^{n\times n}$, where P is the permutation matrix $P = [2, 1, 3, \ldots, n]$, $D \in \mathbb{R}^{n\times n}$ and $U \in \mathbb{R}^{n\times m}$, square matrices $L(\delta) \in \mathbb{R}^{m\times m}$, $D(\delta) \in \mathbb{R}^{m\times m}$ and $U(\delta) \in \mathbb{R}^{m\times m}$ satisfying the conditions of [6, Theorem 2], and such that the (m, m) entry of matrix $B(\delta) = L(\delta)D(\delta)U(\delta)$ is non positive and $B = B(\delta)[1, 2, \ldots, n|1, 2, \ldots, m]$. Then, by [6, Theorem 2] the matrix $A(\delta) = PB(\delta)$ is t.n.p. and $A = PB(\delta)[1, 2, \ldots, n|1, 2, \ldots, m]$ is also t.n.p.

First, for all $\delta > 0$ we extend the diagonal matrix $D \in \mathbb{R}^{n\times n}$ in the following way

$$\begin{aligned} D(\delta) &= \operatorname{diag}(-d_1, -d_2, d_3, \ldots, d_n, \delta^3, \delta^4, \ldots, \delta^{m-n+2}) \\ &= \left[\begin{array}{c|c} D & O \\ \hline O & D(\delta)_{22} \end{array}\right] \in \mathbb{R}^{m\times m}. \end{aligned}$$

The next procedures show how to construct the matrices $L(\delta)$ and $U(\delta)$. We begin extending the unit lower triangular matrix $L = P\tilde{L}$.

Procedure 1. *Let $L \in \mathbb{R}^{n\times n}$ be a unit lower triangular matrix*

$$L = \begin{bmatrix} I_2 & O \\ L_{21} & L_{22} \end{bmatrix}$$

where the entries in the first column of L_{21} are positive, in the second one are nonpositive, L_{22} is a unit lower triangular TN matrix with positive entries under the main diagonal, and for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathcal{Q}_{k,n}$, $k = 2, 3, \ldots, n$,

$$\det L[\alpha|1, 2, \ldots, k] = \begin{cases} \geq 0 & \text{if} \quad \alpha_1 = 1,\ \alpha_2 = 2 \\ \leq 0 & \text{if} \quad 1 \text{ or } 2 \notin \alpha. \end{cases}$$

This procedure allows us to construct an $m \times m$, unit lower triangular matrix $L(\delta)$ such that

$$L(\delta) = \begin{bmatrix} L & O \\ L(\delta)_{21} & L(\delta)_{22} \end{bmatrix} \in \mathbb{R}^{(n+(m-n))\times(n+(m-n))}$$

where the entries $l(\delta)_{i1}$ and $l(\delta)_{i2}$, for $i = 3, 4, \ldots, m$, are positive and nonpositive, respectively, and the submatrix $L(\delta)[3, 4, \ldots, m]$ is a unit lower triangular TN matrix with positive entries under the main diagonal.

The matrix L can be written as

$$L = E_{(1)}^{-1} E_{(2)}^{-1} \cdots E_{(n-1)}^{-1},$$

where $E_{(i)}^{-1} = E_n(m_{n,i})E_{n-1}(m_{n-1,i})\dots E_{i+1}(m_{i+1,i})$, for $i = 2,3,\dots,n-1$, with m_{ij} the multipliers of the Neville elimination of L. For $i = 1$

$$E_{(1)}^{-1} = E_n(m_{n,1})E_{n-1}(m_{n-1,1})\dots \tilde{E}_3(m_{3,1}),$$

where the multiplier $m_{3,1}$ is the $(3,1)$ entry of $\tilde{E}_3$.

For $i = 1,2,\dots,n-1$, we construct

$$E_{(1)}(\delta) = \left[\begin{array}{ccc|ccccc} & E_{(1)} & & O & O & \cdots & O & O \\ \hline 0 & \cdots & -\delta & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -\delta & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & -\delta & 1 \end{array}\right], \quad \hat{E}_{(i)} = \left[\begin{array}{c|c} E_{(i)} & O \\ \hline O & I_{m-n} \end{array}\right].$$

From these matrices and for all δ we compute the $m \times m$ unit lower triangular matrix

$$\begin{aligned} L(\delta) &= E_{(1)}^{-1}(\delta)\hat{E}_{(2)}^{-1}\cdots\hat{E}_{(n-1)}^{-1} \\ &= \begin{bmatrix} L & O \\ L(\delta)_{21} & L(\delta)_{22} \end{bmatrix} \in \mathbb{R}^{(n+(m-n))\times(n+(m-n))}. \end{aligned}$$

By construction it is not difficult to see that

$$\begin{aligned} L(\delta)_{21} &= L(\delta)[n+1,n+2,\dots,m|1,2,\dots,n] \\ &= \begin{bmatrix} l_{n1}\delta & l_{n2}\delta & \cdots & l_{nn-1}\delta & \delta \\ l_{n1}\delta^2 & l_{n2}\delta^2 & \cdots & l_{nn-1}\delta^2 & \delta^2 \\ \vdots & \vdots & & \vdots & \vdots \\ l_{n1}\delta^{m-n} & l_{n2}\delta^{m-n} & \cdots & l_{nn-1}\delta^{m-n} & \delta^{m-n} \end{bmatrix}, \end{aligned}$$

where $[l_{n1}\, l_{n2}\, \dots\, l_{nn-1}\ 1]$ is the last row of L, and

$$L(\delta)_{22} = L(\delta)[n+1,n+2,\dots,m] = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \delta & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \delta^{m-n-2} & \delta^{m-n-3} & \cdots & 1 & 0 \\ \delta^{m-n-1} & \delta^{m-n-2} & \cdots & \delta & 1 \end{bmatrix}.$$

Moreover, for all $\delta > 0$, the entries $l(\delta)_{i1}$ and $l(\delta)_{i2}$, for $i = 3,4,\dots,m$, are positive and nonpositive, respectively, and the submatrix $L(\delta)[3,4,\dots,m]$ is a unit lower triangular TN matrix with positive entries under the main diagonal.

Lemma 1. *For all $\delta > 0$, $\alpha \in \mathcal{Q}_{k,n}$ and $k = 2,3,\dots,n$, the matrix $L(\delta) \in \mathbb{R}^{m\times m}$ from Procedure 1 verifies that*

$$\det L(\delta)[\alpha|1,2,\dots,k] = \begin{cases} \geq 0 & \text{if} \quad \alpha_1 = 1,\ \alpha_2 = 2 \\ \leq 0 & \text{if} \quad 1 \text{ or } 2 \notin \alpha \end{cases}$$

Proof. By construction of the matrix $L(\delta)$ and by [6, Remark 1] the result is straightforward. $\square$

Example 2. *Consider the unit lower triangular matrix*

$$L = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 2 & 0 & 1 & 0 \\ 6 & 0 & 5 & 1 \end{array}\right] = \left[\begin{array}{cc} I_2 & O \\ L_{21} & L_{22} \end{array}\right].$$

By Procedure 1 we construct a 6×6 *unit lower triangular matrix.*

First, we factorize L as

$$\begin{aligned} L = E_{(1)}^{-1} E_{(2)}^{-1} E_{(3)}^{-1} &= \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]^{-1} (I_4)^{-1} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]^{-1} \\ &= \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 6 & 0 & 3 & 1 \end{array}\right] I_4 \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]. \end{aligned}$$

Then, for δ and $i = 1, 2, 3$, we construct the matrices

$$\begin{aligned} E_{(1)}(\delta) &= \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -\delta & 1 & 0 \\ 0 & 0 & 0 & 0 & -\delta & 1 \end{array}\right], \quad \hat{E}_{(2)} = I_6, \\ \hat{E}_{(3)} &= \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]. \end{aligned}$$

Finally, from these matrices we compute the 6×6 unit lower triangular matrix

$$\begin{aligned} L(\delta) &= E_{(1)}^{-1}(\delta) \hat{E}_{(2)}^{-1} \hat{E}_{(3)}^{-1} \\ &= \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 5 & 1 & 0 & 0 \\ \hline 6\delta & 0 & 5\delta & \delta & 1 & 0 \\ 6\delta^2 & 0 & 5\delta^2 & \delta^2 & \delta & 1 \end{array}\right]. \end{aligned}$$

Let us now extend the matrix U. For that, it is necessary to consider two cases.

CASE 1. The n first columns of the matrix U are linearly independent.

Procedure 2. *Let $U = [U_{11}\ U_{12}] \in \mathbb{R}^{n\times(n+(m-n))}$ be a unit upper echelon TN matrix, with positive entries from the leading entry in each row and where U_{11} is a unit upper triangular matrix. This Procedure constructs from U a unit upper triangular TN matrix $\hat{U} \in \mathbb{R}^{m\times m}$ with the following structure*

$$\hat{U} = \begin{bmatrix} U_{11} & U_{12} \\ O & \hat{U}_{22} \end{bmatrix} \in \mathbb{R}^{(n+(m-n))\times n+(m-n))}$$

where $\hat{U}_{22}$ is a unit upper triangular TN matrix.

Matrix U can be factorized as follows

$$U = [\ I_{n\times n}\ O_{n\times(m-n)}\ ]F_{(n)}^{-1}F_{(n-1)}^{-1}\dots F_{(1)}^{-1},$$

where, for $i = 1, 2, \dots, n$,

$$F_{(i)}^{-1} = F_{i+1}(m_{i,i+1})F_{i+2}(m_{i,i+2})\dots F_m(m_{i,m}),$$

with m_{ij} the multipliers of the Neville elimination of U.

We construct the unit upper triangular matrix $\tilde{U} \in \mathbb{R}^{m\times m}$ by the product

$$\tilde{U} = F_{(n)}^{-1}F_{(n-1)}^{-1}\dots F_{(1)}^{-1}.$$

Note that by construction,

$$\tilde{U} = \begin{bmatrix} U_{11} & U_{12} \\ O & \tilde{U}_{22} \end{bmatrix} \in \mathbb{R}^{(n+(m-n))\times n+(m-n))}$$

is a TN matrix.

Example 3. *Consider the unit upper echelon TN matrix*

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} = [\ U_{11}\ U_{12}\ ] \in \mathbb{R}^{3\times(3+2)}.$$

By Procedure 2 we construct a 5×5 unit upper triangular TN matrix $\hat{U}$.

The matrix U can be written as

$$U = [\ I_{3\times 3}\ O_{3\times 2}\ ]F_{(3)}^{-1}F_{(2)}^{-1}F_{(1)}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{F_{(1)}^{-1}},$$

with $F_{(2)}^{-1} = F_{(3)}^{-1} = I_{5\times 5}$. By Procedure 2, the unit upper triangular TN matrix $\hat{U}$ is

$$\hat{U} \quad = \quad F_{(3)}^{-1}F_{(2)}^{-1}F_{(1)}^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ O & \tilde{U}_{22} \end{bmatrix} \in \mathbb{R}^{(3+2)\times 5}.$$

From the previous procedures we obtain the following result.

Theorem 2. *Let $A = \tilde{L}DU \in \mathbb{R}^{n\times m}$ be an $n \times m$ matrix, with negative entries except for $a_{11} = 0$ and $a_{nm} \leq 0$. Consider that $U = [U_{11}\ U_{12}] \in \mathbb{R}^{n\times(n+(m-n))}$ is an upper echelon TN matrix with positive entries above the main diagonal and U_{11} is a unit upper triangular matrix, $D = diag(-d_1, -d_2, d_3, \ldots, d_n)$ with $d_i > 0$, $i = 1, 2, \ldots, n$ and $\tilde{L} \in \mathbb{R}^{n\times n}$ is a unit lower triangular matrix*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad with \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where the entries in the first column of $\tilde{L}_{21}$ are positive, in the second one are nonpositive, $\tilde{L}_{22}$ is unit lower triangular TN matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \leq 0,\ \forall \alpha \in \mathcal{Q}_{k,n},\ \ k = 2, 3, \ldots, n.$$

Then, A is a t.n.p. matrix.

Proof. First of all suppose that $a_{nm} < 0$. Consider the matrix $B = PA = P\tilde{L}DU = LDU$, where the lower triangular matrix L satisfies the conditions of Lemma 1.

Consider the $m\times m$ matrices $L(\delta)$ obtained by Procedure 1, $D(\delta) = \mathrm{diag}(-d_1, -d_2, d_3, \ldots, d_n, \delta^3, \delta^4, \ldots, \delta^{m-n+2})$ and $\hat{U}$ obtained by Procedure 2.

From these matrices we construct

$$\begin{aligned} B(\delta) &= L(\delta)D(\delta)\hat{U} = \begin{bmatrix} L & O \\ L(\delta)_{21} & L(\delta)_{22} \end{bmatrix} \begin{bmatrix} D & O \\ O & D(\delta)_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ O & \hat{U}_{22} \end{bmatrix} \\ &= \begin{bmatrix} LDU_{11} & LDU_{12} \\ L(\delta)_{21}DU_{11} & L(\delta)_{21}DU_{12} + L(\delta)_{22}D(\delta)_{22}\hat{U}_{22} \end{bmatrix}. \end{aligned}$$

Note that,

$$B(\delta)(m, m) = a_{nm}\,\delta^{m-n} + k\,\delta^{m-n+2}, \quad k > 0.$$

Then, since $a_{nm} < 0$, there exists $\delta_0 > 0$ such that $B(\delta)(m, m) < 0$ for all $\delta < \delta_0$. Hence, by the permutation matrix $\tilde{P} = [2,\ 1,\ 3,\ \ldots,\ m]$ we obtain

$$\begin{aligned} A(\delta) &= \tilde{P}B(\delta) = (\tilde{P}L(\delta))D(\delta)\hat{U} \\ &= \begin{bmatrix} \tilde{L}DU_{11} & \tilde{L}DU_{12} \\ L(\delta)_{21}DU_{11} & L(\delta)_{21}DU_{12} + L(\delta)_{22}D(\delta)_{22}\hat{U}_{22} \end{bmatrix}, \end{aligned}$$

which is a t.n.p. matrix by [6, Theorem 2]. Thus, $A = A(\delta)[1, 2, \ldots, n|1, 2, \ldots, m]$ is t.n.p.

Suppose now that $a_{nm} = 0$, and consider

$$B_x = L \begin{bmatrix} -d_1 & & & & \\ & -d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_n - x \end{bmatrix} U,$$

where its (n, m) entry is $-x$. Therefore, for $0 < x < d_n$, we can apply the results obtained when $a_{nm} < 0$.

Nevertheless, if $A_x = PB_x$ we have, for all $0 < x < d_r$, that

$$\det A_x[\alpha|\beta] = s_{\alpha,\beta}x + t_{\alpha,\beta} \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n},\ \beta \in \mathcal{Q}_{k,m},\ k = 1, 2, \ldots, n,$$

where

$$s_{\alpha,\beta} = \det A[\alpha|\beta] \quad \forall \alpha \in \mathcal{Q}_{k,n},\ \beta \in \mathcal{Q}_{k,m},\ k = 1, 2, \ldots, n.$$

If $t_{\alpha,\beta} = 0$ then $\det A[\alpha|\beta] < 0$. Otherwise, i.e. $t_{\alpha,\beta} \neq 0$, since $\det A_x[\alpha|\beta] \leq 0$ for all positive $x < d_r$ by continuity $s_{\alpha,\beta} = \det A[\alpha|\beta] \leq 0$. Thus, A is t.n.p. □

CASE 2. The n first columns of U are not linearly independent. Remind that since $a_{12} < 0$, the first and second columns are linearly independent.

Procedure 3. *Let $U \in \mathbb{R}^{n \times m}$ be a unit upper echelon TN matrix, with positive entries from the leading entry in each row and with the first and second columns linearly independent. This Procedure constructs for all $\delta > 0$ an upper echelon TN matrix $U(\delta) \in \mathbb{R}^{n \times m}$ with its n first columns linearly independent and $\lim_{\delta \to 0} U(\delta) = U$.*

Consider the matrix $Q = U^T \in \mathbb{R}^{m \times n}$ and suppose that we can apply it the Neville elimination process with no pivoting until the kth iteration. Then,

$$E_{(k)}E_{(k-1)} \ldots E_{(2)}E_{(1)}Q = Q_k = \begin{bmatrix} I_k & O \\ O & Q_{k_{22}} \end{bmatrix},$$

where the first nonzero entry in its $(k+1)$st column is below the main diagonal. Now, before applying the $(k+1)$st iteration of Neville elimination to matrix Q_k, we replace in its $(k+1)$ column the zero entries from the $(k+1, k+1)$ position to the first nonzero entry by δ^{j-s}, where j is the row index of this nonzero entry and $s = k+1, k+2, \ldots, j-1$. We call the new matrix $\tilde{Q}_k(\delta)$ and apply the Neville elimination process with no pivoting to obtain $E_{(k+1)}\tilde{Q}_k(\delta) = Q_{k+1}(\delta)$. From $Q_{k+1}(\delta)$ we construct, if it is necessary and in a similar way, the matrix $\tilde{Q}_{k+1}(\delta)$, and apply Neville to obtain $Q_{k+2}(\delta)$ and so on to matrix $Q_n(\delta)$. Then,

$$Q(\delta) = E_{(1)}^{-1}E_{(2)}^{-1} \ldots E_{(k)}^{-1}E_{(k+1)}^{-1}(\delta)E_{(k+2)}^{-1}(\delta) \ldots E_{(n)}^{-1}(\delta)Q_n(\delta) \in \mathbb{R}^{m \times n}$$

is a lower echelon TN matrix with the n first rows linearly independent for all $\delta > 0$, and

$$Q(\delta) = Q + \left[\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & p_{ij}(\delta) & \\ 0 & 0 & & & \end{array}\right]$$

where $p_{ij}(\delta)$ are polynomials in δ with nonnegative coefficients and satisfying

$$\lim_{\delta \to 0} p_{ij}(\delta) = 0 \quad \Longrightarrow \quad \lim_{\delta \to 0} Q(\delta) = Q.$$

Therefore, $U(\delta) = Q(\delta)^T$ is an upper echelon TN matrix with the n first columns linearly independent for all $\delta > 0$.

Example 4. *Consider the following upper echelon TN matrix*

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By Procedure 3 we construct an upper echelon TN matrix of size 4×6, $U(\delta)$, with its four first columns linearly independent.

Applying Procedure 3 to matrix $Q = U^T$ we obtain,

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix} \quad \xrightarrow{E_{(1)}} \quad Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{E_{(2)}=I}$$

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \xrightarrow{\text{column } 3} \quad \tilde{Q}_2(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta^2 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \xrightarrow{E_{(3)}(\delta)}$$

$$Q_3(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{column } 4} \tilde{Q}_3(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta^2 & 0 \\ 0 & 0 & 0 & \delta^2 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_{(4)}(\delta)}$$

$$Q_4(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta^2 & 0 \\ 0 & 0 & 0 & \delta^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The lower echelon TN matrix $Q(\delta)$ is

$$\begin{aligned} Q(\delta) &= E_{(1)}^{-1}E_{(2)}^{-1}E_{(3)}^{-1}(\delta)E_{(4)}^{-1}(\delta)Q_4(\delta) \qquad (1) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & \delta^2 & 0 \\ 1 & 1 & \delta^2+\delta & \delta^2 \\ 1 & 1 & \delta^2+\delta+1 & \delta^2+2\delta \\ 1 & 1 & \delta^2+\delta+2 & \delta^2+4\delta+1 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}}_{Q} + \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \delta^2 & 0 \\ 0 & 0 & \delta^2+\delta & \delta^2 \\ 0 & 0 & \delta^2+\delta & \delta^2+2\delta \\ 0 & 0 & \delta^2+\delta & \delta^2+4\delta \end{array}\right]. \end{aligned}$$

Then, the upper echelon TN matrix with its four first columns linearly independent is

$$U(\delta) = Q(\delta)^T = \left[\begin{array}{cc|cccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & \delta^2 & \delta^2+\delta & \delta^2+\delta+1 & \delta^2+\delta+2 \\ 0 & 0 & 0 & \delta^2 & \delta^2+2\delta & \delta^2+4\delta+1 \end{array}\right].$$

From Procedure 3 and Theorem 2 the following result is deduced.

Theorem 3. *Consider $A = \tilde{L}DU \in \mathbb{R}^{n\times m}$ with negative entries except for $a_{11} = 0$ and $a_{nm} \leq 0$, where $U \in \mathbb{R}^{n\times m}$ is a unit upper echelon TN matrix with positive entries from the leading entry in each row and with the first and second columns linearly independent, $D = diag(-d_1, -d_2, d_3, \ldots, d_n)$ with $d_i > 0$, $i = 1, 2, \ldots, n$ and $\tilde{L} \in \mathbb{R}^{n\times n}$ is a block lower triangular matrix*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \textit{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where the entries in the first column of $\tilde{L}_{21}$ are positive, in the second one are nonpositive, $\tilde{L}_{22}$ is a unit lower triangular TN matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}[\alpha|1,2,\ldots,k] \leq 0,\ \forall\alpha \in \mathcal{Q}_{k,n},\ k=2,3,\ldots,n.$$

Then, A is a t.n.p. matrix.

Proof. From U and by applying Procedure 3 we construct an upper echelon TN matrix, with positive entries above the main diagonal,

$$U(\delta) = U + \underbrace{\left[\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & p_{ij}(\delta) & \\ 0 & 0 & & & \end{array}\right]}_{P(\delta)} = [U(\delta)_{11}\ U(\delta)_{12}],$$

where $p_{ij}(\delta)$ are polynomials in δ with nonnegative coefficients and such that $\lim_{\delta\to 0} p_{ij}(\delta) = 0$ and $U(\delta)_{11}$ is an upper triangular matrix.

Now, we construct

$$A(\delta) = \tilde{L}DU(\delta) = \tilde{L}DU + \tilde{L}DP(\delta),$$

whose (n,m) entry is $A(\delta)(n,m) = a_{nm} + p_s(\delta)$, being $p_s(\delta)$ a polynomial in δ, of degree $s \geq 1$, with nonnegative coefficients and such that $\lim_{\delta\to 0} p_s(\delta) = 0$.

If $a_{nm} < 0$, then there exists δ_0 such that $A(\delta)(n,m) \leq 0$, for all $\delta \leq \delta_0$. Hence, by Theorem 2, $A(\delta)$ is a t.n.p. matrix, that is

$$\det A(\delta)[\alpha|\beta] \leq 0, \quad \forall\alpha \in \mathcal{Q}_{k,n},\ \forall\beta \in \mathcal{Q}_{k,m},\ k=1,2,\ldots,n$$

But, since

$$\det A[\alpha|\beta] = \lim_{\delta\,\to\,0} \det A(\delta)[\alpha|\beta] \leq 0,\ \forall\alpha \in \mathcal{Q}_{k,n},\ \forall\beta \in \mathcal{Q}_{k,m},\ k=1,2,\ldots,n$$

we have that A is also t.n.p.

On the other hand, when $a_{nm} = 0$, we can suppose, without loss of generality, that $a_{nj} < 0$, $\forall j < m$ (otherwise, we know that these columns are linearly dependent) and proceed as in Theorem 2. □

Combining Theorems 1, 2 and 3 we characterize the rectangular t.n.p. matrices with the $(1,1)$ entry equal to zero and full row rank.

Theorem 4. *Let A be an $n \times m$ matrix with negative entries except for $a_{11} = 0$ and $a_{nm} \leq 0$, and full row rank. Then, A is a t.n.p. matrix if and only if A has a unique factorization $\tilde{L}DU$, where $U \in \mathbb{R}^{n\times m}$ is a unit upper echelon TN matrix with positive entries from the leading entry in each row and the two first* *columns linearly independent, $D = diag(-d_1, -d_2, d_3, \ldots, d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$ and $\tilde{L} \in \mathbb{R}^{n \times n}$ is a block lower triangular matrix*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \textit{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the entries in the first column of $\tilde{L}_{21}$ are positive and in the second one nonpositive, $\tilde{L}_{22}$ is a unit lower triangular TN matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n}, \; k = 2, 3, \ldots, n.$$

4. Matrices without full row rank

Now, we extend the results obtained in the previous section for rectangular t.n.p. matrices with the $(1, 1)$ entry equal to zero and full row rank to matrices $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ with $\operatorname{rank}(A) = r < n < m$.

Theorem 5. *Let A be an $n \times m$ t.n.p. matrix with negative entries except for $a_{11} = 0$ and $a_{nm} \leq 0$, and rank$(A) = r < n < m$. Then, A has a unique full rank factorization $\tilde{L}DU$, where $U \in \mathbb{R}^{r \times m}$ is a unit upper echelon TN matrix with positive entries from the leading entry in each row, $D = diag(-d_1, -d_2, d_3, \ldots, d_r)$ with $d_i > 0$, for $i = 1, 2, \ldots, r$ and $\tilde{L} \in \mathbb{R}^{n \times r}$ is a block lower echelon matrix*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \textit{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the entries in the first column of $\tilde{L}_{21}$ are positive and in the second one nonpositive, $\tilde{L}_{22}$ is a unit lower echelon TN matrix with positive entries under the leading entry in each column, and such that

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n}, \; k = 2, 3, \ldots, r.$$

Proof. Let $A_1 = A[1, 2, i_3, \ldots, i_r|1, 2, \ldots, m] \in \mathbb{R}^{r \times m}$ be the matrix formed by the r first linear independent rows of A. Then, there exists a unique unit lower echelon matrix F_1 such that $A = F_1 A_1$. Since A_1 is a t.n.p. matrix, with the entry in position $(1, 1)$ equal to zero and full row rank, by Theorem 1 it has a unique full rank factorization $\tilde{L}_{A_1} D_{A_1} U_{A_1}$, where $U_{A_1} \in \mathbb{R}^{r \times m}$ is a unit upper echelon TN matrix, $D_{A_1} = \operatorname{diag}(-d_1, -d_2, d_3, \ldots, d_r)$ with $d_i > 0$, for $i = 1, 2, \ldots, r$ and $\tilde{L}_{A_1} \in \mathbb{R}^{r \times r}$ is a block lower triangular matrix

$$\tilde{L}_{A_1} = \begin{bmatrix} \tilde{L}_{A_{1_{11}}} & O \\ \tilde{L}_{A_{1_{21}}} & \tilde{L}_{A_{1_{22}}} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{A_{1_{11}}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the entries in the first column of $\tilde{L}_{A_{1_{21}}}$ are positive, in the second one are nonpositive, $\tilde{L}_{A_{1_{22}}}$ is a unit lower triangular TN matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}_{A_1}[\alpha|1, 2, \ldots, k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,r}, \; k = 2, 3, \ldots, r.$$

Thereby, A admits the unique full rank factorization

$$A = F_1 A_1 = \left(F_1 \tilde{L}_{A_1}\right) D_{A_1} U_{A_1} = \tilde{L} D U,$$

where $\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix} \in \mathbb{R}^{n \times r}$, with $\tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\tilde{L}_{22}$ is a unit lower echelon matrix, $D = D_{A_1} \in \mathbb{R}^{r \times r}$ and $U = U_{A_1} \in \mathbb{R}^{r \times m}$. Since $a_{21} < 0$ and $a_{12} < 0$, the matrix $\tilde{L}$ satisfies

- For $i = 3, 4, \ldots, n$,

$$\begin{aligned} a_{i1} = \det A[i|1] &= \sum_{j=1}^{n} \det \tilde{L}[i|j] \det(DU)[j|1] = -d_1 \det \tilde{L}[i|1] \\ &= -d_1 \tilde{L}(i,1) < 0 \quad \Longrightarrow \quad \tilde{L}(i,1) > 0. \end{aligned}$$

- For all $\{i_1, i_2\} \in \mathcal{Q}_{2,n}$,

$$\begin{aligned} \det A[i_1, i_2|1,2] &= \sum_{\forall \gamma \in \mathcal{Q}_{2,r}} \det \tilde{L}[i_1, i_2|\gamma] \det(DU)[\gamma|1,2] \\ &= (-d_1)(-d_2) \det \tilde{L}[i_1, i_2|1,2] \leq 0 \\ &\Longrightarrow \det \tilde{L}[i_1, i_2|1,2] \leq 0. \end{aligned}$$

  In particular, if $i_1 = 2$ we have for $i_2 = 3, 4, \ldots, n$, that

$$\det \tilde{L}[2, i_2|1,2] = \tilde{L}(i_2, 2) \leq 0.$$

- Since $U \in \mathbb{R}^{r \times m}$ is an upper echelon matrix with rank r, we suppose that its linear independent columns are $\{1, 2, j_3, j_4, \ldots, j_r\}$, with $3 \leq j_3 \leq j_4 \leq \ldots \leq j_r \leq m$. Then, for all $\alpha \in \mathcal{Q}_{s,n}$ and for $3 \leq s \leq r$, we have that

$$\begin{aligned} \det A[\alpha|1, 2, j_3, \ldots, j_s] &= \sum_{\forall \gamma \in \mathcal{Q}_{s,r}} \det \tilde{L}[\alpha|\gamma] \det(DU)[\gamma|1, 2, j_3, \ldots, j_s] \\ &= (-d_1)(-d_2) d_3 \ldots d_s \det \tilde{L}[\alpha|1, 2, 3, \ldots, s] \leq 0 \\ &\Longrightarrow \det \tilde{L}[\alpha|1, 2, 3, \ldots, s] \leq 0. \end{aligned}$$

- The submatrix $\tilde{L}_{22} \in \mathbb{R}^{(n-2) \times (r-2)}$, with full column rank satisfies $\forall \alpha \in \mathcal{Q}_{k,n-2}$ and $k = 1, 2, \ldots, r-2$,

$$\begin{aligned} &\det \tilde{L}_{22}[\alpha|1, 2, \ldots, k] = \det \tilde{L}_{22}[\alpha_1, \alpha_2, \ldots, \alpha_k|1, 2, \ldots, k] \\ &= \ - \det \tilde{L}[1, 2, \alpha_1 + 2, \alpha_2 + 2, \ldots, \alpha_k + 2|1, 2, \ldots, k+2] \geq 0, \end{aligned}$$

  which implies that $\tilde{L}_{22}$ is TN [5]. □

The converse of Theorem 5 is not true in general, as the next example shows.

Example 5. *The matrix*

$$A = \tilde{L}DU \quad = \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 & -1 & -1 & -1 \\ -2 & -4 & -2 & -2 & -2 \\ -4 & -7 & -3 & -3 & -2 \\ -6 & -10 & -4 & -4 & 2 \end{bmatrix}.$$

is not a t.n.p. matrix although the matrices $\tilde{L}$, D and U satisfy the conditions of Theorem 5.

To prove the converse of Theorem 5 we need the following procedure.

Procedure 4. *Let $Q \in \mathbb{R}^{n\times r}$ be a unit lower echelon TN matrix, with rank $r < n$ and the first and second rows linearly independent,*

$$Q = \begin{bmatrix} Q_{11} & O \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathbb{R}^{(2+(n-2))\times(2+(r-2))}.$$

For all $\delta > 0$, this procedure allows us to construct a lower echelon TN matrix $Q(\delta)$ of size $n \times p$, $r \leq p \leq n$, such that its p first rows are linearly independent and

$$\lim_{\delta \to 0} Q(\delta) = [Q \ O] \in \mathbb{R}^{n\times(r+(p-r))}.$$

Applying Procedure 3 to matrix Q we obtain $Q_r(\delta)$. If $p > r$, matrix $Q_{r+1}(\delta)$ is made up from $Q_r(\delta)$ adding to this matrix a new column which differs from the zero column only in its $(r+1)$ entry δ^{n-r}. If $p > r+1$, we proceed in a similar way, that is, matrix $Q_{r+2}(\delta)$ is made up from $Q_{r+1}(\delta)$ adding to this matrix a new column which differs from the zero column only in its $(r+2)$ entry δ^{n-r-1}, and so on to arrive to matrix $Q_p(\delta)$.

Then,

$$Q(\delta) = E^{-1}_{(1)}E^{-1}_{(2)} \cdots E^{-1}_{(k)}E^{-1}_{(k+1)}(\delta)E^{-1}_{(k+2)}(\delta) \cdots E^{-1}_{(r)}(\delta)Q_p(\delta) \in \mathbb{R}^{n\times p}$$

is a lower echelon TN matrix with the p first rows linearly independent for all $\delta > 0$. Moreover, $Q(\delta) = [Q(\delta)_1 \ Q(\delta)_2] \in \mathbb{R}^{n\times(r+(p-r))}$ satisfies that

$$Q(\delta)_1 = Q + \left[\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & p_{ij}(\delta) & \\ 0 & 0 & & & \end{array}\right]$$

where $p_{ij}(\delta)$ are polynomials in δ with nonnegative coefficients and satisfying

$$\lim_{\delta\to 0} p_{ij}(\delta) = 0 \quad \Longrightarrow \quad \lim_{\delta\to 0} Q(\delta)_1 = Q$$

and

$$Q(\delta)_2 = \begin{bmatrix} O \\ Q(\delta)_{2_2} \end{bmatrix} \in \mathbb{R}^{(r+(n-r))\times(p-r)}$$

where $Q(\delta)_{2_2} = [h_{ij}(\delta)] \in \mathbb{R}^{(n-r)\times(p-r)}$ is a lower echelon matrix, with its $(n-p)$ first rows linearly independent and $\lim_{\delta\to 0} h_{ij}(\delta) = 0$. Therefore,

$$\lim_{\delta \to 0} Q(\delta) = [Q\ O] \in \mathbb{R}^{n\times(r+(p-r))}$$

Example 6. *Consider the following lower echelon TN matrix*

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

Following Procedure 4 we construct a nonsingular lower triangular TN matrix $Q(\delta)$ *of size* 6×6.

In example 4, by applying Procedure 3 to Q, we obtained the matrix,

$$Q_4(\delta) = E_{(4)}(\delta)E_{(3)}(\delta)E_{(2)}E_{(1)}Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta^2 & 0 \\ 0 & 0 & 0 & \delta^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{6\times 4}.$$

Now, applying Procedure 4 to $Q_4(\delta)$ we have

$$Q_6(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$

Then,

$$\begin{aligned} Q(\delta) &= E_{(1)}^{-1}E_{(2)}^{-1}E_{(3)}^{-1}(\delta)E_{(4)}^{-1}(\delta)Q_6(\delta) \\ &= \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & \delta^2 & 0 & 0 & 0 \\ 1 & 1 & \delta^2+\delta & \delta^2 & 0 & 0 \\ 1 & 1 & \delta^2+\delta+1 & \delta^2+2\delta & \delta^2 & 0 \\ 1 & 1 & \delta^2+\delta+2 & \delta^2+4\delta+1 & 2\delta^2+\delta & \delta \end{array}\right] \\ &= [Q(\delta)_1\ Q(\delta)_2] \in \mathbb{R}^{6\times(4+2)} \end{aligned}$$

is a nonsingular lower triangular TN matrix, for all $\delta > 0$ and such that $\lim_{\delta \to 0} Q(\delta) = [Q\ O] \in \mathbb{R}^{6\times(4+2)}$.

Note that, $Q(\delta)_1$, which is equal to the matrix of the equation (1) given in Example 4, verifies that $\lim_{\delta\to 0} Q(\delta)_1 = Q$. Moreover, the matrix

$$Q(\delta)_2 = \begin{bmatrix} O \\ Q(\delta)_{2_2} \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline \delta^2 & 0 \\ 2\delta^2+\delta & \delta \end{array}\right] \in \mathbb{R}^{(4+(2))\times(2)}$$

verifies that $\lim_{\delta\to 0} Q(\delta)_2 = O$.

Remark 1. *Consider the following block lower echelon matrix $\tilde{L} \in \mathbb{R}^{n\times r}$*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \textit{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the entries in the first column of $\tilde{L}_{21}$ are positive and in the second one nonpositive, $\tilde{L}_{22}$ is unit lower echelon TN matrix with positive entries under the leading entry in each column, and such that

$$\det \tilde{L}[\alpha|1,2,\ldots,k] \le 0, \quad \forall \alpha \in \mathcal{Q}_{k,n},\ k = 2,3,\ldots,r.$$

By applying Procedure 4 to the matrix

$$L = P\tilde{L} = \begin{bmatrix} I_2 & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix},$$

we can obtain a nonsingular lower triangular matrix $L(\delta) \in \mathbb{R}^{n\times n}$ and then the matrix $\tilde{L}(\delta) = PL(\delta) = [\tilde{L}(\delta)_1\ \tilde{L}(\delta)_2] \in \mathbb{R}^{n\times(r+(n-r))}$ with

$$\tilde{L}(\delta)_1 = \tilde{L} + \left[\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & p_{ij}(\delta) & \\ 0 & 0 & & & \end{array}\right] = \begin{bmatrix} \tilde{L}(\delta)_{1_1} \\ \tilde{L}(\delta)_{1_2} \end{bmatrix} \in \mathbb{R}^{(r+(n-r))\times r},$$

and

$$\tilde{L}(\delta)_2 = \begin{bmatrix} O \\ \tilde{L}(\delta)_{2_2} \end{bmatrix} \in \mathbb{R}^{(r+(n-r))\times(n-r)}, \quad \tilde{L}(\delta)_{2_2} = [h_{ij}(\delta)].$$

Moreover, $\tilde{L}(\delta)[3,4,\ldots,n]$ is a lower triangular TN matrix with positive entries under the main diagonal and such that

$$\det \tilde{L}(\delta)[\alpha|1,2,\ldots,k] \le 0, \quad \forall \alpha \in \mathcal{Q}_{k,n},\ k = 2,3,\ldots,n.$$

and

$$\lim_{\delta\to 0} \tilde{L}(\delta) = [\tilde{L}\ O].$$

Taking into account Procedure 4 and Remark 1 we give the following result.

Theorem 6. *Consider* $A = \tilde{L}DU \in \mathbb{R}^{n\times m}$ *with negative entries except for* $a_{11} = 0$ *and* $a_{nm} \le 0$, *and rank*$(A) = r < n < m$, *where* $D = diag(-d_1, -d_2, d_3, \ldots, d_r)$, $d_i > 0$, $i = 1, 2, \ldots, r$, $U \in \mathbb{R}^{r\times m}$ *is a unit upper echelon TN matrix with positive entries from the leading entry in each row and* $\tilde{L} \in \mathbb{R}^{n\times r}$ *is a unit lower echelon matrix*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the entries in the first column of $\tilde{L}_{21}$ *are positive, in the second one are nonpositive,* $\tilde{L}_{22}$ *is unit lower echelon TN matrix with positive entries under the leading entry in each column and such that*

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \le 0, \ \forall \alpha \in \mathcal{Q}_{k,n}, \ k = 2, 3, \ldots, r.$$

and rank$(U) =$ *rank*$(L) = r < n < m$. *Then,* A *is a t.n.p. matrix.*

Proof. Suppose that $a_{nm} < 0$. Following Remark 1 we construct the nonsingular lower triangular matrix $\tilde{L}(\delta) = [\tilde{L}(\delta)_1 \ \tilde{L}(\delta)_2] \in \mathbb{R}^{n\times(r+(n-r))}$, with

$$\tilde{L}(\delta)_1 = \tilde{L} + \left[\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & p_{ij}(\delta) & \\ 0 & 0 & & & \end{array}\right] = \begin{bmatrix} \tilde{L}(\delta)_{1_1} \\ \tilde{L}(\delta)_{1_2} \end{bmatrix} \in \mathbb{R}^{(r+(n-r))\times r}$$

lower echelon with the r first rows linearly independent and $p_{ij}(\delta)$ polynomials in δ with nonnegative coefficients such that $\lim_{\delta\to 0} p_{ij}(\delta) = 0$, and

$$\tilde{L}(\delta)_2 = \begin{bmatrix} O \\ \tilde{L}(\delta)_{2_2} \end{bmatrix} \in \mathbb{R}^{(r+(n-r))\times(n-r)}, \quad \tilde{L}(\delta)_{2_2} = [h_{ij}(\delta)]$$

with $\lim_{\delta\to 0} h_{ij}(\delta) = 0$ and $\tilde{L}(\delta)[3, 4, \ldots, n]$ lower triangular TN matrix with positive entries under the main diagonal and

$$\det \tilde{L}(\delta)[\alpha|1, 2, \ldots, k] \le 0, \quad \forall \alpha \in \mathcal{Q}_{k,n}, \ k = 2, 3, \ldots, n.$$

Now, applying Procedure 4 to matrix U^T we construct a lower echelon TN matrix $U(\delta)^T \in \mathbb{R}^{n\times m}$, such that its n first rows are linearly independent. Then

$$U(\delta) = \begin{bmatrix} U(\delta)_1 \\ U(\delta)_2 \end{bmatrix} \in \mathbb{R}^{(r+(n-r))\times m}$$

where

$$U(\delta)_1 = U + \left[\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & q_{ij}(\delta) & \\ 0 & 0 & & & \end{array}\right] = [U(\delta)_{1_1} \ U(\delta)_{1_2}] \in \mathbb{R}^{r\times(r+(m-r))}$$

is an upper triangular matrix, $q_{ij}(\delta)$ are polynomials in δ with nonnegative coefficients such that $\lim_{\delta\to 0} q_{ij}(\delta) = 0$, and

$$U(\delta)_2 = [O\ U(\delta)_{2_2}] \in \mathbb{R}^{(n-r)\times(r+(m-r))}$$

with $U(\delta)_{2_2} = [g_{ij}(\delta)]$ upper echelon TN matrix with its $(n-r)$ first columns linearly independent and $\lim_{\delta\to 0} g_{ij}(\delta) = 0$.

Consider $D(\delta) = \mathrm{diag}(-d_1, -d_2, d_3, \dots, d_r, \delta, \delta, \dots, \delta)$. Then,

$$\begin{aligned} A(\delta) &= \tilde{L}(\delta)D(\delta)U(\delta) \\ &= \begin{bmatrix} \tilde{L}(\delta)_{1_1} & O \\ \tilde{L}(\delta)_{1_2} & \tilde{L}(\delta)_{2_2} \end{bmatrix} \begin{bmatrix} D & O \\ O & \delta I_{n-r} \end{bmatrix} \begin{bmatrix} U(\delta)_{1_1} & U(\delta)_{1_2} \\ O & U(\delta)_{2_2} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{L}(\delta)_{1_1} DU(\delta)_{1_1} & \tilde{L}(\delta)_{1_1} DU(\delta)_{1_2} \\ \tilde{L}(\delta)_{1_2} DU(\delta)_{1_1} & \tilde{L}(\delta)_{1_2} DU(\delta)_{1_2} + \delta \tilde{L}(\delta)_{2_2} U(\delta)_{2_2} \end{bmatrix}. \end{aligned}$$

It is not difficult to see that $A(\delta)(n,m) = a_{nm} + H_{nm}(\delta)$, with $H_{nm}(\delta)$ a polynomial in δ with nonnegative coefficients such that $\lim_{\delta\to 0} H_{nm}(\delta) = 0$. Since $a_{nm} < 0$, there exists δ_0 such that $A(\delta)(n,m) < 0$ for all $\delta < \delta_0$. By Theorem 2, $A(\delta)$ is t.n.p. for all $\delta < \delta_0$, then

$$\det A(\delta)[\alpha|\beta] \le 0 \quad \forall \alpha \in \mathcal{Q}_{k,n},\ \beta \in \mathcal{Q}_{k,m},\ k = 1, 2, \dots, n.$$

Since $\det A[\alpha|\beta] = \lim_{\delta\to 0} \det A(\delta)[\alpha|\beta] \le 0$ for all $\alpha \in \mathcal{Q}_{k,n},\ \beta \in \mathcal{Q}_{k,m},\ k = 1, 2, \dots, n$, hence A is t.n.p. and obviously with $\mathrm{rank}(A) = r$.

Finally, in the case $a_{nm} = 0$, proceeding as in the proof of Theorem 2 we deduce that A is t.n.p. □

Combining Theorems 5 and 6 we characterize the rectangular t.n.p. matrices with the $(1,1)$ entry equal to zero and arbitrary rank.

Theorem 7. *Let A be an $n \times m$ matrix with negative entries except for $a_{11} = 0$ and $a_{nm} \le 0$, and rank(A) $= r < n < m$. Then, A is a t.n.p. matrix if and only if A has a unique full rank factorization $\tilde{L}DU$, where $U \in \mathbb{R}^{r\times m}$ is a unit upper echelon TN matrix with positive entries from the leading entry in each row, $D = diag(-d_1, -d_2, d_3, \dots, d_r)$ with $d_i > 0$, for $i = 1, 2, \dots, r$ and $\tilde{L} \in \mathbb{R}^{n\times r}$ is a block lower echelon matrix*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \textit{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the entries in the first column of $\tilde{L}_{21}$ are positive and in the second one nonpositive, $\tilde{L}_{22}$ is a unit lower echelon TN matrix with positive entries under the leading entry in each column, and such that

$$\det \tilde{L}[\alpha|1, 2, \dots, k] \le 0, \quad \forall \alpha \in \mathcal{Q}_{k,n},\ k = 2, 3, \dots, r.$$

5. Quasi-bidiagonal factorization of a t.n.p. matrix

In [6] the authors obtain a unique quasi-LDU factorization of nonsingular t.n.p. matrices with the $(1,1)$ entry equal to zero. In the previous sections we have obtained the extension of this factorization for the rectangular case. Now, from these results we construct a quasi-bidiagonal factorization of nonsingular and rectangular t.n.p. matrices with the $(1,1)$ entry equal to zero.

5.1. Nonsingular t.n.p. matrices

Theorem 8. *Let A be an $n \times n$ nonsingular matrix with negative entries except for $a_{11} = 0$ and $a_{nn} \leq 0$. Then, A is a t.n.p. matrix if and only if A admits a unique quasi-bidiagonal factorization*

$$A = PF_{n-1}F_{n-2}\dots F_1 D G_1 G_2 \dots G_{n-1} \tag{2}$$

where $D = diag(-d_1, -d_2, d_3, \dots, d_n)$, with $d_i > 0$ for $i = 1, 2, \dots, n$ and P is the permutation matrix $P = [2,\ 1,\ 3,\ \dots,\ n]$. G_k, for $k = 1, 2, \dots, n-1$, are the unit upper bidiagonal TN matrices defined by

$$G_k = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1 & \alpha_{1,k+1} & & & \\ & & & & \ddots & & & \\ & & & & & 1 & \alpha_{n-k-1,n-1} & \\ & & & & & & 1 & \alpha_{n-k,n} \\ & & & & & & & 1 \end{pmatrix} \tag{3}$$

with $\alpha_{1,2} > 0, \dots, \alpha_{1,n} > 0$ and if $\alpha_{s,t} = 0$ then $\alpha_{s,h} = 0,\ \forall\, h > t$.

For $k = 3, 4, \dots, n-1$, F_k are the unit lower bidiagonal TN matrices,

$$F_k = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & \beta_{1,k+1} & 1 & & & \\ & & & & \ddots & & \\ & & & & \beta_{n-k-1,n-1} & 1 & \\ & & & & & \beta_{n-k,n} & 1 \end{pmatrix} \tag{4}$$

and

$$F_2 = \begin{pmatrix} 1 & & & & & & & & & \\ 0 & 1 & & & & & & & & \\ \beta_{13} & 0 & 1 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & 1 & & & & & \\ & & & & \beta_{k-1,k+1} & & 1 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & \beta_{n-3,n-1} & 1 & \\ & & & & & & & & \beta_{n-2,n} & 1 \end{pmatrix}, \tag{5}$$

$$F_1 = \begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & -\beta_{23} & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & & & & \\ & & & & \beta_{k,k+1} & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & \beta_{n-2,n-1} & 1 & \\ & & & & & & & \beta_{n-1,n} & 1 \end{pmatrix} \tag{6}$$

with $\beta_{1,3} > 0, \dots, \beta_{1,n} > 0$ *and if* $\beta_{s,t} = 0$ *then* $\beta_{s,h} = 0,\ \forall\, h > t$.

Proof. By [6], A admits a unique factorization $\tilde{L}DU$, where $U \in \mathbb{R}^{n\times n}$ is a unit upper triangular TN matrix with positive entries from the leading entry in each row and the two first columns linearly independent, $D = \mathrm{diag}(-d_1, -d_2, d_3, \dots, d_n)$ with $d_i > 0$, for $i = 1, 2, \dots, n$ and $\tilde{L} \in \mathbb{R}^{n\times n}$ is a block lower triangular matrix

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the entries in the first column of $\tilde{L}_{21}$ are positive and in the second one nonpositive, $\tilde{L}_{22}$ is a unit lower triangular TN matrix with positive entries under the main diagonal, and such that

$$\det \tilde{L}[\alpha|1, 2, \dots, k] \le 0, \quad \forall \alpha \in \mathcal{Q}_{k,n},\ k = 2, 3, \dots, n.$$

Since $U \in \mathbb{R}^{n\times n}$ is a unit upper triangular TN matrix, from [9] it admits the unique nonnegative bidiagonal factorization

$$U = [E_n^T(\alpha_{n-1,n}) \dots E_3^T(\alpha_{2,3}) \dots E_2^T(\alpha_{1,2})] \dots [E_n^T(\alpha_{2,n}) E_{n-1}^T(\alpha_{1,n-1})][E_n^T(\alpha_{1,n})],$$

where $\alpha_{1,2} > 0, \dots, \alpha_{1,n} > 0$ and if $\alpha_{s,t} = 0$ then $\alpha_{s,h} = 0,\ \forall\, h > t$.

The matrix $P\tilde{L}$ is not a TN matrix but, taking into account its properties, we can assure that it also admits a unique quasi-bidiagonal factorization in the following form

$$\begin{aligned} P\tilde{L} &= [E_n(\beta_{1,n})][E_{n-1}(\beta_{1,n-1})E_n(\beta_{2,n})]\dots[\tilde{E}_3(\beta_{1,3})E_4(\beta_{2,4})\dots E_n(\beta_{n-2,n})] \\ & \quad [E_3(-\beta_{2,3})E_4(\beta_{3,4})\dots E_n(\beta_{n-1,n})] \end{aligned}$$

where $\beta_{1,3} > 0, \dots, \beta_{1,n} > 0$ and if $\beta_{s,t} = 0$ then $\beta_{s,h} = 0$, $\forall\, h > t$.

Let us define the matrices

$$\begin{aligned} G_k &= E_n^T(\alpha_{n-k,n})\dots E_{k+2}^T(\alpha_{2,k+2})E_{k+1}^T(\alpha_{1,k+1}),\ k = 1,2,\dots,n-1 \\ F_k &= E_{k+1}(\beta_{1,k+1})E_{k+2}(\beta_{2,k+2})\dots E_n(\beta_{n-k,n}),\ k = 3,4,\dots,n-1 \\ F_2 &= \tilde{E}_3(\beta_{1,3})E_4(\beta_{2,4})\dots E_n(\beta_{n-2,n}) \\ F_1 &= E_3(-\beta_{2,3})E_4(\beta_{3,4})\dots E_n(\beta_{n-1,n}) \end{aligned}$$

which expressions are given by (3), (4), (5) and (6), respectively. Since the factorization $A = \tilde{L}DU$ is unique, then A admits the unique quasi-bidiagonal factorization given by (2).

Conversely, if A has the factorization

$$A = PF_{n-1}F_{n-2}\dots F_2F_1DG_1G_2\dots G_{n-2}G_{n-1},$$

and we denote by

$$\begin{aligned} \tilde{L} &= PF_{n-1}F_{n-2}\dots F_2F_1 \\ U &= G_1G_2\dots G_{n-2}G_{n-1} \end{aligned}$$

then $A = \tilde{L}DU$, where $U \in \mathbb{R}^{n\times n}$ is a unit upper triangular TN matrix with positive entries from the leading entry in each row, $D = \mathrm{diag}(-d_1, -d_2, d_3, \dots, d_n)$ with $d_i > 0$, for $i = 1, 2, \dots, n$ and $\tilde{L} \in \mathbb{R}^{n\times n}$ is a block lower triangular matrix

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which verifies that

$$\det \tilde{L}[\alpha|1,2,\dots,k] \le 0, \quad \forall \alpha \in \mathcal{Q}_{k,n},\ k = 2,3,\dots,n.$$

Consequently, by [6, Theorem 2] we conclude that A is a t.n.p. matrix. $\square$

5.2. Rectangular t.n.p. matrices

For rectangular t.n.p. matrix A, its quasi-LDU factorization may be obtained by applying the quasi-Neville elimination process. In general, we do not know the positions of the zero pivots in the process, hence unfortunately it is not always possible to obtain a quasi-bidiagonal factorization of A in a compact form as we get in Theorem 8 for nonsingular t.n.p. matrices. For this reason, from now on, we work with matrices that satisfy the WRC condition, that is, matrices for which it is possible to apply the complete Neville elimination process with no pivoting [12].

Theorem 9. *Let A be an $n \times m$ matrix with negative entries except for $a_{11} = 0$ and $a_{nm} \leq 0$, and full row rank. If A satisfies the WRC condition then, A is a t.n.p. matrix if and only if A admits a unique quasi-bidiagonal factorization*

$$A = PF_{n-1}F_{n-2} \ldots F_1[D\ O_{n\times(m-n)}]G_1G_2 \ldots G_{m-1} \tag{7}$$

where $D = diag(-d_1, -d_2, d_3, \ldots, d_n)$, with $d_i > 0$ for $i = 1, 2, \ldots, n$, P is the permutation matrix $P = [2,\ 1,\ 3,\ \ldots,\ n]$, F_j, for $j = 1, 2, \ldots, n-1$, are given by expressions (4), (5) and (6) and G_j, for $j = 1, 2, \ldots, m-1$, are the unit upper bidiagonal TN matrices

$$G_j = E^T_{j+n}(\alpha_{n,j+n})E^T_{j+n-1}(\alpha_{n-1,j+n-1}) \ldots E^T_{j+1}(\alpha_{1,j+1}),\ \ j = 1, 2, \ldots, m-n,$$
$$G_j = E^T_{m}(\alpha_{m-j,m})E^T_{m-1}(\alpha_{m-j-1,m-1}) \ldots E^T_{j+1}(\alpha_{1,j+1}),\ \ j = m-n+1, \ldots, m-1,$$

with $\alpha_{1,2} > 0, \ldots, \alpha_{1,n} > 0$ and if $\alpha_{s,t} = 0$ then $\alpha_{s,h} = 0$, $\forall\, h > t$.

Proof. By Theorem 4, A admits a quasi-LDU factorization, $A = \tilde{L}DU$. Although $P\tilde{L} \in \mathbb{R}^{n\times n}$ is not a TN matrix, it admits the unique quasi-bidiagonal factorization

$$P\tilde{L} = F_{n-1}F_{n-2} \ldots F_2F_1$$

where F_i, for $i = 1, 2, \ldots, n-1$, are given by expressions (4), (5) and (6).

Since A satisfies the WRC condition then, the unit upper echelon TN matrix U also satisfies this condition. Then it can be factorized, without interchange of its columns, in the following form

$$U = [I_{n\times n}\ O_{n\times(m-n)}]G_1G_2 \ldots G_{m-1},$$

where G_i, for $j = 1, 2, \ldots, m-1$, are unit upper bidiagonal TN matrices given by

$$\begin{aligned} G_j &= E^T_{j+n}(\alpha_{n,j+n})E^T_{j+n-1}(\alpha_{n-1,j+n-1}) \ldots E^T_{j+1}(\alpha_{1,j+1}),\ \ j = 1, 2, \ldots, m-n \\ G_j &= E^T_{m}(\alpha_{m-j,m})E^T_{m-1}(\alpha_{m-j-1,m-1}) \ldots E^T_{j+1}(\alpha_{1,j+1}),\ \ j = m-n+1, \ldots, m-1 \end{aligned}$$

with $\alpha_{1,2} > 0, \ldots, \alpha_{1,n} > 0$ and if $\alpha_{s,t} = 0$ then $\alpha_{s,h} = 0$, $\forall\, h > t$. Therefore, A admits the unique quasi-bidiagonal factorization

$$A = PF_{n-1}F_{n-2} \ldots F_1[D\ O_{n\times(m-n)}]G_1G_2 \ldots G_{m-1}.$$

Conversely, if A has the quasi-bidiagonal factorization given by (7), and we denote by

$$\begin{aligned} \tilde{L} &= PF_{n-1}F_{n-2} \ldots F_1 \\ U &= [I\ O_{n\times(m-n)}]G_1G_2 \ldots G_{m-1} \end{aligned}$$

then, $A = \tilde{L}DU$, where $U \in \mathbb{R}^{n\times m}$ is a unit upper echelon TN matrix with positive entries from the leading entry in each row and the n first linearly independent columns, $D = \text{diag}(-d_1, -d_2, d_3, \ldots, d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$ and $\tilde{L} \in \mathbb{R}^{n\times n}$ is a block lower triangular matrix

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

such that,

$$\det \tilde{L}[\alpha|1,2,\ldots,k] \leq 0, \quad \forall \alpha \in \mathcal{Q}_{k,n},\ k = 2,3,\ldots,n.$$

Therefore, by Theorem 3, A is a t.n.p. matrix. $\square$

For rectangular matrices with arbitrary rank the following result is obtained.

Theorem 10. *Let A be an $n\times m$ matrix with negative entries except for $a_{11} = 0$ and $a_{nm} \leq 0$, and rank$(A) = r$. If A satisfies the WRC condition then, A is a t.n.p. matrix if and only if A admits a unique quasi-bidiagonal factorization*

$$A = PF_{n-1}F_{n-2}\ldots F_1 \begin{bmatrix} D & O_{r\times(m-r)} \\ O_{(n-r)\times r} & O_{(n-r)\times(m-r)} \end{bmatrix} G_1G_2\ldots G_{m-1}$$

where $D = diag(-d_1, -d_2, d_3, \ldots, d_r)$, with $d_i > 0$ for $i = 1,2,\ldots,r$ and P is the permutation matrix $P = [2,\ 1,\ 3,\ \ldots,\ n]$. Moreover, F_i, for $i = 3,4,\ldots,n-1$, are unit lower bidiagonal TN matrices given by

$$\begin{aligned} F_i &= E_{i+1}(\beta_{1,i+1})E_{i+2}(\beta_{2,i+2})\ldots E_{i+r}(\beta_{r,i+r}),\ \text{for } i = 3,4\ldots,n-r, \\ F_i &= E_{i+1}(\beta_{1,i+1})E_{i+2}(\beta_{2,i+2})\ldots E_n(\beta_{n-j,n}),\ \text{for } i = n-r+1,\ldots,n-1, \end{aligned}$$

and

$$\begin{aligned} F_2 &= \tilde{E}_3(\beta_{1,3})E_4(\beta_{2,4})\ldots E_r(\beta_{r-2,r}) \\ F_1 &= E_3(-\beta_{2,3})E_4(\beta_{3,4})\ldots E_r(\beta_{r-1,r}) \end{aligned}$$

with $\beta_{1,3} > 0$, $\ldots$, $\beta_{1,n} > 0$ and if $\beta_{s,t} = 0$ then $\beta_{s,h} = 0$, for all $h > t$, and G_j, for $j = 1,2,\ldots,m-1$, are the unit upper bidiagonal TN matrices defined by

$$\begin{aligned} G_j &= E^T_{j+n}(\alpha_{n,j+n})E^T_{j+n-1}(\alpha_{n-1,j+n-1})\ldots E^T_{j+1}(\alpha_{1,j+1}),\ \text{for } j = 1,2,\ldots,m-r, \\ G_j &= E^T_m(\alpha_{m-j,m})E^T_{m-1}(\alpha_{m-j-1,m-1})\ldots E^T_{j+1}(\alpha_{1,j+1}),\ \text{for } j = m-r+1,\ldots,m-1, \end{aligned}$$

with $\alpha_{1,2} > 0$, $\ldots$, $\alpha_{1,m} > 0$ and if $\alpha_{s,t} = 0$ then $\alpha_{s,h} = 0$, for all $h > t$.

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